Question 1  (BN 5.8) For \( j \in \mathbb{Z} \), let \( V_j \) be the space of all finite energy signals \( f \) for which the Fourier transform \( \hat{f}(k) \) vanishes outside of the interval \([-2^j \pi, 2^j \pi]\).

(a) Show that \( \{V_j\}_{j \in \mathbb{Z}} \) satisfies properties 1–4 in the definition of a multiresolution analysis, Definition 5.1.

(b) Show that \( \phi(x) = \text{sinc}(x) \) defined in Eq. 5.1 satisfies property 5 of Definition 5.1, and is thus a scaling relation for the \( V_j \)'s.

(c) Show that \( \phi \) satisfies the scaling relation

\[
\phi(x) = \phi(2x) + 2 \sum_{k \in \mathbb{Z}} (-1)^{k} \frac{1}{(2k+1)\pi} \phi(2x - 2k - 1).
\]

(d) Find an expansion for the wavelet \( \psi \) associated with \( \phi \).

(e) Find the high- and low-pass decomposition filters \( h \) and \( l \).

(f) Find the high- and low-pass reconstruction filters \( \tilde{h} \) and \( \tilde{l} \).

Question 2  (BN 4.9) For \( j \in \mathbb{Z} \) let \( V_j \) be the space of all finite energy signals \( f \) that are continuous and piecewise linear, with possible corners occurring only at the points \( k/2^j \) for \( k \in \mathbb{Z} \).

(a) Show that \( \{V_j\}_{j \in \mathbb{Z}} \) satisfies properties 1–4 in the definition of a multiresolution analysis, Definition 5.1.

(b) Let \( \phi(x) \) be the tent function \( \text{max}(0, 1-|x|) \). Show that \( \{\phi(x-k)\}_{k \in \mathbb{Z}} \) is a (nonorthogonal) basis for \( V_j \). Find the scaling relation for \( \phi \).

Question 3  (BN 5.10) Let \( \phi(x) \) be the tent function \( \text{max}(0, 1-|x|) \).

(a) Compute the Fourier transform \( \hat{\phi}(\xi) \) of \( \phi \).

(b) Show that

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(\xi + 2\pi k)^4} = \frac{3 - 2\sin^2(\xi/2)}{48\sin^4(\xi/2)}
\]

(Hint: Differentiate Eq. 5.28 twice and simplify).

(c) Verify that the function \( g \) with Fourier transform

\[
\hat{g}(\xi) = 2\sqrt{2/\pi} \frac{\sin^2(\xi/2)}{\xi^2 \sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}
\]
is such that \( \{g(x - k)\}_{k \in \mathbb{Z}} \) is an orthonormal set. (Hint: Use (b) to show that

\[
\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2\pi k)|^2 = \frac{1}{2\pi}
\]

and apply Theorem 5.18.