Question 1  Prove the Weierstrass Approximation Theorem: Every continuous function \( f : [-1, 1] \to \mathbb{C} \) can be uniformly approximated by polynomials. I.e. given \( \epsilon > 0 \), there exists a degree \( n \geq 0 \) and a degree-\( n \) polynomial \( p(x) = p_0 + p_1 x + \cdots + p_n x^n \) such that
\[
|f(x) - p(x)| \leq \epsilon \quad \text{for } |x| \leq 1.
\]
(a) Define
\[
g(t) = f(\cos t) \quad \text{for } |t| \leq \pi.
\]
Show that \( g \) is even, periodic and continuous for \( |t| \leq \pi \).
(b) Find a sequence of even trigonometric polynomials
\[
q_n(t) = \sum_{|k| \leq n} q_{nk} \cos(kt)
\]
converging uniformly to \( g \) as \( n \to \infty \).
(c) Prove by induction that
\[
T_n(x) = \cos(nt)
\]
is a polynomial in the variable \( x = \cos t \).
(d) Prove the Weierstrass Approximation Theorem.
(e) Find the approximating polynomials explicitly for \( f(x) = |x| \).

Question 2  Solve the classical moment problem: is every continuous function \( f : [-1, 1] \to \mathbb{C} \) uniquely determined by the sequence \( \{m_0, m_1, \ldots\} \) of its moments
\[
m_k = \int_{-1}^{1} x^k f(x) \, dx?
\]

Question 3  (a) Compute all the moments \( m_k \) over \([0, \infty)\)
\[
m_k = \int_{0}^{\infty} x^k f(x) \, dx
\]
for \( f(x) = \exp(-x^{1/4}) \sin(x^{1/4}) \).
(b) Discuss in view of your answer to Question 2.
**Question 4**  
(a) Compute the coefficients $\hat{f}(k)$ of the Fourier sine series

$$\sum_{k=1}^{\infty} \hat{f}(k) \sin kx$$

over the interval $|x| \leq \pi$ for the odd function $f(x) = \frac{1}{2}\text{sign}(x)$.

(b) Find an explicit formula for the first critical point $x_N > 0$ of the partial sum error

$$g_N(x) = \sum_{k=1}^{N} \hat{f}(k) \sin kx - \frac{1}{2}.$$  

(I.e. find the smallest positive solution $x_N$ of the equation $g_N'(x) = 0$.)

(c) Evaluate the limiting overshoot

$$\lim_{N \to \infty} g_N(x_N)$$

(d) Explain Gibbs’ phenomenon quantitatively.