Question 1 Suppose $A$ is a complex $n \times n$ matrix. Show that the following are equivalent:

(a) The columns of $A$ form an orthonormal basis in $C^n$.

(b) $A^*A = I$.

(c) $\|Ax\| = \|x\|$ for all $x \in C^n$.

Solution: (a) $\iff$ (b) Let $a_j = Ae_j$ be the columns of $A$. Then

$$A^*A = \begin{pmatrix} a_1^* & \ldots & a_n^* \end{pmatrix} \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} = \begin{pmatrix} \langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \ldots \\ \langle a_1, a_2 \rangle & \langle a_2, a_2 \rangle & \ldots \\ \ldots & \ldots & \langle a_n, a_n \rangle \end{pmatrix}$$

is the identity matrix iff the columns of $A$ are orthonormal. Note that our inner products have the complex conjugate on the second factor so $x^*y = \langle x, y \rangle$.

(b) $\implies$ (c) Suppose $A^*A = I$. Then

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle = \langle Ix, x \rangle = \langle x, x \rangle = \|x\|^2.$$

Taking the square root gives $\|Ax\| = \|x\|$ for all $x \in C^n$.

(c) $\implies$ (b) Suppose

$$\langle Ax, Ax \rangle = \langle A^*Ax, x \rangle = \langle x, x \rangle$$

for all $x \in C^n$.

Note that $\langle Bx, x \rangle = 0$ for all $x$ does not imply $B = 0$ unless it is already known that $B$ is selfadjoint. E.g. take

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

with the usual inner product on $R^2$.

Since $A^*A$ is selfadjoint it is unitarily diagonalizable, $A^*A = QAQ^*$ where $QQ^* = I$. Let $y = Q^*x$. Then

$$\langle y, y \rangle = \langle Q^*x, Q^*x \rangle = \langle x, QQ^*x \rangle = \langle x, x \rangle.$$  

Hence

$$\langle y, y \rangle = \langle x, x \rangle = \langle A^*Ax, x \rangle = \langle QAQ^*x, x \rangle = \langle AQ^*x, x \rangle = \langle Ay, y \rangle.$$
Since $Q^*$ is invertible, we can choose any $y \in C^n$. By letting $y = e_j$, we can conclude that

$$1 = \langle y, y \rangle = \langle \Lambda y, y \rangle = \Lambda_j.$$  

Thus $\Lambda = I$ and $A^*A = QQ^* = I$.

Alternate proof: By the parallelogram identity, $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y$. Hence $A^*A = I$. This proof avoids using the big machinery of orthogonal diagonalization and is thus to be preferred over the previous one.

**Question 2** Suppose $A : V \to W$ is a linear map between two inner product spaces. Show that the nullspace of $A$ is exactly the perpendicular complement of the range of $A^*$.

**Solution**: First suppose $x \in \text{Null}(A)$. Then $Ax = 0$. Suppose also that $y = A^*z$, so $y \in \text{Range}(A^*)$. Then

$$\langle x, y \rangle = \langle x, A^*z \rangle = \langle Ax, z \rangle = \langle 0, z \rangle = 0.$$  

Thus $\text{Null}(A) \subset \text{Range}(A^*)\perp$.

Conversely suppose $x \in \text{Range}(A^*)\perp$. That means for any $y = A^*z$, we have $\langle x, y \rangle = 0$. We want to show $Ax = 0$. Consider for an arbitrary $z \in W$ the inner product

$$\langle Ax, z \rangle = \langle x, A^*z \rangle = 0.$$  

Then since $z$ is arbitrary, it must be that $Ax = 0$. So $\text{Range}(A^*)\perp \subset \text{Null}(A)$, and thus $\text{Null}(A) = \text{Range}(A^*)\perp$.

**Question 3** Prove the Fredholm Alternative: Suppose $A : V \to W$ is a linear map between two inner product spaces. Let $b \in W$. Then either

(a) $Ax = b$ for some $x \in V$ or  
(b) There is $w \in W$ with $A^*w = 0$ and $\langle b, w \rangle \neq 0$.

**Solution**: If $Ax = b$ for some $x \in V$, then we’re done, so suppose not. Now suppose for every $w \in \text{Range}(A)^\perp$, we have $\langle w, b \rangle = 0$. Then $b \in (\text{Range}(A)^\perp)^\perp = \text{Range}(A)$. So there must be a (necessarily) nonzero vector $w \in \text{Range}(A)^\perp = \text{Null}(A^*)$ such that $\langle w, b \rangle \neq 0$, which is what we wanted to show.
Question 4 Use the Fredholm Alternative and the Fundamental Theorem of Algebra to prove the existence and uniqueness of polynomial interpolation: given \( n + 1 \) distinct real numbers \( x_0, x_1, \ldots, x_n \) and \( n + 1 \) complex numbers \( f_0, f_1, \ldots, f_n \), there exists a unique degree-\( n \) polynomial \( P(x) = p_0 + p_1 x + \cdots + p_n x^n \) such that \( P(x_j) = f_j \) for \( 0 \leq j \leq n \).

Solution: The problem of interpolation can be cast as trying to solve the system

\[
\begin{align*}
    p_0 + p_1 x_0 + \cdots + p_n x_0^n &= f_0 \\
    \vdots \\
    p_0 + p_1 x_n + \cdots + p_n x_n^n &= f_n
\end{align*}
\]

which we can write as \( Xp = f \). We must have \( \text{Null}(X) = 0 \) since any polynomial of degree \( n \) vanishing at \( n + 1 \) distinct points is 0 by the fundamental theorem of algebra. Since \( \det(X) = \det(X^*) \) we must also have \( \text{Null}(X^*) = 0 \). By the Fredholm alternative, \( \text{Range}(X) = \text{Null}(X^*)^\perp = 0^\perp = C^{n+1} \). Thus we can interpolate any set of coefficients, and can do so uniquely since \( \text{Null}(X) = 0 \).

Note that the FTA is actually unnecessary since a difference of two interpolants \( q = p_1 - p_2 \) has \( n + 1 \) real zeroes. By Rolle’s Theorem, its first derivative \( q' \) therefore has \( n \) real zeroes, and similarly its \( n \)th derivative is a constant with 1 zero. Since the highest-order term is therefore 0, \( q \) has degree \( \leq n - 1 \). Repeating the argument \( n - 1 \) times shows \( q \) is zero.

Question 5 Prove that a projection \( P \) on an inner product space is an orthogonal projection if and only if \( P^* = P \).

Solution: \( P \) being an orthogonal projection means for all \( x, y \in V \)

\[
\langle Px - x, Py \rangle = \langle P^*(Px - x), y \rangle = 0
\]

which is true iff (since \( x \in V \) and \( y \in V \) and arbitrary)

\[
P^*(P - I) = 0.
\]

Expanding gives that this is equivalent to

\[
P^*P = P^*.
\]
Taking adjoints gives
\[ P^* P = P. \]
Thus this is equivalent to
\[ P = P^* P = P^*. \]

**Question 6** Let
\[ K_t(x) = \frac{t}{\pi(t^2 + x^2)} \]
for \( t > 0 \) and \( x \in \mathbb{R} \).

**(a)** Evaluate
\[ \int_{-\infty}^{\infty} K_t(x - y) \, dy. \]

**Solution**: This integral is
\[ \int_{-\infty}^{\infty} K_t(x - y) \, dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + (y - x)^2} \, dy \]
since \((x - y)^2 = (y - x)^2\). Now we change our variable to \( z = y - x \), so \( dx = dy \). Then we have
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t(1 + \frac{z^2}{t^2})} \, dz = \frac{1}{\pi t} \int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{z}{t}\right)^2} \, dz = \frac{1}{\pi} \arctan\left(\frac{z}{t}\right)\bigg|_{-\infty}^{\infty} = \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1. \]

**(b)** Use the Dominated Convergence Theorem to show that
\[ \int_{-\infty}^{\infty} K_t(x - y)f(y) \, dy \rightarrow f(x) \]
as \( t \rightarrow 0 \), for all bounded continuous functions \( f \).

**Solution**: We make the change of variables \( y = x + ts \), so \( dy = t \, ds \) to transform the integral as follows
\[ \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2 + (x - y)^2} f(y) \, dy = \frac{t^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2 + s^2} f(x + ts) \, ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} f(x + ts) \, ds. \]
Now since \( f \) is bounded, we have
\[ \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} f(x + ts) \right| \leq \frac{M}{1 + s^2} \]
which is integrable. Since \( f \) is continuous,

\[
\frac{1}{\pi} \frac{1}{1 + s^2} f(x + ts) \to \frac{1}{\pi} \frac{1}{1 + s^2} f(x)
\]

for each \( s \) as \( t \to 0 \). Thus the dominated convergence theorem gives

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} f(x + ts) ds \to \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} f(x) ds = \frac{\pi}{\pi} f(x) = f(x).
\]

**Question 7** Show that

\[
\int_{-\infty}^{\infty} \frac{e^{-|x-y|/t}}{2t} f(y) dy \to f(x)
\]

as \( t \to 0 \), for all bounded continuous functions \( f \).

**Solution**: We proceed as in the previous problem. First let \( y = x + ts \). Then our integral becomes

\[
\int_{-\infty}^{\infty} \frac{e^{-|ts|/t}}{2t} f(x + ts) t ds = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|s|} f(x + ts) ds.
\]

Since

\[
e^{-|s|} f(x + ts) \leq Me^{-|s|}
\]

we can apply the dominated convergence theorem to get that

\[
\frac{1}{2} \int_{-\infty}^{\infty} e^{-|s|} f(x + ts) ds \to \frac{1}{2} \int_{-\infty}^{\infty} e^{-|s|} f(x) ds = \frac{1}{2} 2f(x) = f(x)
\]