Question 1  (a) Use Fourier transform to find a bounded solution \( u \) of
\[ u_{xx} + u_{tt} = 0 \]
in the upper half plane \( x \in R, t > 0 \), with boundary conditions
\[ u(x, 0) = g(x) \]
where \( g \in L^2(R) \) is bounded and continuous.

(b) Show that \( u \) attains its boundary values in the sense that
\[ u(x, t) \to g(x) \]
as \( t \to 0 \).

(c) Assume that \( g' \in L^2(R) \) is also bounded and continuous. Argue directly from the Laplace equation that if
\[ u_x(x, t) \to \Lambda g(x) \]
then the Dirichlet-Neumann operator \( \Lambda \) must satisfy
\[ \Lambda^2 g(x) = -g''(x). \]

(d) Find the kernel of the Hilbert transform operator \( H \) such that
\[ \Lambda g = H(g'). \]

Question 2  (a) Use Fourier transform to show that the bounded solution \( u \) of the free-space heat equation
\[ u_t = u_{xx} \]
for \( x \in R \) and \( t > 0 \), with bounded continuous initial conditions \( u(x, 0) = u_0(x) \), is given by
\[ u(x, t) = K_t * u_0(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} u_0(y) dy \]
for \( t > 0 \).

(b) Show that \( u \) attains its initial conditions in the sense that
\[ u(x, t) \to u_0(x) \]
as \( t \to 0 \).
Question 3  Solve the integral equation

\[ D^{1/2} h(t) = \int_0^t \frac{1}{\sqrt{4\pi (t-s)}} h(s) \, ds = g(t) \]

where \( g \) is a nice function with \( g(0) = 0 \). (Hint: Square \( D^{1/2} \).)

Question 4  (a) Solve the initial-boundary value problem for the heat equation

\[ u_t = u_{xx} \]

for \( x > 0, \ t > 0 \), with homogeneous initial conditions

\[ u(x, 0) = 0 \]

and boundary conditions

\[ u(0, t) = g(t) \]

where \( g \) is a nice function with \( g(0) = 0 \). (Hint: Try \( u(x, t) = \int_0^t K_{t-s}(x) h(s) \, ds \) and solve an integral equation for \( h \).)

(b) Assume that \( g' \in L^2(\mathbb{R}) \) is also bounded and continuous. Argue directly from the heat equation that if

\[ u_x(x, t) \to \Lambda g(t) \]

as \( x \to 0 \), then the Dirichlet-Neumann operator \( \Lambda \) must satisfy

\[ \Lambda^2 g(t) = -g'(t). \]

(c) Find the Dirichlet-Neumann operator \( \Lambda \).

Question 5  Use Fourier transform in the variable \( t \) to solve the problem of Question 4. (Hint: Extend \( g \) and \( u \) continuously to be zero for negative \( t \). Be careful when taking square roots of complex numbers.)