(1a) Find an orthonormal basis $e_1, e_2$ for the range of the matrix

$$A = \begin{bmatrix}
-2 & 2 & 0 \\
2 & 7 & 3 \\
1 & 8 & 3 \\
\end{bmatrix} = \begin{bmatrix} a_1 | a_2 | a_3 \end{bmatrix}$$
(1b) Find the $3 \times 3$ matrix $P$ which projects orthonormally onto the range of $A$. 
(1c) Find the closest point $y$ in the range of $A$ to

$$b = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}.$$
(2a) Let $u(x,t)$ be the solution of the wave equation

$$u_t = u_x$$

which is $2\pi$-periodic in $x$ and satisfies the initial condition $u(x,0) = g(x)$ where $g \in L^2(-\pi,\pi)$. Find the complex Fourier coefficients $\hat{u}(k,t)$ in terms of $\hat{g}$. 
(2b) Show that

$$\int_{-\pi}^{\pi} |u(x, t)|^2 dx = \int_{-\pi}^{\pi} |g(x)|^2 dx$$

for all $t \geq 0$. 
(2c) Sum the Fourier series to express $u(x, t)$ directly in terms of $g$. 
(2d) Show that $u$ is $2\pi$-periodic in $t$:

$$u(x, t + 2\pi) = u(x, t)$$

for all $t \geq 0$. 
(3a) Compute the complex Fourier coefficients on the interval $-\pi < x < \pi$ of the function $f(x) = x(\pi^2 - x^2)$. (Hint: $f(x)e^{-ikx} = (iD)(\pi^2 + D^2)e^{-ikx}$ where $D = d/dk$ is independent of $x$.)
(3a continued)
(3a continued)
(3b) Show that

\[ S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}. \]
(3c) State a theorem justifying (3b) and verify its hypotheses on $f(x) = x(\pi^2 - x^2)$. 