Question 1: Suppose you can only afford to evaluate 11 terms of either side of the PSF

\[
\frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} e^{-(x-2\pi k)^2/4t} = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-tk^2} e^{ikx}.
\]

Find \( \delta \) such that the error in the right-hand side (truncated after 11 terms) is smaller than \( 10^{-14} \) for \( t \geq \delta \) and \( |x| \leq \pi \) and the relative error in the left hand side (truncated after 11 terms) is smaller than \( 10^{-14} \) for \( 0 < t \leq \delta \) and \( |x| \leq \pi \).

Question 2: Use the PSF to prove the Euler-Maclaurin summation formula

\[
\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_{0}^{\infty} f(x) \, dx - \frac{1}{12} f'(0) + \frac{1}{720} f''(0) - \cdots
\]

for a smooth function \( f \). Find formulas for the rest of the coefficients \( B_{2k} \) in

\[
\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_{0}^{\infty} f(x) \, dx - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0)
\]

by applying the formula to a suitable test function like \( f(x) = e^{-tx} \).

Question 3: In Problem Set 08, you proved that the Hermite functions \( h_n \) were eigenfunctions of the Fourier transform. Apply the PSF to \( h_n \) and choose parameters \( x \) and \( T \) to find formulas for eigenvectors \( f_p \in C^N \) and eigenvalues \( \lambda_p \in C \) of the \( N \times N \) discrete Fourier transform matrix \( F \) with elements

\[
F_{jk} = \frac{1}{\sqrt{N}} e^{-2\pi ijk/N}
\]

for \( 0 \leq j,k \leq N-1 \). Are all the vectors \( f_p \) orthogonal to each other? Why or why not?

Question 4: (a) Use the obvious identity

\[
\frac{1}{x} = \int_{0}^{\infty} e^{-tx} \, dx
\]

to evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx.
\]

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(b) Use the double angle formula and integrate by parts to evaluate the integral
\[ \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx. \]
(c) Prove that both of these integrals converge as improper Riemann integrals (albeit for different reasons). (d) Use scaling to evaluate the integral
\[ \int_{-\infty}^{\infty} \frac{\sin tx}{x} \, dx \]
for \( t \in \mathbb{R} \).