Math 118  Solutions 05  Spring 2016

1. (a) Since \( \cos(-t) = \cos(t) \),
g is even. Since \( \cos(t+2\pi) = \cos(t) \),
g is \( 2\pi \)-periodic. Since \( f \) is continuous and \( \cos \) is continuous,
g is the composition of continuous functions and therefore continuous.

(b) By Fejer's theorem, the averaged Fourier sums
\[
\varphi_n(f)(t) = \frac{1}{n+1} \sum_{k=-n}^{n} \hat{f}(k) \frac{\sin((n+1)k)}{(n+1)k} \sum_{j=-n}^{n} \hat{g}(j) e^{ikj}
\]
converge uniformly to \( g \) on \([-\pi, \pi]\).

\[
\sigma_n g(t) = \sum_{1k|\leq n} \left( \frac{n+1-1k}{n+1} \hat{g}(k) \right) e^{ikt}
\]
is an trigonometric polynomial with
\[
\hat{g}(n) = \frac{1}{\sqrt{2\pi}} \frac{n+1-1k}{n+1} \hat{g}(\pm k).
\]

(c) Since \( \cos(a+b) = \cos a \cos b + \sin a \sin b \),
addition gives
\[
\cos(a+b) + \cos(a-b) = 2 \cos a \cos b.
\]
Putting \( a = (n-1)t \) and \( b = t \) gives
\[ T_n(x) + T_{n-2}(x) = 2xT_{n-1}(x). \]

Since \( T_0(x) = 1 \) and \( T_1(x) = x \) are polynomials in \( x \), so are all the \( T_n \)'s, and

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \]

Since the coefficients of this two-term recurrence are independent of \( n \), it can be solved explicitly. The characteristic equation has roots

\[ r^2 - 2xr + 1 = 0 \]

has roots

\[ r = x \pm \sqrt{x^2 - 1} = r_+ \]

so

\[ T_n(x) = A r_+^n + B r_-^n \]

where \( A \) and \( B \) are determined so that \( T_0 = 1, \ T_1 = x \).

Hence \( A = B = \frac{1}{2} \) and

\[ T_n(x) = \frac{1}{2} \left[ (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right] \]

\[ = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \frac{n-k}{2} \]

\[ = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left( x - 1 \right)^{\frac{n-k}{2}} \]
is an even trigonometric polynomial.

\[ \int_{-1}^{1} (x^2 - 1) \sin(\pi x) \, dx = 0 \]

So for even values of \( n \), choose \( m = 0 \).

The polynomial is a polynomial of degree \( n \).

in which the triangle functions \( T(x) \).
2. Let $f$ and $g$ be continuous functions on $[-1, 1]$ with the same moments. Then $h(x) = f(x) - g(x)$ has all moments equal to zero. If

$$p(x) = \sum_{j=0}^{n} p_j x^j$$

is any degree $n$ polynomial, then

$$\langle h, p \rangle = \int_{-1}^{1} h(x) \overline{p(x)} \, dx$$

$$= \sum_{j=0}^{n} p_j \int_{-1}^{1} h(x) \, x^j \, dx$$

$$= 0.$$ 

By Weierstrass, let $\|h(x) - p(x)\| \leq \varepsilon$ for $|x| \leq 1$. Then

$$0 = \langle h, p \rangle = \langle h - p, p \rangle + \langle p, p \rangle$$

$$\langle h, h \rangle = |\langle h, h - p \rangle| \leq \|h\| \|h - p\|$$

$$\leq 2 \varepsilon \|h\|$$

by Cauchy-Schwarz.

So

$$\|h\| (\|h\| - 2\varepsilon) \leq 0.$$ 

Hence $\|h\| \leq 2\varepsilon$ and since $\varepsilon$ was arbitrary, $\|h\| = 0$. Since $h$ is continuous, $h(x) \equiv 0 \quad |x| \leq 1.$
3. \( \int_0^{\infty} x e^{-x^4} \sin(x^{1/4}) \, dx \)

\[
= 4 \int_0^{\infty} y e^{y (4k+3)} \sin(y) \, dy
\]

Integration by parts gives

\[
\int_0^{\infty} y e^{y (4k+3)} \sin(y) \, dy = \frac{d}{dy} \frac{e^{-(1+i)y}}{(-1+i)} \, dy
\]

\[
= \frac{(4k+3)!}{(-1+i)^{4k+3}} \int_0^{\infty} e^{-(1+i)y} \, dy
\]

\[
= \frac{(4k+3)!}{(-1+i)^{4k+3}} \cdot \left( \frac{1}{i\pi/4} \right)
\]

But

\[
(-1+i) = \sqrt{2} e^{i\pi/4}
\]

so \((-1+i)^4 = 4 e^{-i\pi} = -4 \) and the result is therefore correct.

Hence \( \int_0^{\infty} x e^{-x^{1/4}} \sin(x^{1/4}) \, dx = 0 \) for \( k = 0, 1, \ldots \)