1. Use Gram-Schmidt orthogonalization to find an orthonormal basis for the span of \( \{ e^{-x}, e^{-2x}, e^{-3x} \} \) in \( L^2(0,\infty) \) with inner product
\[
\langle f, g \rangle = \int_0^\infty f(x) g(x) \, dx.
\]

2. Find the projection of \( f(x) = x e^{-x/2} \) onto the subspace of problem 1.

3. Let \( D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \} \),
\[ L^2(D) = \{ f : D \to \mathbb{C} \mid \int_D |f|^2 \, dx \, dy < \infty \}, \]
\[
\langle f, g \rangle = \int_D f \overline{g} \, dx \, dy.
\]
Show that
\[
\varphi_n(x,y) = (x+iy)^n
\]
\( n \in \mathbb{N} \)
is an orthogonal set in \( L^2(D) \). Normalize them. Project \( f(x,y) = \sqrt{x+iy} \) onto
\( \text{Span} \{ \varphi_0, ..., \varphi_N \} \).
4. Let $A$ be an $n \times n$ complex matrix. Show that

(a) the columns of $A$ are orthonormal
(b) $A^*A = I$
(c) $\|A\| = \|x\| \quad \forall x \in \mathbb{C}^n$

are equivalent.

5. Prove the Fredholm alternative:
If $A : V \to W$ is a linear map between inner product spaces $V$ and $W$ and $b \in W$, then either

$Ax = b$ has a solution $x \in V$

or

$\exists w \in W$ with $A^*w = 0 \neq \langle b, w \rangle$.

6. Find a sequence $f_n \in L^2(0, 1)$ such that $f_n \to 0$ in $L^2(0, 1)$ but not uniformly.
7. Let $K_+(x) = \frac{t}{\pi(t^2 + x^2)}$ for $t > 0$ and $x \in \mathbb{R}$.

Use DCT to show that

$$\int_{-\infty}^{\infty} K_+(x-y) f(y) dy \to f(x) \quad \forall x \in \mathbb{R}$$

for bounded continuous $f$. 