Remarks on a multiverse language

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January 2021

Introduction

We shall discuss

References

(1) Gödel's program, in *Interpreting Gödel*, Juliette Kennedy ed., Cambridge Univ. Press 2014.

(2) A philosophical reconstruction of Steel's multiverse project, P. Maddy and T. Meadows, Bulletin of Symbolic Logic.

(3) Normalizing iteration trees and comparing iteration strategies, available at www.math.berkeley.edu/∼steel

Plan:

I. Some philosophy.
II. Maximize.
III. A multiverse language and theory.
IV. HOD in models of the Axiom of Determinacy.

I. Some philosophy

Naturalism, Holism

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(3) Epistemology should analyze what has worked in the past, as a guide to what might work in the future.

(4) Mathematical thought goes back into pre-history, and can be studied from many angles.

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(6) Language and theory (conceptual systems) belong primarily to social groups, not individuals. "Self-evidence" is too individual-dependent to count much. Extrinsic evidence today leads to self-evidence tomorrow.

(7) Mathematics is ahead of the rest of science in the degree to which its language and theory have been made clear and explicit, i.e. formalized. *Metamathematics* is possible, and essential to the epistemology of set theory.

Indispensability

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(3) Powerful unifying principles are preferable to "good enough for now". While there are basic unsolved problems in pure set theory, we should continue to develop it, regardless of applications.
(4) In an important sense, all the ways we have found to strengthen ZFC are consistent with one another. This argues for investigating them further.

Meaning skepticism

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(3) Philosophers have written *a lot* about theories of meaning. It's called *analytic* philosophy.

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(7) One need not have a theory of a notion in order to employ it.

II. Maximize

Let LST be the language of set theory, i.e. its syntax coupled with the meaning we currently assign to that syntax.

(1) All mathematical language can be translated into LST.
(2) Not all mathematically interesting statements are decided by ZFC.

LST is semantically complete, but ZFC is proof-theoretically incomplete.
The key methodological maxim that epistemology can contribute to the search for a stronger foundation for mathematics is:

maximize interpretative power.

Our foundational language and theory should enable us to say as much as possible, as efficiently as possible.

Maximize sets or maximize set theory?

Maximize is probably more often taken to apply to sets: "there are as many sets as possible".

(i) The intuition that the V_α's "go on as long as possible" is behind many large cardinal hypotheses.

(ii) Maximizing reals was once taken as an argument against CH.

(iii) Forcing axioms are motivated by this idea.

... the set theoretic arena in which mathematics is to be modelled should be as generous as possible; the set theoretic axioms from which mathematical theorems are to be proved should be as powerful and fruitful as possible. This desire to found mathematics without incumbering it generates the set theoretic maxim I call MAXIMIZE.

Maddy, *$V = L$ and Maximize*.

This first clause led to the idea of theories "providing isomorphism types", and some attempts to make Maximize more precise in that direction.

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Morals:

(a) Avoid "premature ontologizing".

(b) Maximize is a principle of rationality, not a law of nature.

The consistency strength hierarchy

Definition

Let T and U be axiomatized theories extending ZFC; then $T \leq_{\mathrm{Con}} U$ iff ZFC proves $\mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$. If $T \leq_{\mathrm{Con}} U$ and $U \leq_{\mathrm{Con}} T$, then we write $T \equiv_{\mathrm{Con}} U$, and say that T and U have the same consistency strength, or are *equiconsistent*.

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These days, the way a set theorist convinces people that T is consistent is to show by forcing that $T \leq_{\mathrm{Con}} H$ for some large cardinal hypothesis H.

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Inner model theory provides the strongest such evidence. You get an exhaustive, detailed account of a minimal universe in which the hypothesis holds.

Natural consistency strengths wellordered: If T is a natural extension of ZFC, then there is an extension H axiomatized by large cardinal hypotheses such that $T \equiv_{\mathrm{Con}} H$. Moreover, $\leq_{\mathrm{Con}}$ is a prewellorder of the natural extensions of ZFC. In particular, if T and U are natural extensions of ZFC, then either $T \leq_{\mathrm{Con}} U$ or $U \leq_{\mathrm{Con}} T$.

If T is the natural theory, and H the large cardinal hypothesis:

(a) $T \leq_{\mathrm{Con}} H$ by forcing,

(b) $H \leq_{\mathrm{Con}} T$ by inner model theory.

Remark. This is a *phenomenon*, not a theorem. There are many theorems along these lines, but also many open questions, some of them pretty fundamental. Especially in inner model theory. Even at middling large cardinal levels, there are no nontrivial theorems of the form (b).

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As natural theories proceed up the large cardinal hierarchy in consistency strength, they agree on an ever-increasing class of mathematical statements.

Maddy, Meadows BSL.

Definition

Let Γ be a set of sentences in the syntax of LST, and T a theory; then $\Gamma_T = \{\varphi \mid \varphi \in \Gamma \wedge T \vdash \varphi\}$.

A theory of the natural numbers:

Phenomenon: If T and U are natural extensions of ZFC, then

$$\begin{aligned} T \leq_{\mathrm{Con}} U &\Leftrightarrow (\Pi^0_1)_T \subseteq (\Pi^0_1)_U \\ &\Leftrightarrow (\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U \end{aligned}$$

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Thus the wellordering of natural consistency strengths corresponds to a wellordering by inclusion of theories of the natural numbers. There is no divergence at the arithmetic level, if one climbs the consistency strength hierarchy in any natural way we know of.

A theory of the reals:

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strength at least that of "there are infinitely many Woodin cardinals"; then either $(\Pi^1_\omega)_T \subseteq (\Pi^1_\omega)_U$, or $(\Pi^1_\omega)_U \subseteq (\Pi^1_\omega)_T$.

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In other words, the second-order arithmetic generated by natural theories is an eventually monotonically increasing function of their consistency strengths.

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Our model-producing methods lead to eventual Γ-monotonicity because in order to produce a model for a theory T that is sufficiently strong with respect to Γ, we must produce a Γ-correct model.

The metamathematical indicator of this is a *generic absoluteness theorem*. E.g., if $M \models \text{ZFC}+$"there are arbitrarily large Woodin cardinals", then $L(\mathbb{R})^M \equiv L(\mathbb{R})^N$ for all set-generic extensions N of M.

IV. The Levy-Solovay boundary

None of our current large cardinal axioms decide CH, because they are preserved by small forcing, whilst CH can both be made true and made false by small forcing. Because CH is provably not generically absolute, it cannot be decided by large cardinal hypotheses that are themselves generically absolute.

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Theorem (Levy, Solovay)

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CH, is a Σ^2_1 statement. It is the simplest sort of statement large cardinals do not decide. There are many more of them in general set theory.

III. A multiverse language and theory

Large cardinal hypotheses are cofinal in the part of the interpretability hierarchy we know about. But, like ZFC itself, they are set-forcing-invariant, so they cannot decide CH and the many other statements that are not set-forcing-invariant.

Can we isolate a sublanguage of LST in which the mathematics based on set-forcing-invariant principles can be carried out?

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC.)

(2) (a) Every world is a transitive proper class. An object is a set just in case it belongs to some world.
(b) If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
(c) If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world.
(d) (Amalgamation.) If U and W are worlds, then there are G, H set generic over them such that $W[G] = U[H]$.

The natural way to get a model of MV is as follows.

Let M be a transitive model of ZFC, and let G be M-generic for $\mathrm{Col}(\omega, < \mathsf{OR}^M)$. The worlds of the multiverse M^G are all those W such that

$$W[H] = M[G \restriction \alpha],$$

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It follows from a result of Laver that the full first order theory of M^G is independent of G, and present in M, uniformly over all M.

That is, there is a recursive translation function t such that whenever M is a model of ZFC and G is $\mathrm{Col}(\omega, < \mathsf{OR}^M)$-generic over M, then

$$M^G \models \varphi \Leftrightarrow M \models t(\varphi),$$

for all sentences φ of the multiverse language. $t(\varphi)$ just says "φ is true in some (equvalently all) multiverse(s) obtained from me".

If $\mathcal{W}$ is a model of MV, then for any world $M \in \mathcal{W}$, there is a G such that $\mathcal{W} = M^G$. Thus assuming MV indicates then that we are using the multiverse language as a sublanguage of the standard one, in the way described above. Also, it is clear that if φ is any sentence in the multiverse language, then MV proves

$$\varphi \Leftrightarrow \text{ for all worlds } M,\ t(\varphi)^M \ \Leftrightarrow \ \text{for some world } M,\ t(\varphi)^M.$$

Thus everything that can be said in the multiverse language can be said using just one world-quantifier.

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There is no obvious way to state CH in the multiverse language.

A reply Maddy-Meadows

Why wouldn't it be sufficient to isolate a set of axioms that captures this central idea well enough to generate a mathematically successful theory, even if it wasn't complete for some natural collection of toy models? Without a satisfactory answer to this question, we have no reason to adopt the axiomatizabilty requirement, and we're left without a principled argument for Amalgamation.

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Reply. I wanted to capture an existing successful theory, generically invariant set theory, not invent a new one. The existing theory has been developed in $\mathcal{L}_\in$, but $\mathcal{L}_{MV}$ and its translation t into $\mathcal{L}_\in$ seem useful in isolating it. Amalgamation is true under this translation. It records the intention to be translatable via t into $\mathcal{L}_\in$. Amalgamation is not meant to be an independent insight into the nature of multiverses. It clarifies how you want to be understood.

Have we lost expressive power?

One can think of the standard language as the multiverse language, together with a constant symbol $\dot{V}$ for a reference universe. Statements like CH are intended as statements about the reference universe. To what extent is this constant symbol meaningful? Does one lose anything by retreating to the superficially less expressive multiverse language? We distinguish three answers to this question:

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Weak relativist thesis: Every proposition that can be expressed in the standard language LST can be expressed in the multiverse language.

Strong absolutist thesis: "$\dot{V}$" makes sense, and that sense is not expressible in the multiverse language.

Finally, perhaps weak relativism and the absolutist's idea of a distinguished reference world can be combined, in that that there is an individual world that is definable in the multiverse language.

An elementary forcing argument shows that if the multiverse has a definable world, then it has a unique definable world, and this world is included in all the others. (An observation due to Woodin.) In this case, we call this unique world included in all others the *core* of the multiverse.

Weak absolutist thesis: There are individual worlds that are definable in the multiverse language; that is, the multiverse has a core.

Why weak relativism?

The strongest evidence for the weak relativist thesis is that the mathematical theory based on large cardinal hypotheses that we have produced to date can be naturally expressed in the multiverse sublanguage.

Perhaps we lose something when we do that, some future mathematics built around an understanding of the symbol $\dot{V}$ that does not involve defining $\dot{V}$ in the multiverse language. But at the moment, it's hard to see what that is.

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The weak relativist thesis can be considered as a piece of advice: don't go looking for it.

Some replies to Maddy-Meadows

The substance of Steel's thought can be formulated more effectively in philosophically innocent mathematical terms. By these means, we steer away from the vagaries of mathematical meaning, truth, and existence and toward the methodologically central questions: how exactly do we select our theories and by what right?

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Reply. We select our theories based on how they are interpreted. Our framework theory should be such that all others can be translated into it. Paraphrasing Hilbert: depriving a philosopher of the notion of meaning is like depriving a boxer of the use of his fists.

We're assuming that our examination of the various candidates [for a foundational theory] shows them all to be on equal footing and that our best response is to trim the syntax of $\mathcal{L}_\in$. On those assumptions, consider the state of two imaginary set theorists, a universe theorist and a multiverse theorist.

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Maddy, Meadows, BSL.

Some replies:

(1) The terminology universe theorist/multiverse theorist is somewhat misleading. Our multiverse theorist could adopt the syntax of $\mathcal{L}_\in$, and just be careful to stay in the range of the translation function.

(2) The question for both of them is "Does the multiverse have a core?" If they agree that it does, the question for the universe theorist/strong absolutist is "Is V the core?".

(3) If all answers are "yes", there is no important disagreement.

(4) Weak relativism and Strong Absolutism are views on the semantics of $\mathcal{L}_\in$. They are not expressible in $\mathcal{L}_\in$. Weak Absolutism is a thesis regarding sets, expressible in $\mathcal{L}_\in$.

VI. Does the multiverse have a core?

Whatever one thinks of the semantic completeness of the multiverse language, it does bring the weak absolutist thesis to the fore, as a fundamental question. Because the multiverse language is a sublanguage of the standard one, this is a question for everyone. If the multiverse has a core, then surely it is important, whether it is the denotation of the absolutist's $\dot{V}$ or not!

VI. Does the multiverse have a core?

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Neither MV nor its extensions by large cardinal hypotheses up to the level of supercompact cardinals decides whether there is a core to the multiverse, or the basic theory of this core if it exists (Fuchs, Hamkins, Reitz). But

Theorem

(Usuba 2016, 2019) If there is an extendible cardinal, then the generic multiverse V^G has a core.

The Fuchs-Hamkins-Reitz work shows that nothing follows from extendible cardinals concerning the basic theory of the core.

Is the core a canonical inner model?

The canonical inner model M_H for a large cardinal hypothesis H is its most concrete realization. Its construction yields a thorough *fine structure theory* for the model. We have constructed M_H for many H. Do they have a general form?

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The sets in any M_H are ordinal definable in a certain generically absolute way.

Definition

Let $A \subseteq \omega^\omega$; then A is *homogeneously Suslin* (Hom_∞) iff for all κ, there is a system $\langle M_s, i_{s,t} \mid s, t \in \omega^{<\omega} \rangle$ such that

(1) $M_\emptyset = V$, and each M_s is closed under κ-sequences,

(2) for $s \subseteq t$, $i_{s,t} \colon M_s \to M_t$,

(3) if $s \subseteq t \subseteq u$, then $i_{s,u} = i_{t,u} \circ i_{s,t}$, and

(4) for all x, $x \in A$ iff $\lim_n M_{x \upharpoonright n}$ is wellfounded.

Theorem

(Martin, S., Woodin 1985) Assume there are arbitrarily large Woodin cardinals; then for any $A \in Hom_\infty$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

Theorem

(Woodin 1987?) If there are arbitarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ *truth is generically absolute.*

Remark. The generically absolute statements in our "theory of the concrete" are all $(\Sigma^2_1)^{Hom_\infty}$. CH is Σ^2_1, but definitely not $(\Sigma^2_1)^{Hom_\infty}$.

Recall that a set is *ordinal definable* (OD) iff it is definable over the universe of sets from ordinal parameters, and is *hereditarily ordinal definable* (HOD) just in case it and all members of its transitive closure are OD.

V looks like the HOD of a determinacy model

Theorem

(Woodin, late 1980s) Assume there are arbitrarily large Woodin cardinals; then for any $A \in Hom_\infty$,

(a) $HOD^{L(A,\mathbb{R})}$ *is* Σ^2_1 *correct in* $L(A,\mathbb{R})$, *and*

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Definition

(Woodin) $V = $ *ultimate* L is the statement: There are arbitrarily large Woodin cardinals, and for any Σ_2 sentence φ of LST: if if φ is true, then for some $A \in Hom_\infty$, $\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi$.

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Theorem

(Woodin) If $V = \textit{ultimate } L$*, then*

(1) V *is the core of its multiverse* V^G*.*

(2) V *is "generically absolute HOD".*

One can state the axiom in the multiverse sublanguage.

Ultimate? Like L?

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At the same time, one hopes that $V =$ ultimate L will yield a detailed fine structure theory for V, removing the incompleteness that large cardinal hypotheses by themselves can never remove. It is known that $V =$ ultimate L implies the CH, and many instances of the GCH. Whether it implies the full GCH is a crucial open problem.

Definition

(AD^+) A pointclass Γ is *long* iff there is an $A \in \Gamma$ such that A codes an (ω_1, ω_1) iteration strategy for a pure extender premouse with a long extender on its sequence. Otherwise Γ is *short*.

Theorem

(Folk?) Suppose there are arbitrarily large Woodin cardinals, and that there is a supercompact cardinal. Assume also that V is uniquely iterable; *then there is a long Γ in Hom_∞.*

Theorem

(S. 2015-16) Assume $\mathrm{AD}^+ +$ "there is a long pointclass"; then

(1) *for any short $\Gamma \subseteq P(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}}+$, $HOD^{L(\Gamma,\mathbb{R})}$ is a least branch premouse (so satisfies* GCH, *and has a fine structure), and*

(2) *there is a short $\Gamma \subseteq P(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}}+$ and $HOD^{L(\Gamma,\mathbb{R})} \models$ "there is a subcompact cardinal".*

Morals: Granted an iterability hypothesis:

(1) $V = \text{Ult} - L$ is consistent with subcompacts.

(2) $V = \text{Ult} - L$ implies V has a fine structure, e.g. satisfies GCH.

Big open problems:

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Big open problems:

(1) Can one remove the iterability hypothesis?

(2) Can one replace "subcompact" by "supercompact" in the conclusion?

Final remarks

(1) The big open problems have been open for 25-50 years. They are central to inner model theory. Solving them is the most important project in this neighborhood.

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(4) Adopting $V = \mathrm{Ult} - L$ would not, and should not, mean ending the further development of theories like the forcing axioms. What can be forced is of permanent interest in set theory. (See Douglas Blue, *The generic multiverse is not going away*, preprint.)

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(5) If it works out, $V = \mathrm{Ult} - L$ would be a clarificatory axiom, like the Axiom of Extensionality, or the Axiom of Regularity.

(6) Transcending this framework would probably mean finding some (Σ_2) large cardinal hypothesis that cannot hold in the HOD of a determinacy model.

Thank You!