The Comparison Lemma

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Plan

(I) Context and preliminaries.

(II) Iteration trees and iteration strategies. A general comparison process for short-extender mice. The Dodd-Jensen Lemma and the mouse order.

(III) Comparing iteration strategies. Dodd-Jensen and the mouse pair order.

References:

(1) *The comparison lemma*, J. Steel, APAL 2024, 46 pp.

(2) *A comparison process for mouse pairs*, J. Steel, Lecture Notes in Logic v. 51, (CUP) 2022, 536 pp.

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Three wellordered hierarchies

(1) The consistency strength hierarchy on natural extensions of ZFC:

$$T \leq_{con} U \quad \text{iff} \quad \mathrm{PA} \vdash \mathrm{Con(U)} \Rightarrow \mathrm{Con(T)}.$$

(2) The Wadge order on homogeneously Suslin sets of reals:

$$A \leq_w B \quad \text{iff} \quad \exists f\colon \mathbb{R} \to \mathbb{R}(f \text{ is continuous and } \forall x(x \in A \leftrightarrow f(x) \in B)).$$

(3) The mouse order on canonical inner models for large cardinal hypotheses. Roughly:

$$M \leq^* N \quad \text{iff} \quad M \text{ iterates to an initial segment of an iterate of } N.$$

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These hierarchies turn out to be closely related. Our focus is the mouse order, whose linearity is established by the Comparison Lemma, and whose wellfoundedness is established by the Dodd-Jensen Lemma.

The mouse order underlies the other two hierarchies.

The Wadge hierarchy

Definition

Let $A \subseteq \omega^\omega$; then A is *κ-homogeneously Suslin* (Hom_κ) iff there is a system $\langle M_s, i_{s,t} \mid s, t \in \omega^{<\omega}\rangle$ such that

(1) $M_\emptyset = V$, and each M_s is closed under κ-sequences,

(2) for $s \subseteq t$, $i_{s,t} \colon M_s \to M_t$,

(3) if $s \subseteq t \subseteq u$, then $i_{s,u} = i_{t,u} \circ i_{s,t}$, and

(4) for all x, $x \in A$ iff $\lim_n M_{x \restriction n}$ is wellfounded.

Theorem (Martin 1968)

If $A \in Hom_\kappa$ for some κ, then G_A is determined.

Definition

A *boldface pointclass* is a $\Gamma \subset P(\mathbb{R})$ that is closed under complements, and closed downward under $\leq_w$.

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Theorem (Wadge, Martin, late 1960s)

Assume AD*; then the collection of boldface pointclasses is wellordered by inclusion.*

Definition

$\mathrm{Hom}_\infty = \bigcap_\kappa \mathrm{Hom}_\kappa$.

Theorem

(Martin, S., Woodin 1985) Assume there are arbitrarily large Woodin cardinals; then

(1) for any $A \in Hom_\infty$, $A^\sharp \in Hom_\infty$, so

(2) for any boldface pointclass Γ properly included in Hom_∞, then $L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$.

Remark. Martin-Solovay showed in 1968 that all Hom_∞ sets are Universally Baire (uB). The proof of the 1985 result above shows that if there are arbitrarily large Woodin cardinals, then every uB set is Hom_∞.

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Theorem

(Woodin 1987?) If there are arbitarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ truth is set-generically absolute.

Question. Does any large cardinal hypothesis (e.g. the existence of supercompacts) imply that $L(\mathrm{Hom}_\infty, \mathbb{R}) \models \mathrm{AD}$?

A positive answer would be an "anti-inner-model" theorem for the hypothesis is question, since it seems to be a basic feature of the known inner models that the reasonably closed ones satisfy "There is a $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ wellorder of the reals".

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The mouse order vs. Hom_∞

The Comparison Lemma, for the canonical inner models (mice) M to which it currently applies, shows that if $x \in \mathbb{R} \cap M$, then x is ordinal definable in a generically absolute way. More precisely, x is $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ definable from a countable ordinal α:

$$y = x \text{ iff } \quad \exists A \in \mathrm{Hom}_\infty[(\mathrm{HC}, \in, A) \models \varphi[y, \alpha]],$$

where in our case

$\varphi =$" A codes an iteration strategy for some countable mouse N such that y is the α-th real of N.

So in practice, mice are certified as standard by having Hom_∞-coded iteration strategies. (More later!) The mouse order then corresponds to the Wadge order on the codesets for these strategies.

Mouse existence

What about the converse? Are the iteration strategies for mice Wadge cofinal in Hom_∞? Does every real that is $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ in a countable ordinal belong to a mouse? This question is not precise, because we have no general definition of *mouse* and *iteration strategy* as of now. With the notions we have now, we can give positive answers for a nontrivial initial segment of the Wadge hierarchy.

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Theorem (Sargsyan, S., Woodin 1994–2009)

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R}))$ *+ "there is no boldface pointclass* Γ *such that* $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} + \theta$ *is regular"; then*

(1) HOD is a canonical inner model (a mouse), so

(2) HOD $\models$ GCH.

The theorem has been extended, and there are many open questions on the frontier of that effort.

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Determinacy models from hod pairs

Woodin has shown (early 1990s) that every model M of $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$ is a symmetric extension of its HOD. So HOD^M can "see" the surrounding determinacy model. It is natural to conjecture that HOD^M is always a mouse, a canonical inner model with a fine structure like that of the ones we know. This motivates

Conjecture. Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$; then $\mathrm{HOD} \models \mathsf{GCH}$.

Presumably proof of the conjecture requires a completely general notion of mouse and iteration strategy, and a proof that the iteration strategies are cofinal in the Wadge hierarchy.

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Presumably proof of the conjecture requires a completely general notion of mouse and iteration strategy, and a proof that the iteration strategies are cofinal in the Wadge hierarchy.
Our focus is on the comparison problem, not the existence problem. But they are intertwined.

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Extenders and ultrapowers

Any $j\colon M \to N$ can be captured by system of measures on M. To capture j and N up to λ, for $a \subseteq \lambda$ finite put

$$X \in E_a \text{ iff } a \in j(X).$$

One has the diagram

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i_E"'] & & N \\ & \lim_a \mathrm{Ult}(M, E_a) \arrow[ur, "k"'] & \end{tikzcd}$$

with $k \restriction \lambda = \text{ id}$. We write $\mathrm{Ult}_0(M, E)$ for $\lim_a \mathrm{Ult}(M, E_a)$.

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Definition

An *extender E over M with support ν* is a system of compatible measures $\langle E_a \mid a \in [\nu]^{<\omega} \rangle$ on M coding an elementary $i^M_E \colon M \to \lim_a \mathrm{Ult}(M, E_a)$. We say E is *short* if $\nu \leq i_E(\mathrm{crit}(E))$.

Short extenders can represent superstrong embeddings, but not extendibility embeddings. We write $\kappa_E = \mathrm{crit}(i_E)$ and $\lambda_E = i^M_E(\kappa_E)$. We set

$$\begin{aligned} \mathrm{Ult}_0(M, E) &= \lim_a \mathrm{Ult}(M, E_a) \\ &= \{[a, f]^M_E \mid a \in [\nu]^{<\omega} \wedge f \in M\}, \end{aligned}$$

and for $x \in M$,

$$i^M_E(x) = [a, \lambda x.x]^M_E \quad (\text{ for any } a).$$

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Our M will always be rudimentarily closed and satisfy the Axiom of Choice, so we have Los' theorem for Σ_0 formulae, and the canonical embedding i_E^M is cofinal and Σ_0 elementary, and hence Σ_1 elementary. We have

$$a = [a, \mathrm{id}]_E,$$
$$[a, f]_E^M = i_E^M(f)(a),$$

so E is derived from i_E:

$$X \in E_a \quad \text{iff} \quad a \in i_E^M(X).$$

If E is an extender over M and $P(\kappa_E)^N = P(\kappa_E)^M$, then E is also an extender over N. $\mathrm{Ult}_0(M, E)$ and $\mathrm{Ult}_0(N, E)$ then have the same subsets of λ_E, and $i_E^M \restriction P(\kappa_E)^M = i_E^N \restriction P(\kappa_E)^N$. But one of the ultrapowers may be wellfounded and the other illfounded.

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Pure extender premice

A *pure extender premouse* is a structure M constructed from a coherent sequence $\dot{E}^M$ of extenders.

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Pure extender premice

A *pure extender premouse* is a structure M constructed from a coherent sequence $\dot{E}^M$ of extenders. *Coherence* means the extenders are added in order of increasing strength (increasing Mitchell order), without leaving gaps.

The notions are essentially due to W. Mitchell (1974, 1978).

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More detail: A *potential pure extender premouse* is an amenable J-structure

$$M = \langle J_\alpha^{\vec{E}}, \in, \vec{E}, \gamma, F \rangle$$

with various properties. $o(M) = \mathrm{OR} \cap M = \omega\alpha$. The language $\mathcal{L}_0$ of M has $\in$, predicate symbols $\dot{E}$ and $\dot{F}$, and a constant symbol $\dot{\gamma}$.

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If M is a potential pure extender premouse, then $\dot{E}^M$ is a sequence of extenders, and either $\dot{F}^M$ is empty (i.e. M is *passive*), or $\dot{F}^M$ codes a new extender F being added to our sequence by M.

In the active case, F is an extender over M with support λ_F. We set $\mathrm{lh}(F) = o(M)$.

If M is a proper initial segment of some premouse N, then $F = \dot{E}^N_{\mathrm{lh}(F)}$. It may be that N has subsets of κ_F that are not in M, in which case F is *not* an extender over N! F only measures the subsets of κ_F that got into the model before we added F. (The Baldwin-Mitchell idea.)

To make this work we need a fine structural analysis of the first level of N over which a new subset of κ_F is definable.

The main further requirements on F are

(1) (λ-indexing) $M \models \kappa^+$ exists, and $o(M) = \mathrm{lh}(F) = \lambda_F^{+,M}$. $\dot{F}^M$ is the graph of $i_F^M \restriction (M|\kappa^{+,M})$.

(2) (Coherence) $i_F^M(\dot{E}^M) \restriction o(M) + 1 = (\dot{E}^M)^\frown\langle\emptyset\rangle$.

(3) (Initial segment condition, J-ISC) If G is a whole proper initial segment of F, then G must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is its index in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$.

(4) If N is a proper initial segment of M, then N is a potential premouse.

Here an initial segment

$$G = F \restriction \eta =_{df} F \cap ([\eta]^{<\omega} \times M)$$

of F is *whole* iff $\eta = \lambda_G$.

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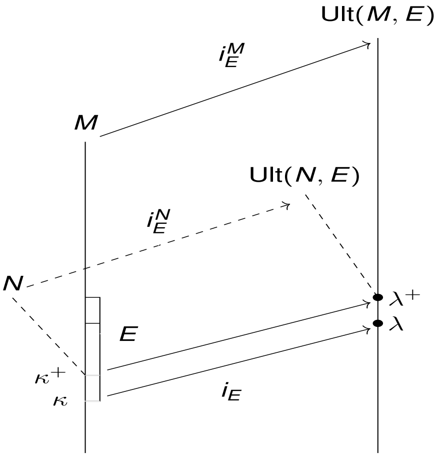

M agrees with Ult(M, E) and Ult(N, E) to $(\lambda^+)^{\mathrm{Ult}(M,E)}$.

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Soundness

The basic fine structural notions apply to potential premice.

$$\begin{aligned}\rho_1(M) = & \text{ least } \alpha \text{ s.t. } \exists r \in M(\mathrm{Th}^M_{\Sigma_1}(\alpha \cup \{r\}) \notin M), \\ p_1(M) = & \text{ first standard parameter of } M \\ = & \text{ lex-least descending sequence of ordinals } r \\ & \text{such that } \mathrm{Th}^M_{\Sigma_1}(\rho_1(M) \cup r) \notin M.\end{aligned}$$

We allow $\rho_1(M) = o(M)$ and $p_1(M) = \emptyset$. One can define $\rho_n(M)$ and $p_n(M)$ in a similar way.
Premice are acceptable J-structures, and the key to their fine structure is that if they are sufficiently iterable, then their standard parameters are solid and universal.

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Approximate definitions: for $\rho = \rho_1(M)$ and $p = p_1(M)$,

(1) p is *1-solid* iff the function $q \mapsto \mathrm{Th}_1^M(\rho \cup q)$, defined on parameters $q <_{\text{lex}} p$, belongs to M.

(2) p is *1-universal* iff $P(\rho)^M \subseteq \mathrm{cHull}_1^M(\rho \cup p)$.

1-solidity is used to show that if $i \colon M \to N = \mathrm{Ult}_0(M, E)$ is the canonical embedding, then $i(p_1(M)) = p_1(N)$.

Definition

M is *1-solid* iff $p_1(M)$ is 1-solid and $1 - \textit{universal}$. M is *1-sound* iff in addition $M = \mathrm{Hull}_1^M(\rho_1(M) \cup p_1(M))$.

Remark. Suppose M is 1-sound, and $N = \mathrm{Ult}_0(M, E)$ where $\rho_1(M) \leq \kappa_E$; then M is the 1-core of N, and i_E^M is the anticore map. That is, $M = \mathrm{cHull}_1^N(\rho_1(N) \cup p_1(N))$, and i_E^M is the anticollapse.

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Definition

A *pure extender premouse* is a potential premouse all of whose proper initial segments are n-sound for all n.

If M is a premouse, then for all κ, $P(\kappa) \cap M \subseteq M|(\kappa^+)^M$. This is a strong, local form of GCH.

Remark For the mice we construct, solidity and universality are proved by comparison arguments, in an induction that keeps pace with the construction.

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Remark For the mice we construct, solidity and universality are proved by comparison arguments, in an induction that keeps pace with the construction. To obtain soundness, we simply replace the current M by the appropriate hull (core) of M. (I.e. *core down*.) Universality is used to see that we don't lose too much when we core down; for example, we never lose subsets of ω.

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Further remarks.

(1) The indexing scheme itself can be varied, but there are abstract results on “inner model operators” that imply that any two reasonable schemes should produce intertranslatable hierarchies. The main two indexing schemes in use (λ-indexing and ms-indexing) were shown to be intertranslatable by G. Fuchs.

(2) Within a given indexing scheme, there is some freedom in choosing the premouse axioms. They should be simple enough so as to be preserved by Σ_0-ultrapowers and Σ_1-Skolem hulls. (These are.)

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The iteration game

Let M be a premouse. In $\mathcal{G}(M, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $M = \mathcal{M}_0^{\mathcal{T}}$ at the base.

Round $\alpha + 1$*:* I picks an extender $E_\alpha^{\mathcal{T}}$ from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ with support $\geq$ the supports of all earlier extenders chosen . Let β be least such that crit$(E_\alpha^{\mathcal{T}}) <$ support $(E_\beta^{\mathcal{T}})$. We set

$$\beta = T\text{-pred}(\alpha + 1),$$

and

$$\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}_k(\mathcal{M}_\beta^{\mathcal{T}} | \eta, E_\alpha^{\mathcal{T}}),$$

where $\langle \eta, k \rangle$ is as large as possible. If E_α is not applied to all of $\mathcal{M}_\beta$, we say that $\mathcal{T}$ *drops* at $\alpha + 1$, and put $\alpha + 1 \in D^{\mathcal{T}}$.
If $\mathcal{M}_{\alpha+1}^{\mathcal{T}}$ is illfounded, then the game is over and I wins.

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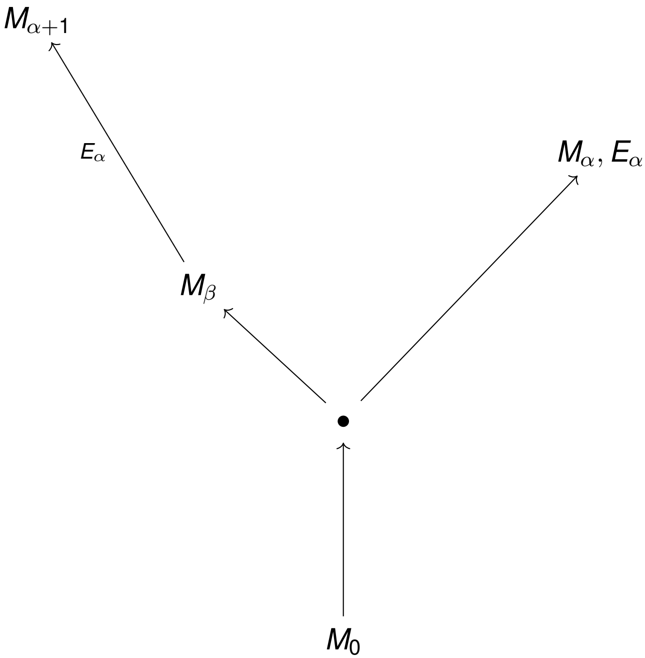

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Round λ, for λ limit: II must pick a branch b of $\mathcal{T}$ which is cofinal in λ such that $D^{\mathcal{T}} \cap b$ is finite, and

$$\mathcal{M}_b^{\mathcal{T}} =_{df} \text{ dirlim }_{\alpha \in (b - supD^{\mathcal{T}})} \mathcal{M}_\alpha^{\mathcal{T}}$$

is wellfounded. If he fails to do so, I wins and the game is over. If he succeeds, then we set

$$\mathcal{M}_\lambda^{\mathcal{T}} = \text{ dirlim }_{\alpha \in (b - \sup D^{\mathcal{T}})} \mathcal{M}_\alpha^{\mathcal{T}}$$

and continue.
If I has not won at some round $\alpha < \theta$, then II wins.
A play of $\mathcal{G}(M, \theta)$ in which II has not yet lost is called a *normal iteration tree* on M.

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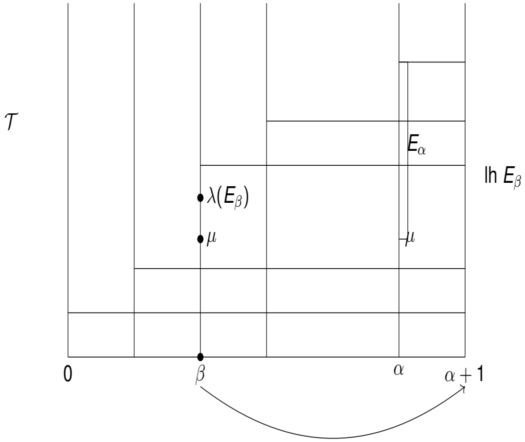

The vertical lines represent the models of $\mathcal{T}$, and the horizontal ones their agreement with one another. β is the $\mathcal{T}$-predecessor of $\alpha + 1$.

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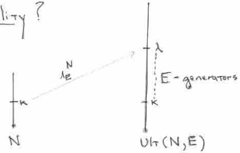
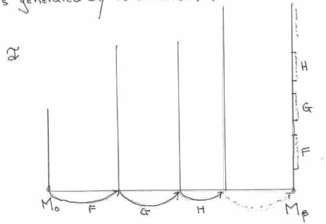
Why normality?



$\mathrm{Ult}(N,E) = \{ i(f)(a) \mid f \in N \text{ and } a \in [\lambda]^{<\omega} \}$.

It's generated by $\lambda \cup \mathrm{ran}(i)$.



The individual extenders used going from M_0 to M_β can be recovered from $i^{\mathcal{T}}_{0\beta}$.

e.g. $X \in F_a$ iff $a \in i_F(X)$ iff $a \in i^{\mathcal{T}}_{0\beta}(X)$.

Definition

A *θ-iteration strategy for M* is a winning strategy for II in $\mathcal{G}(M, \theta)$. We say *M* is *θ-iterable* just in case there is such a strategy. If Σ is a strategy for II in $\mathcal{G}(M, \theta)$, and $P = \mathcal{M}_\alpha^\mathcal{T}$ for some $\mathcal{T}$ played by Σ, then we call *P* a *Σ-iterate* of *M*.

Definition

$M \trianglelefteq N$ iff *M* is an initial segment of *N*, 'and $M \triangleleft N$ iff $M \trianglelefteq N$ and $M \neq N$.

Lemma (Comparison, Martin, Mitchell, S. 1985-89)

Let Σ and Γ be $\theta + 1$ iteration strategies for P and Q respectively, where $\theta = \max(|P|, |Q|)^+$; then there are a Σ-iterate R of P and a Γ-iterate S of Q such that either

(a) $R \trianglelefteq S$ and the branch P-to-R does not drop, or

(b) $S \trianglelefteq R$ and the branch Q-to-S does not drop.

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Corollary

If P and Q are $\omega_1 + 1$-iterable, then $P|\alpha = Q|\alpha$, where $\alpha = \inf(\omega_1^P, \omega_1^Q)$. That is, their canonical wellorders of the reals by stage-of-construction are compatible.

Proof. If P-to-R does not drop, then $P|\omega_1^P = R|\omega_1^R$. So we can apply the comparison lemma. $\square$

Corollary

If P is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap P$, then x is ordinal definable.

Proof. Let x be the α-th real in P. Then $y = x$ iff y is the α-th real in some $\omega_1 + 1$- iterable premouse. $\square$

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Corollary

Assume AD*. Let $x \in \mathbb{R} \cap P$, where P is an ω_1-iterable countable mouse; then x is Σ^2_1 in a countable ordinal.*

Proof.

Under AD, ω_1 is measurable, so countable ω_1-iterable mice are $\omega_1 + 1$-iterable. If Σ is an ω_1 strategy, then Σ can be coded by set of reals, so "x is the α-th real in some ω_1-iterable mouse" is Σ^2_1. □

Corollary

Assume there are arbitrarily large Woodin cardinals, and let P be a countable mouse with an ω_1-iteration strategy that is coded by a Hom_∞ set of reals; then every real is P is $(\Sigma^2_1)^{Hom_\infty}$ is a countable ordinal.

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What about the converse–how far do the mice go?

Definition

Mouse Capturing (MC) is the statement: for all reals x and y, if x is $\Sigma_1^2(y)$ in a countable ordinal, then there is an ω_1-iterable y-premouse M such that $x \in M$.

Theorem (Woodin 1990s, Sargsyan 2009)

Assume AD^+*, and suppose there is no boldface pointclass* Γ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} +$ *"θ is regular"; then Mouse Capturing holds.*

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Proof of comparison lemma

For definiteness, let P and Q be countable, so that Σ and Γ are $\omega_1 + 1$-iteration strategies.
We build padded iteration trees $\mathcal{T}$ on P and $\mathcal{U}$ on Q inductively, by "iterating away the least disagreement" at successor steps, and using our iteration strategies at limit steps. Notation:

$\mathcal{T}$: models P_α, extenders E_α, branch embeddings $i_{\alpha,\beta}$,
$\mathcal{U}$: models Q_α, extenders F_α, branch embeddings $j_{\alpha,\beta}$.

At step $\alpha + 1$, let

$$\gamma = \text{ least } \beta \text{ such that } P_\alpha | \beta \neq Q_\alpha | \beta.$$

If there is no such β, the comparison terminates.

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Otherwise,

$$E_\alpha = \dot{E}_\gamma^{P_\alpha}, \text{ and}$$
$$F_\alpha = \dot{E}_\gamma^{Q_\alpha}.$$

The rest of step $\alpha + 1$ is determined by the rules for normal, padded iteration trees. (If e.g. $F_\alpha = \emptyset$, then $P_{\alpha+1} = P_\alpha$.)
At limit steps we let $\mathcal{T} \upharpoonright \lambda + 1$ be $\bigcup_{\alpha<\lambda} \mathcal{T} \upharpoonright \alpha$, extended by the branch $\Sigma(\bigcup_{\alpha<\lambda} \mathcal{T} \upharpoonright \alpha)$ if this tree has limit length.
Similarly on the $\mathcal{U}$ side.

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Claim. The comparison terminates at some stage $\alpha < \omega_1$.

Proof. Suppose not, and let $\mathcal{T}$ and $\mathcal{U}$ be the normal trees of length $\omega_1 + 1$ that result. Let $\pi \colon H \to V_\xi$ be elementary, where ξ is large, everything relevant is in ran(π), and H is countable and transitive.

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Let $\alpha = \text{crit}(\pi)$. We have $\pi \restriction (\mathcal{T} \restriction \alpha) = \text{ id}$ and $\pi(\mathcal{T} \restriction \alpha) = \mathcal{T} \restriction \omega_1$. Since the branch $[0, \omega_1]_T$ is club, $\alpha \in [0, \omega_1]_T$. Similarly on the $\mathcal{U}$ side. Moreover,

$$\mathcal{T} \restriction \alpha + 1 = \pi^{-1}(\mathcal{T})$$

and

$$\mathcal{U} \restriction \alpha + 1 = \pi^{-1}(\mathcal{U}).$$

This implies

$$i_{\alpha,\omega_1} = \pi \restriction P_\alpha$$

and

$$j_{\alpha,\omega_1} = \pi \restriction Q_\alpha.$$

To see this, for $y \in P_\alpha$, let $y = i_{\beta,\alpha}(x)$ where $\beta < \alpha$; then π fixes β and x, so

$$\begin{aligned} \pi(y) &= \pi(i_{\beta,\alpha}(x)) \\ &= i_{\beta,\omega_1}(x) \\ &= i_{\alpha,\omega_1} \circ i_{\beta,\alpha}(x) \\ &= i_{\alpha,\omega_1}(y), \end{aligned}$$

as desired. Also,

$$P(\alpha)^{P_\alpha} = P(\alpha)^{P_{\omega_1}} = P(\alpha)^{Q_{\omega_1}} = P(\alpha)^{Q_\alpha}.$$

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Thus π, i_{α,ω_1}, and j_{α,ω_1} all generate the same (α, ω_1)-extender; call it G.

Let E be the first extender used in $\mathcal{T}$ along $[\alpha, \omega_1]_T$. Because generators are not moved, E is an initial segment of G. That is, for $X \subseteq \text{crit}(E)$ in P_α and $a \subseteq \lambda_E$ finite, letting $E = E_\gamma^{\mathcal{T}}$,

$$\begin{aligned} X \in E_a &\Leftrightarrow a \in i_E(X) \\ &\Leftrightarrow a \in i_{\gamma+1,\omega_1} \circ i_E(X) \\ &\Leftrightarrow a \in i_{\alpha,\omega_1}(X) \\ &\Leftrightarrow X \in G_a. \end{aligned}$$

Line 2 follows from line 1 because $a \in [\lambda_E]^{<\omega}$ and $\text{crit}(i^{\mathcal{T}}_{\gamma+1,\omega_1}) \geq \lambda_E$.

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Remark. This is the raison de' etre for iteration trees! Our comparison process could enter a loop if we let generators be moved. Linear iterations are provably insufficient for comparison once we reach mice with Woodin cardinals.

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Remark. Comparing mice with long extenders seems to require moving the long generators.Comparing iteration strategies seems to require moving generators.

Remark Even if $E = E_\alpha$ is total over M_α, we may have $\kappa_E < \lambda(E_\beta)$ where T-pred$(\alpha + 1) = \beta < \alpha$, and E *not* be total over M_β. What we do then is take strongest E-ultrapower of an initial segment of M_β that we can. This is the only reasonable thing to do. It means that once we go beyond linear iterations, we need fine structure theory from the beginning.

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By the Jensen initial segment condition, E is then the first whole initial segment of G that is not on the sequence of the common lined up part $N = P_{\omega_1}|\omega_1 = Q_{\omega_1}|\omega_1$. For the same reasons, F is the first whole initial segment of G that is not on the N-sequence. Thus $E = F$. Since $\mathrm{lh}(E) = \mathrm{lh}(F)$, they were used at the same stage in the comparison. But we were iterating away disagreements, so $E \neq F$, contradiction. $\square$

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Thus the comparison terminates at some $\alpha < \omega_1$, with $R = P_\alpha$ and $S = Q_\alpha$ such that $R \trianglelefteq S$ or $S \trianglelefteq R$. If $R \triangleleft S$, then R is sound, and therefore the branch P-to-R did not drop, so we have an iteration map P-to-R, yielding conclusion (a). Similarly, if $S \triangleleft R$ we get conclusion (b).

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Thus we may assume $R = S$. It is now enough to show

that one of the two branches P-to-R and Q-to-S did not drop. Assume otherwise, and let C be the common core,

$$C = \text{core}(R) = \text{core}(S)$$

and let

$$\pi \colon C \to R$$

be the anticore map. C occurs on both branches, and the fact that iteration maps preserve fine structure implies that π is the iteration map of both the branch C-to-R of $\mathcal{T}$, and the branch C-to-S of $\mathcal{U}$. But as in the termination proof, this means the first extenders used in these two branches are the same, a contradiction.
This proves the Comparison Lemma. $\square$

Strategy uniqueness

We don't have a full comparison lemma or a mouse order yet, because how two mice compare might depend on the iteration strategies that are used to compare them. This can happen if the mice have Woodin cardinals. But otherwise, their iteration strategies are unique, and we do have a full comparison, and a mouse order.

Definition

A premouse M is *Q-full* iff M has a largest cardinal, and whenever $M \models \delta$ is Woodin, then $\rho_\omega(M) < \delta$.

Examples:

(1) Any mouse cut at a successor cardinal below its bottom Woodin.

(2) Any active premouse that projects to ω (e.g. $M_n^\sharp$, $M_\omega^\sharp$).

(3) Not M_n or M_ω.

Theorem

Suppose M is sound and Q-full; then M has at most one $|M|^+ + 1$-iteration strategy.

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Proof. Suppose Σ and Γ are distinct strategies, and $\Sigma(\mathcal{T}) = b$ and $\Gamma(\mathcal{T}) = c$ where $b \neq c$. Set

$$\delta = \delta(\mathcal{T}) = \sup(\{\mathrm{lh}(E_\alpha^{\mathcal{T}}) \mid \alpha < \mathrm{lh}(\mathcal{T})\}).$$

Lemma (Martin, S., 1985)

δ is Woodin in $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ with respect to all $f \colon \delta \to \delta$ such that $f \in \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$.

We now compare $\mathcal{T}^\frown b$ with $\mathcal{T}^\frown c$, using Σ and Γ to continue the two trees.

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This results in trees $\mathcal{U}$ and $\mathcal{V}$ extending $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ with last models R and S. We may assume wlog that $R \trianglelefteq S$ and M-to-R has not dropped. Then

$$P(\delta)^R \subseteq \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}},$$

so

$$R \models \delta \text{ is Woodin.}$$

So M has Woodin cardinals, and projects below the least of them. Thus R is unsound, so $R = S$. Letting $i \colon M \to R$ and $j \colon M \to S$ be the branch embeddings of $\mathcal{U}$ and $\mathcal{V}$, we get $i = j =$ anticore map. As in the proof of the comparison lemma, this leads to a contradiction. □

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Corollary

Assume AD^+*, and let* M *be a countable, Q-full,* ω_1*-iterable premouse; then its unique* ω_1*-iteration strategy is coded by a* $\Delta^2_1(\{M\})$ *of reals.*

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Stacks of normal trees

Definition

$G(M, \lambda, \theta)$ is the game in which the players play λ rounds, the α-th round being a play of $G(N, \theta)$, where N is the last model of round $\alpha - 1$, or the direct limit along the branch produced by the prior rounds if α is a limit.

In $G(M, \lambda, \theta)$ I moves at successor stages, by playing an extender or starting a new round if he wishes. If the current round lasts θ moves, then there are no further rounds, and the game is over.
II picks branches at limit stages, and his obligation is just to insure all models are wellfounded, including the direct limit of the base models in the final stack of length λ.

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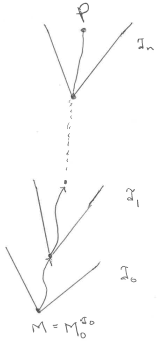
M-stacks

s a stack on M.

$i_s : M \to P$

each $\mathcal{T}_i$ normal



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Definition

A (λ, θ)*-iteration strategy* is a M is a winning strategy for II in $G(M, \lambda, \theta)$.

A Q-full M can have at most one $(\lambda, |M|^+ + 1)$-iteration strategy, by the proof for $\lambda = 1$ given above.

Definition

Let M be a premouse; then M is *countably iterable* iff every countable elementary submodel of M is $(\omega_1, \omega_1 + 1)$-iterable.

Countable iterability is what one needs to prove that M is well-behaved in a fine structural sense; for example, that its standard parameter is solid and universal.

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Definition

s is an M-stack iff s is a position in some $G(M, \lambda, \theta)$ with I to move that is not yet a loss for II.

Iterates of an iterable structure are iterable, via a *tail strategy*.

Definition (Tail strategy)

Let Ω be a (λ, θ) iteration strategy for M, and s be an M-stack according to Ω with $\mathrm{lh}(s) < \lambda$, and N be the last model of s; then then Ω_s is the $(\lambda - \mathrm{lh}(s), \theta)$ strategy for N

$$\Omega_s(t) = \Omega(s^\frown t).$$

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Let $\pi: M \to N$ be elementary, $\mathcal{T}$ on M with models M_α and extenders E_α. We lift $\mathcal{T}$ to a copied tree $\pi\mathcal{T}$ on N with the same tree order as $\mathcal{T}$, models N_α and extenders F_α. The construction produces elementary copy maps

$$\pi_\alpha : M_\alpha \to N_\alpha,$$

such that

(1) if $\beta \leq \alpha$, then $\pi_\alpha \restriction \mathrm{lh}(E_\beta) = \pi_\beta \restriction \mathrm{lh}(E_\beta)$ and $N_\alpha | \mathrm{lh}(F_\beta) = N_\beta | \mathrm{lh}(F_\beta)$, and

(2) if $\beta \leq_T \alpha$, then $\pi_\alpha \circ i^{\mathcal{T}}_{\beta,\alpha} = i^{\pi\mathcal{T}}_{\beta,\alpha} \circ \pi_\beta$.

Set $\pi_0 = \pi$. The successor step is as follows: let $E = E_\alpha$, $\beta = T\text{-pred}(\alpha + 1)$, and

$$F = \pi_\alpha(E),$$
$$P = \mathcal{M}^{*,\mathcal{T}}_{\alpha+1},$$
$$Q = \pi_\beta(P).$$

$F_\alpha = F$, and for $k = k(P)$, let

$$\pi_{\alpha+1} \colon \mathrm{Ult}_k(P, E) \to \mathrm{Ult}_k(Q, F)$$

be the completion of the map

$$\pi_{\alpha+1}([a, f]^{P^k}_E) = [\pi_\alpha(a), \pi_\beta(f)]^{Q^k}_F,$$

for $a \in [\lambda(E)]^{<\omega}$.

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Since the copy maps commute with the branch embeddings, at limit steps λ we have a unique elementary $\pi_\lambda \colon M_\lambda \to N_\lambda$ that commutes with the branch embeddings of $\mathcal{T}$ and $\pi\mathcal{T}$ along $[0,\lambda)_T$. It is easy to check (1) and (2).

If $\pi\mathcal{T}$ ever reaches an illfounded model, we stop the construction.

We can copy stacks of plus trees by successively copying the individual plus trees in the stack.

Definition

If Ω is an iteration strategy for N, and $\pi \colon M \to N$ is elementary, then Ω^π is the pullback strategy for M, given by

$$\Omega^\pi(s) = \Omega(\pi s),$$

for all s such that $\pi s \in \mathrm{dom}(\Omega)$.

If Ω is a (λ, θ)-iteration strategy for N, then Ω^π is a (λ, θ) iteration strategy for M.

Corollary

Every elementary submodel of a mouse is also a mouse.

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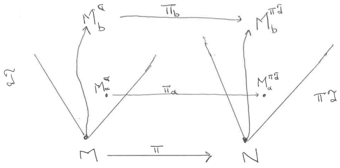
Pullback strategies

Given Σ for N, and $\pi : M \to N$



if $b = \Sigma(\pi\mathcal{T})$

then $\Sigma^\pi(\mathcal{T}) = b$

The Dodd-Jensen Lemma

Lemma (Dodd-Jensen)

Let M be Q-full, and $(\theta, \theta + 1)$-iterable, where $\theta = |M|^+$. Let Σ be the unique $(\theta, \theta + 1)$ strategy for M, and s a stack by Σ with last model N, and let

$$\pi \colon M \to P \trianglelefteq N$$

be elementary; then

(1) the branch M-to-N of s does not drop, $P = N$, and

(2) for $i \colon M \to N$ the iteration map, $i(\eta) \leq \pi(\eta)$ for all $\eta < o(M)$.

Proof. On the board. It is crucial that $(\Sigma_s)^\pi = \Sigma$. This follows from the uniqueness of Σ. □

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So the iteration maps on Q-full, iterable mice are pointwise minimal. This implies they are unique.

Corollary

Let M be Q-full, and $(\theta, \theta+1)$-iterable, where $\theta = |M|^+$. Let Σ be the unique $(\theta, \theta+1)$ strategy for M, and s and t stacks by Σ with last models P and Q such that $P \trianglelefteq Q$. Then $P = Q$, and if M-to-P does not drop, then M-to-Q does not drop, and the two iteration maps are equal.

Remark There can be distinct non-dropping stacks going from M-to-N. It's just the iteration maps that must be equal.

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The mouse order on Q-full mice.

Because it is the most important context for us, we shall assume AD^+ and consider only countable mice. Let M and N be countable, Q-full, (ω_1, ω_1)-iterable premice then

$$M \leq^* N \text{ iff } \exists R, S, \pi (S \text{ is a countable iterate of } N \\ \wedge R \trianglelefteq S \wedge \pi\colon M \to R \text{ is elementary}).$$

Remarks.

(a) $\leq^*$ is a prewellorder of order-type $\leq$ boldface δ^2_1.

(b) $M \leq^* N$ iff when you compare them via least disagreement, with last models R on the M side and S on the N side, then M-to-R doesn't drop and$R \trianglelefteq S$. Show $M <^* N$ iff either $R \triangleleft S$ or N-to-S drops.

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Remarks. Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}) + \mathsf{MC})$; then

(i) the codesets for ω_1-iteration strategies for Q-full mice are Wadge cofinal in the boldface $\boldsymbol{\Delta}^2_1$ sets of reals, and

(ii) the order type of $\leq^*$ is exactly boldface $\boldsymbol{\delta}^2_1$.

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Mouse limits and HOD

Assume AD^+, and let P be countable, Q-full, and (ω_1, ω_1)-iterable. We define a direct limit system $(\mathcal{F}_P, \prec)$ by

$$Q \in \mathcal{F}_P \text{ iff } Q \text{ is a countable, non-dropping iterate of } P,$$

and for $Q, R \in \mathcal{F}_P$,

$$Q \prec R \text{ iff } R \text{ is a non-dropping iterate of } Q.$$

For $Q \prec R$, let $\pi_{Q,R} \colon Q \to R$ be the (unique) iteration map; then we set

$$M_\infty(P) = \text{dirlim}(\mathcal{F}_P, \prec),$$

where the direct limit is under the $\pi_{Q,R}$ for $Q \prec R$. $(\mathcal{F}_P, \prec)$ is countably directed, so $M_\infty(P)$ is wellfounded, and we then take it to be transitive.

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Exercise. Assume AD^+.

(a) $P \equiv^* Q$ iff $M_\infty(P) = M_\infty(Q)$.

(b) $M_\infty(P) \in \mathrm{HOD}$.

(c) $o(M_\infty(P)) <$ boldface δ^2_1.

(d) $|o((M_\infty(P))|$ is a Suslin cardinal. (Hint: the Kunen-Martin theorem is relevant.)

Definition

P is *full* iff P is Q-full, and whenever $P \leq^* Q$, then $M_\infty(P) \trianglelefteq M_\infty(Q)$.

Theorem (S. 1994)

Assume AD^+*, and suppose Mouse Capturing holds; then*

$$HOD|\delta^2_1 = \bigcup_{P \text{ full}} M_\infty(P).$$

So under these hypotheses, $\mathrm{HOD}|\delta^2_1$ is a premouse, and in particular satisfies GCH.

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Mouse pairs

We'd like a full comparison theorem for mice with Woodin cardinals, but their iteration strategies are not unique. We need to compare the strategies too, i.e. to compare *pairs* consisting of a mouse and an iteration strategy for it. This requires that the iteration strategies have certain regularity properties: strong hull condensation, normalizing well, and internal lift consistency.

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Strong hull condensation

Roughly, Σ has *strong hull condensation* iff whenever $\mathcal{U}$ is a normal tree on P by Σ, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is appropriately elementary, then $\mathcal{T}$ is by Σ.

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Strong hull condensation

Roughly, Σ has *strong hull condensation* iff whenever $\mathcal{U}$ is a normal tree on P by Σ, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is appropriately elementary, then $\mathcal{T}$ is by Σ.

One must be careful about the elementarity required of Φ, and in particular, the extent to which Φ is required to preserve exit extenders. There are several possible condensation properties here: hull condensation (Sargsyan), strong hull condensation, and still stronger ones.

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Strong hull condensation means condensing under *tree embeddings*.

Definition

A *tree embedding* of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, v, \langle s_\beta \mid \beta < \mathrm{lh}\mathcal{T}\rangle, \langle t_\beta \mid \beta + 1 < \mathrm{lh}\mathcal{T}\rangle\rangle$$

with various properties, including:

$$E^{\mathcal{U}}_{u(\alpha)} = t_\alpha(E^{\mathcal{T}}_\alpha).$$

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The diagram related to successor steps in $\mathcal{T}$ is:

$$\begin{tikzcd}
M^{\mathcal{T}}_{\alpha+1} \arrow[rr, "s_{\alpha+1}"] & & M^{\mathcal{U}}_{v(\alpha+1)} \\
& M^{\mathcal{U}}_{u(\beta)} & \\
& & M^{\mathcal{U}}_{\beta^*} \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \arrow[ul] \\
M^{\mathcal{T}}_{\beta} \arrow[uuu, "E^{\mathcal{T}}_{\alpha}"] \arrow[uur, "t_\beta"] \arrow[urr, "\rho"] \arrow[rr, "s_\beta"] & & M^{\mathcal{U}}_{v(\beta)} \arrow[u] \\
M^{\mathcal{T}}_{\alpha} \arrow[rr, "t_\alpha"] & & M^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

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Normalizing well

For $\langle \mathcal{T}, \mathcal{U} \rangle$ a stack on P, we can re-order the use of extenders so as to produce a normal tree $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$, and an embedding of the last model of $\mathcal{U}$ into the last model of $\mathcal{W}$. We call $W(\mathcal{T}, \mathcal{U})$ the *embedding normalization* of the stack $\langle \mathcal{T}, \mathcal{U} \rangle$.

$$\begin{tikzcd} P \arrow[r, "i^{\mathcal{T}}"] \arrow[drr, "i^{\mathcal{W}}"'] & Q \arrow[r, "i^{\mathcal{U}}"] & R \arrow[d, "\pi"] \\ & & S \end{tikzcd}$$

Then Σ *2-normalizes well* iff for all $\langle \mathcal{T}, \mathcal{U} \rangle$,

$$\langle \mathcal{T}, \mathcal{U} \rangle \text{ is by } \Sigma \text{ iff } W(\mathcal{T}, \mathcal{U}) \text{ is by } \Sigma,$$

and

$$\Sigma^{\pi}_{\langle \mathcal{W} \rangle} = \Sigma_{\langle \mathcal{T}, \mathcal{U} \rangle}.$$

Σ *normalizes well* iff all its tails 2-normalize well.

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Internal lift consistency

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Given $M \trianglelefteq N$ and s a stack on N, we can lift s to a stack s^+ on N. (Like copying under the identity map, but the ultrapowers on the N side can use more functions.) A strategy Σ for N is *internally lift consistent* iff whenever $M \trianglelefteq N$ and s^+ is the lift of s on M, then

$$s \text{ is by } \Sigma_M \text{ iff } s^+ \text{ is by } \Sigma.$$

Strategy mice

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Definition

A *least branch premouse* (lpm) is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders, and a predicate $\dot{\Sigma}^{\mathcal{M}}$ for an iteration strategy for $\mathcal{M}$.

Remarks

(a) $\mathcal{M}$ has a hierarchy, and a fine structure.

(b) We use Jensen indexing for the extenders in $\dot{E}^{\mathcal{M}}$.

(c) At strategy-active stages in an lpm, we tell $\mathcal{M}$ the value of $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$, where $\mathcal{T}$ is the $\mathcal{M}$-least tree such that $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$ is currently undefined.

Mouse pairs

Definition

A *mouse pair* is a pair (P, Σ) such that

(1) P is a countable premouse (pure extender or least branch),

(2) Σ is an iteration strategy defined on all countable stacks on P,

(3) Σ normalizes well, has strong hull condensation, and is internally lift consistent,

(4) if P is an lpm, then Σ is pushforward consistent; i.e. whenever Q is a Σ-iterate of P via s, then $\dot{\Sigma}^{Q} \subseteq \Sigma_s$.

We have limited the notion to countable mice and (ω_1, ω_1)-strategies to smooth some statements later.

Elementary properties of mouse pairs

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Definition

$\pi \colon (P, \Sigma) \to (Q, \Psi)$ is *elementary* iff $\pi \colon P \to Q$ is Σ_k elementary, where $k = k(P)$, and $\Sigma = \Psi^\pi$.

Lemma

An elementary submodel of a mouse pair is a mouse pair.

Elementary properties of mouse pairs

Definition

$\pi\colon (P, \Sigma) \to (Q, \Psi)$ is *elementary* iff $\pi\colon P \to Q$ is Σ_k elementary, where $k = k(P)$, and $\Sigma = \Psi^\pi$.

Lemma

An elementary submodel of a mouse pair is a mouse pair.

Definition

(Q, Ψ) is an *iterate of* (P, Σ) iff there is a stack s by Σ with last model Q, and $\Psi = \Sigma_s$.

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Lemma

(Iteration maps are elementary) Let (P, Σ) be a mouse pair, and let s be a stack by Σ giving rise to the iteration map $\pi\colon P \to Q$; then $(\Sigma_s)^\pi = \Sigma$.

This property of Σ is called *pullback consistency*.

Lemma

(Dodd-Jensen) The Σ-iteration map from (P, Σ) to (Q, Ψ) is the pointwise minimal elementary embedding of (P, Σ) into (Q, Ψ).

Remark. The concept of mouse pair lets us state the Dodd-Jensen in its proper generality.

Comparison

Theorem (Comparison)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable; then they have a common iterate* (R, Φ) *such that* R *is countable and at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

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Comparison

Theorem (Comparison)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable; then they have a common iterate* (R, Φ) *such that* R *is countable and at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

Definition

(Mouse order) $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) embeds elementarily into some iterate of (Q, Ψ).

Corollary

Assume AD^+*; then the mouse order* $\leq^*$ *on mouse pairs of a fixed type is a prewellorder.*

Remark. Again, there is no mouse order on mice with Woodin cardinals.

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Mouse pair constructions

Theorem (Woodin, late 1980s)

(AD$^+$)) For any Suslin-co-Suslin set B, there is an $(N, \tau, \delta, \Sigma)$ that coarsely captures B.

This means:

(a) N is countable, $N \models \mathrm{ZFC} +$ "δ is Woodin",

(b) Σ is an iteration strategy for N defined on all $s \in \mathrm{HC}$, and $\Sigma \restriction V_\delta^N \in N$, and

(c) if $i \colon N \to M$ is an iteration map by Σ, and g is $\mathrm{Col}(\omega, i(\delta))$-generic over M, then $i(\tau)_g = B \cap M[g]$.

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Inside N, we have the maximal (pure extender, lbr hod) pair construction $\langle (M_{\nu,k}, \Omega_{\nu,k}) \mid \langle \nu, k \rangle \leq_{\text{lex}} \langle \delta, 0 \rangle \rangle$:

(a) each $(M_{\nu,k}, \Omega_{\nu,k})$ is a mouse pair,

(b) an E gets added to the sequence of $M_{\nu,0}$ whenever doing so produces a premouse, and E extends to a nice extender E^* in N,

(c) $\Omega_{\nu,k}$ is the strategy for $M_{\nu,k}$ that is induced by Σ,

(d) information about $\Omega_{\nu,k}$ is inserted at strategy-active stages, and

(e) $(M_{\nu,k+1}, \Omega_{\nu,k+1}) = \text{core}(M_{\nu,k}, \Omega_{\nu,k})$.

Comparison arguments show that the construction never breaks down; all levels are mouse pairs whose cores exist, and the E added in (b) is unique.

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The main lemma is

Lemma

Assume AD^+, *let* (P, Σ) *be a mouse pair, and let* (N, Ψ) *be a coarse* Γ*-Woodin pair such that* $P \in HC^N$ *and* (N, Ψ) *captures Code*(Σ)*. Let* $\mathbb{C}$ *be the maximal full background construction of* N *for pairs of the same type; then there is a level* (M, Ω) *of* $\mathbb{C}$ *such that*

(a) (P, Σ) *iterates to* (M, Ω)*, and*

(b) (P, Σ) *iterates strictly past all levels of* $\mathbb{C}$ *that are strictly earlier than* (M, Ω)*.*

Let us sketch why no strategy disagreements show up when we compare (P, Σ) with a level (R, Λ) of $\mathbb{C}$. Let $\mathcal{T}$ from P to R be by Σ. Let $\mathcal{U}$ on R be by both $\Sigma_{\langle \mathcal{T} \rangle}$ and Λ, and let $\Lambda(\mathcal{T}) = b$. We must show $\langle \mathcal{T}, \mathcal{U}^\frown b \rangle$ is by Σ. Let $\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b)$ be its embedding normalization.

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$$
\begin{tikzcd}
M_b^{\mathcal{U}} \arrow[r] & M_\infty^{\mathcal{W}_b} \arrow[r] & i_b^*(R) & N_b \\
R \arrow[u, "\mathcal{U}"] & & R \arrow[u, "i_b^*"'] & N \arrow[u, "i_b^*"'] \\
P \arrow[u, "\mathcal{T}"] \arrow[uur, "\mathcal{W}_b"] \arrow[uurr, "i_b^*(\mathcal{T})"'] & & &
\end{tikzcd}
$$

(i) $\mathcal{T}$ is by Σ, so $i_b^*(\mathcal{T})$ is by Σ.

(ii) There is a tree embedding of $\mathcal{W}_b$ into $i_b^*(\mathcal{T})$, so $\mathcal{W}_b$ is by Σ by strong hull condensation.

(iii) Since Σ normalizes well, $\langle \mathcal{T}, \mathcal{U}^\frown b\rangle$ is by Σ.

Least disagreement comparison

A more effective comparison process yields

Theorem (Sargsyan, S. 2024)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable and coded by reals* x_P, x_Q*. Let* T_Σ *and* T_Ψ *be Suslin representations of the codesets of the two strategies; then* (P, Σ) *and* (Q, Ψ) *have a common iterate* (R, Φ) *such that* R *is countable in* $L[x_p, x_Q, T_\Sigma, T_\Psi]$*.*

That the more effective process succeeds relies on results proved using the less effective one.

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Hod pair capturing

Least branch hod pairs can be used to compute HOD, provided that there are enough of them.

Definition

(AD^+) *HOD pair capturing* (HPC) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P, Σ) with scope HC such that A is definable over $(\mathrm{HC}, \in, \Sigma)$.

Remarks.

(a) Under AD^+, if (P, Σ) is a mouse pair, then Code(Σ) is Suslin and co-Suslin.

(b) HPC implies that every Suslin-co-Suslin set of reals A is in a symmetric extension of some hod pair (P, Σ). So the theory of $L(A, \mathbb{R})$ is definable over P.

(c) MC implies HPC. We would guess the converse is true, but do not have a proof.

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HPC holds in the minimal model of $\mathrm{AD}_{\mathbb{R}} + \theta$ is regular, and somewhat beyond, by Sargsyan's work.

Theorem (Sargsyan 2023)

Assume AD^{+} + $\neg$HPC*; then there is an lbr hod pair* (P, Σ) *such that*
$P \models \mathrm{ZFC} +$ *"there is a Woodin limit of Woodin cardinals".*

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Definition

NLE is the statement: there is no ω_1-iteration strategy for a premouse with a long extender on its sequence.

There is no general notion of premice with long extenders yet, but we do have a theory for premice with "not too many" long extenders. NLE says we are in the initial segment of the Wadge hierarchy below the first iteration strategy for such a premouse.

Theorem

Assume AD^+*, and that there is an iterable premouse with a long extender. Let* $\Gamma \subseteq P(\mathbb{R})$ *be such that* $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$ *; then* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*.*

In light of this theorem, the following is almost certainly true:

Conjecture. $(\mathsf{AD}^+ + \mathsf{NLE}) \Rightarrow \mathsf{HPC}$.

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HOD as a mouse limit

Definition
(AD^+) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

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HOD as a mouse limit

Definition
(AD^+) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

$M_\infty(P, \Sigma)$ is well-defined by the Dodd-Jensen lemma. Moreover, it is OD from the rank of (P, Σ) in the mouse order. Thus $M_\infty(P, \Sigma) \in \mathrm{HOD}$.

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(AD^+) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

$M_\infty(P, \Sigma)$ is well-defined by the Dodd-Jensen lemma. Moreover, it is OD from the rank of (P, Σ) in the mouse order. Thus $M_\infty(P, \Sigma) \in \mathrm{HOD}$. It is an initial segment of the lpm hierarchy of HOD *if* (P, Σ) is "full".

Definition
A mouse pair (P, Σ) is full iff for all mouse pairs (Q, Ψ) such that $(P, \Sigma) \leq^* (Q, \Psi)$, we have $M_\infty(P, \Sigma) \trianglelefteq M_\infty(Q, \Psi)$.

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Theorem

Assume $\mathrm{AD}_{\mathbb{R}} + \mathrm{HPC}$*; then* $HOD|\theta$ *is the union of all* $M_\infty(P,\Sigma)$ *such that* (P,Σ) *is a full lbr hod pair.*

Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then* $HOD|\theta$ *is an lpm. Thus* $HOD \models \mathrm{GCH}$*.*

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Characterizing the Woodins of HOD

Recall the *Solovay sequence*: θ_0 is the sup of the lengths of OD prewellorders of $\mathbb{R}$, $\theta_{\alpha+1}$ is the sup of the OD(A) prewellorders, for any and all A of Wadge rank θ_α, and $\theta_\lambda = \bigcup_{\alpha<\lambda} \theta_\alpha$ for λ a limit.

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Definition

κ is a *cutpoint* of a premouse $\mathcal{M}$ iff there is no extender E on the $\mathcal{M}$-sequence such that $\mathrm{crit}(E) < \kappa \leq \mathrm{lh}(E)$.

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Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then equivalent are:*

(a) *δ is a cutpoint Woodin cardinal of HOD,*

(b) *$\delta = \theta_0$, or $\delta = \theta_{\alpha+1}$ for some α.*

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κ is a *cutpoint* of a premouse $\mathcal{M}$ iff there is no extender E on the $\mathcal{M}$-sequence such that $\mathrm{crit}(E) < \kappa \leq \mathrm{lh}(E)$.

Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then equivalent are:*

(a) δ *is a cutpoint Woodin cardinal of HOD,*

(b) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

Thus θ_0 *is the least Woodin cardinal of HOD.*

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Characterizing the Woodins of HOD

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Definition

κ is a *cutpoint* of a premouse $\mathcal{M}$ iff there is no extender E on the $\mathcal{M}$-sequence such that $\mathrm{crit}(E) < \kappa \leq \mathrm{lh}(E)$.

Theorem

Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then equivalent are:*

(a) δ *is a cutpoint Woodin cardinal of HOD,*

(b) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

Thus θ_0 *is the least Woodin cardinal of HOD.*

Remark. Woodin showed θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD. He proved an approximation to their being cutpoints.

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Theorem

Assume $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$*, and let* κ *be a successor cardinal of HOD such that* $\kappa < \theta$*. Let*

$$\delta = \sup(\{|S| \mid S \text{ is an OD prewellorder of } {}^{\omega}\kappa\}).$$

Then δ *is the least Woodin cardinal of HOD above* κ*.*

Remark. This was conjectured by Sargsyan.
The construction of Suslin representations for the iteration strategies in mouse pairs plays an important role in many of the proofs above.

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Suslin representations for mouse pairs

Let (P, Σ) be a mouse pair. A tree $\mathcal{T}$ by Σ is *M_∞-relevant* iff there is a normal $\mathcal{U}$ by Σ extending $\mathcal{T}$ with last model Q such that the branch P-to-Q does not drop. Σ^{rel} is the restriction of Σ to M_∞-relevant trees.

Recall that A is *κ-Suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$.

Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

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Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

Remark. Code(Σ^{rel}) is not α-Suslin, for any $\alpha < |M_\infty(P, \Sigma)|$, by Kunen-Martin. So $|M_\infty(P, \Sigma)|$ is a Suslin cardinal.

Proof sketch. $M_\infty(P,\Sigma)$ is the direct limit along a generic stack s of trees by Σ.

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Proof sketch. $M_\infty(P, \Sigma)$ is the direct limit along a generic stack s of trees by Σ. But s can be fully normalized, so there is a single normal tree $\mathcal{W}$ on P with last model $M_\infty(P, \Sigma)$ such that every countable "weak hull" of $\mathcal{W}$ is by Σ.

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$$\mathcal{T} \text{ is by } \Sigma \Leftrightarrow \quad \exists \Phi \colon \mathcal{T} \to \mathcal{U}$$
$$(\Phi \text{ is a weak tree embedding.})$$

The right-to-left direction follows from very strong hull condensation for Σ.

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(Φ is a weak tree embedding.)

The right-to-left direction follows from very strong hull condensation for Σ.
For left-to-right direction, we may assume $\mathcal{T}$ has last model Q, and P-to-Q does not drop. We then have a normal $\mathcal{U}$ on Q with last model $M_\infty(P,\Sigma)$ such that all countable weak hulls of $\mathcal{U}$ are by Σ.

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We have

$$
\begin{tikzcd}
P \arrow[rr, "\mathcal{W}"] \arrow[dr, "\mathcal{T}"'] & & M_\infty(P,\Sigma) \\
& Q \arrow[ur, "\mathcal{U}"'] &
\end{tikzcd}
$$

Then

$$\mathcal{W} = X(\mathcal{T},\mathcal{U})$$

is the full normalization of $\langle \mathcal{T},\mathcal{U}\rangle$. The construction of $X(\mathcal{T},\mathcal{U})$ produces a weak tree embedding from $\mathcal{T}$ into $X(\mathcal{T},\mathcal{U})$, which is what we want.

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Thus our Suslin representation verifies that $\mathcal{T}$ is in the M_∞-relevant part of Σ by producing a weak tree embedding of $\mathcal{T}$ into $\mathcal{W}$.

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Suslin cardinals and mouse limits

Recall that κ is a Suslin cardinal iff there is a set of reals that is κ-Suslin, but not α-Suslin for any $\alpha < \kappa$.

Theorem (Jackson, Sargsyan, S. 2018-2019)

Assume AD^+*. Let* (P, Σ) *be a mouse pair, and let* $\kappa < o(M_\infty(P, \Sigma))$*; then equivalent are*

(a) κ *is a Suslin cardinal,*

(b) $\kappa = |\tau|$ *for some cutpoint* τ *of* $M_\infty(P, \Sigma)$*.*

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Corollary

Assume $\mathsf{AD}^+ + \mathsf{HPC}$*; then equivalent are*

(a) κ *is a Suslin cardinal,*

(b) $\kappa = |\tau|$*, for some cutpoint* τ *of HOD.*

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Woodin limits of Woodins have more strength than one might guess.

Theorem (Sargsyan, S. 2018)

Assume AD^+*, and that there is an lbr hod pair* (P, Σ) *such that* $P \models \mathsf{ZFC}$ *+ "*δ *is a Woodin limit of Woodin cardinals + "there are infinitely many Woodin cardinals above* δ*". Then there is a pointclass* Γ *such that*

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(1) $L(\Gamma, \mathbb{R}) \models$ *"the largest Suslin cardinal exists, and belongs to the Solovay sequence" (*LSA*), and*

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(1) $L(\Gamma, \mathbb{R}) \models$ "the largest Suslin cardinal exists, and belongs to the Solovay sequence" (LSA), and

(2) $L(\Gamma, \mathbb{R}) \models$ "if A is a set of reals that is $OD(s)$ for some $s \colon \omega \to \theta$, then A is Suslin and co-Suslin".

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Part (1) is due to Sargsyan, and requires weaker hypotheses on *P*. The insight that Woodin limits of Woodins are what you need for (2) is due to Sargsyan.

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HOD pairs and Chang models

Relatives of the following theorems were proved earlier by Woodin.

Theorem (Gappo, Sargsyan 2022)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) *such that* P *is countable,* Σ *is coded by a uB set, and* $P \models \mathrm{ZFC}+$ *"there is a Woodin limit of Woodin cardinals"; then the Chang model* $L({}^{\omega}OR)$ *satisfies* AD*.*

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Let $F(\alpha, X)$ iff $X \subseteq P_{\omega_1}({}^\omega\alpha)$ and contains a club in $P_{\omega_1}({}^\omega\alpha)$.

Corollary (to proof)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) such that P is countable, Σ is coded by a uB set, and $P \models \mathrm{ZFC}+$"there is a measurable Woodin cardinal". Let $F(\alpha, X)$ iff X contains a club in $P_{\omega_1}({}^\omega\alpha)$; then

(1) $L({}^\omega OR)[F] \models \mathrm{AD}_{\mathbb{R}}$, *and*

(2) $L({}^\omega OR)[F] \models$ *"for all α, $\{X \mid F(\alpha, X)\}$ is an ultrafilter".*

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Remarks

(i) The model of the corollary satisfies $\mathsf{AD}_{\mathbb{R}}$ plus "ω_1 is X-supercompact, for all sets X.

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(i) The model of the corollary satisfies $AD_{\mathbb{R}}$ plus "ω_1 is X-supercompact, for all sets X.

(ii) We don't see how to reduce the mouse-existence hypothesis in the corollary to that in the theorem. Both proofs lean heavily of the theory of hod mice, and on the proofs of approximations to HPC that we have now.

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(ii) We don't see how to reduce the mouse-existence hypothesis in the corollary to that in the theorem. Both proofs lean heavily of the theory of hod mice, and on the proofs of approximations to HPC that we have now.

(iii) Woodin had already found a proof of the same conclusions from a proper class of Woodin limits of Woodins, using results of Neeman on iterability and long game determinacy at that level.

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(iv) In the Gappo-Sargsyan proof, initial segments of the Chang model in question get realized as generalized derived models associated to iterates of (P, Σ).

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(v) Douglas Blue and Sargsyan have very recently shown that these generalized derived models ("Nairian models") satisfy a natural extension of $\mathsf{AD}_{\mathbb{R}}$ to objects of higher type known as AD_2.

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Thank you!

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