Question 1  (a) Compute the complex exponential Fourier coefficients \( \hat{f}(k) \) of 
\[ f(x) = e^{rx} \]
for the interval \( |x| \leq \pi \).
(b) For the case \( r = 1/2 \) plot partial sums versus \( f \) for \( N = 10, 20, 30 \) on the larger interval \( |x| \leq 2\pi \). Explain the regions of your plot where convergence appears to be fast versus slow.

Question 2  (a) Compute the complex exponential Fourier coefficients \( \hat{f}(k) \) of 
\[ f(x) = \pi^2 - x^2 \]
for the interval \( |x| \leq \pi \).
(b) Show that the Fourier series converges uniformly for \( |x| \leq \pi \).
(c) Evaluate
\[ \sum_{n=1}^{\infty} \frac{1}{n^2}. \]
(d) Evaluate
\[ \sum_{n=1}^{\infty} \frac{1}{n^4}. \]

Question 3  (a) Solve the heat equation 
\[ u_t = u_{xx} \]
for \(-\pi \leq x \leq \pi\) with periodic boundary conditions \( u(-\pi, t) = u(\pi, t) \) and initial condition \( u(x, 0) = \pi^2 - x^2 \).
(b) Express the solution as an integral operator 
\[ u(x, t) = \int_{-\pi}^{\pi} K_t(x,y)u(y,0) \, dy \]
and find the kernel \( K_t(x,y) \).

Question 4  Let \(-\pi < a < b < \pi \) and \( Q(x) \) be a polynomial of degree \( d \). Evaluate the complex exponential Fourier coefficients of \( f(x) = Q(x) \) for \( a < x < b \) and \( f(x) = 0 \) otherwise.
Question 5  (a) Compute the complex exponential Fourier coefficient
\[
\hat{\varphi}_j(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_j(x) e^{-\pi i k x} dx
\]
over the interval $[-1, 1]$ of the four functions $\varphi_j$ defined in Question 5 of Problem Set 02.

(b) Explain the relations between the four sequences $\hat{\varphi}_j(k)$ in terms of the scaling and shifting relations between the functions $\varphi_j$.

(c) Express the projection $P$ from Question 5 of Problem Set 02 in the form
\[
Pf(x) = \sum_{-\infty}^{\infty} \hat{P}(x, k) \hat{f}(k)
\]
and find the coefficient functions $\hat{P}(x, k)$.

Question 6  (a) Let $f$ and $g$ be $2\pi$-periodic piecewise smooth functions such that
\[
f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k) e^{i k x}
\]
and
\[
g(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k) e^{i k x}.
\]
Define $h = f \ast g$ by
\[
h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k) \hat{g}(k) e^{i k x}.
\]
Express $\hat{f}$ and $\hat{g}$ as integrals, combine them, and reverse the order of integration and summation to obtain an integral formula for $h$ in terms of $f$ and $g$.

(b) Let $g \in L^2(-\pi, \pi)$ have complex exponential Fourier coefficients $\hat{g}(k)$. Show that (cf. https://arxiv.org/abs/0806.0150)
\[
\sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k)
\]
if and only if
\[
g(0) = \frac{1}{2a} \int_{-a}^{a} g(y) \, dy.
\]
Note that \( \frac{\sin(ka)}{ka} \to 1 \) as \( a \to 0 \).

(c) Show that

\[
\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \frac{\pi - 1}{2}.
\]