Scales in $K(\mathbb{R})$ at the end of a weak gap

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In this note we shall prove

Theorem 0.1 *Let $\mathcal{M}$ be a countably ω-iterable $\mathbb{R}^{\mathcal{M}}$-mouse which satisfies* AD*, and $[\alpha, \beta]$ a weak gap of $\mathcal{M}$. Suppose $\Sigma_1^{\mathcal{M}|\alpha}$ is captured by mice with iteration strategies in $\mathcal{M}|\alpha$.*[1] *Let n be least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$; then we have that $\mathcal{M}$ believes that $\boldsymbol{\Sigma}_{\mathbf{n}}^{\mathcal{M}|\beta}$ has the Scale Property.*

This complements the work of [2] on the construction of scales of minimal complexity on sets of reals in $K(\mathbb{R})$. Theorem 0.1 was proved there under the stronger hypothesis that all sets definable over $\mathcal{M}$ are determined, although without the capturing hypothesis. (See [2, Theorem 4.14].) Unfortunately, this is more determinacy than would be available as an induction hypothesis in a core model induction. The capturing hypothesis, on the other hand, is available in such a situation. Since core model inductions are one of the principal applications of the construction of optimal scales , it is important to prove 0.1 as stated.

Our proof will incorporate a number of ideas due to Woodin which figure prominently in the weak gap case of the core model induction. It relies also on the connection between scales and iteration strategies with the Dodd-Jensen property first discovered in [7]. Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})^{\mathcal{M}}}$ be the pointclass at the beginning of the weak gap referred to in 0.1. In section 1, we use Woodin's ideas to construct a Γ-*full* a mouse $\mathcal{Q}$ having ω Woodin cardinals cofinal in its ordinals, together with an iteration strategy Σ which *condenses well* in the sense of [5, Def. 1.13]. In section 2, we construct the desired scale from $\mathcal{Q}$ and Σ.

The reader should see sections 1 and 2 of [2] for an elementary discussion of $K(\mathbb{R})$. Sections 4.2 and 4.3 of [2] introduce the notion of a Σ_1-*gap* in $K(\mathbb{R})$, and use it to describe the pattern of scales in $K(\mathbb{R})$. We shall assume the reader is familiar with the definitions and statements of results in section 4 of [2]. It is not necessary to know any proofs there.

[1] This capturing hypothesis is explained below.

1 A fullness-preserving iteration strategy

For the rest of this note we adopt the hypotheses of 0.1. We may as well also assume $\omega\beta = o(\mathcal{M})$. To make some smaller points easier to handle, we shall assume β is a limit ordinal, $\rho_1(\mathcal{M}) = \mathbb{R}$, and $\mathcal{M}$ is passive. (A little extra care is needed if $\mathcal{M}$ is active of type II.) We may also assume

$$\mathcal{M} \models \Theta \text{ exists },$$

and $\mathcal{M}$ has a nontrivial extender on its sequence with index (hence critical point) above $\Theta^{\mathcal{M}}$. This is because the contrary case is handled in [2, Theorem 4.15]. Let us write θ for $\Theta^{\mathcal{M}}$. We have $\alpha < \theta < \beta$, and $\mathcal{M}|\alpha \prec_1^{\mathbb{R}} \mathcal{M} = \mathcal{M}|\beta$.[2]

Let $\Sigma = \Sigma_a^{1,\mathcal{M}}$ be our nonreflecting Σ_1-type, realized by a in $\mathcal{M}$, but not realized by any b in any $\mathcal{M}|\gamma$ for $\gamma < \beta$.[3] We may assume $a = \langle G, w_1\rangle$, where G is a finite subset of β and $w_1 \in \mathbb{R}$, and that G is Brouwer-Kleene minimal, in the sense that whenever $H \in [\beta]^{<\omega}$ and $H <_b G$ then $\langle H, w_1\rangle$ does not realize Σ in $\mathcal{M}$. (Here $H <_b G$ iff $\max\,(H \triangle G) \in G$.)

We define a canonical sequence of initial segments $\mathcal{M}|\beta_i$ of $\mathcal{M}$. Let

$$\beta_0 = \text{ least } \eta > \theta \text{ s.t. } G \in \mathcal{M}|\eta \wedge \exists\lambda < \eta(\mathcal{M}|\lambda \models \mathsf{ZFC}).$$

Given $\beta_i < \beta$, let

$$\psi_i = \text{ least } \psi \in \Sigma \text{ s.t. } \mathcal{M}|\beta_i \not\models \psi[\langle G, w_1\rangle],$$

and set

$$\beta_{i+1} = \text{ least } \gamma \text{ s.t. } \mathcal{M}|\gamma \models \psi_i[\langle G, w_1\rangle].$$

Let

$$Y_i = \{c \mid c \text{ is } \Sigma_1^{\mathcal{M}|\beta_i}\text{-definable from parameters in } \mathbb{R} \cup \{G, \mathbb{R}\}\},$$

and

$$\mathcal{N}_i = \text{ transitive collapse of } Y_i.$$

By condensation, each $\mathcal{N}_i$ is a proper initial segment of $\mathcal{M}|\theta$.

Claim 1. $\bigcup_{i<\omega} Y_i = \mathcal{M}$.

Proof. Easy; see the proof of 4.14 in [2]. □

[2] If $\mathcal{P}$ is a premouse with ordinal height $\omega\gamma$, then $\mathcal{P}|\gamma = \mathcal{P}$, and if $\xi < \gamma$, $\mathcal{P}|\xi$ is the initial segment $\mathcal{Q}$ of $\mathcal{P}$ with ordinal height $\omega\xi$, such that $\mathcal{Q}$ has last extender $E_\xi^{\mathcal{P}}$. So $\mathcal{P}|\xi$ might be active. Some authors use $\mathcal{P}||\xi$ here. In this particular case, both $\mathcal{M}|\alpha$ and $\mathcal{M}|\beta$ happen to be passive.

[3] The Σ_1 formulae in Σ are in the language $\mathcal{L}^*$ of $\mathbb{R}$-premice; see [2, §2].

As in [2], the Coding Lemma gives us Skolem functions which can be used to verify the identity of G. More precisely, let

$$f_i\colon \mathbb{R} \overset{\text{onto}}{\to} Y_i$$

be the natural map, which is $\Sigma_1^{\mathcal{M}|\beta_i}$ in the parameter $\langle G, w_1, \Sigma\rangle$. For $x \in \mathbb{R}$, let

$$\sigma_i^x = \begin{cases} 0, & \text{when } f_i(x) \not<_b G \\ \text{least } \sigma \in \Sigma \text{ s.t.} \mathcal{M} \not\models \sigma[\langle f_i(x), w_1\rangle], & \text{otherwise} \end{cases}$$

and

$$h_i(x) = \begin{cases} 0, & \text{when } f_i(x) \not<_b G \text{ or } \sigma_i^x \text{ is } \Sigma_1 \\ \text{least } k \geq i \text{ s.t. } \mathcal{M}|\beta_k \models \neg\sigma_i^x[\langle f_i(x), w_1\rangle]\ , & \text{otherwise.} \end{cases}$$

It is not hard to see, using the Coding Lemma as is done in [8] or [2], that there is a real w_2 such that the maps $x \mapsto \sigma_i^x$ and h_i are $\Sigma_2^{\mathcal{M}|\beta_i}$ in $\langle G, w_1, w_2\rangle$, uniformly in i. The key here is that we can make do with the Coding Lemma applied to sets in $\mathcal{M}|\theta$, since each $\mathcal{N}_i$ is a proper initial segment of $\mathcal{M}|\theta$. Since $\mathcal{M} \models \mathsf{AD}$, we have enough determinacy for this use of the Coding Lemma.

Let $z = \langle w_1, w_2\rangle$. We now introduce a theory which describes $\mathcal{M}$ as the union of the Y_k. Its language $\mathcal{L}$ has $\in, =$, and constant symbols $\dot{z}$, $\dot{G}$, and $\dot{\mathcal{M}}_k$ and $\dot{\beta}_k$ for all $k < \omega$. If φ is an $\mathcal{L}$-formula involving no constants $\dot{\mathcal{M}}_k$ or $\dot{\beta}_k$ for $k \geq m$, then we say φ has support m. Let B_0^m be the collection of Σ_0 formulae of $\mathcal{L}$ with support m, and $B_0 = \bigcup_{k<\omega} B_0^k$. Let

$$S = \{\phi \in B_0 \mid \phi \text{ is a sentence and } \mathcal{M} \models \phi\}.$$

Here the symbols $\dot{z}, \dot{G}$, etc., are allowed to occur in ϕ, and are to be interpreted in $\mathcal{M}$ in the obvious way. Let also

$$S_k = \{\langle \phi, \langle x_1, ..., x_n\rangle\rangle \mid \phi \in B_0^k \text{ and } x_1, ..., x_n \in \mathbb{R} \text{ and } \mathcal{M} \models \phi[x_1, ..., x_k]\}.$$

Clearly, each S_k is $\mathrm{OD}^{\mathcal{M}|\gamma}(z)$, for some $\gamma < \beta$.

We seek now a suitable z-premouse with term relations capturing the S_k, and an iteration strategy which moves these term relations correctly.

Let

$$\Gamma = \Sigma_1^{\mathcal{M}|\alpha} \cap P(\mathbb{R}),$$

where α begins our gap, and for b countable transitive, let

$$C_\Gamma(b) = \{c \subseteq b \mid \exists \xi < \alpha (c \in \mathrm{OD}^{\mathcal{M}|\xi}(b))\}.$$

Definition 1.1 *For b countable transitive, we let $Lp^\Gamma(b)$ be the union of all b-premice projecting to b, and having ω_1-iteration strategies in $\mathcal{M}|\alpha$ (or equivalently, in $\mathcal{M}$). We regard $Lp^\Gamma(b)$ as a b-mouse.*

Our capturing hypothesis is just that

$$C_\Gamma(b) = P(b) \cap \mathrm{Lp}^\Gamma(b),$$

for all countable transitive b.

Definition 1.2 *Let $t \in \mathbb{R}$, and let $1 \leq k \leq \omega$; then a t-premouse $\mathcal{N}$ is* k-suitable *iff there is a strictly increasing sequence $\langle \delta_n \mid n < k\rangle$ such that*

(a) $\forall \delta$, $\mathcal{N} \models \delta$ is Woodin if and only if $\exists i < k(\delta = \delta_i)$,

(b) if $k = \omega$, then the δ_i are cofinal in the ordinals of $\mathcal{N}$; if $k < \omega$, then $OR^{\mathcal{N}} = \sup(\{(\delta_{k-1}^{+n})^{\mathcal{N}} \mid n < \omega\})$;

(c) $Lp^\Gamma(\mathcal{N}|\xi) \trianglelefteq \mathcal{N}$, for all cutpoints ξ of $\mathcal{N}$, and

(d) if $\xi \in OR \cap \mathcal{N}$ and $\forall i(\xi \neq \delta_i)$, then $Lp^\Gamma(\mathcal{N}|\xi) \models \xi$ is not Woodin.

For $k < \omega$, this is just the definition of [4] or [1]. An ω-suitable mouse is just one of the form $\bigcup_{k<\omega} \mathcal{N}_k$, where $\mathcal{N}_k$ is k-suitable. It follows easily from (a) and (d) that for ξ as in (d), there is an η such that $\mathcal{N}|\eta \trianglelefteq \mathrm{Lp}^\Gamma(\mathcal{N}|\xi)$ and $\mathcal{N}|\eta \models \xi$ is not Woodin. If $\mathcal{N}$ is k-suitable and $i < k$, then $\delta_i^{\mathcal{N}}$ is the i^{th} Woodin of $\mathcal{N}$, and $\mathcal{N}|(i-1)$ is the unique i-suitable initial segment of $\mathcal{N}$. Thus $\delta_i^{\mathcal{N}}$ is the largest Woodin cardinal of $\mathcal{N}|i$, for all $i \geq 0$.

We call a term τ for a set of reals in $V^{\mathrm{Col}(\omega,\nu)}$ *standard* if it is equal to its own forcing relation. In this context, a real is a subset of ω. So τ is standard iff $\tau = \{\langle p, \sigma\rangle \mid p \in col(\omega,\nu) \wedge \sigma \subseteq (\mathrm{Col}(\omega,\nu)) \times \{\check{n} \mid n \in \omega\}) \wedge p \vdash \sigma \in \tau\}$. See [6], from which we also take the following definition.

Definition 1.3 *Let $\mathcal{N}$ be k-suitable, and ν a cardinal of $\mathcal{N}$. Let $A \subseteq \mathbb{R}$. Then $\tau_{A,\nu}^{\mathcal{N}}$ is the unique standard term $\sigma \in \mathcal{N}$ such that $\sigma^g = A \cap \mathcal{N}[g]$ for all g generic over $\mathcal{N}$ for $\mathrm{Col}(\omega,\nu)$, if there is such a term. Otherwise, $\tau_{A,\nu}^{\mathcal{N}}$ is undefined. We say that $\mathcal{N}$* captures A *iff $\tau_{A,\nu}^{\mathcal{N}}$ exists, for all cardinals ν of $\mathcal{N}$. If $A = S_i$ and $\nu = \delta_j^{\mathcal{N}}$, then we write $\tau_{i,j}^{\mathcal{N}}$ for $\tau_{A,\nu}^{\mathcal{N}}$.*

Definition 1.4 *Let $\mathcal{N}$ be a premouse, and $\mathcal{T}$ an iteration tree on $\mathcal{N}$; then we say $\mathcal{T}$ is* Γ-guided *iff for all limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}([0,\lambda]_T, \mathcal{T})$ exists, and is a proper initial segment of $Lp^\Gamma(\mathcal{M}(\mathcal{T}))$. We say that $\mathcal{T}$ is* maximal *iff $Lp^\Gamma(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin, and say that $\mathcal{T}$ is* short *otherwise.*

Note that there is at most one branch with the properties of $[0,\lambda]_T$ in 1.4. For if b and c are cofinal branches of $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq \mathrm{Lp}^\Gamma(\mathcal{M}(\mathcal{T}))$ and $\mathcal{Q}(c,\mathcal{T}) \trianglelefteq \mathrm{Lp}^\Gamma(\mathcal{M}(\mathcal{T}))$, then both $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ are satisfied to be ω_1+1-iterable in $\mathcal{M}|\alpha$, so we can compare them there. We get $\mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(c,\mathcal{T})$, and hence $b = c$. (See [3, 6.11, 6.12].)

We are only going to apply the notions of 1.4 to finite compositions of normal iteration trees. One should understand $\delta(\mathcal{T})$ as the sup of the lengths of the extenders used in the last normal tree.

Definition 1.5 *Let $\mathcal{T}$ be a maximal tree on a suitable $\mathcal{N}$; then $\mathcal{M}(\mathcal{T})^+$ is the unique suitable $\mathcal{P}$ such that $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{P}$, and $\delta(\mathcal{T})$ is the largest Woodin cardinal of $\mathcal{P}$.*

Thus $\mathcal{M}(\mathcal{T})^+$ represents what Γ can tell us about the "true" branch model $\mathcal{M}_b^{\mathcal{T}}$, if there is one.

Definition 1.6 *Let $\mathcal{N}$ be k-suitable, and $A \subseteq \mathbb{R}$, and suppose $\mathcal{N}$ captures A. Then we say an iteration tree $\mathcal{T}$ on $\mathcal{N}$* respects A *iff whenever $\mathcal{P} = \mathcal{M}_\xi^{\mathcal{T}}$ for some $\xi < \mathrm{lh}(\mathcal{T})$, then*

(a) if $[0, \xi]_T$ does not drop, and $i = i_{0,\xi}^{\mathcal{T}}$, then $\mathcal{P}$ is k-suitable and captures A, moreover

$$i(\tau_{A,\nu}^{\mathcal{N}}) = \tau_{A,i(\nu)}^{\mathcal{P}}$$

for all cardinals ν of $\mathcal{N}$.

(b) if ξ is a limit ordinal and $[0, \xi]_T$ drops, then $\mathcal{T}$ is short.

We have built some of what is called *fullness-preserving* in [5] into definition 1.6. We might have strengthened (b) further, by keeping track of just which Γ-Woodin cardinals can remain after a drop (only images of the ones you started with), but this seems unnecessary.

Definition 1.7 *Let $\mathcal{N}$ be k-suitable, where $k < \omega$, and $A \subseteq \mathbb{R}$. We say that $\mathcal{N}$ is* normally A-iterable *iff $\mathcal{N}$ captures A, and whenever $\mathcal{T}$ is a normal, Γ-guided tree on $\mathcal{N}$, then*

(a) $\mathcal{T}$ respects A,

(b) if $\mathcal{T}$ has successor length, it can be freely extended one more step (i.e., the relevant ultrapowers are wellfounded),

(c) if $\mathcal{T}$ has limit length and is short, then there is a cofinal branch b of $\mathcal{T}$ such that $\mathcal{T}^\frown b$ is Γ-guided, and

(d) if $\mathcal{T}$ is maximal, then there is a cofinal, wellfounded, nondropping branch b of $\mathcal{T}$ such that $\mathcal{T}^\frown b$ respects A.

Definition 1.8 *Let $\mathcal{N}$ be ω-suitable, and $A \subseteq \mathbb{R}$; then $\mathcal{N}$ is normally A-iterable iff for each $j < \omega$, $\mathcal{N}|j$ is normally A-iterable.*

Full A-iterability is defined by a game. Let $\mathcal{N}$ be k-suitable and capture A. The game $G(A, \mathcal{N})$ is played as follows: I plays $\mathcal{T}_0$ a normal, maximal Γ-guided tree on N, II plays a nondropping cofinal wellfounded branch b_0 of $\mathcal{T}_0$ such that $\mathcal{T}_0{}^\frown b_0$ respects A, I plays a normal, maximal Γ-guided tree $\mathcal{T}_1$ on $\mathcal{M}_{b_0}^{\mathcal{T}_0}$, II plays a nondropping, cofinal, wellfounded branch b_1 of $\mathcal{T}_1$ such that $\mathcal{T}_1{}^\frown b_1$ respects A, ..., and so on for ω rounds. In addition to the requirements listed above, II must maintain that if $\mathcal{P}$ is a model of the composition of the $\mathcal{T}_i$ such that $\mathcal{N}$-to-$\mathcal{P}$ does not drop, then $\mathcal{P}$ is normally A-iterable.

Definition 1.9 *Let $\mathcal{N}$ be k-suitable, and $A \subseteq \mathbb{R}$; then $\mathcal{N}$ is* almost A-iterable *iff*

(a) $k < \omega$, and II has a winning strategy in $G(A, \mathcal{N})$, or

(b) $k = \omega$, and for all $j < \omega$, II has a winning strategy in $G(A, \mathcal{N}|j)$.

Although almost-iterability only gives us iteration strategies for proper initial segments of an ω- suitable $\mathcal{N}$, this is enough for a version of the comparison lemma.

Lemma 1.9.1 *Let $\mathcal{R}$ and $\mathcal{N}$ be ω-suitable and almost A-iterable, then there is an ω-suitable $\mathcal{P}$ such that for each $i < \omega$, there are iteration trees $\mathcal{T}_i$ on $\mathcal{R}|i$ and $\mathcal{U}_i$ on $\mathcal{N}|i$ which respect A, have common last model $\mathcal{P}|i$, and are such that the branches $\mathcal{R}|i$-to-$\mathcal{P}|i$ and $\mathcal{R}|i$-to-$\mathcal{P}|i$ do not drop.*

Sketch of proof. Fix almost A-iteration strategies Σ_i for $\mathcal{R}|i$ and Γ_i for $\mathcal{N}|i$. We now simply coiterate all the $\mathcal{R}|i$ and $\mathcal{N}|i$ using these strategies. More precisely, set

$$\mathcal{H}_i^0 = \mathcal{R}|i \text{ and } \mathcal{G}_i^0 = \mathcal{N}|i$$

for all i. In the first round, we simultaneously compare all $\mathcal{H}_i^0|0$ and $\mathcal{G}_j^0|0$. This means iterating normally in Γ-guided fashion until one of the trees being produced is maximal, then going one move deep in each strategy. See [4] for a similar comparison argument. This round of the simultaneous comparison must terminate at some stage $\leq \omega_1^{L[T,x]}$, where T is a tree of a Γ-scale and x codes $\langle (\mathcal{H}_i^0, \mathcal{G}_i^0) \mid i < \omega \rangle$, because its Γ-guided portion can be done within $L[T, x]$. Now let $\mathcal{H}_i^1$ be the last model of the tree on $\mathcal{H}_i^0$ we produced in round 1, and similarly for $\mathcal{G}_j^1$, so that

$$\mathcal{H}_i^1|0 = \mathcal{G}_j^1|0,$$

for all i and j. We now move to the second round, in which we simultaneously compare all $\mathcal{H}_i^1|1$ and $\mathcal{G}_j^1|1$. ($\mathcal{H}_0^1$ and $\mathcal{G}_0^1$ are initial segments of all $\mathcal{H}_i^1$ and $\mathcal{G}_j^1$, so will not move from here on.) Again, this means iterating normally in Γ-guided fashion until one of the trees being produced is maximal, then going a second move deep in each strategy with index ≥ 1. This round of the simultaneous comparison must terminate at some stage $\leq \omega_1^{L[T,x]}$, where x codes $\langle (\mathcal{H}_i^1, \mathcal{G}_i^1) \mid i < \omega \rangle$, because its Γ-guided portion can be done within $L[T, x]$. Now let $\mathcal{H}_i^2$ be the last model of the tree on $\mathcal{H}_i^1$ we produced in round 2, and similarly for $\mathcal{G}_j^2$, so that

$$\mathcal{H}_i^2|1 = \mathcal{G}_j^2|1,$$

for all i and j. Etc.

Let $\mathcal{T}_i$ be the tree leading from $\mathcal{H}_i^0$ to $\mathcal{H}_i^i$, and $\mathcal{U}_i$ the tree leading from $\mathcal{G}_i^0$ to $\mathcal{G}_i^i$. It is clear that

$$\mathcal{H}_i^i = \mathcal{G}_i^i$$

for all i, and that the $\mathcal{H}_i^i$ fit together into a single ω-suitable $\mathcal{P}$, as described. $\square$

It should be noted that the embeddings from the $\mathcal{R}|i$ to $\mathcal{P}|i$ given by 1.9.1 may not fit together into an embedding from $\mathcal{R}$ to $\mathcal{P}$.

For full A-iterability, we demand also a certain Dodd-Jensen property of II's winning strategy in $G(A, \mathcal{N})$. In order to describe this, we need the following notions.

Definition 1.10 *Let $\mathcal{N}$ be i-suitable and capture A, and let $k < i$. We set*

$$\gamma_{A,k}^{\mathcal{N}} = \sup(Hull_1^{\mathcal{N}}(\{\tau_{A,k}^{\mathcal{N}}\}) \cap \delta_k^{\mathcal{N}})).$$

If $A = S_j$, then we write $\gamma_{j,k}^{\mathcal{N}}$ for $\gamma_{A,k}^{\mathcal{N}}$. Finally, if $\mathcal{N}$ captures all the S_j, we set

$$\gamma_k^{\mathcal{N}} = \sup_{j<\omega} \gamma_{j,k}^{\mathcal{N}}.$$

It should be pointed out that the Σ_1-hull used to define $\gamma_{A,k}^{\mathcal{N}}$ is taken over all of $\mathcal{N}$[4], rather than just $\mathcal{N}|k$.

The following lemma comes from Woodin's analysis of $\mathrm{HOD}^{L(\mathbb{R})}$; see [4]. It is basically the "zipper argument".

Lemma 1.10.1 *Let $\mathcal{N}$ be j-suitable for some j, and capture A. Let $\mathcal{T}$ be a normal, maximal, Γ-guided iteration tree on $\mathcal{N}|\delta_k^{\mathcal{N}}$, with all critical points above $\delta_{k-1}^{\mathcal{N}}$ if $k \geq 1$. Suppose b and c are cofinal branches of $\mathcal{T}$ such that $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ respect A, and such that $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ are almost A-iterable*[5]*; then $i_b^{\mathcal{T}} \restriction \gamma_{A,k}^{\mathcal{N}} = i_c^{\mathcal{T}} \restriction \gamma_{A,k}^{\mathcal{N}}$.*

We want an $\mathcal{N}$ such that lemma 1.10.1 holds for compositions of normal, maximal trees, and to that end make the following definition.

Definition 1.11 *Let $\mathcal{N}$ be i-suitable, and let $k < i$. Suppose $\mathcal{N}$ captures A.*

(1) We call a finite sequence $\langle(\mathcal{T}_l, \mathcal{M}_l, \pi_l) \mid 1 \leq l \leq m\rangle$ A-good at k *iff setting $\mathcal{M}_0 = \mathcal{N}$, we have that for all $l \leq m$*

(i) $\mathcal{T}_l$ is a normal iteration tree on $\mathcal{M}_{l-1}$ with last model $\mathcal{M}_l$, and canonical embedding $\pi_l \colon \mathcal{M}_{l-1} \to \mathcal{M}_l$ (so that $\mathcal{M}_{l-1}$-to-$\mathcal{M}_l$ does not drop),

(ii) $\mathcal{T}_l$ is based on $\mathcal{M}_{l-1}|\delta_k^{\mathcal{M}_{l-1}}$, and has all critical points above $\delta_{k-1}^{\mathcal{M}_{l-1}}$ if $k > 1$,

(iii) $\mathcal{M}_l$ is almost A-iterable,

(iv) $\mathcal{T}_l$ respects A, and

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[4]Or equivalently, over all $\mathcal{N}|j$ for $j < \omega$.

[5]The point of this condition is that it implies that $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ can be compared as in 1.9.1. This is important because $\gamma_{A,k}$ was defined using definability over arbitrary $\mathcal{N}|i$, not just over $\mathcal{N}|k$.

(v) either $\mathcal{T}_l$ is Γ-guided, or $\mathcal{T}_l = \mathcal{S}^\frown b$ where $\mathcal{S}$ is maximal and Γ-guided.

(2) *We say that an A-good sequence as above* gives rise to π, *if $\pi = \pi_l \circ ... \circ \pi_1$.*

(3) *We say that $\mathcal{N}$ is* locally A-iterable at k *iff whenever s and t are A-good sequences at k giving rise to $\pi: \mathcal{N} \to \mathcal{P}$ and $\sigma: \mathcal{N} \to \mathcal{Q}$ respectively, and $\mathcal{P}|k = \mathcal{Q}|k$, then*

$$\pi \restriction \gamma^{\mathcal{N}}_{A,k} = \sigma \restriction \gamma^{\mathcal{N}}_{A,k}.$$

Definition 1.12 *Let $\mathcal{N}$ be k-suitable, and $A \subseteq \mathbb{R}$; then $\mathcal{N}$ is A-*iterable *iff*

(a) *$k < \omega$, and II has a winning strategy Θ in $G(A, \mathcal{N})$ such that whenever $\mathcal{P}$ is a non-dropping Θ-iterate of $\mathcal{N}$, then $\mathcal{P}$ is locally A iterable at all $i < k$, or*

(b) *$k = \omega$, and for all $j < \omega$, $\mathcal{N}|j$ is A-iterable.*

*We call a strategy Θ as in (a) an A-*iteration strategy *for $\mathcal{N}$. We call $\mathcal{P}$ an A-*iterate *of $\mathcal{N}$ if $\mathcal{P}$ lies on a non-dropping branch of an iteration tree (i.e. finite composition of normal iteration trees) played according to an A-iteration strategy.*

A-iterability implies normal and local A-iterability. Because $G(A, \mathcal{N})$ is a closed game on HC, A-iterability is absolute to models containing $\mathrm{HC} \cup \{U, A\}$, where U is any universal Γ-set.[6] The reader may wonder why we didn't define A-iterablity for ω-suitable $\mathcal{N}$ as something stronger, namely the existence of a winning strategy for II in $G(A, \mathcal{N})$. The reason is that it seems possible in the abstract that the result of a play according to such a strategy could fail to be A-iterable.

The following is a minor variant of a result of Woodin.

Lemma 1.12.1 *Let $A \subseteq \mathbb{R}$ be $OD^{\mathcal{M}}(t)$, where $t \in \mathbb{R}$; then for a cone of $s \geq_T t$, there is an ω-suitable, A-iterable s-premouse.*

Sketch of proof. Call A Γ-*bad* if it is a counterexample to the lemma. Suppose there is a Γ-bad A; then noting that A is actually $\mathrm{OD}^{\mathcal{M}|\theta}(t)$, we have that $\mathcal{M}|\xi \models A$ is Γ-bad, where $\xi > \theta$ is least such that $\mathcal{M}|\xi \models \mathsf{ZF}$. Since $\xi < \beta$ is inside the gap, we get $\xi < \alpha$ such that $\mathcal{M}|\xi \models \mathsf{ZF}$ and $\mathcal{M}|\xi \models$ there is a $\bar{\Gamma}$-bad A. Here $\bar{\Gamma} = \Sigma_1^{\mathcal{M}|\bar{\alpha}}$, where $\bar{\alpha}$ begins the Σ_1-gap which ends at ξ or later. Let A be $\bar{\Gamma}$-bad in the sense of $\mathcal{M}|\xi$, and note that then A really is $\bar{\Gamma}$-bad.

By [2], there is a self-justifying system $\mathcal{B}$ at the end of the gap which begins at $\bar{\alpha}$ such that $A \in \mathcal{B}$. Let s be a real coding t, and such that all the sets $\mathcal{B}$ are $\mathrm{OD}^{\mathcal{M}|\xi}(s)$. Now let P to be a coarse, iterable Γ^*- Woodin mouse, where $\Gamma^* = \Sigma_1^{\mathcal{M}|\eta}$ is a scaled pointclass well

[6] We need U to define the Lp^{Γ}-operator, and hence identify the Γ-guided portions of iterations.

beyond the gap that begins at $\bar{\alpha}$. (See [6].) Using P as a background universe for a full background extender contstruction, we build $L[\vec{E}, s]$. An argument like that in [6] shows that this construction reaches a fully (ω_1, ω_1)-iterable, $\bar{\Gamma}$- ω-suitable $\mathcal{N}$ whose iteration strategy produces only trees which respect all $B \in \mathcal{B}$. It follows that $\mathcal{N}$ is A-iterable, so that A is not in fact $\bar{\Gamma}$-bad, a contradiction.[7] $\square$

Remark 1.13 Lemma 1.12.1 can be improved to read: Let $A \subseteq \mathbb{R}$ be $\mathrm{OD}^{\mathcal{M}}(t)$, where $t \in \mathbb{R}$; then there is an ω-suitable, A-iterable t-premouse. For by minimizing the ordinals from which a bad A is defined, we can arrange that A is definable from t over $\mathcal{M}$. We can then take $\mathcal{M}|\xi$ to be the least level of $\mathcal{M}$ satisfying $\mathsf{ZF} + \varphi(t)$, for some formula φ. In this situation, the closed game representation of [2] yields a sjs $\mathcal{B}$ all of whose members are definable over $\mathcal{M}|\xi$ from t. We can then proceed to a contradiction as above. (This argument is due to Woodin.)

Corollary 1.14 *For a cone of $s \geq_T z$, there is an ω- suitable s-premouse $\mathcal{N}$ such that for all $i < \omega$, $\mathcal{N}$ is S_i-iterable.*

Sketch of proof. Intersecting the cones we have for each i from 1.12.1, we get a cone of s such that for each i, there is an i-suitable s-premouse $\mathcal{N}_i$ which is S_i-iterable. Fixing s in this cone, we can coiterate all the $\mathcal{N}_i$ and produce $\mathcal{N}$ which is ω- suitable, and S_i-iterable for all i. (Note here that S_i is simply definable from S_j whenever $i < j$.) See the proof of theorem 1.9.1. The main point is that the coiteration terminates. To see that is does, notice that as long as all the iteration trees being produced are Γ-guided, the coiteration lies in $L[T_\Gamma, x_0]$, where T_Γ is the tree of a scale on a universal Γ set, and x_0 is a real coding $\langle \mathcal{N}_i \mid i < \omega \rangle$. So at some stage less than or equal to $\omega_1^{L[T_\Gamma, x_0]}$, one of the trees in the coiteration is maximal. But then at this point, all the trees are maximal, and we have lined up the images of the $\mathcal{N}_i|k_0$, for some fixed $k_0 \geq 1$. Letting x_1 code the sequence of models we have at this point, and working in $L[T_\Gamma, x_1]$, we line up the images of the $\mathcal{N}_i|k_1$, for some $k_1 > k_0$. Etc. $\square$

Let us fix s_0 in the cone given by 1.14. From now on, the reader should assume that, unless otherwise specified, all suitable premice are s_0-premice.

Notice that if $\mathcal{N}$ is ω-suitable and captures all S_i, and $\mathbb{R}^*$ is the set of reals of a symmetric collapse below $\mathrm{OR}^{\mathcal{N}}$ over $\mathcal{N}$, say as given by generics g_i on $\mathrm{Col}(\omega, \delta_i^{\mathcal{N}})$, then

$$S_i \cap \mathbb{R}^* = \bigcup_{j<\omega} \tau_{i,j}^{g_j}$$

[7] The main point in addition to those covered in [6] is that on definability grounds, the full background construction must produce a level $\mathcal{N}_\gamma$ with ω-many $\bar{\Gamma}$-Woodins before it produces any collapsing structures beyond $\bar{\Gamma}$. Further, the realization strategy which the construction provides respects $\mathcal{B}$, by condensation for self-justifying systems. Finally, it has the Dodd-Jensen property, which implies that all models it reaches are locally A-iterable everywhere, because of 1.10.1. (For this last point, see [4].)

for all i. So $\mathcal{N}$ knows the theories $S_i \cap \mathbb{R}^*$, in a certain sense. These theories may not piece together into anything reasonable, however.[8] In contrast, if $\mathcal{N}$ is S_i-iterable for all i, then the $S_i \cap \mathbb{R}^*$ collectively describe an iterable $\mathbb{R}^*$-mouse which thinks it is the first place Σ is realized, and that $\dot{G}$ is the canonical G relative to $(\dot{z})_0$. We now show this.

Definition 1.15 *Let $\mathcal{N}$ be ω-suitable and capture all S_i. We say that $\mathcal{N}$ is* $\vec{S}$-iterable *iff $\mathcal{N}|i$ is S_i-iterable for all i. We call $\mathcal{P}$ an* $\vec{S}$-iterate of $\mathcal{N}$ *iff $\mathcal{P}$ is ω-suitable, and for all $i < \omega$, $\mathcal{P}|i$ is an S_i-iterate of $\mathcal{N}|i$.*

Lemma 1.15.1 *Let $\mathcal{N}$ be ω- suitable and $\vec{S}$-iterable; then there is a countable $\bar{\mathcal{M}}$ which is elementarily (with respect to the language $\mathcal{L}$) embeddable into $\mathcal{M}$, and an $\vec{S}$-iterate $\mathcal{P}$ of $\mathcal{N}$, such that $\mathbb{R} \cap \bar{\mathcal{M}}$ is the reals of a symmetric collapse over $\mathcal{P}$.*

Proof. Let Σ_i be an S_i-iteration strategy for $\mathcal{N}|i$. Let $\pi \colon H \to V_\gamma$ be elementary, where γ is large, H is countable and transitive, and everything relevant is in $\mathrm{ran}(\pi)$. We take

$$\bar{\mathcal{M}} = \pi^{-1}(\mathcal{M}).$$

Letting $\langle x_i \mid i < \omega \rangle$ enumerate $\mathbb{R} \cap \bar{\mathcal{M}}$, we can now simultaneously coiterate all the $\mathcal{N}|i$, using $\bar{\Sigma}_i = \Sigma_i \restriction \mathrm{HC}^{\bar{\mathcal{M}}}$ to pick branches at maximal stages, while dovetailing in the genericity iterations which guarantee that for the final ω-suitable $\mathcal{P}$ produced, x_i is generic over $\mathcal{P}$ for $\mathrm{Col}(\omega, \delta_i^{\mathcal{P}})$. As usual, the proper initial segment of the process that produces $\mathcal{P}|i$ is done in H. (The argument of 1.14 shows that the process of making the first n reals generic terminates at some countable-in-H stage.) Thus $\mathbb{R}^H = \mathbb{R}^{\bar{\mathcal{M}}}$ is the set of reals of a symmetric collapse over $\mathcal{P}$. □

Corollary 1.16 *Let $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable, and let HC^* be the hereditarily countable sets of a symmetric collapse over $\mathcal{N}$; then*

$$(HC^*, \in, S_i \cap HC^*)_{i<\omega} \prec (HC, \in, S_i)_{i<\omega}.$$

Proof. This is really a corollary to the proof of 1.15.1. For $\mathcal{N}$ ω-suitable, we set $\mathrm{Col}_i = \mathrm{Col}_i^{\mathcal{N}} = \mathrm{Col}(\omega, \delta_i)^{\mathcal{N}}$. If h is $\mathrm{Col}_0 \times ... \times \mathrm{Col}_k$-generic over $\mathcal{N}$, then $\mathcal{G}(h) = \mathcal{G}(h)^{\mathcal{N}} = \{g \mid g$ is Col_{k+1}-generic over $\mathcal{N}\}$. The main point is

Lemma 1.16.1 *Let $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable, and let h be $\mathrm{Col}_0 \times ... \times \mathrm{Col}_k$ generic over $\mathcal{N}$. Let ρ be a Σ_0 formula in $\mathsf{LST} \cup \{\dot{S}_i \mid i < \omega\}$, and let $x \in \mathcal{N}[h]$. For $1 \le i \le n$, let Q_i be one of the quantifiers $\exists$ or $\forall$; then the following are equivalent:*

[8] They may not even yield a model of "I am a level of $K(\mathbb{R}^*)$", because there may be existential statements in $S_i \cap \mathbb{R}^*$ which are not provided with witnesses in any $S_j \cap \mathbb{R}^*$.

1.$(HC, \in, S_i)_{i\in\omega} \models Q_1v_1...Q_nv_n\rho[x]$,

2.
$\forall g_1 \in \mathcal{G}(h) Q_1 v_1 \in HC^{\mathcal{N}[h,g_1]}$
$\forall g_2 \in \mathcal{G}(h^\frown\langle g_1\rangle) Q_2 v_2 \in HC^{\mathcal{N}[h,g_1,g_2]}$
....
....
$\forall g_n \in \mathcal{G}(h^\frown\langle g_1...g_{n-1}\rangle) Q_n v_n \in HC^{\mathcal{N}[h,g_1...g_n]}$
$(HC^{\mathcal{N}[h,g_1,...,g_n]}, \in, S_i \cap \mathcal{N}[h,, g_1...g_n])_{i\in\omega} \models \rho[x, v_1...v_n]$.

Proof. Let $\psi = Q_1v_1...Q_nv_n\rho$. Letting p be so large that $k+n \leq p$ and $i < p$ whenever $\dot{S}_i$ occurs in ρ, notice that condition (2) is uniformly first order over the structure $(\mathcal{N}|p, \tau_{i,p}^{\mathcal{N}})_{i<p}$. That is, there is a formula θ_ψ in the language of such structures such that whenever $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable, and h is $\mathrm{Col}_0 \times ... \times \mathrm{Col}_k$ generic over $\mathcal{N}$, and $x \in \mathcal{N}[h]$, then

$$(\mathcal{N}|p)[h] \models \theta_\psi[h, x] \Leftrightarrow \text{ clause (2) holds.}$$

We now prove the equivalence of (1) and (2) by induction on n (simultaneously for all $\mathcal{N}$,h, and x). If $n = 0$, then $\psi = \rho$ is Σ_0, and the equivalence is obvious. Now let $n \geq 1$, and suppose first that $Q_1 = \exists$, so that $\psi = \exists v_1\tau$.

Suppose $(\mathrm{HC}, \in, S_i)_{i\in\omega} \models \exists v_1\tau[x]$, and pick a real y coding some witness z in HC. Using the proof of 1.15.1, we form an $\vec{S}$-iterate $\mathcal{R}$ of $\mathcal{N}$ such that the $\langle S_i \mid i < p\rangle$-iteration map

$$i: \mathcal{N}|p \to \mathcal{R}|p$$

has critical point $> \delta_k^{\mathcal{N}}$, and therefore extends to

$$i: (\mathcal{N}|p)[h] \to \mathcal{R}[h],$$

and moreover,

$$y \in \mathcal{R}[h, g]$$

for some g which is $\mathcal{R}$-generic on $\mathrm{Col}_{k+1}^{\mathcal{R}}$. By our induction hypothesis,

$$(\mathcal{R}|p)[h, g] \models \theta_\tau[h^\frown\langle g\rangle, \langle x, y\rangle].$$

Moreover, the same is true with g replaced by any finite variant g_s of g, so that $\exists v_1\tau$ is forced over $\mathcal{R}[h]$ in $\mathrm{Col}_{k+1}^{\mathcal{R}}$. Pulling back using i, we get that

$$(\mathcal{N}|p)[h] \models \overset{\mathrm{Col}_{k+1}}{\Vdash} \exists v_1\tau(\hat{h}, \hat{x}),$$

which is equivalent to (2).

Conversely, if (2) holds, then we can pick any g which is $\mathcal{N}[h]$-generic over $\mathrm{Col}^{\mathcal{N}}_{k+1}$, and we have some $z \in \mathrm{HC}^{\mathcal{N}[h,g]}$ such that (2) holds for $\mathcal{N}$, $h^\frown\langle g\rangle$, $\langle x, z\rangle$, and τ. By induction, this gives $(\mathrm{HC}, \in, S_i)_{i\in\omega} \models \tau[x, z]$, as desired.

Now suppose $Q_1 = \forall$, so that $\psi = \forall v_1 \tau$. If (2) holds, but (1) does not, then we get a contradiction by $\vec{S}$-iterating $\mathcal{N}$ above $\delta^{\mathcal{N}}_k$ so as to make some z such that $(\mathrm{HC}, \in, S_i)_{i\in\omega} \models \neg\tau[x, z]$ generic over $\mathrm{Col}^{\mathcal{R}}_{k+1}$. We get $\mathcal{R}[h, g] \models \theta_\tau[h^\frown\langle g\rangle, x, z]$ using the elementarity of $i\colon \mathcal{N}|p \to \mathcal{R}|p$. But our induction hypothesis applied to $\mathcal{R}, h^\frown\langle g\rangle, \langle x, z\rangle$, and τ gives some g_s which is a finite variant of g such that $\mathcal{R}[h, g_s] \models \neg\theta_\tau[h^\frown\langle g_s\rangle, x, z]$. Inspecting τ, we see that this is a contradiction.

The proof that (1) implies (2) is like the proof that (2) implies (1) in the case $Q_1 = \exists$. □

We now prove corollary 1.16. Let $\langle g_i \mid i < \omega\rangle$ be the generics for the $\mathrm{Col}(\omega, \delta^{\mathcal{N}}_i)$ giving rise to HC^*. Let $x \in \mathrm{HC}^*$, and suppose

$$(\mathrm{HC}, \in, S_i)_{i<\omega} \models \exists v \tau[x].$$

By Tarski-Vaught, it will be enough to show that there is some $z \in \mathrm{HC}^*$ such that $(\mathrm{HC}, \in, S_i)_{i<\omega} \models \tau[x, z]$. For this, let $h = \langle g_1, ..., g_k\rangle$ be such that $x \in \mathrm{HC}^{\mathcal{N}[h]}$. By the lemma, we have

$$\mathcal{N}[h] \models \theta_{\exists v\tau}[h, x],$$

which easily implies that

$$\mathcal{N}[h, g_{k+1}] \models \tau[h^\frown\langle g_{k+1}\rangle, x, z]$$

for some $z \in \mathrm{HC}^{\mathcal{N}[h,g_{k+1}}$. Applying the lemma again, we have $(\mathrm{HC}, \in, S_i)_{i<\omega} \models \tau[x, z]$, as desired. □

By building on the proof of corollary 1.16, we can obtain a crucial condensation result. This is just the condensation one would get if the S_i constituted a self-justifying system. (See [6].) Although we do not know that, the S_i do code a nonreflecting type, and this will be enough for the argument. Our first step is

Lemma 1.16.2 *Let $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable, and let*

$$\pi\colon \mathcal{Q} \to \mathcal{N}$$

be Σ_1-elementary, with $\tau^{\mathcal{N}}_{i,j} \in \mathrm{ran}(\pi)$ for all $i, j < \omega$; then for all but finitely many k, $\mathrm{ran}(\pi)$ is cofinal in $\delta^{\mathcal{N}}_k$. In other words

$$\gamma^{\mathcal{N}}_k = \delta^{\mathcal{N}}_k$$

for all but finitely many k.

Proof. Assume not. Let n_0 be such that

$$\sup(\mathrm{ran}(\pi)) \cap \delta^{\mathcal{N}}_{n_0} = \gamma_0 < \delta^{\mathcal{N}}_{n_0}.$$

Letting

$$X = \mathrm{Hull}^{\mathcal{N}}_1(\mathrm{ran}(\pi) \cup \gamma_0),$$

a familiar argument shows that for all $j \geq n_0$,

$$\sup(X \cap \delta^{\mathcal{N}}_j) = \sup(\mathrm{ran}(\pi) \cap \delta^{\mathcal{N}}_j).$$

So letting $\mathcal{Q}^*$ be the transitive collapse of X, and π^* the collapse map, we have that $\mathcal{Q}^*$ and π^* satisify the hypotheses of the lemma on $\mathcal{Q}$ and π, moreover $\mathrm{crit}(\pi^*) = \gamma_0 < \delta^{\mathcal{N}}_{n_0}$, and $\pi^*(\gamma_0) = \delta^{\mathcal{N}}_{n_0}$. Let us re-initialize our notation by setting $\mathcal{Q} = \mathcal{Q}^*$ and $\pi = \pi^*$.

Let $\phi\colon H \to V_\gamma$ be elementary, where H is countable and transitive, γ is large, and everything relevant in in $\mathrm{ran}(\phi)$. Let

$$\phi(\bar{\mathcal{M}}) = \mathcal{M},$$

and let $\bar{\mathbb{R}} = \mathbb{R}^H = \mathbb{R}^{\bar{\mathcal{M}}}$. We now do a genericity iteration with all critical points above $\mu_0 = \delta^{\mathcal{Q}}_{n_0}$ so as to produce an iterate $\mathcal{P}$ of $\mathcal{Q}$ such that $\bar{\mathbb{R}}$ is the set of reals of a symmetric collapse over $\mathcal{P}$. This is done as follows. Let $\mathcal{Q}_0 = \mathcal{Q}, \mathcal{N}_0 = \mathcal{N}$, and $\pi_0 = \pi$. Now suppose that after making the first n reals in $\bar{\mathbb{R}}$ generic over $\mathcal{Q}_n$ below $i(\delta^{\mathcal{Q}}_k)$, where $i\colon \mathcal{Q} \to \mathcal{Q}_n$ is the iteration map given by the normal iteration tree $\mathcal{T}$ on $\mathcal{Q}$ we are producing, we have a copy map

$$\pi_n\colon \mathcal{Q}_n \to \mathcal{N}_n,$$

where $\mathcal{N}_n$ is the last model of $\pi\mathcal{T}$, which is a Γ-guided, normal iteration tree on $\mathcal{N}$. Let $j > k$ be least such that $\mathrm{ran}(\pi)$ is bounded in $\delta^{\mathcal{N}}_j$. Because π_n commutes with the tree embeddings, and i is continuous at $\delta^{\mathcal{Q}}_j$, we have

$$\mathrm{ran}(\pi_n) \text{ is bounded in } \delta^{\mathcal{N}_n}_j.$$

Let η be the bound. We now extend $\mathcal{T}$ by doing the genericity iteration to make the next real in $\bar{\mathbb{R}}$ generic for the extender algebra at $\delta^{\mathcal{Q}_{n+1}}_j$, making sure all critical points are above $\delta^{\mathcal{Q}_n}_{j-1}$. The whole point is that if $\mathcal{U}$ is the tree we are producing in this phase, then $\pi_n\mathcal{U}$ is a tree on $\mathcal{N}_n|\eta$ having all critical points above $\delta^{\mathcal{N}_n}_{j-1}$, and therefore $\pi_n\mathcal{U}$ is Γ-guided, and can always be continued in Γ-guided fashion.

Let $i\colon \mathcal{Q} \to \mathcal{P}$ be the iteration map, let $\mathcal{R}$ be the normal, Γ-guided iterate of $\mathcal{N}$ produced in the construction, and let $\psi\colon \mathcal{P} \to \mathcal{R}$ be the copy map. Note that $\mathcal{R}$ is still $\vec{S}$-iterable.[9]

[9]We need to maintain in this construction that the tree leading to $\mathcal{R}$ is Γ-guided, so that its initial segments are according to all the S_i-iteration strategies for the $\mathcal{N}|i$. If this broke down, we would have to choose one of the S_i-iteration strategies to pull back in constructing our tree on $\mathcal{Q}$, whereas we need the maps of the lifted tree on $\mathcal{N}$ to be an S_i-iteration maps, for *all* i.

Let $\sigma_{i,j} = \psi^{-1}(\tau^{\mathcal{R}}_{i,j})$. Let $\langle g_j \mid j < \omega \rangle$ be a sequence of generics over $\mathcal{P}$ giving rise to $\bar{\mathbb{R}}$ as the reals of a symmetric collapse over $\mathcal{P}$, and set

$$S^*_i = \bigcup_{j<\omega} \sigma^{g_j}_{i,j}.$$

Claim. For all $i < \omega$, $S^*_i = S_i \cap \bar{\mathbb{R}}$.

Proof. Since $\mathcal{R}$ is $\vec{S}$-iterable, by 1.16 its symmetric collapses, via the interpretations of the $\tau^{\mathcal{R}}_{i,j}$, yield countable elementary submodels of $\mathcal{M}$. Part of this can be reduced to first order properties of the $\tau^{\mathcal{N}}_{i,j}$ inside the $\mathcal{R}|k$, and these properties pass down to $\mathcal{P}$ and the $\sigma_{i,j}$. It follows that $S^* = \bigcup_i S^*_i$ is the B_0-theory of an $\bar{\mathbb{R}}$-premouse $\mathcal{M}^*$ (which has been expanded to an $\mathcal{L}$-structure).

Subclaim. $\mathcal{M}^*$ is $\omega_1 + 1$-iterable.

Sketch of proof. We first show that $\mathcal{M}^*$ is iterable with respect to iteration trees with all critical points above $\theta^{\mathcal{M}^*}$. Let X be the set of reals of a symmetric collapse over $\mathcal{R}$. By 1.16, there is a unique $\mathcal{L}$-expanded X-mouse $\mathcal{M}^-$ described by the $S_n \cap X$, and a B_0-elementary

$$\rho \colon \mathcal{M}^- \to \mathcal{M}.$$

Let $\mathcal{H}, \mathcal{H}^-$, and $\mathcal{H}^*$ be the local HOD's of $\mathcal{M}$, $\mathcal{M}^-$, and $\mathcal{M}^*$ respectively, as defined in [2]. We can write

$$\mathcal{H} = \lim_{n<\omega} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the part of $\mathcal{H}$ described in S_n, and similarly for $\mathcal{H}^-$ and $\mathcal{H}^*$ vis-a-vis the $S_n \cap X$ and the S^*_n. The individual maps of these direct limit systems are also described in the S_n, $S_n \cap X$, and S^*_n. Note that the $\mathcal{H}^*$ and $\mathcal{H}^-$ systems are piecewise "in" $\mathcal{P}$ and $\mathcal{R}$ respectively, by the homogeneiety of the symmetric collapse forcing. (The models and maps are actually proper classes of $\mathcal{P}$ and $\mathcal{R}$ whose initial segments are definable from the $\sigma_{i,j}$ in the case of $\mathcal{P}$, and $\tau^{\mathcal{R}}_{i,j}$ in the case of $\mathcal{R}$.) It follows that ψ induces a cofinal, Σ_1-elementary map

$$\psi^* \colon \mathcal{H}^* \to \mathcal{H}^-.$$

Similarly, ρ induces a cofinal, Σ_1-elementary map

$$\rho^* \colon \mathcal{H}^- \to \mathcal{H}.$$

Since countable elementary submodels of $\mathcal{H}$ are $\omega_1 + 1$- iterable, we have that $\mathcal{H}^*$ is $\omega_1 + 1$ iterable. As verified in [2], this implies that $\mathcal{M}^*$ is $\omega_1 + 1$-iterable above $\theta^{\mathcal{M}^*}$.

To handle trees on $\mathcal{M}^*$ which drop below its θ, we simply replace $\mathcal{H}^*$ and $\mathcal{H}$ by the local HOD's corresponding to the level of $\mathcal{M}^*$ being iterated. This proves the subclaim. □

In order to prove the claim, it suffices to show that $\mathcal{M}^* = \bar{\mathcal{M}}$, where each is regarded as an $\mathcal{L}$-structure. Now each is an $\omega_1 + 1$-iterable $\bar{\mathbb{R}}$-mouse which believes it is the first level at which the type Σ is realized, and thus the restrictions of $\mathcal{M}^*$ and $\bar{\mathcal{M}}$ to the language of $\mathbb{R}$-premice are the same. We leave it to the reader to see that there are enough sentences in S^* to pin down the interpretations of the remaining symbols of $\mathcal{L}$, forcing them to have the intended interpretations given by $\bar{\mathcal{M}}$. For example, letting $G^* = \dot{G}^{\mathcal{M}^*}$, our theory S^* guarantees that G^* is Brouwer-Kleene least in the sense of $\mathcal{M}^*$ such that $\langle w_1, G^* \rangle$ realizes the type Σ. (It was in order to be able to record this fact about G in B_0-sentences that we appealed to the Coding Lemma, and introduced the real w_2. This is the point where we cash in on that manuever.) This completes the proof of the claim. □

Let $\varphi(v)$ be the formula of $\mathcal{L}$ which asserts that v is a real coding a Γ-Woodin premouse. Then

$$\mathcal{N}|(n_0+1) \models \mathrm{Col}(\omega, \delta^{\mathcal{N}}_{n_0}) \Vdash \langle \varphi, \rho \rangle \in \tau^{\mathcal{N}}_{0,n_0},$$

where ρ is the canonical name for a real coding $\mathcal{N}|\delta^{\mathcal{N}}_{n_0}$. It follows that $\mathcal{Q}|(n_0+1)$ satisfies the the same statement, with $\bar{\rho}$, the canonical name for a real coding $\mathcal{Q}|\mu_0$, replacing ρ, and μ_0 replacing $\delta^{\mathcal{N}}_{n_0}$, and $\pi^{-1}(\tau_{0,n_0})$ replacing τ_{0,n_0}. But $i\colon \mathcal{Q} \to \mathcal{P}$ is Σ_1-elementary, and has critical point strictly greater than μ_0. It follows that $\mathcal{P}|(n_0+1)$ satisfies the same statement concerning $\bar{\rho}$ and σ_{0,n_0}. However, by the claim, σ_{0,n_0} is interpreted by S_0 in collapses of μ_0 over $\mathcal{P}$. Thus $\mathcal{Q}|\mu_0$ really is a Γ-Woodin premouse. Since $\mathcal{Q}|\mu_0 = \mathcal{N}|\mu_0$, and μ_0 is not one of the Woodins of $\mathcal{N}$, we have a contradiction. This proves the lemma. □

Definition 1.17 *Let $\mathcal{N}$ be ω- suitable and $\vec{S}$-iterable, and let $k < \omega$. We say $\mathcal{N}$ is k-*stable *iff whenever $\mathcal{R}$ is an $\vec{S}$-iterate of $\mathcal{N}$ via iterations with all critical points above $\delta^{\mathcal{N}}_k$, and $j > k$, then $Hull^{\mathcal{R}}_1(\{\tau^{\mathcal{R}}_{i,l} \mid i \in \omega \wedge j \leq l\})$ is unbounded in $\delta^{\mathcal{R}}_j$.*

Corollary 1.18 *For some $k < \omega$, there is an ω-suitable, $\vec{S}$-iterable, k-stable mouse $\mathcal{N}$.*

Proof. We begin with any ω-suitable, $\vec{S}$-iterable $\mathcal{N}_0$. Suppose we have constructed $\mathcal{N}_i$, which is not i-stable; then let $\mathcal{N}_{i+1}$ be an $\vec{S}$-iterate of $\mathcal{N}_i$ which witnesses this. This process must stop at some stage $i < \omega$, as otherwise, since $i \to \infty$, there is a unique ω-suitable $\mathcal{N}$ such that $\mathcal{N}|i = \mathcal{N}_i|i$ for all i. This $\mathcal{N}$ is $\vec{S}$-iterable, and hence a counterexample to 1.16.2. □

We can now prove our condensation result.

Lemma 1.18.1 *Let $\mathcal{N}$ be ω-suitable, $\vec{S}$-iterable, and k-stable. Let*

$$\mathcal{Q} = \mathcal{H}^{\mathcal{N}}_1(\delta^{\mathcal{N}}_k \cup \{\tau^{\mathcal{N}}_{i,j} \mid i, j < \omega\}),$$

be the transitive collapse of the Σ_1-Skolem hull. Then $\mathcal{Q}$ is ω-suitable, and

$$\pi(\tau^{\mathcal{Q}}_{i,j}) = \tau^{\mathcal{N}}_{i,j},$$

for all $i, j < \omega$. Moreover, the symmetric collapse of $\mathcal{Q}$ is correct, in that letting $(HC^*)^{\mathcal{Q}}$ be the hereditarily countable sets of a symmetric collapse over $\mathcal{Q}$ determined by generics g_i for $i < \omega$, and $S_i^* = \bigcup_{j<\omega}(\tau_{i,j}^{\mathcal{Q}})^{g_j}$, we have

$$(HC^*, \in, S_i^*)_{i<\omega} \prec (HC, \in, S_i)_{i<\omega}.$$

Proof. Let $\sigma_{i,j} = \pi^{-1}(\tau_{i,j}^{\mathcal{N}})$. Let $(\mathrm{HC}^*)^{\mathcal{Q}}$ be the hereditarily countable sets of a symmetric collapse over $\mathcal{Q}$ determined by generics g_i for $i < \omega$, and let

$$S_i^* = \bigcup_{j<\omega} \sigma_{i,j}^{g_j}.$$

Claim.$(\mathrm{HC}^*, \in, S_i^*)_{i<\omega} \prec (\mathrm{HC}, \in, S_i)_{i<\omega}$.

Proof. Let Σ_i be an S_i-iteration strategy for $\mathcal{N}|i$, for each i. Let $\phi\colon H \to V_\gamma$ be elementary, where H is countable and transitive, γ is large, and everything relevant in in $\mathrm{ran}(\phi)$. Let

$$\phi(\bar{\mathcal{M}}) = \mathcal{M},$$

and let $\bar{\mathbb{R}} = \mathbb{R}^H = \mathbb{R}^{\bar{\mathcal{M}}}$. Inspecting the proofs of 1.16 and 1.16.2, we see that it is enough to prove

Subclaim. For any j, there is a genericity iteration

$$i\colon \mathcal{Q} \to \mathcal{P}$$

with $\mathrm{crit}(i) > \delta_j^{\mathcal{Q}}$ such that $\bar{\mathbb{R}}$ is the reals of a symmetric collapse over $\mathcal{P}$, together with an $\vec{S}$-iterate $\mathcal{W}$ of $\mathcal{N}$ and a Σ_1-elementary

$$\psi\colon \mathcal{P} \to \mathcal{W},$$

such that

$$\psi(i(\sigma_{i,j})) = \tau_{i,j}^{\mathcal{W}},$$

for all i, j.

Proof. Fix j. We may assume $\mathcal{N}$ is j-stable. We now order the reals in $\bar{\mathbb{R}}$ in type ω, and do the standard genericity iteration of $\mathcal{Q}$ using the extender algebras at the $\delta_l^{\mathcal{Q}}$ for $l > j$. Our first worry is that we may not have enough iterability to carry out this iteration. We shall overcome this worry by adapting the proof of [5, Theorem 1.28].

Let us consider the first stage of the process. Letting $\bar{\delta} = \delta_{j+1}^{\mathcal{Q}}$ and $\delta = \delta_{j+1}^{\mathcal{N}}$, we are iterating $\mathcal{Q}$ below $\bar{\delta}$ to make the first real in $\bar{\mathbb{R}}$ generic over the image of the extender algebra of $\mathcal{Q}$ at $\bar{\delta}$. Let $\pi\colon \mathcal{Q} \to \mathcal{N}$ be the collapse map. Letting $\mathcal{T}$ be the tree on $\mathcal{Q}|\bar{\delta}$ being produced, we have at any given stage $\pi\mathcal{T}$ on $\mathcal{N}|\delta$. So long as $\pi\mathcal{T}$ is Γ-guided and short, it can be freely extended in a Γ-guided way, and this extension is according to all the Σ_i. So the only problem that can arise in this first stage is that we may reach a tree $\mathcal{T}$ such that $\mathcal{U} = \pi\mathcal{T}$ is Γ-guided, normal, and maximal. Suppose this is the case.

For any $l < \omega$, let

$$b_l = \Sigma_l(\mathcal{U}), \text{ where } \mathcal{U} = \pi\mathcal{T},$$

and let

$$i_l \colon \mathcal{N} \to \mathcal{N}_l = \mathcal{M}^{\mathcal{U}}_{b_l}$$

be the canonical embedding. Note that $i_l(\delta) = \delta(\mathcal{U})$, for all l. Let $\mathcal{R}$ be a common $\vec{S}$-iterate of all the $\mathcal{N}_l$, with the critical points of the iteration maps from the $\mathcal{N}_l|p$ to $\mathcal{R}|p$ being all $> \delta(\mathcal{U})$. Set

$$T_l = \mathrm{Th}^{\mathcal{N}|l}(\delta \cup \{\tau^{\mathcal{N}}_{i,j} \mid i, j \leq l\}),$$

for $j + 1 \leq l$, where we regard T_l as a subset of δ. The fact that $\mathcal{U}^\frown b_m$ preserves all S_l for $l \leq m$ guarantees that whenever $j + 1 \leq l \leq m$,

$$i_l(T_l) = i_m(T_l) = \mathrm{Th}^{\mathcal{R}}(\delta(\mathcal{U}) \cup \{\tau^{\mathcal{R}}_{i,j} \mid i, j \leq l\}).$$

Now since $\mathcal{N}$ was j-stable, $\mathrm{Hull}^{\mathcal{R}}_1(\{\tau^{\mathcal{R}}_{i,j} \mid i, j < \omega\})$ is cofinal in $\delta(\mathcal{U})$. From the proof of Theorem 1.28 of [5], we see that the b_l converge to a cofinal branch b of $\mathcal{U}$ given by

$$\eta \in b \Leftrightarrow \exists l \forall m \geq l (\eta \in b_m).$$

(This definitely uses that the hull of the $\tau_{i,j}$ in $\mathcal{R}$ is cofinal in $\delta(\mathcal{U})$.) The same argument also gives that $i^{\mathcal{U}}_b(\delta) = \delta(\mathcal{U})$, and for all $l \geq j + 1$,

$$i^{\mathcal{U}}_b(T_l) = i_l(T_l).$$

Moving down to $\mathcal{Q}$ and $\mathcal{T}$, let

$$\bar{T}_l = \pi^{-1}(T_l) = \mathrm{Th}^{\mathcal{Q}|l}(\bar{\delta} \cup \{\sigma_{i,j} \mid i, j \leq l\}).$$

Let

$$\pi_b \colon \mathcal{M}^{\mathcal{T}}_b \to \mathcal{M}^{\mathcal{U}}_b$$

be the copy map. Since $\delta(\mathcal{T})$ is the sup of the $\mathrm{crit}(E^{\mathcal{T}}_\eta)$ for $\eta + 1 \in b$, and similarly for $\mathcal{U}$, and since $\pi_b(\mathrm{crit}(E^{\mathcal{T}}_\eta)) = \mathrm{crit}(E^{\mathcal{U}}_\eta)$ for all η, we have that π_b maps $\delta(\mathcal{T})$ cofinally into $\delta(\mathcal{U})$. Now if $\xi < \bar{\delta}$, then

$$\pi_b(i^{\mathcal{T}}_b(\xi)) = i^{\mathcal{U}}_b(\pi(\xi)) < \delta(\mathcal{U}),$$

so that $i^{\mathcal{T}}_b(\xi) < \delta(\mathcal{T})$. On the other hand, $i^{\mathcal{T}}_b$ is continuous at $\bar{\delta}$. Thus

$$i^{\mathcal{T}}_b(\bar{\delta}) = \delta(\mathcal{T}).$$

Clearly, we also have

$$\pi_b(i_b^{\mathcal{T}}(\bar{T}_l)) = i_b^{\mathcal{U}}(T_l) = \mathrm{Th}^{\mathcal{R}|l}(\delta(\mathcal{U}) \cup \{\tau_{i,j} \mid i,l \leq l\}).$$

But notice that the whole of $\mathcal{M}_b^{\mathcal{T}}$ is coded by the $i_b^{\mathcal{T}}(\bar{T}_l)$. This means that

$$\mathcal{M}_b^{\mathcal{T}} = \mathcal{H}_1^{\mathcal{R}}(\delta_{j+1}^{\mathcal{R}} \cup \{\tau_{i,j}^{\mathcal{R}} \mid i,j \in \omega\}),$$

with the $i_b^{\mathcal{T}}(\sigma_{i,j})$ being the collapses of the $\tau_{i,j}^{\mathcal{R}}$.

We are now done with the first stage. The first real in $\bar{\mathbb{R}}$ has been made generic over $\mathcal{M}_b^{\mathcal{T}}$ at $i_b^{\mathcal{T}}(\bar{\delta})$. Moreover, $M_b^{\mathcal{T}}$ is related to $\mathcal{R}$ as $\mathcal{Q}$ was to $\mathcal{N}$. In the case that we never reach a maximal $\mathcal{T}$ while making the first real generic, our copied tree $\mathcal{U}$ is Γ-guided, and thus according to all $\vec{S}$-iteration strategies. So in either case, we are in the situation we began with after the first real has been made generic.

We now simply repeat this process with the later reals in $\bar{\mathbb{R}}$, producing $\vec{S}$-iterable structures $\mathcal{R}_k$ as we go. Our $\mathcal{W}$ at the end is defined by: $\mathcal{W}|i =$ eventual value of $\mathcal{R}_k|i$ as $k \to \infty$. We leave it to the reader to check the remaining details. □

This proves the subclaim, and thereby the claim. The claim implies that $\mathcal{Q}$ is ω-suitable, by the argument in the proof of 1.16.2. The rest of lemma 1.18.1 follows easily. □

Clearly, the proof of 1.18.1 shows that the hull $\mathcal{Q}$ is iterable in a certain sense. It does not quite show that $\mathcal{Q}$ is $\vec{S}$-iterable, since $\vec{S}$-iterations are allowed to involve trees of the form $\mathcal{T}^\frown\mathcal{U}$, where for some k, $\mathcal{T}$ lives above δ_k and $\mathcal{U}$ lives below it, whereas the soundness of $\mathcal{Q}$ above δ_k was needed to iterate below δ_k in the argument of 1.18.1. We now isolate the iterability which does follow from the proof.

Definition 1.19 *Let $\mathcal{Q}$ be ω-suitable and capture all the S_i, and let $j < \omega$.*

*(a) $\mathcal{Q}$ is j-*sound *iff $\mathcal{Q} = \ Hull_1^{\mathcal{Q}}(\delta_j^{\mathcal{Q}} \cup \{\tau_{i,l}^{\mathcal{Q}} \mid i,l < \omega\})$.*

*(b) $\mathcal{Q}$ is j-*realizable *iff there is a j-stable, $\vec{S}$-iterable, ω-suitable $\mathcal{N}$, and a Σ_1 elementary $\pi\colon \mathcal{Q} \to \mathcal{N}$ such that $\pi(\tau_{i,l}^{\mathcal{Q}}) = \tau_{i,l}^{\mathcal{N}}$ for all $i,l < \omega$. If in addition, $\pi \restriction \delta_j^{\mathcal{Q}}$ =identity, then we say that $\mathcal{Q}$ is* strongly j-realizable *via π.*

From 1.18.1 we get

Corollary 1.20 *There is a $j < \omega$ and a j-sound, strongly j-realizable $\mathcal{Q}$; moreover if $\pi\colon \mathcal{Q} \to \mathcal{N}$ is a strong j-realization, then $\mathcal{Q}|j = \mathcal{N}|j$, and $\pi \restriction \mathcal{Q}|j$ =identity.*

We also get

Lemma 1.20.1 *Let $\mathcal{Q}$ be ω-suitable and capture all S_i, and suppose $\mathcal{Q}$ is j-sound and j-realizable. Let $\mathcal{T}$ be a normal, Γ-guided putative iteration tree on $\mathcal{Q}|\delta_j^{\mathcal{Q}}$ with all critical points above $\delta_{j-1}^{\mathcal{Q}}$ if $j \geq 1$. Then $\mathcal{T}$ respects all S_i, and*

(a) if $\mathcal{T}$ has a last model $\mathcal{P}$, then either

(i) $\mathcal{Q}$-to-$\mathcal{P}$ does not drop, and $\mathcal{P}$ is j-sound and j-realizable, or

(ii) $\mathcal{Q}$-to-$\mathcal{P}$ drops, and $\mathcal{P}$ has an ω_1-iteration strategy in Γ for trees with all critical points above $\delta_{j-1}^{\mathcal{Q}}$;

(b) if $\mathcal{T}$ has limit length, then there is a unique cofinal branch b of $\mathcal{T}$ such that $\mathcal{T}^\frown b$ respects all S_i and

(i) b does not drop, and $\mathcal{M}_b^{\mathcal{T}}$ is j-sound and j-realizable, or

(ii) b drops, and $\mathcal{M}_b^{\mathcal{T}}$ has an ω_1-iteration strategy in Γ for trees with all critical points above $\delta_{j-1}^{\mathcal{Q}}$.

(c) if $\mathcal{Q}$ is strongly j-realizable, then the realization maps of (a)(i) and (b)(i) can be taken to be strong.

Lemma 1.20.1 is proved by picking a j-realization $\pi\colon \mathcal{Q} \to \mathcal{N}$ of $\mathcal{Q}$, and noting that $\mathcal{U} = \pi\mathcal{T}$ is according to all S_i-iteration strategies for the $\mathcal{N}|i$ because $\mathcal{T}$ is normal and Γ-guided. This and the rest of the argument follow the proof of 1.18.1, so we omit further detail. (The proof that b is unique is a comparison argument.)

The iteration strategy for $\mathcal{Q}$ which is implicitly described in Lemma 1.20.1 acts on finite compositions of normal iteration trees on $\mathcal{Q}|\delta_j^{\mathcal{Q}}$.[10] It is clear that it condenses well, in the sense of [5]. This would be enough to go straight to the determinacy of the $\mathcal{M}$-definable sets in a typical core model induction. Here we shall use the direct limit of all the non-dropping iterates of $\mathcal{Q}$ under this strategy to produce a scale.

2 Scales from iteration strategies

We shall use a method for constructing scales discovered in [7]; see especially the last paragraph. The method has been exploited by Woodin to obtain self-justifying systems in a context similar to the present one.

[10]To see that we indeed have an iteration strategy, note that if $\mathcal{T}$ as in 1.20.1 is short, then the extension of $\mathcal{T}$ provided by (b) must be Γ-guided. On the other hand, if $\mathcal{T}$ is maximal, then (b)(i) must apply, and $i_b(\delta_j^{\mathcal{Q}}) = \delta(\mathcal{T})$, so $\mathcal{T}$ has no proper normal extension $\mathcal{U}$ on $\mathcal{Q}|\delta_j^{\mathcal{Q}}$. Thus we must move on to the next normal tree in our finite composition, and (b)(i) tells us we are in the position we began in.

Let us now fix a j-sound, strongly j-realizable $\mathcal{Q}$ for the remainder of the proof. In order to save notation, let us assume that $j = 0$; in the general case, all iterations to follow should be taken to be above $\delta_{j-1}^{\mathcal{Q}}$. Let Σ be the iteration strategy for $\mathcal{Q}$ described implicitly in 1.20.1, so that Σ acts on finite compositions of normal trees on $\mathcal{Q}|\delta_0^{\mathcal{Q}}$. [11]

We show first that Σ has the Dodd-Jensen property.

Definition 2.1 *Suppose that $\mathcal{T}_0$ is a normal iteration tree on $\mathcal{Q} = \mathcal{M}_0$, and for all $i \leq n$, $\mathcal{T}_i$ is a normal iteration tree on $\mathcal{M}_i$ with last model $\mathcal{M}_{i+1}$. Suppose none of the branches $\mathcal{M}_i$-to-$\mathcal{M}_{i+1}$ drop, and let $j_i \colon \mathcal{M}_i \to \mathcal{M}_{i+1}$ be the canonical embedding. Suppose further that each $\mathcal{T}_i$ is on $\mathcal{M}_i|\delta_0^{\mathcal{M}_i}$, and that the composition of the $\mathcal{T}_i$ is by Σ. Then for all $l < i \leq n+1$, we say that $\mathcal{M}_i$ is a* Σ-iterate *of $\mathcal{M}_l$, and $j_{i-1} \circ ... \circ j_l$ is a* Σ-iteration map.

Lemma 2.1.1 *Let $i \colon \mathcal{P} \to \mathcal{R}$ and $j \colon \mathcal{P} \to \mathcal{R}$ be Σ-iteration maps; then $i = j$.*

Proof. Iteration trees by Σ respect all S_i, by the proof of 1.18.1, so it is enough to show that $i \restriction \delta_0^{\mathcal{P}} = j \restriction \delta_0^{\mathcal{P}}$. It is easy to see that since $\mathcal{P}$ is 0-realizable,

$$\delta_0^{\mathcal{P}} = \sup_{k<\omega} \gamma_{k,0}^{\mathcal{P}}.$$

So fix k; we shall show that i and j agree below $\gamma_{k,0}^{\mathcal{R}}$.

Let $\langle \mathcal{T}_i \mid i \leq n \rangle$ and $\langle \mathcal{S}_i \mid i \leq m \rangle$ be the finite sequences of normal trees by Σ giving rise to i and j respectively. Set $\mathcal{M}_0 = \mathcal{P} = \mathcal{W}_0$, and let $\mathcal{M}_{i+1}$, $\mathcal{W}_{i+1}$ be the last models of $\mathcal{T}_i, \mathcal{S}_i$ respectively, so that $\mathcal{M}_{n+1} = \mathcal{R} = \mathcal{W}_{m+1}$. Let $i_l \colon \mathcal{M}_l \to \mathcal{M}_{l+1}$ and $j_l \colon \mathcal{W}_l \to \mathcal{W}_{l+1}$ be the tree embeddings. Let $\pi_0 \colon \mathcal{P} \to \mathcal{N}_0$ witness that $\mathcal{P}$ is strongly 0-realizable, so that $\mathcal{N}_0|0 = \mathcal{P}|0$ and $\pi_0 \restriction \mathcal{P}_0|0$ is the identity. Let Θ be an S_k-iteration strategy for $\mathcal{N}_0|k$.

If $\mathcal{T}_0$ is Γ-guided, then regarded as a tree on $\mathcal{N}_0$, $\mathcal{T}_0$ is according to all $\vec{S}$-iteration strategies. Letting $\mathcal{N}_1$ be the last model of $\mathcal{T}_0$ regarded as being on $\mathcal{N}_0$, we have a copy map $\pi_1 \colon \mathcal{M}_1 \to \mathcal{N}_1$ which is a strong 0-realization and which commutes fully with the tree embeddings. Let $\mathcal{U}_0$ be $\mathcal{T}_0$, regarded as a tree on $\mathcal{N}_0|k$, and let i_0^* be the embedding associated to $\mathcal{U}_0$. We have

$$i_0 \restriction \gamma_{k,0}^{\mathcal{M}_0} = i_0^* \restriction \gamma_{k,0}^{\mathcal{M}_0},$$

indeed the maps agree up to $\delta_0^{\mathcal{M}_0}$ in this case.

If $\mathcal{T}_0$ is not Γ-guided, then $\mathcal{T}_0 = \mathcal{T}^\frown b$ where $\mathcal{T}$ is maximal. By the proof of 1.18.1, there is a strong 0-realization $\pi_1 \colon \mathcal{M}_1 \to \mathcal{N}_1$. Now let $c = \Theta(\mathcal{T})$, where we regard $\mathcal{T}$ now as a tree on $\mathcal{N}_0|k$, so that $\mathcal{M}_c^{\mathcal{T}}|0 = \mathcal{N}_1|0$. Note that

$$i_c^{\mathcal{U}} \restriction \gamma_{k,0}^{\mathcal{N}_0} = i_b^{\mathcal{T}} \restriction \gamma_{k,0}^{\mathcal{N}_0},$$

[11] The nonreflecting type Σ is behind us now.

because b was chosen as a limit of branches each of which moved the term for S_k correctly, and had S_k-iterable direct limit models. (See 1.10.1.) Let $\mathcal{U}_0 = \mathcal{T}^\frown c$ regarded as a tree on $\mathcal{N}_0|k$, and let $i_0^* = i_c^{\mathcal{T}}$. We have again

$$i_0 \restriction \gamma_{k,0}^{\mathcal{M}_0} = i_0^* \restriction \gamma_{k,0}^{\mathcal{M}_0},$$

though this time there may be no further agreement.

Repeating this construction we obtain an S_k-good sequence giving rise to

$$i^* = i_{n-1}^* \circ .... \circ i_0^* : \mathcal{N}_0|k \to \mathcal{S},$$

such that

$$i^* \restriction \gamma_{k,0}^{\mathcal{P}} = i \circ \gamma_{k,0}^{\mathcal{P}}.$$

Similarly, there is an S_k-good sequence giving rise to

$$j^* = j_{m-1}^* \circ ... \circ j_0^* : \mathcal{N}|k \to \mathcal{Y}$$

such that

$$j^* \restriction \gamma_{k,0}^{\mathcal{P}} = j \restriction \gamma_{k,0}^{\mathcal{P}}.$$

We have that $\mathcal{S}|0 = \mathcal{Y}|0 = \mathcal{R}|0$. Since $\mathcal{P}$ is locally S_k-iterable,

$$i^* \restriction \gamma_{k,0}^{\mathcal{P}} = j^* \restriction \gamma_{k,0}^{\mathcal{P}},$$

which completes the proof. $\square$

We can now define

$$\begin{aligned} \mathcal{F} \quad = \quad & \{\mathcal{R} \mid \mathcal{R} \text{ is a } \Sigma\text{-iterate of } \mathcal{Q} \\ & \quad \text{via an iteration tree based on } \mathcal{Q}|\delta_0^{\mathcal{Q}}.\}, \end{aligned}$$

and for $\mathcal{R}, \mathcal{S} \in \mathcal{F}$,

$$\mathcal{R} \prec^* \mathcal{S} \Leftrightarrow \ \mathcal{S} \text{ is a } \Sigma\text{-iterate of } \mathcal{R},$$

and for $\mathcal{R} \prec^* \mathcal{S}$, let

$$\pi_{\mathcal{R},\mathcal{S}} : \mathcal{R} \to \mathcal{S}$$

be the unique embedding given by 2.1.1. Let $\mathcal{Q}_\infty$ be the direct limit of the system $(\mathcal{F}, \prec^*)$ under the $\pi_{\mathcal{R},\mathcal{S}}$, and $\pi_{\mathcal{R},\infty}\mathcal{R} \to \mathcal{Q}_\infty$ the direct limit map. The system is countably directed, because we can simultaneously compare any family $\mathcal{R}_i, i < \omega$, and thereby obtain an upper bound. Thus $\mathcal{Q}_\infty$ is wellfounded, and we identify it with its transitive collapse.

We need the following strengthening of 2.1.1.

Lemma 2.1.2 *Let $\mathcal{P} \prec^* \mathcal{R}$, and suppose $\sigma\colon \mathcal{P} \to \mathcal{R}$ is Σ_1-elementary, and such that $\sigma(\tau_{i,j}^{\mathcal{P}}) = \tau_{i,j}^{\mathcal{R}}$ for all i, j; then for all $\eta \in \mathcal{P}$,*

$$\pi_{\mathcal{P},\mathcal{R}}(\eta) \leq \sigma(\eta).$$

Proof. The well-known proof of the Dodd-Jensen lemma works here. The main point is that $\mathcal{T}$ is a finite stack of normal trees leading from $\mathcal{P}$ to $\mathcal{R}$ and played according to Σ, then $\sigma\mathcal{T}$ is according to Σ. That is because realizable branches of $\sigma\mathcal{T}$ are realizable as branches of $\mathcal{T}$, and because $\mathcal{T}$ picks unique realizable branches. Letting $\mathcal{R}_1$ be the last model of $\sigma\mathcal{T}$ and $\sigma_1\colon \mathcal{R} \to \mathcal{R}_1$, we can now repeat the process, as in the Dodd-Jensen proof. In the end, we have $\mathcal{R} \prec^* \mathcal{R}_1 \prec^* \mathcal{R}_2 \prec^*$ ..., with the direct limit of the $\mathcal{R}_i$ under the $\pi_{\mathcal{R}_i,\mathcal{R}_{i+1}}$ being illfounded. This contradicts the wellfoundedness of $\mathcal{Q}_\infty$. [12] □

Corollary 2.2 *Let $\mathcal{R} \in \mathcal{F}$, and suppose $\sigma\colon \mathcal{R} \to \mathcal{Q}_\infty$ is such that $\sigma(\tau_{i,j}^{\mathcal{R}}) = \pi_{\mathcal{R},\infty}(\tau_{i,j}^{\mathcal{R}})$ for all i, j; then*

$$\pi_{\mathcal{R},\infty}(\eta) \leq \sigma(\eta)$$

for all $\eta \in \mathcal{R}$.

Proof. By countable directedness, we can find $\mathcal{S}$ such that $\mathcal{R} \prec^* \mathcal{S}$ and $\mathrm{ran}(\sigma) \subseteq \mathrm{ran}(\pi_{\mathcal{S},\infty})$. Define $\bar{\sigma}\colon \mathcal{R} \to \mathcal{S}$ by

$$\bar{\sigma}(a) = \pi_{\mathcal{S},\infty}^{-1}(\sigma(a)).$$

Fix $\eta \in \mathcal{R}$. We have $\pi_{\mathcal{R},\mathcal{S}}(\eta) \leq \bar{\sigma}(\eta)$ by 2.1.2. Applying $\pi_{\mathcal{S},\infty}$ to both sides, we get $\pi_{\mathcal{R},\infty}(\eta) \leq \sigma(\eta)$, as desired. □

One could use the arguments of [4] to see that $\mathcal{Q}_\infty$ is ordinal definable from the real s_0 over $\mathcal{M}$ (our original $\mathbb{R}$-mouse), and perhaps also generalize the other results of [4], but we have not checked this. We shall need none of this for our purpose here.

For our purpose, it suffices to show that each S_i admits a scale, all of whose norms belong to $\mathcal{M}$. Since the S_i are Wadge cofinal in the sets of reals in $\mathcal{M}$, this easily implies boldface $\Sigma_1^{\mathcal{M}} \cap P(\mathbb{R})$, which is just the class of countable unions of sets of reals in $\mathcal{M}$ in our present situation, has the scale property. So fix i.

The idea is that the tree of our scale on S_i will verify $S_i(y)$ by producing an $\mathcal{R} \in \mathcal{F}$ such that $y \in (\tau_{i,0}^{\mathcal{R}})^g$ for some g generic over $\mathcal{R}$. Part of this consists of Γ properties, which can

[12] The reader may wonder why we went to the trouble of defining local A-iterability, and proving lemma 2.1.1, instead of going directly to the full Dodd-Jensen lemma 2.1.2. The reason is that in order to do this, one seems to need to strengthen A-iterability in another way, requiring that the direct limit associated to a play of $G(\mathcal{N})$ be wellfounded. But then $G(\mathcal{N})$ is no longer closed, and our absoluteness argument in the proof of 1.12.1 won't work as stated. One still has enough abssoluteness if $\mathcal{M}$ has an active extender at some index strictly below its length, and one could probably handle the contrary case by the game methods of [2], but the present route seems simpler.

be verified using a tree for Γ, and the rest can be verified by embedding $\mathcal{R}$ into $\mathcal{Q}_\infty$ in a way which extends the embedding $\pi_{\mathcal{Q},\infty}$.

Let $\langle \varphi_n \mid n < \omega$ enumerate the sentences of $\mathcal{L}^* = \mathcal{L} \cup \{\dot{a}_i \mid i < \omega\}$, where the $\dot{a}_i$ are new contants. We say $x \in \omega^\omega$ codes a premouse iff

$$T_x = \{\varphi_n \mid x(n) = 0\}$$

is a complete, Henkinized theory of a premouse. In this case, we let

$$\begin{aligned} \mathcal{R}_x &= \{\dot{a}_i^x \mid i < \omega\} \\ &= \text{premouse whose theory is } T_x. \end{aligned}$$

Here $\dot{a}_i^x$ is the element of $\mathcal{R}_x$ named by $\dot{a}_i$. Let us also adopt some simple (projective) coding of iteration trees on $\mathcal{Q}$; let $\mathcal{T}_z$ be the iteration tree coded by the real z. We define $G^- \subseteq \mathbb{R}^3$ by letting $G^-(y,z,x)$ if and only if z codes an iteration tree on $\mathcal{Q}$, x codes a premouse, and for some g

(a) $\mathcal{T}_z$ is a normal, Γ-guided iteration tree based on $\mathcal{Q}|\delta_0^{\mathcal{Q}}$ which does not drop anywhere,

(b) $\mathcal{R}_x \trianglelefteq \mathcal{M}(\mathcal{T}_z)^+$, and $\mathcal{R}_x \models \delta(\mathcal{T}_z)$ is Woodin, and

(c) g is $\mathcal{R}_x$-generic over $\mathrm{Col}(\omega, \delta(\mathcal{T}_z))$, and $y \in (\dot{a}_0^x)^g$.

It is easy to check that G^- is Γ in the parameter $\mathcal{Q}$. Let us put

$$G(y,z,x) \Leftrightarrow G^-(y,z,x) \wedge \mathcal{R}_x = \mathcal{M}(\mathcal{T}_z)^+ \wedge \dot{a}_0^x = \tau_{i,0}^{\mathcal{R}_x}.$$

It is not hard to see, using genericity iterations, that for all y,

$$S_i(y) \Leftrightarrow \exists z \exists x G(y,z,x).$$

Since the inf-norm propagation of scales won't take us out of $\mathcal{M}$, it suffices to see that G has a scale all of whose norms are in $\mathcal{M}$.

Let $\vec{\psi}$ be a Γ-scale on G^-. We now define some additional norms on G. Fix y,z,x such that $G(y,z,x)$. Since $\mathcal{T}_z$ is Γ-guided, it is according to Σ, and since $\mathcal{R}_x = \mathcal{M}(\mathcal{T})^+$, $\mathcal{T}_z$ is maximal. Letting $\Sigma(\mathcal{T}_z) = b$, we have that

$$\mathcal{R}_x \trianglelefteq \mathcal{M}_b^{\mathcal{T}} =_{\mathrm{df}} \mathcal{P},$$

and $\mathcal{P} \in \mathcal{F}$. Let $\mathcal{P}^+$ be the expansion of $\mathcal{P}$ in the language $\mathcal{L}^{**}$, which is $\mathcal{L}^*$ (with $\dot{a}_n$ naming $\dot{a}_n^x$ again) together with names for $\delta(\mathcal{T}_z)$ and the $\tau_{i,j}^{\mathcal{P}}$ and the $\mathcal{P}|j$. Our additional norms on G will record the first order theory of $\mathcal{P}^+$, as well as information regarding the embedding $\pi_{\mathcal{P},\infty} \restriction \delta(\mathcal{T}_z)$.

Let $\langle \theta_n \mid n < \omega \rangle$ enumerate the Σ_0 sentences of $\mathcal{L}^{**}$. Let

$$T_x^+ = \{\theta \mid \theta \text{ is } \Sigma_0 \text{ and } \mathcal{P}^+ \models \theta\}.$$

Put

$$\begin{aligned} \phi_n^0(y,z,x) &= 0, \text{ if } \theta_n \in T_x^+ \\ &= 1, \text{ if } \theta_n \notin T_x^+. \end{aligned}$$

If $\theta_n = \exists v < \dot{\delta}\psi(v)$, then we put

$$\phi_n^1(y,z,x) = \text{ least k such that } (\dot{a}_k = \text{ least v s.t.} \psi(v)) \in T_x^+$$

if $\theta_n \in T_x^+$, and set $\phi_n^1(y,z,x) = 0$ otherwise. Finally, if $(\dot{a}_n < \gamma_{k,0}^{\dot{\mathcal{P}}|k}) = \mu_{n,k} \in T_x^+$, then we set

$$\phi_{n,k}^2(y,z,x) = \pi_{\mathcal{P},\infty}(\dot{a}_n^x),$$

and otherwise, we let $\phi_{n.k}^2(y,z,x) = 0$. (Here $\dot{\mathcal{P}}|k$ is the $\mathcal{P}^+$-name for $\mathcal{P}|k$. The sentence $\mu_{n,k}$ involves $\dot{\tau}_{k,k}$ in addition to $\dot{\mathcal{P}}|k$.) Let $\vec{\rho}$ be the putative scale on G whose norms are those of $\vec{\psi}$,$\vec{\phi}^0$, $\vec{\phi}^1$, and $\vec{\phi}^2$.

Claim 1. $\vec{\rho}$ is a scale on G.

Proof. We first verify the limit property. Let

$$(y_i, z_i, x_i) \to (y,z,x)(\text{ mod } \vec{\rho}).$$

Since $\vec{\psi}$ is a scale, we have $G^-(y,z,x)$, so that $\mathcal{T}_z$ is Γ-guided and $\mathcal{R}_x \trianglelefteq \mathcal{M}(\mathcal{T}_z)^+$. Since the (y_i, z_i, x_i) converge mod $\vec{\phi}^0$,

$$T^+ = \lim_i T_{x_i}^+$$

exists, and has a unique pointwise definable model $\mathcal{P}^+$. By convergence mod $\vec{\phi}^1$, we have that $\mathcal{R}_x = \mathcal{P}|0$, although this is a slight abuse of notation, since we do not yet know that $\mathcal{P}$ is wellfounded, much less suitable. For this, we shall use convergence mod $\vec{\phi}^2$.

Let

$$\gamma = \sup(\{\xi < \dot{\delta}^{\mathcal{P}^+} \mid \exists k (\xi \text{ is definable over } \mathcal{P}|k \text{ from } \dot{\tau}_{k,k}^{\mathcal{P}^+}\}).$$

Since $\gamma \leq \dot{\delta}^{\mathcal{P}^+} = \delta(\mathcal{T}_z)$, it is contained in the wellfounded part of $\mathcal{P}^+$, and the notation is justified. Let $\bar{\mathcal{P}} = \mathcal{H}_1^{\mathcal{P}}(\gamma \cup \{\dot{\tau}_{i,j}^{\mathcal{P}^+}\})$ be the Σ_1 Skolem hull, transitivised on its wellfounded part. Letting

$$\sigma : \bar{\mathcal{P}} \to \mathcal{P}$$

be the canonical embedding, we have that $\sigma \restriction \gamma = $ identity. Let $\pi_i : \mathcal{P}_{x_i} \to \mathcal{Q}_\infty$ be the canonical embedding given by the fact $\mathcal{P}_{x_i} \in \mathcal{F}$. We define $\pi : \mathcal{P}|\gamma \to \mathcal{Q}_\infty$ by

$$\pi(\dot{a}_n^x) = \text{ eventual value of } \pi_i(\dot{a}_n^{x_i}) \text{ as } i \to \infty.$$

This eventual value exists because if $\dot{a}_n^x < \gamma$, then there is a sentence $\varphi \in T_x^+$ expressing this, and $\varphi \in T_{x_i}^+$ for all sufficiently large i, which gives a k such that $\dot{a}_n^{x_i} < \gamma_{k,0}^{\mathcal{P}_{x_i}|k}$ for all sufficiently large i. The eventual value of $\phi_{n,k}^2(y_i, z_i, x_i)$ is then what we want. We can extend π to an embedding of $\bar{\mathcal{P}}$ into $\mathcal{Q}_\infty$, which we also call π, as follows. Let

$$\mathcal{Q}_\infty|j = \pi_{\mathcal{Q},\infty}(\mathcal{Q}|j)$$

and

$$\tau_{i,j}^\infty = \pi_{\mathcal{Q},\infty}(\tau_{i,j}^{\mathcal{Q}}).$$

Let $c \in \bar{\mathcal{P}}$. We can find a $j < \omega$, a Σ_0 formula of the language of premice φ, and parameters $\dot{a}_{i_0}^x, ..., \dot{a}_{i_n}^x < \gamma_{j,0}^{\dot{\mathcal{P}}|j}$, such that

$$c = \text{ unique } v \text{ such that } \mathcal{P}|j \models \varphi[v, \dot{a}_{i_0}^x, ..., \dot{a}_{i_n}^x, \dot{\tau}_{j,j}^{\mathcal{P}^+}].$$

(This uses the fact that the $T_{x_i}^+$ converge to T_x^+, and that c is definable in the language of T_x^+, and convergence mod $\vec{\phi}^1$ to fix the $i_0, ..., i_n$ relevant to T_{x_i} for sufficiently large i.) We then set

$$\pi(c) = \text{ unique } v \text{ such that } \mathcal{Q}_\infty|j \models \varphi[v, \pi(\dot{a}_{i_0}^x), ..., \pi(\dot{a}_{i_n}^x), \tau_{j,j}^\infty].$$

We leave it to the reader to check that $\pi\colon \bar{\mathcal{P}} \to \mathcal{Q}_\infty$ is well-defined and Σ_1 elementary, and that

$$\pi((\tau_{i,j})^{\bar{\mathcal{P}}^+}) = \tau_{i,j}^\infty$$

for all i, j. As in the proof of 2.2, we can find an $\mathcal{S} \in \mathcal{F}$ and a map $\bar{\pi}\colon \bar{\mathcal{P}} \to \mathcal{S}$ such that $\pi = \pi_{\mathcal{S},\infty} \circ \bar{\pi}$. Letting $j\colon \mathcal{S} \to \mathcal{N}$ be a 0-realization, and applying 1.18.1 to $j \circ \bar{\pi}$, we see that $\bar{\mathcal{P}}$ is ω-suitable. This immediately implies $\gamma = \delta(\mathcal{T}_z)$, as $\mathcal{T}_z$ was Γ-guided. This gives $\bar{\mathcal{P}} = \mathcal{P}$, and $\sigma =$ identity. We also have that $\mathcal{P}$ and captures all S_k, and $\tau_{i,j}^{\mathcal{P}} = \dot{\tau}_{i,j}^{\mathcal{P}^+}$ for all i, j. But now $\dot{a}_0^x = \dot{\tau}_{i,0}^{\mathcal{P}^+}$, since the corresponding fact is recorded in all $T_{x_i}^+$. Thus $\mathcal{R}_x = \mathcal{P}|0 = \mathcal{M}(\mathcal{T})^+$, and $\dot{a}_0^x = \tau_{i,0}^{\mathcal{R}_x}$, so that $G(y, z, x)$, as desired.

In order to verify the lower semi-continuity property of scales, it suffices to show that for $\bar{\pi}\colon \mathcal{P} \to \mathcal{Q}_\infty$ defined as above, we have $\pi_{\mathcal{P},\infty}(\eta) \leq \bar{\pi}(\eta)$ for all η. This is an immediate consequence of 2.2. This proves claim 1. □

Claim 2. Each norm ρ_n is in $\mathcal{M}$.

Proof. It is clear that $G \in \mathcal{M}$, since it can be defined from Γ and S_i in a simple way. We show that $\phi_{n,k}^2 \in \mathcal{M}$, and leave the other norms to the reader. Let $(y_0, z_0, x_0), (y_1, z_1, x_1) \in G$. We shall describe informally how to determine whether $\phi_{n,k}^2(y_0, z_0, x_0) \leq \phi_{n,k}^2(y_1, z_1, x_1)$. Let

$$P_k(\mathcal{S}) \Leftrightarrow \mathcal{S} \text{ is } k+1\text{-suitable and } S_k\text{- iterable.}$$

Note that $P_k \in \mathcal{M}$. Our informal procedure will make use of P_k.

From x_0 and x_1 we can compute $\mathcal{R}_0 = \mathcal{R}_{x_0}$ and $\mathcal{R}_1 = \mathcal{R}_{x_1}$, as well as $a_0 = \dot{a}_n^{x_0}$ and $a_1 = \dot{a}_n^{x_1}$. Pick $\mathcal{S}_i$ for $i = 0, 1$ such that

$$\mathcal{R}_i \trianglelefteq \mathcal{S}_i \text{ and } P_k(\mathcal{S}_i).$$

A simple comparison argument shows that

$$\gamma_i = \gamma_{k,0}^{\mathcal{S}_i}$$

is independent of our choice for $\mathcal{S}_i$. We may therefore assume that $a_i < \gamma_i$ for $i = 0, 1$, as in the other cases we can easily determine whether $\phi_{n,k}(y_0, z_0, x_0) \leq \phi_{n,k}(y_1, z_1, x_1)$.

Pick Σ_i an S_k-iteration strategy for $\mathcal{S}_i$. (Again, the output of our procedure will be independent of the particular choices for the Σ_i.[13]) Let us now compare $\mathcal{R}_0$ with $\mathcal{R}_1$. We obtain iteration trees $\mathcal{T}_i$ on $\mathcal{R}_i$ giving rise to iteration maps to a common model

$$\sigma_i \colon \mathcal{R}_i \to \mathcal{W}.$$

Here either $\mathcal{T}_i$ is Γ-guided, or $\mathcal{T}_i = \mathcal{U}_i{}^\frown b_i$, where $\mathcal{U}_i$ is Γ-guided and $b_i = \Sigma_i(\mathcal{U}_i)$. In either case, $\mathcal{W}$ is independent of our choice of the Σ_i, as is $\sigma_i \restriction \gamma_i$. (Notice that the $\mathcal{R}_i$ are locally S_k-iterable.) We claim that

$$\phi_{n,k}(y_0, z_0, x_0) \leq \phi_{n,k}(y_1, z_1, x_1) \Leftrightarrow \sigma_0(a_0) \leq \sigma_1(a_1).$$

Since we can determine the truth of the right hand side in $\mathcal{M}$, and it is independent of the choices made, it suffices to prove this claim.

From the proof of claim 1, we see that there are unique $\mathcal{P}_0, \mathcal{P}_1 \in \mathcal{F}$ such that

$$\mathcal{R}_i = \mathcal{P}_i | 0.$$

The comparison of $\mathcal{P}_0$ with $\mathcal{P}_1$ leads to Σ-iterates $\mathcal{P}_i^*$ of $\mathcal{P}_i$ such that $\mathcal{W} = \mathcal{P}_i^* | 0$ for $i = 0, 1$. Let

$$\pi_i \colon \mathcal{P}_i \to \mathcal{P}_i^*$$

be the Σ-iteration map. Let

$$\tau_i \colon \mathcal{P}_i^* \to \mathcal{N}$$

be the remainder of the comparison. Noting that $\tau_i \restriction \mathcal{W}$ is the identity, we see that

$$\begin{aligned}
\pi_{\mathcal{P}_0,\infty}(a_0) \leq \pi_{\mathcal{P}_1,\infty}(a_1) \quad &\Leftrightarrow \quad \tau_0(\pi_0(a_0)) \leq \tau_1(\pi_1(a_1)) \\
&\Leftrightarrow \quad \pi_0(a_0) \leq \pi_1(a_1) \\
&\Leftrightarrow \quad \sigma_0(a_0) \leq \sigma_1(a_1).
\end{aligned}$$

[13] We should note that the S_k- iteration game is determined in $\mathcal{M}$, as $\mathcal{M}$ had an active extender indexed above $\theta^{\mathcal{M}}$.

Here the last equivalence uses the agreement between π_i and σ_i given by lemma 1.10.1. This completes the proof of claim 2. $\square$

These claims complete the proof of our main theorem in the case that $\mathcal{M}$ is passive, and $o(\mathcal{M})$ is a limit ordinal, and $\rho_1(\mathcal{M})$. The other cases are similar. The only case with much additional difficulty is the case that $\mathcal{M}$ is active of type II. The problem here is that the $\mathcal{M}||\beta_i$ will not be premice, since they will have proper fragments of the last extender of $\mathcal{M}$. The reader should see [2] for a method for handling the details in this case. $\square$

References

[1] R.O. Ketchersid, Toward $\mathsf{AD}_{\mathbb{R}}$ from the Continuum Hypothesis and an ω_1-dense ideal, Ph.D. thesis, U.C. Berkeley, June 2000.

[2] J.R. Steel, Scales in $K(\mathbb{R})$, in The Cabal Seminar I; Games, Scales, and Suslin Cardinals, A.S. Kechris, B. Loewe, and J. Steel eds., to appear.

[3] J.R. Steel, An outline of inner model theory, in The Handbook of Set Theory, M. Foreman, A. Kanamori, and M. Magidor eds., to appear.

[4] J.R. Steel, Woodin's analysis of $\mathrm{HOD}^{L(\mathbb{R})}$, available at http://www.math.berkeley.edu/ steel.

[5] J.R Steel, PFA implies $\mathsf{AD}^{L(\mathbb{R})}$, Journal of Symb. Logic, vol. 70, no. 4 (2005), pp.1255–1296.

[6] J. R. Steel, A theorem of Woodin on mouse sets, available at http://www.math.berkeley.edu/∼steel

[7] J.R. Steel, $\mathrm{HOD}^{L(\mathbb{R})}$ is a core model below Θ, Bulletin of Symbolic Logic vol. 1, March 1995, pp. 75-84..

[8] J.R. Steel, Scales in $L(\mathbb{R})$, in Cabal Seminar 79-81, Lecture Notes in Math, Springer-Verlag, 1983, 107-156, A.S. Kechris, D.A. Martin, and Y.N. Moscovakis eds.