Question 1: Show that $K = D^2 - x^2$ is a symmetric operator on $L^2(R)$: for nice smooth functions $f, g \in L^2(R)$ we have
\[ \int_{-\infty}^{\infty} f(x)Kg(x)^*dx = \langle f, Kg \rangle = \langle Kf, g \rangle. \]

Question 2: Show that $\|h_n\|^2 = \frac{\sqrt{\pi}n!}{2^n}$. (Hint: Square the expansion $\sum_{n=0}^{\infty} y^n h_n(x) = e^{x^2/2}e^{-(x-y)^2}$ and integrate.)

Question 3: Calculate the first three Hermite polynomials and use them to compute
\[ \int_{-\infty}^{\infty} x^2 e^{-x^2}dx. \]

Question 4: (a) Show that $-\langle Kf, f \rangle = \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2dx = \sum_{n=0}^{\infty} (2n + 1) \frac{\langle f, h_n \rangle^2}{\|h_n\|^2}$ for real-valued $f \in L^2(R)$. (b) Prove the weak Heisenberg inequality
\[ \int_{-\infty}^{\infty} f'(x)^2 + x^2 f(x)^2dx \geq \int_{-\infty}^{\infty} f(x)^2dx \]
for such $f$.

Question 5: Show that
\[ e^{2it\sqrt{s}} = e^{-t^2} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H_n(s) \]
(Hint: Seek an expansion of the form $e^{2it\sqrt{s}} = \sum_{n=0}^{\infty} f_n(t)H_n(s)$ and use orthogonality of the $H_n$'s.)
**Question 6:** Use Cramer’s inequality

\[ |H_n(s)| \leq 1.09 \, 2^{n/2} \sqrt{n!e^{s^2/2}} \]

and Stirling’s approximation to show that the error in \( N \) terms of the approximation in Question 5 is bounded by

\[ |e^{2its} - \sum_{n=0}^{N-1} f_n(t) H_n(s)| \leq 10 \left( \frac{2e}{N} \right)^{N/2} \]

for \( N > 10 \), \(|t| \leq 1\), and \(|s| \leq 2\). How many terms are required to get 10-digit accuracy?