Mouse pairs and Suslin cardinals

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VIG, Feb. 2019

Introduction

Problem: Analyze HOD in models of determinacy.

Post-1970 work done by Becker, Harrington, Kechris, Martin, Moschovakis, Sargsyan, Solovay, Steel, Woodin, and others.Main methods: descriptive set theory (games and definable scales) and inner model theory (mice and iteration strategies).

Conjecture 1. Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$; then $\mathrm{HOD} \models \mathsf{GCH}$.

Conjecture 2. There is $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$ such that $\mathrm{HOD}^M \models$ "there is a huge cardinal".

Definition

"No long extenders" (NLE) is the assertion: there is no countable, iterable pure extender mouse with a long extender on its sequence.

Theorem

(S. 2015) Assume AD^+*, and suppose there is a countable, iterable pure extender mouse with a long extender on its sequence; then*

1. *for any boldface pointclass* Γ *such that* $L(\Gamma,\mathbb{R}) \models \mathsf{NLE}$*,* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH}$*, and*
2. *there is a bolface pointclass* Γ *such that* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ *"there is a subcompact cardinal".*

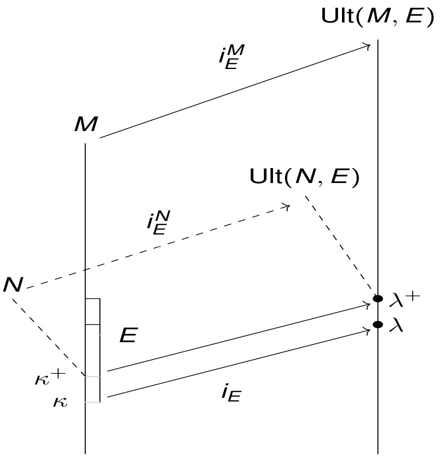
Moral: Below long extenders, there is a simple general notion of *mouse pair*, and a general comparison theorem for them. They have a fine structure. *Modulo the existence of iteration strategies*, they can be used to analyze HOD, and they can have subcompact cardinals.

A Glossary

(a) An *extender E over M* is a system of measures on M coding an elementary $i_E \colon M \to \mathrm{Ult}(M, E)$. E is *short* iff all its component measures concentrate on $\mathrm{crit}(i_E)$.

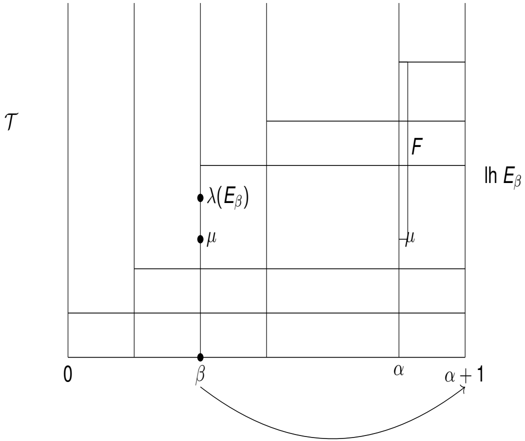
$$\mathrm{Ult}(M, E) = \{[a, f]^M_E \mid f \in M \text{ and } a \in [\lambda]^{<\omega}\},$$

where $\lambda = \lambda(E) = i_E(\mathrm{crit}(E))$.



M agrees with Ult(M, E) and Ult(N, E) to $(\lambda^+)^{\mathrm{Ult}(M,E)}$.

(b) A *normal iteration tree on M* is an iteration tree $\mathcal{T}$ on M in which the extenders used have increasing strengths, and are applied to the longest possible initial segment of the earliest possible model. (So along branches of $\mathcal{T}$, generators are not moved.)


$\mathcal{T}$
F
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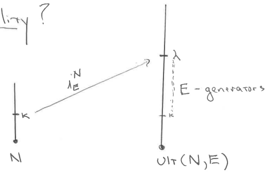
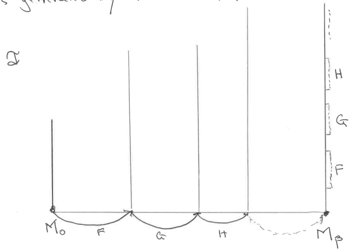
Why normality?



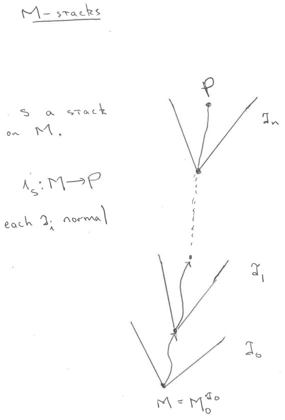
$$\mathrm{Ult}(N,E) = \{ i(f)(a) \mid f \in N \text{ and } a \in [\lambda]^{<\omega} \}.$$

It's generated by $\lambda \cup \mathrm{ran}(i)$.



The individual extenders used going from M_0 to M_β can be recovered from $i^{\mathcal{T}}_{0\beta}$.

(c) An *M-stack* is a sequence $s = \langle \mathcal{T}_0, ..., \mathcal{T}_n \rangle$ of normal trees such that $\mathcal{T}_0$ is on M, and $\mathcal{T}_{i+1}$ is on the last model of $\mathcal{T}_i$.



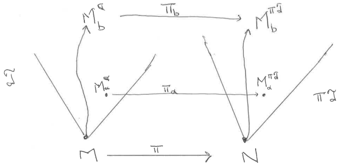
(d) An *iteration strategy* Σ for M is a function that is defined on M-stacks s that are by Σ whose last tree has limit length, and picks a cofinal wellfounded branch of that tree.

(e) If s is an M-stack, then Σ_s is the *tail strategy* given by $\Sigma_s(t) = \Sigma(s^\frown t)$.

(f) It $\pi\colon M \to N$ is elementary, and Σ is an iteration strategy for N, then Σ^π is the *pullback strategy* given by: $\Sigma^\pi(s) = \Sigma(\pi s)$.

Pullback strategies

Given Σ for N, and $\pi : M \to N$



if $b = \Sigma(\pi\mathcal{T})$

then $\Sigma^\pi(\mathcal{T}) = b$

Mouse pairs

Definition

(a) A *pure extender premouse* is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders.

(b) A *least branch premouse* (lpm) is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders, and a predicate $\dot{\Sigma}^{\mathcal{M}}$ for an iteration strategy for $\mathcal{M}$.

Remarks

(a) $\mathcal{M}$ has a hierarchy, and a fine structure. By convention, there is a $k = k(\mathcal{M})$ such that $\mathcal{M}$ is *k-sound*. (I.e., $\mathcal{M} = \text{Hull}_k(\rho_k^{\mathcal{M}} \cup p_k^{\mathcal{M}})$.)

(b) We use Jensen indexing for the extenders in $\dot{E}^{\mathcal{M}}$.

(c) At strategy-active stages in an lpm, we tell $\mathcal{M}$ the value of $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$, where $\mathcal{T}$ is the $\mathcal{M}$-least tree such that $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$ is currently undefined. (Woodin, Schlutzenberg-Trang.)

Definition

A *mouse pair* is a pair (P, Σ) such that

(1) P is a countable premouse (pure extender or least branch),

(2) Σ is an iteration strategy defined on all countable stacks on P,

(3) Σ has strong hull condensation, and

(4) if P is an lpm, then whenever Q is a Σ-iterate of P via s, then $\dot{\Sigma}^{\mathcal{Q}} \subseteq \Sigma_s$.

Strong hull condensation

Roughly, Σ has *strong hull condensation* iff $\mathcal{T}$ and $\mathcal{U}$ are normal trees on P, and $\mathcal{U}$ is by Σ, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is appropriately elementary, then $\mathcal{T}$ is by Σ.
One must be careful about the elementarity required of Φ, and in particular, the extent to which Φ is required to preserve exit extenders. There are several possible condensation properties here: hull condensation (Sargsyan), strong hull condensation, and still stronger ones.

Strong hull condensation means condensing under *tree embeddings*.

Definition

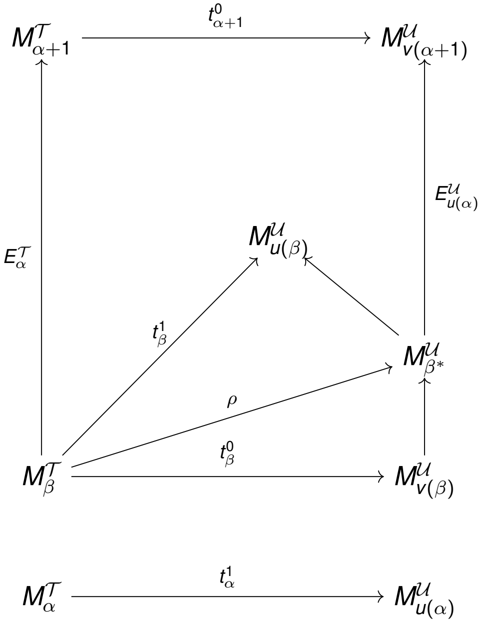
A *tree embedding* of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, \langle t^0_\beta \mid \beta < \operatorname{lh}\mathcal{T}\rangle, \langle t^1_\beta \mid \beta + 1 < \operatorname{lh}\mathcal{T}\rangle\rangle$$

with various properties, including:

$$t^1_\alpha(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}.$$

The diagram related to successor steps in $\mathcal{T}$ is:



Elementary properties of mouse pairs

Definition

$\pi\colon (P, \Sigma) \to (Q, \Psi)$ is *elementary* iff $\pi\colon P \to Q$ is Σ_k elementary, where $k = k(P)$, and $\Sigma = \Psi^\pi$.

Lemma

An elementary submodel of a mouse pair is a mouse pair.

Definition

(Q, Ψ) is an *iterate of* (P, Σ) iff there is a stack s by Σ with last model Q, and $\Psi = \Sigma_s$.

Lemma

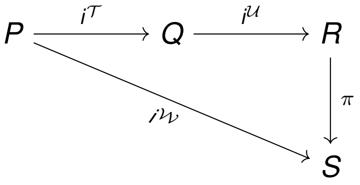
(Iteration maps are elementary) Let (P, Σ) be a mouse pair, and let s be a stack by Σ giving rise to the iteration map $\pi\colon P \to Q$; then $(\Sigma_s)^\pi = \Sigma$.

Lemma

(Dodd-Jensen) The Σ-iteration map from (P, Σ) to (Q, Ψ) is the pointwise minimal elementary embedding of (P, Σ) into (Q, Ψ).

Normalizing well

For $\langle \mathcal{T}, \mathcal{U} \rangle$ a stack on P, there is a natural normal tree $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ obtained by inserting the extenders of $\mathcal{U}$ into $\mathcal{T}$. We have



Then Σ *2-normalizes well* iff

$$\langle \mathcal{T}, \mathcal{U} \rangle \text{ is by } \Sigma \text{ iff } W(\mathcal{T}, \mathcal{U}) \text{ is by } \Sigma,$$

and

$$\Sigma^{\pi}_{\langle \mathcal{W} \rangle} = \Sigma_{\langle \mathcal{T}, \mathcal{U} \rangle}.$$

for all such stacks $\langle \mathcal{T}, \mathcal{U} \rangle$.

One can extend the construction of $W(\mathcal{T}, \mathcal{U})$ so as to define the embedding normalization $W(s)$ of a countable stack of normal trees. One has an elementary π from the last model of s to the last model of $W(s)$. If one has

$$s \text{ is by } \Sigma \text{ iff } W(s) \text{ is by } \Sigma,$$

and

$$\Sigma^{\pi}_{\langle \mathcal{W}(s) \rangle} = \Sigma_s.$$

for all such stacks $\langle \mathcal{T}, \mathcal{U} \rangle$, and the same is true for all tails of Σ, then we say that Σ *normalizes well*.

Theorem

(Schlutzenberg 2015) Let (P, Σ) be a mouse pair; then Σ normalizes well.

Comparison

Theorem (Comparison)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type; then they have a common iterate* (R, Φ) *such that at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

Definition

(Mouse order) $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) embeds elementarily into some iterate of (Q, Ψ).

Corollary

Assume AD^+*; then the mouse order* $\leq^*$ *on mouse pairs of a fixed type is a prewellorder.*

Phalanx comparisons work too. From this we get

Theorem

Assume AD^+*, and let* (P, Σ) *be a mouse pair; then the standard parameter of* P *is solid and universal, and hence* (P, Σ) *has a core.*

Theorem

Assume AD^+*, and let* N *be a countable, iterable, coarse* Γ*-Woodin model; then the hod pair construction of* N *does not break down.*

Theorem

Suppose that V *is uniquely iterable, and there are arbitrarliy large Woodin cardinals; then the hod pair construction of* V *does not break down.*

Phalanx comparisons also yield Condensation, and

Theorem

(Trang, S., 2017) Assume AD^+*, and let* (P, Σ) *be a mouse pair; then* $P \models \forall\kappa(\Box_\kappa \Leftrightarrow \kappa$ *is not subcompact).*

Phalanx comparisons also give

Theorem

Assume AD^+*, and let* (P, Σ) *be a mouse pair; then*

(1) Σ *is positional,*

(2) Σ *has very strong hull condensation, and*

(3) Σ *fully normalizes well.*

Hod pair capturing

Least branch hod pairs can be used to compute HOD, provided that there are enough of them.

Definition

(AD^+) HOD *pair capturing* (HPC) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P, Σ) with scope HC such that A is Wadge reducible to $\mathrm{Code}(\Sigma)$.

Remark. Under AD^+, if (P, Σ) is a mouse pair, then $\mathrm{Code}(\Sigma)$ is Suslin and co-Suslin.

Theorem

Assume AD^+, and that there is an iterable premouse with a long extender. Let $\Gamma \subseteq P(\mathbb{R})$ be such that $L(\Gamma, \mathbb{R}) \models \mathrm{NLE}$; then $L(\Gamma, \mathbb{R}) \models \mathrm{HPC}$.

In light of this theorem, the following is almost certainly true:

Conjecture. $(\mathsf{AD}^+ + \mathsf{NLE}) \Rightarrow \mathsf{HPC}$.

HPC holds in the minimal model of $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular, and somewhat beyond, by Sargsyan's work. In fact,

Theorem

(Sargsyan, S., 2018) Assume AD^+*, and let* Δ *be the pointclass of all sets Wadge reducible to the code of an lbr hod pair; then* $L(\Delta, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} +$ "θ *is regular".*

HPC localizes:

Theorem

Assume $\mathsf{AD}^+ + \mathsf{HPC}$*, and let* $\Gamma \subseteq P(\mathbb{R})$*; then* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$.

Definition

(AD$^+$) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

$M_\infty(P, \Sigma)$ is well-defined by the Dodd-Jensen lemma. Moreover, it is OD from the rank of (P, Σ) in the mouse order. Thus $M_\infty(P, \Sigma) \in \mathrm{HOD}$. It is an initial segment of the lpm hierarchy of HOD *if* (P, Σ) is "full".

Definition

A mouse pair (P, Σ) is full iff for all mouse pairs (Q, Ψ) such that $(P, \Sigma) \leq^* (Q, \Psi)$, we have $M_\infty(P, \Sigma) \trianglelefteq M_\infty(Q, \Psi)$.

Theorem

Assume $\mathrm{AD}_{\mathbb{R}}$ + HPC*; then* $\mathrm{HOD}\,|\theta$ *is the union of all* $M_\infty(P,\Sigma)$ *such that* (P,Σ) *is a full lbr hod pair.*

Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then* $\mathrm{HOD}\,|\theta$ *is an lpm. Thus* $\mathrm{HOD} \models \mathrm{GCH}$*.*

The construction of Suslin representations for the iteration strategies in mouse pairs plays an important role in many of the proofs above.

Suslin representations for mouse pairs

Let (P, Σ) be a mouse pair. A tree $\mathcal{T}$ by Σ is *M_∞-relevant* iff there is a normal $\mathcal{U}$ by Σ extending $\mathcal{T}$ with last model Q such that the branch P-to-Q does not drop. Σ^{rel} is the restriction of Σ to M_∞-relevant trees.
Recall that A is *κ-Suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$.

Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

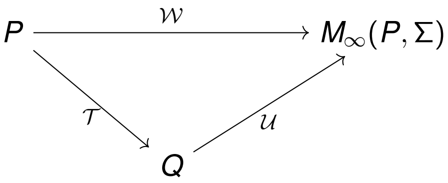
Remark. Code(Σ^{rel}) is not α-Suslin, for any $\alpha < |M_\infty(P, \Sigma)|$, by Kunen-Martin. So $|M_\infty(P, \Sigma)|$ is a Suslin cardinal.

Proof sketch. $M_\infty(P, \Sigma)$ is the direct limit along a generic stack s of trees by Σ.But s can be fully normalized, so there is a single normal tree $\mathcal{W}$ on P with last model $M_\infty(P, \Sigma)$ such that every countable "weak hull" of $\mathcal{W}$ is by Σ.But then for $\mathcal{T}$ countable and M_∞-relevant,

$$\mathcal{T} \text{ is by } \Sigma \Leftrightarrow \mathcal{T} \text{ is a weak hull of } \mathcal{W}.$$

The right-to-left direction follows from very strong hull condensation for Σ.
For left-to-right direction, we may assume $\mathcal{T}$ has last model Q, and P-to-Q does not drop. We then have a normal $\mathcal{U}$ on Q with last model $M_\infty(P, \Sigma)$ such that all countable weak hulls of $\mathcal{U}$ are by Σ.

We have



Then

$$\mathcal{W} = X(\mathcal{T}, \mathcal{U})$$

is the full normalization of $\langle \mathcal{T}, \mathcal{U} \rangle$. The construction of $X(\mathcal{T}, \mathcal{U})$ produces a weak hull embedding from $\mathcal{T}$ into $X(\mathcal{T}, \mathcal{U})$, which is what we want.

Thus our Suslin representation verifies that $\mathcal{T}$ is in the M_∞-relevant part of Σ by producing a weak hull embedding of $\mathcal{T}$ into $\mathcal{W}$.

Characterizing the Woodins of HOD

Recall the *Solovay sequence*: θ_0 is the sup of the lengths of OD prewellorders of $\mathbb{R}$, $\theta_{\alpha+1}$ is the sup of the OD(A) prewellorders, for any and all A of Wadge rank θ_α, and $\theta_\lambda = \bigcup_{\alpha<\lambda} \theta_\alpha$ for λ a limit.

Definition

κ is a *cutpoint* of a premouse $\mathcal{M}$ iff there is no extender E on the $\mathcal{M}$-sequence such that $\mathrm{crit}(E) < \kappa \leq \mathrm{lh}(E)$.

Theorem

Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then equivalent are:*

(a) δ *is a cutpoint Woodin cardinal of* HOD*,*

(b) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

Thus θ_0 *is the least Woodin cardinal of* HOD*.*

Remark. Woodin showed θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD. He proved an approximation to their being cutpoints.

Theorem

Assume $\mathsf{AD}_{\mathbb{R}}$ + HPC, *and let* κ *be a successor cardinal of* HOD *such that* $\kappa < \theta$. *Let*

$$\delta = \sup(\{|S| \mid S \text{ is an OD prewellorder of } {}^{\omega}\kappa\}).$$

Then δ *is the least Woodin cardinal of* HOD *above* κ.

Remark. This was conjectured by Sargsyan.

Suslin cardinals and mouse-limits

Theorem

Let (P, Σ) be a mouse pair, and let κ be a cutpoint of $M_\infty(P, \Sigma)$; then $|\kappa|$ is a Suslin cardinal.

We conjecture the following converse holds:

Conjecture. Let (P, Σ) be a mouse pair, and κ be a Suslin cardinal such that $\kappa < o(M_\infty(P, \Sigma))$; then κ is a cutpoint of $M_\infty(P, \Sigma)$.

The conjecture implies that under $\mathsf{AD}^+ + \mathsf{HPC}$, the Suslin cardinals of V are precisely the cardinals of V that are cutpoints in HOD.

With Jackson and Sargsyan, we have proved the conjecture when

(i) κ is a limit of Suslin cardinals,

(ii) κ is the next Suslin after a limit of Suslins,

(iii) κ is one of the first ω Suslin cardinals, or

(iv) the largest limit of Suslins below κ has cofinality ω.

Some of the arguments use

Theorem

(Sargsyan 2018) Let (P, Σ) be a mouse pair, and $\kappa = \mathrm{crit}(E)$, where E is a total extender on the sequence of $M = M_\infty(P, \Sigma)$; then there is a countably complete V-ultrafilter U on κ such that $i_E^M(\kappa) \leq i_U^V(\kappa)$.

Thank you!