Question 1  (BN 4.8) Let \( n \) be a positive integer, and let \( f \) be a continuous function defined on \([0, 1]\). Let \( h_k(t) = \sqrt{n} \phi(nt - k) \) where \( \phi(t) \) is the Haar scaling function (which is 1 on the interval \([0, 1]\) and 0 elsewhere). Form the \( L^2 \) projection of \( f \) onto the span of the \( h_k \)'s,

\[
f_n = <f, h_0> h_0 + \cdots + <f, h_{n-1}> h_{n-1}.
\]

(a) Show that \( f_n \) converges uniformly to \( f \) on \([0, 1]\).

(b) For \( f(t) = 1 - t^2 \), compute the Haar wavelet decomposition on \([0, 1]\) for \( n = 4, 8 \) and \( 16 \). Plot the results.

Question 2  (BN 4.11) Fix \( a \) with \( 0 < a < 1 \). Let \( g(t) = 1 - t^2 \) for \( a \leq t < 1 \) and 0 otherwise.

(a) Show that the Haar wavelet approximation \( g_n \) does not converge uniformly to \( g \) as \( n \to \infty \).

(b) Discretize \( g \) over \([0, 1]\) with \( n = 7 \). Plot the magnitudes of the level-6 wavelet coefficients. Estimate the discontinuity for \( a = 7/17, a = 8/9 \) and \( a = 2/7 \). Explain.

Question 3  Let

\[
H^1(R) = \{ f \in L^2(R) | f' \in L^2(R) \}
\]

with inner product

\[
<f, g> = \int_{-\infty}^{\infty} f(x)g(x) + f'(x)g'(x)dx.
\]

(a) Fix \( x_0 \in R \). Find \( g \in H^1 \) such that

\[
<f, g> = f(x_0)
\]

for all \( f \in H^1 \).

(b) Fix \( h > 0 \) and find \( g_j \in H^1 \) such that

\[
<f, g_j> = f(jh)
\]

for \( j \in \mathbb{Z} \) and all \( f \in H^1 \).

(c) Evaluate the Gram matrix elements \(<g_i, g_j>\).

(d) Given the values \( f_j = f(jh) \) of a function \( f \in H^1 \), design a scheme for computing the best approximation of \( f \) in the \( H^1 \) norm by exponentials
\[ a_je^x + b_je^{-x} \] on each interval \(jh \leq x \leq (j+1)h\). (Hint: Use (a-c) and sum up coefficients from left and right intervals.)

(e) Implement your scheme and test it on \(f(x) = e^{-x^2}\cos(10x)\), using \(N = 2, 4, 8, 16, \ldots, 1024\) equidistant samples on \([-\pi, \pi]\). Measure the errors in the \(H^1\) norm as \(N\) varies.