Question 1 (a) Show that the Hermite polynomial \( H_n(x) \) satisfies
\[
H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (x - ik)^n \, dk.
\]
(b) Show that
\[
|(x + ik)^n| \leq 2^n (|x|^n + |k|^n).
\]
(c) Use Stirling’s approximation \( n! \approx (n/e)^n \) to show
\[
\frac{|h_n(x)|}{\|h_n\|} \leq 2^{n+1} e^{-x^2/2} + 2 \left( \frac{8e}{n} \right)^{n/2} e^{-x^2/2} |x|^n.
\]
(d) Show that
\[
\frac{|h_n(x)|}{\|h_n\|} \leq O(2^{-n})
\]
for \(|x| \geq \sqrt{2\pi n}\).
(e) Explain why scaled Hermite functions \( h_0(cx), h_1(cx), \ldots, h_n(cx) \) might form a suitable basis for approximating functions \( f \in L^2 \) which are approximately band- and time-limited in the sense that
\[
\int_{|x| \geq T} |f(x)|^2 \, dx \leq \epsilon^2 \|f\|^2
\]
and
\[
\int_{|k| \geq K} |\hat{f}(k)|^2 \, dk \leq \epsilon^2 \|\hat{f}\|^2.
\]
How should \( n \) and \( c \) relate to \( K \) and \( T \)?

Question 2 (a) Show that
\[
FDf(k) = \hat{f}'(k) = ik\hat{f}(k) = ikFf(k)
\]
and
\[
F(xf)(k) = \hat{xf}(k) = i\hat{f}'(k) = iDFf(k).
\]
(b) Show that the differential operator
\[
D_{ab}f(x) = (a^2 - x^2)f'(x) - b^2 x^2 f(x)
\]
satisfies
\[
FD_{ab} = D_{ba}F.
\]
(c) Show that $D_{ab}$ commutes with the orthogonal projection onto time-limited functions

$$P_a f(t) = f(t)$$

for $|t| \leq a$ and

$$P_a f(t) = 0$$

for $|t| > a$.

(d) Use (b) and (c) to show that $D_{ab}$ commutes with the integral operator

$$S_{ab} f(t) = P_a Q_b P_a f(t) = \frac{1}{\pi} \int_{-a}^{a} \frac{\sin(b(t - s))}{t - s} f(s) \, ds$$

where $Q_b = F^* P_b F$ is the orthogonal projection onto bandlimited functions.

(e) Explain why the eigenfunctions of $D_{ab}$ might be useful in representing approximately time- and band-limited functions.

**Question 3** Solve the integral equation

$$D^{-1/2} h(t) = \int_0^t \frac{1}{\pi(t-s)} h(s) \, ds = g(t)$$

where $g$ is a nice function with $g(0) = 0$. (Hint: Square $D^{-1/2}$.)

**Question 4** (a) Solve the initial-boundary value problem for the heat equation

$$u_t = u_{xx}$$

for $x > 0$, $t > 0$, with homogeneous initial conditions

$$u(x, 0) = 0$$

and boundary conditions

$$u(0, t) = g(t)$$

where $g$ is a nice function with $g(0) = 0$. (Hint: Try $u(x, t) = \int_0^t K_{t-s}(x) h(s) \, ds$ and solve an integral equation for $h$.)

(b) Assume that $g' \in L^2(R)$ is also bounded and continuous. Argue directly from the heat equation that if

$$u_x(x, t) \to \Lambda g(t)$$

as $x \to 0$, then the Dirichlet-Neumann operator $\Lambda$ must satisfy

$$\Lambda^2 g(t) = g'(t).$$

(c) Find the Dirichlet-Neumann operator $\Lambda$.  

2
**Question 5**  Use Fourier transform in the variable $x$ to solve the problem of Question 4. (Hint: Extend $u$ discontinuously to be zero for negative $x$. Integrate by parts. Use parity.)