Question 1  (a) Compute the complex exponential Fourier coefficients \( \hat{f}(k) \) of 
\[ f(x) = e^{rx} \]
for the interval \( |x| \leq \pi \).

(b) For the case \( r = 1/2 \) plot partial sums versus \( f \) for \( N = 10, 20, 30 \) on the larger interval \( |x| \leq 2\pi \). Explain the regions of your plot where convergence appears to be fast versus slow.

Question 2  (a) Compute the complex exponential Fourier coefficients \( \hat{f}(k) \) of 
\[ f(x) = \pi^2 - x^2 \]
for the interval \( |x| \leq \pi \).

(b) Show that the Fourier series converges uniformly for \( |x| \leq \pi \).

(c) Evaluate 
\[ \sum_{n=1}^{\infty} \frac{1}{n^2}. \]

(d) Evaluate 
\[ \sum_{n=1}^{\infty} \frac{1}{n^4}. \]

Question 3  (a) Solve the heat equation 
\[ u_t = u_{xx} \]
for \(-\pi \leq x \leq \pi \) with periodic boundary conditions \( u(-\pi, t) = u(\pi, t) \) and initial condition \( u(x, 0) = \pi^2 - x^2 \).

(b) Express the solution as an integral operator 
\[ u(x, t) = \int_{-\pi}^{\pi} K_t(x, y)u(y, 0) \, dy \]
and find the kernel \( K_t(x, y) \).

Question 4  Let \(-\pi < a < b < \pi \) and \( Q(x) \) be a polynomial of degree \( d \). Evaluate the complex exponential Fourier coefficients of \( f(x) = Q(x) \) for 
\( a < x < b \) and \( f(x) = 0 \) otherwise.
**Question 5**  (a) Compute the complex exponential Fourier coefficient

\[ \hat{\varphi}_j(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_j(x)e^{-\pi ikx} \, dx \]

over the interval \([-1, 1]\) of the four functions \(\varphi_j\) defined in Question 5 of Problem Set 02.

(b) Explain the relations between the four sequences \(\hat{\varphi}_j(k)\) in terms of the scaling and shifting relations between the functions \(\varphi_j\).

(c) Express the projection \(P\) from Question 5 of Problem Set 02 in the form

\[ Pf(x) = \sum_{-\infty}^{\infty} \hat{P}(x,k) \hat{f}(k) \]

and find the coefficient functions \(\hat{P}(x,k)\).

**Question 6**  (a) Let \(f\) and \(g\) be \(2\pi\)-periodic piecewise smooth functions such that

\[ f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)e^{ikx} \]

and

\[ g(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k)e^{ikx}. \]

Define \(h = f \ast g\) by

\[ h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)e^{ikx}. \]

Express \(\hat{f}\) and \(\hat{g}\) as integrals, combine them, and reverse the order of integration and summation to obtain an integral formula for \(h\) in terms of \(f\) and \(g\).

(b) Let \(g \in L^2(-\pi, \pi)\) have complex exponential Fourier coefficients \(\hat{g}(k)\). Show that (cf. https://arxiv.org/abs/0806.0150)

\[ \sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k) \]

if and only if

\[ g(0) = \frac{1}{2a} \int_{-a}^{a} g(y) \, dy. \]
Note that \( \frac{\sin(ka)}{ka} \to 1 \) as \( a \to 0 \).

(c) Show that

\[
\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \frac{\pi - 1}{2}.
\]