Correctness, definability, and self-iterabilty for ω-small mice

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This is joint work with Ralf Schindler.

We shall show that if $\mathcal{M}$ is a fully iterable ω-small mouse such that $\mathbb{R} \cap L(\mathbb{R} \cap \mathcal{M}) \subseteq \mathcal{M}$, then $L(\mathbb{R} \cap \mathcal{M}) \models$ "there is a wellorder of $\mathbb{R}$". In fact:

Theorem 0.1 *Let $\mathcal{M}$ be a fully iterable, ω-small mouse such that $\mathbb{R} \cap L(\mathbb{R} \cap \mathcal{M}) \subseteq \mathcal{M}$; then there are ordinals $\bar{\beta}$ and β such that*

(a) $J_{\bar{\beta}}(\mathbb{R})^{\mathcal{M}} \models \mathsf{AD}$*, and*

(b) $J_{\bar{\beta}+1}(\mathbb{R})^{\mathcal{M}} \models$ *"there is a wellorder of $\mathbb{R}$".*

As a corollary to the proof, one gets

Theorem 0.2 *Let $\mathcal{M}$ be a fully iterable, ω-small, proper class mouse. Then for some α, $\mathcal{M}$ knows how to iterate itself for set-sized trees with all critical points $> \alpha$.*

This shows that any such $\mathcal{M}$ satisfies "I am K over $\mathcal{M}|\alpha$", for some α. We believe that in fact, one can show any such $\mathcal{M}$ breaks into finitely many intervals $[\alpha, \beta]$ such that $\mathcal{M}$ knows how to iterate itself for iteration trees in $\mathcal{M}|\beta$ with all critical points $> \alpha$, and thus $\mathcal{M}|\beta$ satisfies "I am K over $\mathcal{M}|\alpha$".

The fact that $V = K$ holds in $\mathcal{M}$ (in the sense hinted at above) would let one extend proofs involving covering arguments to $\mathcal{M}$. For example, let $\diamond^{**}_{\kappa,\lambda}$ be the assertion: There is a function F with domain $P_\kappa(H_\lambda)$ such that for all $X \in P_\kappa(H_\lambda)$, $|F(X)| \leq |X|$, and for all $A \subseteq H_\lambda$, there are club many $X \in P_\kappa(\lambda)$ such that $X^\omega \subseteq X \Rightarrow A \cap X \in F(X)$. Kanamori showed L satisfies $\forall \kappa, \lambda \diamond^{*}_{\kappa,\lambda}$, where the single $*$ in the superscript indicates we have dropped the restriction to X such that $X^\omega \subseteq X$. Granted that $\mathcal{M}$ satisfies $V = K$ in the sense hinted at above, one can apply the proof of covering for larger core models inside an $\mathcal{M}$ as above, so as to show $\mathcal{M}$ satisfies $\forall \kappa, \lambda \diamond^{**}_{\kappa,\lambda}$. (We do not know how to prove the full $\diamond^{*}_{\kappa,\lambda}$ holds.)

Finally, we believe these results generalize to tame mice, replacing $L(\mathbb{R})$ by $K(\mathbb{R})$.