Question 1  Use Gram-Schmidt orthogonalization or $QR$ factorization to find an orthonormal basis for the span of $\{e^x, e^{\omega x}, e^{\omega^2 x}, e^{\omega^3 x}\}$ in $L^2(-\pi, \pi)$ with inner product

$$<f, g> = \int_{-\pi}^{\pi} f(x)\bar{g}(x) \, dx$$

and $\omega = (-1 + i)/\sqrt{2}$.

Question 2  (a) Find the orthogonal projection $Pf(x)$ of

$$f(x) = x^2 e^{-ix/2}$$

onto the subspace of Question 1.

(b) Express $P$ in the form of an integral operator

$$Pf(x) = \int_{-\pi}^{\pi} K(x, y) f(y) \, dy$$

and find the kernel $K(x, y)$.

Question 3  Let $D$ be the unit disk in $C$,

$$L^2(D) = \{f : D \to C | \int \int_D |f(x, y)|^2 \, dx \, dy < \infty\},$$

and

$$<f, g> = \int \int_D f(x, y)\bar{g}(x, y) \, dx \, dy.$$ 

(a) Show that

$$\varphi_n(x, y) = (x + iy)^n = r^n e^{in\theta}$$

for $n \in \mathbb{N}$ is an orthogonal set in $L^2(D)$.

(b) Normalize them.

(c) Project

$$f(x, y) = \sqrt{x + iy} = r^{1/2} e^{i\theta/2}$$

onto the span of $\{\varphi_0, \ldots, \varphi_N\}$.

(d) Evaluate

$$\sum_{n=0}^{N} (n + 1)\xi^n$$

for $\xi \in C$ and integers $N \geq 0$. 

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(e) Express the orthogonal projection \( P \) onto the span of \( \{ \varphi_0, \ldots, \varphi_N \} \) as an integral operator

\[
P f(z) = \int \int_D K(z, z') f(z') dx' dy', \quad z = x + iy
\]

and find an explicit formula (i.e. not involving sums from 0 to \( N \)) for the kernel \( K \).

**Question 4** Find a sequence \( f_n \in L^2(0, 1) \) such that \( f_n \to 0 \) in \( L^2(0, 1) \) but not uniformly on \([0, 1]\).

**Question 5** Let \( \varphi_j(x) = 0 \) for all \( j \) whenever \( |x| \geq 1 \) and set

\[
\begin{align*}
\varphi_0(x) &= 1 \\
\varphi_1(x) &= \text{sign}(x) \\
\varphi_2(x) &= \varphi_1(2x - 1) \\
\varphi_3(x) &= \varphi_1(2x + 1).
\end{align*}
\]

(a) Sketch \( \varphi_j \) for \( 0 \leq j \leq 3 \).
(b) Show that these functions are orthogonal in \( L^2(-1, 1) \).
(c) Normalize them.
(d) Compute the orthogonal projection \( Pf \) of \( f(x) = x \) onto the span of \( \{ \varphi_j \mid 0 \leq j \leq 3 \} \).
(e) Express \( P \) in the form of an integral operator

\[
P f(x) = \int_{-1}^1 K(x, y) f(y) \, dy
\]

(f) Sketch the kernel \( K(x, y) \).

**Question 6** Suppose \( f \in L^2(0, 1) \) is differentiable and \( f \) is orthogonal to \( g(x) = e^x + 1 - e \).

(a) Show that \( f' \) is orthogonal to \( G(x) = e^x - 1 - (e - 1)x \).
(b) Explain why.