Definition  A set of real numbers has measure 0 if it can be enclosed in a collection of intervals \([a_n, b_n]\) with arbitrarily small total length \(\sum_0^\infty b_n - a_n\).

Example  The rational numbers \(r_n = p_n/q_n\) in the interval \([0, 1]\) have measure zero, because rational \(r_n\) can be enclosed in an interval of length \(\epsilon 2^{-n}\) and the total length of these intervals is \(2\epsilon\).

Definition  A property of real numbers holds almost everywhere if the set of real numbers where it does not hold has measure zero.

Example  Almost every real number between 0 and 1 is irrational.

Theorem 1  Suppose a sequence of functions \(f_n\) converges to a function \(f\) almost everywhere and is dominated
\[
|f_n(x)| \leq g(x)
\]
for all \(x\) by a function \(g\) with
\[
\int g(x) \, dx < \infty.
\]
Then
\[
\int f_n(x) \, dx \to \int f(x) \, dx
\]
as \(n \to \infty\).

Example 1:  Since
\[
\int_0^1 1 \, dx < \infty,
\]
any sequence \(f_n(x)\) on \(0 \leq x \leq 1\) which converges to \(f\) almost everywhere and is uniformly bounded by 1 satisfies
\[
\int_0^1 f_n(x) \, dx \to \int_0^1 f(x) \, dx.
\]
For example, \(x^n \to 0\) except for \(x = 0\), so
\[
\int_0^1 x^n \, dx = \frac{1}{n+1} \to 0.
\]
But \(f_n(x) = n\) for \(0 \leq x \leq 1/n\) and 0 otherwise is not dominated by any integrable \(g\), so we cannot conclude that \(1 \to 0\).
Math 118, Handout 0: Dominated convergence and applications

Example 2: For $\epsilon > 0$ let

$$G_\epsilon(x) = \frac{1}{\sqrt{\pi \epsilon^2}} e^{-x^2/\epsilon^2}$$

be the Gauss kernel with scale $\epsilon$. Use $G_\epsilon$ to average a continuous bounded $f$:

$$f_\epsilon(x) = \int_{-\infty}^{\infty} G_\epsilon(x-y) f(y) \, dy.$$  

Changing variables gives

$$f_\epsilon(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(x+\epsilon t) \, dt.$$  

As $\epsilon \to 0$, each value of the integrand converges:

$$e^{-t^2} f(x+\epsilon t) \to e^{-t^2} f(x),$$

and it is dominated by $e^{-t^2}$ times that maximum of $|f|$. By Dominated Convergence,

$$f_\epsilon(x) \to f(x)$$

for every $x$.

Theorem 2 If

$$\int |g(x)| \, dx < \infty$$

and

$$\int g(x) \, dx = 1$$

then for every continuous $f : \mathbb{R} \to \mathbb{C}$ with $|f(x)| \leq M$ for all $x$, we have

$$\int \frac{1}{\epsilon} g \left( \frac{x-y}{\epsilon} \right) f(y) \, dy \to f(x)$$

for every $x$ as $\epsilon \to 0$.  