Math 118, Handout 0: Dominated convergence and applications

**Definition** A set of real numbers has measure 0 if it can be enclosed in a collection of intervals \([a_n, b_n]\) with arbitrarily small total length \(\sum_0^\infty b_n - a_n\).

**Example** The rational numbers \(r_n = p_n/q_n\) in the interval \([0, 1]\) have measure zero, because rational \(r_n\) can be enclosed in an interval of length \(\epsilon 2^{-n}\) and the total length of these intervals is \(2\epsilon\).

**Definition** A property of real numbers holds almost everywhere if the set of real numbers where it does not hold has measure zero.

**Example** Almost every real number between 0 and 1 is irrational.

**Theorem 1** Suppose a sequence of functions \(f_n\) converges to a function \(f\) almost everywhere and is dominated \(|f_n(x)| \leq g(x)\) for all \(x\) by a function \(g\) with \(\int g(x) \, dx < \infty\).

Then \(\int f_n(x) \, dx \to \int g(x) \, dx\) as \(n \to \infty\).

**Example 1:** Since \(\int_0^1 1 \, dx < \infty\), any sequence \(f_n(x)\) on \(0 \leq x \leq 1\) which converges to \(f\) almost everywhere and is uniformly bounded by 1 satisfies \(\int_0^1 f_n(x) \, dx \to \int_0^1 f(x) \, dx\).

For example, \(x^n \to 0\) except for \(x = 0\), so \(\int_0^1 x^n \, dx = \frac{1}{n + 1} \to 0\).

But \(f_n(x) = n\) for \(0 \leq x \leq 1/n\) and 0 otherwise is not dominated by any integrable \(g\), so we cannot conclude that \(1 \to 0\).
Example 2: For $\epsilon > 0$ let

$$G_\epsilon(x) = \frac{1}{\sqrt{\pi \epsilon^2}} e^{-x^2/\epsilon^2}$$

be the Gauss kernel with scale $\epsilon$. Use $G_\epsilon$ to average a continuous bounded $f$:

$$f_\epsilon(x) = \int_{-\infty}^{\infty} G_\epsilon(x - y) f(y) \, dy.$$  

Changing variables gives

$$f_\epsilon(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(x + \epsilon t) \, dt.$$  

As $\epsilon \to 0$, each value of the integrand converges:

$$e^{-t^2} f(x + \epsilon t) \to e^{-t^2} f(x),$$  

and it is dominated by $e^{-t^2}$ times that maximum of $|f|$. By Dominated Convergence,

$$f_\epsilon(x) \to f(x)$$  

for every $x$.

Theorem 2  If

$$\int |g(x)| \, dx < \infty$$

and

$$\int g(x) \, dx = 1$$

then for every continuous $f : R \to C$ with $|f(x)| \leq M$ for all $x$, we have

$$\int \frac{1}{\epsilon} g\left(\frac{1}{\epsilon}\right) f(y) \, dy \to f(x)$$

for every $x$ as $\epsilon \to 0$. 