Some fine structure needed to normalize well

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Abstract

This is a continuation of [3] and [4]. There are some fine structural problems in the proof that normalizing commutes with lifting to a background universe that were overlooked in [3]. We solve those problems here. The solution involves restricting the stacks $\langle \mathcal{T}, \mathcal{U} \rangle$ under consideration, by requiring that $\mathcal{T}$ be *well-separated*. It also involves a change to the way iterations are lifted to a background universe. In the last section, we sketch a variant solution, one that involves changing the definition of "premouse" so as to demand a form of projectum free spaces.

1 Introduction

We assume the reader is familiar with [2], and the corrections to it in [3]. Those corrections purport to lead to a proof that if (P, Σ) is a level of a background construction $\mathbb{C}$, and $\langle \mathcal{T}, \mathcal{U} \rangle$ is a maximal stack of normal trees by Σ, and $\mathcal{T}$ is λ-separated, then $W(\mathcal{T}, \mathcal{U})$ is by Σ. The proof is based on the fact that normalization commutes with lifting to the background, that is

$$W(\mathcal{T}, \mathcal{U})^* = W(\mathcal{T}^*, \mathcal{U}^*),$$

in the notation of [2].[1] The hypothesis that $\mathcal{T}$ is λ-separated is used to show that there is enough agreement between lifting maps that one can continue an inductive proof that $W(\mathcal{T}, \mathcal{U})^* = W(\mathcal{T}^*, \mathcal{U}^*)$.

Unfortunately, there is a fine structural case in which the agreement of lifting maps is still not good enough to continue. One can see the source of it without going

[1] Actually, in a dropping case one only gets that $W(\mathcal{T}, \mathcal{U})^*$ is an initial segment of $W(\mathcal{T}^*, \mathcal{U}^*)$, but that is good enough.

into the details of the induction. Suppose we have $Q = M^{\mathbb{C}}_{\nu,k}$, and two extenders E and K on the Q-sequence, with $\mathrm{lh}(K) < \mathrm{lh}(E)$. Let

$$\sigma\colon Q|\,\mathrm{lh}(E) \to R$$

be the map resurrecting E from Q to $R = M^{\mathbb{C}}_{\theta,0}$. That is, $\sigma = \sigma_{\nu,k}[Q|\,\mathrm{lh}(E)]$. Let

$$\tau\colon Q|\,\mathrm{lh}(K) \to S$$

resurrect K, i.e. $\tau = \sigma_{\nu,k}[Q|\,\mathrm{lh}(K)]$. Finally, let

$$\varphi\colon R|\,\mathrm{lh}(\sigma(K)) \to N$$

resurrect $\sigma(K)$ from R, that is, $\varphi = \sigma_{\theta,0}[R|\,\mathrm{lh}(\sigma(K))]$. What we need to keep the induction going is that $S = N$, and

$$\tau = \varphi \circ \sigma \restriction Q|\,\mathrm{lh}(K).$$

In other words, the resurrection of K from Q should begin with the resurrection of E, then continue with the resurrection of the image of K.

However, this need not be the case. For example, it could happen that $\tau(K) = K$, but $\lambda_K = \mathrm{crit}(\sigma)$. More generally, if the $\mathrm{lh}(K)$ and $\mathrm{lh}(E)$ dropdown sequences $\vec{\beta}$ and $\vec{\gamma}$ of Q have some associated projectum ρ in common, then we can have $\tau(K) \neq \varphi \circ \sigma(K)$.

For simplicity, let's assume that ρ is the only projectum associated to $\vec{\beta}$, and the only one associated to $\vec{\gamma}$, so that σ and τ are themselves the uncoring maps acting on two levels $Q|\beta$ and $Q|\gamma$ projecting to ρ. If $\gamma = \beta$ we have no problem. The other case is that $\gamma < \mathrm{lh}(E)$, so K is coded by a subset of ρ belonging to $Q|\,\mathrm{lh}(E)$. We have $\sigma\restriction\rho = \tau\restriction\rho = \mathrm{id}$. If $\sigma(\rho) = \rho$, then $\sigma(K) = K$, and since the resurrection of K from R is just $\tau(K)$, $\varphi(\sigma(K)) = \tau(K)$, which is what we want in this simplified case. But if $\sigma(\rho) > \rho$, then $\sigma(K) \neq K$, so $\varphi(\sigma(K)) \neq \tau(K)$.

In general, the source of a problematic clash between the resurrection maps $\mathrm{res}_{\nu,k}[N]$ and $\mathrm{res}_{\nu,k}[P]$, where $P \lhd N$, lies in some component of $\mathrm{res}_{\nu,k}[N]$ being an uncoring with critical point equal to the current projectum.

We look more closely at such uncorings in the next section. In sections 3 and 4, we re-define resurrection and liftings to a background. In sections 5 and 6, we define well-separated plus trees, and show that $W(\mathcal{T},\mathcal{U})^* = W(\mathcal{T}^*,\mathcal{U}^*)$ when $\mathcal{T}$ is well separated. Finally, in section 7 we outline a variant approach.[2]

[2]We thank Benjamin Siskind for some very helpful conversations on the problems addressed here, and in the earlier corrections to [2].

2 Measurable projecta

Our problem stems from the possibilty of that some core taken in a construction $\mathbb{C}$ might have an anti-core map whose critical point is the projectum. We can characterize precisely when this happens.

Lemma 2.1 *Let $\mathbb{C}$ be a pure extender w-construction, $\pi\colon M^{\mathbb{C}}_{\nu,k+1} \to M^{\mathbb{C}}_{\nu,k}$ be the anti-core map, and $\rho = \rho(M^{\mathbb{C}}_{\nu,k})$; then*

(1) there is no $M_{\nu,k}$-total E on the $M_{\nu,k}$-sequence such that $\mathrm{crit}(E) = \rho$, and

(2) the following are equivalent:

- *(a) $\mathrm{crit}(\pi) = \rho$,*
- *(b) there is an $M_{\nu,k+1}$-total E on the $M_{\nu,k+1}$-sequence such that $\mathrm{crit}(E) = \rho$,*
- *(c) the $M_{\nu,k+1}$ sequence has a total order zero measure D on ρ, and there is a (unique) elementary $\sigma\colon \mathrm{Ult}(M^-_{\nu,k+1}, D) \to M_{\nu,k}$ such that $\sigma(\rho) = \rho$ and $\pi = \sigma \circ i_D$.*[3]

Proof. For (1), we use the amenable closure argument. Let E be a total-on-$M_{\nu,k}$ extender from the $M_{\nu,k}$-sequence such that $\rho = \mathrm{crit}(E)$, and let $A \subset \rho$ be the new Σ_{k+1} set. We have $A \cap \alpha \in M_{\nu,k}$ for all $\alpha < \rho$. Let $\sigma\colon M_{\nu,k}|\,\mathrm{lh}(E) \to M^{\mathbb{C}}_{\theta,0}$ be the resurrection map for E, and $F = \sigma(E)$, and $F^* = F^{\mathbb{C}}_{\theta}$ its background. Since E is total on $M_{\nu,k}$, all cores taken between $\langle\theta, 0\rangle$ and $\langle\nu, k\rangle$ corresponded to projecta $> \rho$, so $\sigma\restriction\rho^{+,M_{\nu,k}} + 1 = \mathrm{id}$. Letting $M = M^{\mathbb{C}}_{\theta,0}$, this means $M_{\nu,k}$ and M agree to their common ρ^+.

But now $A = i_{F^*}(A) \cap \rho$, so $A \in i_{F^*}(M)$ by the elementarity of i_{F^*}. But M and $i_{F^*}(M)$ agree to their common ρ^+. Hence $A \in M$, contradiction.

For (2), clearly (c) implies (b), and (b) implies (a) by part (1). So we must see that (a) implies (c). For this, we just inspect the usual proof that $M_{\nu,k+1}$ is solid and universal. Let $M = M_{\nu,k}$ and $H = M_{\nu,k+1}$. We compare (M, H, ρ) with M. The usual proof shows that the final model on both sides is the same. Call it Q. We also get that Q is above H on the phalanx side, and the branches H-to-Q and M-to-Q do not drop. The branch embeddings $i\colon H \to Q$ and $j\colon M \to Q$ are such that $crit(i) \geq \rho$ and $\mathrm{crit}(j) \geq \rho$.[4] We have also that $i(p_(H)) = j(p(M)) = p(Q)$.

[3] For P a premouse, P^- is the same as P, except that $k(P^-) = k(P) - 1$.

[4] The phalanx side ends with Q above H by Dodd-Jensen. It cannot drop on the branch to Q by Dodd-Jensen. The M side cannot end with P such that $Q \lhd P$ because otherwise the new subset of ρ would be in P, hence M. So $P = Q$, and the M-to-P branch of $\mathcal{U}$ does not drop. Its embedding j has $\mathrm{crit}(j) \geq \rho$ because $\rho = \rho(H) = \rho(Q) = \rho(P)$.

Since $i(p(H)) = j(p(M))$, and $i \restriction \rho = j \restriction \rho = \text{ id}$, we get

$$\pi = j^{-1} \circ i.$$

But $\mathrm{crit}(j) > \rho$ by part (1), and $\mathrm{crit}(\pi) = \rho$, so $\mathrm{crit}(i) = \rho$.

By (1) and $\mathrm{crit}(j) > \rho$, ρ is not the critical point of a total on Q extender from Q. It follows that the first extender used in i is the order zero total measure on ρ from the H-sequence. Call this D. We get the desired $\sigma\colon \mathrm{Ult}(H, D) \to M$ by setting $\sigma = j^{-1} \circ k$, where k is the branch tail of H-to-Q, i.e. $i = k \circ i_D$. □

Remark 2.2 The equivalence of (2)(a) and (2)(b) requires only that we are dealing with a mouse and its core. Part (1), and the equivalent (2)(c), rely on amenable closure. So these only work for the cores taken in a background construction. It is easy to produce a counterexample otherwise, by taking $M = \mathrm{Ult}(H, E)$ for E not of order zero.

Definition 2.3 *For M a premouse, M^+ is the same as M, except that $k(M^+) = k(M) + 1$. If $k(M) > 0$, then M^- is the same as M, except that $k(M^-) = k(M) - 1$. We let $\rho^-(M) = \rho(M^-) = \rho_{k(M)}(M)$.*

Of course, M^+ is only a premouse if M is sound.

Definition 2.4 *For N a premouse,*

(a) N is projectum-critical *iff there is a total-on-N extender E on the N-sequence such that* $\mathrm{crit}(E) = \rho^-(N)$.

(b) (N, D) is a pfs-violation *iff D is a total-on-N extender on the N-sequence, and* $\mathrm{crit}(D) = \rho^-(N)$.

(c) A pfs-violation (N, D) has order zero *iff D has order zero.*

Here "pfs" stands for "projectum-free spaces". Clearly, N is projectum-critical iff there is a (unique) order zero pfs-violation of the form (N, D). We will sometimes say that (N, κ, D) is a pfs-violation, if (N, D) is a pfs-violation and $\kappa = \mathrm{crit}(D) = \rho_{k(N)}(N)$.

3 Resurrection re-defined

Lemma 2.1 says that the uncoring $\pi\colon M^{\mathbb{C}}_{\nu,k+1} \to M^{\mathbb{C}}_{\nu,k}$ has critical point $\rho(M_{\nu,k})$ iff $M_{\nu,k+1}$ is projectum-critical. We are going to change slightly how the resurrection maps work at such points. It will also help a little to change the standard notation for them.

Let $\mathbb{C}$ be a w-construction, let $\langle \eta, l\rangle <_{\text{lex}} \text{lh}(\mathbb{C})$, and let $N \trianglelefteq M_{\eta,l}$. We define analogs of $\text{Res}_{\eta,l}[N]$ and $\sigma_{\eta,l}[N]$. These were the complete resurrection of N from stage $\langle \eta, l\rangle$, along with its resurrection map. We want to also look at intermediate resurrections, from $\langle \eta, l\rangle$ to some $\langle \xi, k\rangle >_{\text{lex}} \text{Res}_{\eta,l}[N]$.

We can reduce the number of subscripts by making use of the fact that in any construction $\mathbb{C}$, $M^{\mathbb{C}}_{\eta,l}$ determines $\langle \eta, l\rangle$. Fixing $\mathbb{C}$, we write

$$P \leq_{\mathbb{C}} Q \Leftrightarrow \exists \eta, l, \xi, k(P = M^{\mathbb{C}}_{\eta,l} \wedge Q = M^{\mathbb{C}}_{\xi,k} \wedge \langle \eta, l\rangle \leq_{\text{lex}} \langle \xi, k\rangle).$$

If Q is a model of $\mathbb{C}$ and $N \trianglelefteq Q$, we define $R = \text{Res}_Q[N]$ and $\sigma = \text{res}_Q[N]$. We shall have $R \leq_{\mathbb{C}} Q$, $k(R) = k(N)$, and $\sigma\colon N \to R$ is elementary.[5] For S such that $R \leq_{\mathbb{C}} S \leq_{\mathbb{C}} Q$, we shall also define the partial resurrection $\text{Res}_{Q,S}[N]$ and its map $\text{res}_{Q,S}[N]$.

For any Q, we let

$$\text{Res}_Q[Q] = Q, \text{ and } \text{res}_Q[Q] = \text{ id}.$$

The remainder of the definition is by induction on the place of Q in $<_{\mathbb{C}}$. Suppose first that $Q = M_{\nu,k+1}$, let $R = M_{\nu,k}$, and let

$$\pi\colon Q^- \to X = M_{\nu,k}$$

be the uncoring map.[6] π is cofinal and elementary. We define $\text{Res}_{Q,X}[N]$ and $\text{res}_{Q,X}[N]$ for $N \triangleleft Q$. If Q is not projectum-critical, the definition is the usual one, which simplifies in that case to

$$\text{Res}_{Q,X}[N] = \pi(N),$$

and

$$\text{res}_{Q,X}[N] = \pi.$$

[5]It may not be cofinal.

[6]For S a premouse with $k(S) > 0$, we let S^- be the same as S, except that $k(S^-) = k(S) - 1$.

If Q is projectum-critical, then let (Q, D) be the unique order zero pfs-violation, so that $\mathrm{crit}(D) = \rho^-(Q) = \rho(X)$. We have the factor map $\sigma\colon \mathrm{Ult}(Q, D) \to X$, with $\pi = \sigma \circ i_D$, and $\sigma \restriction \rho(X)^{+,X} = \mathrm{id}$. We set

$$\mathrm{Res}_{Q,X}[N] = \begin{cases} \sigma(N), & \text{for } o(N) < \mathrm{lh}(D), \\ \pi(N), & \text{otherwise,} \end{cases}$$

and

$$\mathrm{res}_{Q,X}[N] = \begin{cases} \sigma \restriction N, & \text{for } o(N) < \mathrm{lh}(D), \\ \pi \restriction P, & \text{otherwise.} \end{cases}$$

The standard resurrection maps differ from these at N such that $\rho(X)^{+,X} \leq o(N) < \mathrm{lh}(D)$, where the standard maps would have values $\pi(N)$ and $\pi \restriction N$. The maps above reflect the fact that although Q and $\mathrm{Ult}(Q, D)$ agree past N in this case, we shall be thinking of N as an initial segment of $\mathrm{Ult}(Q, D)$ when we lift an ultrapower by D to the background universe.

If $N = Q^-$, then $\mathrm{Res}_{Q,X}[N] = X$ is the complete resurrection of N from Q, and we write $\mathrm{Res}_Q[N]$ for it, and $\mathrm{res}_Q[N]$ for its map. (I.e. π.) If $N \lhd Q^-$ and $Y <_{\mathbb{C}} X$, we then define $\mathrm{Res}_{Q,Y}[N]$ and $\mathrm{res}_{Q,Y}[N]$ for $Y <_{\mathbb{C}} X$ by composing:

$$\mathrm{Res}_{Q,Y}[N] = \mathrm{Res}_{X,Y}[\mathrm{Res}_{Q,X}[N]],$$

and

$$\mathrm{res}_{Q,Y}[N] = \mathrm{res}_{X,Y}[\mathrm{Res}_{Q,X}[N]] \circ \mathrm{res}_{Q,X}[N].$$

The $<_{\mathbb{C}}$ least Y such that $\mathrm{Res}_{Q,Y}[N]$ is defined is the complete resurrection of N from Q, or $\mathrm{Res}_Q[N]$. The complete resurrection map is $\mathrm{res}_Q[N]\colon N \to \mathrm{Res}_Q[N]$.

The limit case, when $k(Q) = 0$, is handled as before.[7]

Some simple observations:

(i) $\mathrm{Res}_Q[N]$ is the $<_{\mathbb{C}}$-least X such that $\mathrm{Res}_{Q,X}[N]$ is defined.

(ii) $k(N) = k(\mathrm{Res}_{Q,X}[N])$, and $\mathrm{res}_{Q,X}[N]$ is elementary.

(iii) If $P \lhd N$, then $\mathrm{Res}_Q[P] <_{\mathbb{C}} \mathrm{Res}_Q[N]$.

(iv) Suppose $k(N) > 0$ and $\mathrm{Res}_{Q,X}[N]$ is defined; then $\mathrm{Res}_{Q,X}[N^-] = (\mathrm{Res}_{Q,X}[N])^-$.

[7] Let $Q = M_{\nu,0}$ and $N \lhd Q$. Let ρ be the minimum value of $\rho_{k(R)}(R)$ for $N \lhd R \lhd Q$, and let $N \lhd R \lhd Q$ be such that $\rho_{k(R)}(R) = \rho$. For $R \leq_{\mathbb{C}} S <_{\mathbb{C}} Q$, $\mathrm{Res}_{Q,S}[N] = N$ and $\mathrm{res}_{Q,S}[N] = \mathrm{id}$. For $S <_{\mathbb{C}} R$, $\mathrm{Res}_{Q,S}[N] = \mathrm{Res}_{R,S}[N]$, etc.

(v) If $\mathrm{Res}_Q[N] = M_{\nu,k+1}$, then $\mathrm{Res}_Q[N^-] = M_{\nu,k}$. Moreover, if $\pi\colon (M_{\nu,k+1})^- \to M_{\nu,k}$ is the uncoring map, then $\pi \circ \mathrm{res}_Q[N] = \mathrm{res}_Q[N^-]$.

These are easy to prove by induction.

The resurrection maps on the way to $\mathrm{Res}_Q[N]$ and $\mathrm{res}_Q[N]$ can also be defined using the *N-dropdown sequence of Q*. This is a good way of organizing things.

Definition 3.1 *Let $N \lhd Q$. The N-*dropdown sequence of *Q is given by*

(1) $A_0 = N$,

(2) A_{i+1} is the least $B \trianglelefteq Q$ such that $A_i \lhd B$ and $\rho^-(B) < \rho^-(A_i)$.

We let $n(Q, N)$ be the largest n such that A_n is defined. We write $A_i = A_i(Q, N)$.

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Let Q be a level of a background construction $\mathbb{C}$, and $N \lhd Q$. Let $\langle A_i \mid i \leq n\rangle$ be the N-dropdown sequence of Q, and set $\kappa_i = \rho^-(A_i)$. For $i > 0$, $k(A_i) > 0$, so set $k_i = k(A_i^-)$. ‘We can analyze the partial resurrections $\mathrm{Res}_Q[A_i]$ and $\mathrm{res}_Q[A_i]$, starting with $i = n$ and working down to $i = 0$, where we reach the complete resurrection of $A_0 = N$. This was done in FSIT; here is a quick summary, adapted to our slightly changed resurrection maps.

(1) $A_n \leq_{\mathbb{C}} Q$, $\mathrm{Res}_Q[A_n] = A_n$, and $\mathrm{res}_Q[A_n]$ is the identity.

(2) Thus if $n = 0$, then $N <_{\mathbb{C}} Q$. In this case, $\mathrm{Res}_Q[N] = N$ and $\mathrm{res}_Q[N]$ is the identity. Moreover, for $P \lhd N$, $\mathrm{Res}_Q[P] = \mathrm{Res}_N[P]$ and $\mathrm{res}_Q[P] = \mathrm{res}_N[P]$.

For the remaining facts, we assume $n > 0$. Then

(3) Let $A_n = M_{\eta,k_n+1}$, so that $\kappa_n = \rho_{k_n+1}(A_n)$. We let $\pi_n\colon (M_{\eta,k_n+1})^- \to M_{\eta,k_n}$ be the uncoring map. We have then that $\mathrm{Res}_Q[A_n^-] = M_{\eta,k_n}$ and $\mathrm{res}_Q[A_n^-] = \pi_n$.

(4) If A_n is not projectum-critical, then for all $P \lhd A_n$, $\mathrm{Res}_{Q,M_{\eta,k_n}}[P] = \pi_n(P)$ and $\mathrm{res}_{Q,M_{\eta,k_n}}[P] = \pi_n \restriction P$. If A_n is projectum-critical, this holds except when $\kappa_n \leq o(P) < \mathrm{lh}(D)$, where D is the order zero measure of A_n on κ_n. In that case, $\mathrm{Res}_{Q,M_{\eta,k_n}}[P] = P$ and $\mathrm{res}_{Q,M_{\eta,k_n}}[P]$ is the identity.

(5) In particular, when $P \lhd N$, then $\mathrm{Res}_{Q,M_{\eta,k_n}}[P] = \mathrm{res}_{Q,M_{\eta,k_n}}[N](P)$ unless A_n is projectum-critical, and for D the order zero measure of A_n on κ_n, $\kappa_n \leq o(P) < \mathrm{lh}(D) \leq o(N)$.

(6) For $i \leq n-1$, let $A_i^1 = \pi_n(A_i)$. Then the $\pi_n(N)$ dropdown sequence of M_{η,k_n} begins with $\langle A_i^1 \mid i \leq n-1 \rangle$. If it is the entire sequence, i.e. $n(M_{\eta,k_n}, \pi_n(N)) = n-1$, then we are back where we began, replacing Q by $Q_1 = M_{\eta,k_n} = \text{Res}_Q[A_n^-]$, and N by $\pi_n(N)$. The projecta associated to the new dropdown sequence are the $\pi_n(\kappa_i)$ for $i \leq n-1$, except that we may have (The projecta are also preserved, except that we may have $\rho^-(A_{n-1}^1) = \sup \pi_n``\kappa_{n-1} < \pi_n(\kappa_{n-1})$, in the case that $A_{n-1} = A_n^-$.)

(7) It can happen that the $\pi_n(N)$ dropdown sequence of M_{η,k_n} has one more term, namely $A_n^1 = M_{\eta,k_n}$. This can happen when $\kappa_{n-1} = \rho_{k_n}(A_n) = \rho^-(A_n^-)$, but $A_{n-1} \lhd A_n^-$, so that A_n^- is not in the N-dropdown sequence of Q. If also π_n is discontinuous at κ_{n-1}, then $\sup \pi_n``\kappa_{n-1} = \rho_{k_n}(M_{\eta,k_n}) < \pi_n(\kappa_n - 1) = \rho^-(A_{n-1}^1)$.

(8) If (7) applies, let $\tau_0 \colon M_{\eta,k_n}^- \to M_{\eta,k_n-1}$ be the uncoring. The $\tau_0 \circ \pi_n(N)$ dropdown sequence of M_{η,k_n-1} starts with $\langle \tau_0(A_i^1) \mid i \leq n-1 \rangle$. If that is all of it, set $Q_1 = M_{\eta,k_n-1}$ and $\sigma_n = \tau_0$. Now go back to step 1, replacing Q by Q_1 and N by $\sigma_n \circ \pi_n(N)$.

(9) If the $\tau_0 \circ \pi_n(N)$ dropdown sequence of M_{η,k_n-1} has another term at the end, it is M_{η,k_n-1} itself, and we have $\tau_0 \circ \pi_n(\kappa_{n-1}) = \rho_{k_n-1}(M_{\eta,k_n-1})$, and the uncoring $\tau_1 \colon M_{\eta,k_n-1} \to M_{\eta,k_n-2}$ is discontinuous at $\tau_0 \circ \pi_n(\kappa_{n-1})$.

(10) Eventually we reach $l \leq k_n$ and $\sigma_n \colon M_{\eta,k_n} \to M_{\eta,l}$ the composition of the uncoring maps τ_j such that the $\sigma_n \circ \pi_n(N)$ dropdown sequence of $M_{\eta,l}$ is $\langle \sigma_n \circ \pi_n(A_i) \mid i \leq n-1 \rangle$. Set $Q_1 = M_{\eta,l}$, $N_1 = \sigma_n \circ \pi_n(N)$, and go back to step 1.

This analysis lets us write

$$\text{res}_Q[A_i] = \sigma_{i-1} \circ \pi_{i-1} \ldots \sigma_n \circ \pi_n,$$

and

$$\text{res}_Q[A_i^-] = \pi_i \circ \text{res}_Q[A_i].$$

Some of the σ_i and π_i may be the identity. In general,

$$\pi_i \colon \text{Res}_Q[A_i] \to \text{Res}_Q[A_i^-]$$

is an uncoring map of $\mathbb{C}$. The projectum associated to π_i is $\rho^-(\text{Res}_Q[A_i])$. We have $\rho^-(\text{Res}_Q[A_i]) \leq \text{res}_Q[A_i](\rho^-(A_i))$, but strict inequality is possible, for the reason described in item (6).

Here are a few more general facts.

Lemma 3.2 *Let $\mathbb{C}$ be a background construction, let Q be a model of $\mathbb{C}$, and let $N \lhd Q$. Suppose that $\text{Res}_Q[N] \leq_{\mathbb{C}} X \leq_{\mathbb{C}} Y \leq_{\mathbb{C}} Q$; then*

(1) $\text{Res}_{Q,X}[N] = \text{Res}_{Y,X}[\text{Res}_{Q,X}[N]]$*, and*

(2) $\text{res}_{Q,X}[N] = \text{res}_{Y,X}[\text{Res}_{Q,Y}[N]] \circ \text{res}_{Q,Y}[N]$.

Moreover, if $P \lhd N$, then $\text{Res}_Q[P] <_{\mathbb{C}} \text{Res}_Q[N]$, and either

(3A) $\text{Res}_{Q,Y}[P] = \text{res}_{Q,Y}[N](P)$*, and* $\text{res}_{Q,Y}[P] = \text{res}_{Q,Y}[N]\restriction P$*, or*

(3B) letting $\langle A_i \mid i \leq n\rangle$ be the N-dropdown sequence of Q, some A_i is projectum-critical, and letting D be the order zero measure of A_i on $\rho^-(A_i)$, $\rho^-(A_i) \leq o(P) < \text{lh}(D) \leq o(N)$.

The consistency-of-resurrections property (3)(A) is what we would like to have in our proposed proof that $W(\mathcal{T},\mathcal{U})^* = W(\mathcal{T}^*,\mathcal{U}^*)$. Unfortunately, it is weakened by the presence of alternative (3)(B). Alternative (3)(B) is needed to take into account the possibility of projectum-critical uncorings. For the standard resurrection maps, we can add to (3)(B) that $\kappa \leq o(P) < \kappa^{+,M}$. We seem to have just made matters worse with our re-defined resurrection maps, as now $\kappa^{+,M} \leq o(P) < \text{lh}(D)$ is a possibility. But these maps are going to be used in a revised lifting procedure, and the two revisions, together with a restriction on $\mathcal{T}$, will yield a proof that $W(\mathcal{T},\mathcal{U})^* = W(\mathcal{T}^*,\mathcal{U}^*)$.

4 $\text{lift}(\mathcal{T}, M, \mathbb{C})$ re-defined

We now need to make a small change to the way plus trees on $M^{\mathbb{C}}_{\nu,k}$ are lifted to the background. The change is: when $D = (E^{\mathcal{T}}_\alpha)^-$ is part of an order zero pfs-violation (N, κ, D) such that $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\alpha$, then at the background level we may not copy the relevant ultrapower, we may realize it instead. When we do realize, (2)(c) of Lemma 2.1 gives us a next lifting map that agrees appropriately with the earlier lifting maps.

More precisely, $\text{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle \psi_\alpha \mid \alpha < \text{lh}(\mathcal{T})\rangle, \langle \eta_\alpha, l_\alpha \mid \alpha < \text{lh}(\mathcal{T})\rangle\rangle$ is defined as before[8], but with the following exception. We say that α *is* $\text{lift}(\mathcal{T}, M, \mathbb{C})$-*anomalous* if this exception applies. The following paragraphs define what it is to be anomalous, and describe $\psi_{\alpha+1}$ in that case.

Set $M_\alpha = \mathcal{M}^{\mathcal{T}}_\alpha$, $E = (E^{\mathcal{T}}_\alpha)^-$, and let $\kappa = \text{crit}(E)$. If there is no B such that $M_\alpha | \text{lh}(E) \trianglelefteq B \trianglelefteq M_\alpha$ and (B, E) is an order zero pfs-violation, then α is not

[8]Using the new resurrection maps defined above.

lift($\mathcal{T}, M, \mathbb{C}$)-anomalous. Suppose now there is such a B, and let B be the first such initial segment of M_α. E is a normal measure, so $\alpha = T$-pred$(\alpha+1)$, and

$$M^{\mathcal{T}}_{\alpha+1} = \text{Ult}(B, E_\alpha),$$

by the rules of normal trees. Here $E_\alpha = E$ or $E_\alpha = E^+$. Now let

$$Q_\alpha = M^{\mathbb{C}_\alpha}_{\eta_\alpha, l_\alpha},$$

where $\mathbb{C}_\alpha = i^{\mathcal{T}^*}_{0,\alpha}(\mathbb{C})$, so that

$$\psi_\alpha \colon M_\alpha \to Q_\alpha.$$

Let $K = \psi_\alpha(E)$, $\kappa = \text{crit}(K)$, and $A = \psi_\alpha(B)$. Thus (A, κ, K) is an order zero pfs-violation. The usual lifting would let $E^{\mathcal{T}^*}_\alpha$ be the background in $\mathbb{C}_\alpha$ for the complete resurrection

$$G = \text{res}_{Q_\alpha}[Q_\alpha | \text{lh}(K)](K),$$

of K, but we may not want to do that.[9] We shall follow the standard procedure iff $\text{res}_{Q_\alpha}[A]$ involves no projectum-critical uncoring.

Let

$$N = Q_\alpha | \text{lh}(K).$$

Notice that A is in the N-dropdown sequence of Q_α. (Let this sequence be $\langle A_i \mid i \leq n\rangle$. Then $A_0 = N$ and $\rho^-(A_0) = o(N)$. $A_1 = N^+$, and $\rho^-(A_1) \leq \kappa^{+,N}$. If $\rho^-(A_1) \leq \kappa$, then $A = A_1$, and otherwise, $A = A_2$.) Moreover, we can factor

$$\text{res}_{Q_\alpha}[A^-] = \pi \circ \text{res}_{Q_\alpha}[A],$$

where

$$\pi \colon \text{ Res}_{Q_\alpha}[A] \to \text{Res}_{Q_\alpha}[A^-]$$

is the uncoring map of $\mathbb{C}_\alpha$ associated to $\rho^-(\text{Res}_{Q_\alpha}[A])$. Let $R = \text{Res}_{Q_\alpha}[A]$, and $S = \text{Res}_{Q_\alpha}[A^-]$.

Let $\tau = \text{res}_{Q_\alpha}[A]$, so that

$$\tau \colon A \to R$$

is elementary, and $\rho^-(R) \leq \tau(\kappa)$. We say α is lift($\mathcal{T}, M, \mathbb{C}$)-anomalous iff $\rho^-(R) = \tau(\kappa)$. If α is not lift($\mathcal{T}, M, \mathbb{C}$) anomalous, and we let $E^{\mathcal{T}^*}_\alpha$ be the $\mathbb{C}_\alpha$-background of G, as usual.

[9] It doesn't matter here whether E_α is E or E^+, since the background for G also backgrounds G^+.

Suppose $\tau(\kappa) = \rho^-(R) =_{\text{df}} \mu$. We must have that $\mu = \text{crit}(\pi)$, because otherwise $\mu = \text{crit}(G)$, and the background extender G^* for G also has critical point μ, so $\text{Res}_{Q_\alpha}[A^-]$ would be amenably closed at $\mu = \rho(\text{Res}_{Q_\alpha}[A^-])$, contradiction. Let

$$D = \tau(K) = \tau \circ \psi_\alpha(E).$$

By Lemma 2.1, we we get a natural realization

$$\sigma\colon \text{Ult}(R, D) \to S$$

by setting $\sigma([\{\mu\}, f]^R_{\tau(K)}) = \pi(f)(\mu)$.[10]

There are now two cases, depending on whether $E_\alpha = E$ or $E_\alpha = E^+$. Suppose first $E_\alpha = E$. Let

$$i\colon \text{Ult}(B, E) \to \text{Ult}(R, D)$$

be the copy map, given by $i([\{\kappa\}, f]^B_E) = [\{\mu\}, \tau \circ \psi_\alpha(f)]^R_D$. We let

$$\psi_{\alpha+1} = \sigma \circ i,$$

and $Q_{\alpha+1} = S = \text{Res}_{Q_\alpha}[A^-]$. We also let $\mathcal{M}^{\mathcal{T}^*}_{\alpha+1} = \mathcal{M}^{\mathcal{T}^*}_\alpha$ in this anomalous case. We set

$$\text{res}_\alpha = \sigma \circ \tau \restriction N.$$

We have that $\psi_{\alpha+1}$ agrees with $\text{res}_\alpha \circ \psi_\alpha$ on $\text{lh}(E^{\mathcal{T}}_\alpha)$. ($i$ agrees with $\tau \circ \psi_\alpha$ on $\text{lh}(E)$, so $\sigma \circ i$ agrees with $\sigma \circ \tau \circ \psi_\alpha$ on $\text{lh}(E)$.) Note that $\text{res}_\alpha \circ \psi_\alpha$ *realizes* the generators of E_α in $S | \text{crit}(G)^+$, rather than copying them into $i_{G^*}(S) | \text{lh}(G^*)$, as would be done in the standard procedure when E_α is not of plus type.

Suppose next that $E_\alpha = E^+$. Let $i\colon \text{Ult}(B, E) \to \text{Ult}(R, D)$ and $\sigma\colon \text{Ult}(R, D) \to S = \text{Res}_{Q_\alpha}[A^-]$ be as above. Let $B_1 = \text{Ult}(B, E)$, $E_1 = i^B_E(E)$, $R_1 = \text{Ult}(R, D)$, and $D_1 = i^R_D(D)$. Thus

$$\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \text{Ult}(B_1, E_1).$$

We are going to set

$$Q_{\alpha+1} = i_{G^*}(S),$$

and let $\psi_{\alpha+1}$ be the natural map, which we now describe.

Let $i_1\colon \text{Ult}(B_1, E_1) \to \text{Ult}(R_1, D_1)$ come from copying under i. Let $D_2 = \pi(D) = \sigma(D_1)$. Let $\sigma_1\colon \text{Ult}(R_1, D_1) \to \text{Ult}(S, D_2)$ come from copying under σ. Finally, let $\theta = \text{res}_S[S | \text{lh}(D_2)]$ so that $\theta(D_2) = G$. If $A = A_1$, then $D_2 = G$ and θ is the

[10] This is a $k(R)$-ultrapower, so the formula applies to f given by Skolem terms.

identity, and if $A = A_2$, then $\mathrm{crit}(\theta) > \mathrm{crit}(D_2) = \mathrm{crit}(G)$.[11] We have a natural map $\varphi\colon \mathrm{Ult}(S, D_2) \to i_{G^*}(S)$ given by setting

$$\varphi([a, f]^S_{D_2}) = [\theta(a), f]^{\mathcal{M}^{\mathcal{T}^*}_\alpha}_{G^*}.$$

We set

$$\psi_{\alpha+1} = \varphi \circ \sigma_1 \circ i_1.$$

Again, we let $\mathrm{res}_\alpha = \sigma \circ \tau \restriction N$. Recall that our convention is that $\mathrm{lh}(E^+) = \mathrm{lh}(E)$. $i \restriction \mathrm{lh}(E) = i_1 \restriction \mathrm{lh}(E)$, $\sigma_1 \restriction \mathrm{lh}(D) = \sigma \restriction lh(D)$, and $\varphi \restriction \sigma(\mathrm{lh}(D)) = \varphi \restriction \mathrm{crit}(D_2)^{+,S}$, and $\theta \restriction \mathrm{crit}(D_2)^{+,S} = \mathrm{id}$. So again, we get

$$\psi_{\alpha+1} \restriction \mathrm{lh}(E_\alpha) = \mathrm{res}_\alpha \circ \psi_\alpha \restriction \mathrm{lh}(E_\alpha).$$

Again, this relies on our convention on the lengths of plus extenders. Note that in the plus case too, $\mathrm{res}_\alpha \circ \psi_\alpha$ maps the the generators of E_α, namely $\lambda(E) \cup \{\lambda(E)\}$, into $S|\,\mathrm{crit}(G)^+$, rather than copying them into $i_{G^*}(S)|\,\mathrm{lh}(G)$ as would be done in the plus case by the standard procedure.

If α is not anomalous, then as usual, we let

$$\mathrm{res}_\alpha = \mathrm{res}_{Q_\alpha}[Q_\alpha|\,\mathrm{lh}(K)].$$

We get that $\psi_{\alpha+1}$ agrees with $\mathrm{res}_\alpha \circ \psi_\alpha$ on $\lambda^0(E_\alpha)$ in general, and on $\mathrm{lh}(E_\alpha) + 1$ in the plus case, as before.

Having redefined our lifting process, we get the following useful agreement lemma.

Lemma 4.1 *Let α be lift-anomalous. Using the notation above ($\psi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to Q_\alpha$, $N = Q_\alpha|\,\mathrm{lh}(\psi_\alpha(E^{\mathcal{T}}_\alpha))$, and resurrection maps computed according to $\mathbb{C}_\alpha = i^{\mathcal{T}^*}_{0,\alpha}(\mathbb{C})$), we have that for $P \lhd N$*

(a) $\mathrm{res}_\alpha(P) = \mathrm{Res}_{Q_\alpha, \mathrm{Res}_{Q_\alpha}[N]}[P]$, *and*

(b) res_α *agrees with* $\mathrm{res}_{Q_\alpha, \mathrm{Res}_{Q_\alpha}[N]}[P]$ *on* P.

The lemma follows easily from the way we have re-defined $\mathrm{Res}^{\mathbb{C}_\alpha}_{Q_\alpha}[P]$ and $\mathrm{res}_{Q_\alpha}[P]$ in the case that the resurrection involves uncoring a pfs-violation, and the way we have re-defined res_α in the lift-anomalous case.

The same agreement conclusions hold if Q_α, N, and P are such that no relevant projectum-critical uncoring is possible:

[11] G and D_2 are unsound order zero extenders. So the fact that they yield the same normal measure does not imply they are equal.

Lemma 4.2 *Using the notation above ($\psi_\alpha\colon \mathcal{M}_\alpha^{\mathcal{T}} \to Q_\alpha$, $N = Q_\alpha|\operatorname{lh}(\psi_\alpha(E_\alpha^{\mathcal{T}}))$, and resurrection maps computed according to $\mathbb{C}_\alpha = i_{0,\alpha}^{\mathcal{T}^*}(\mathbb{C})$), let $P \lhd N$, and suppose there is no pfs violation (A, κ, D) such that (i) A is in the N-dropdown sequence of Q_α, and (ii) $\kappa \leq o(P) < \operatorname{lh}(D)$; then*

(a) $\operatorname{res}_\alpha(P) = \operatorname{Res}_{Q_\alpha, \operatorname{Res}_{Q_\alpha}[N]}[P]$, *and*

(b) $\operatorname{res}_\alpha$ *agrees with* $\operatorname{res}_{Q_\alpha, \operatorname{Res}_{Q_\alpha}[N]}[P]$ *on* P.

Proof. We are not in the anomalous case, so $\operatorname{res}_\alpha = \operatorname{res}_{Q_\alpha}[N]$. Because there are no pfs violations of the sort described, Lemma 3.2(3)(A) must apply, and we have that $\operatorname{res}_{Q_\alpha}[N](P) = \operatorname{Res}_{Q_\alpha, \operatorname{Res}_{Q_\alpha}[N]}[P]$ and $\operatorname{res}_{Q_\alpha}[N]$ agrees with $\operatorname{res}_{Q_\alpha, \operatorname{Res}_{Q_\alpha}[N]}[P]$ on P. □

Note that (a) and (b) of Lemma 4.1 would fail in the anomalous case if we defined $\operatorname{res}_\alpha$ to be $\operatorname{res}_{Q_\alpha}[N]$ in that case. This would be a problem for normalizing in the case $P = \psi_\alpha(M_\alpha|\operatorname{lh}(F))$, where F is being inserted into $\mathcal{T}$ as part of the normalizing process. By lifting as we have done in the $\operatorname{lift}(\mathcal{T}, M, \mathbb{C})$-anomalous case, we have avoided this problem.

Neither Lemma 4.1 nor Lemma 4.2 covers the case that there is a pfs-violation (A, κ, H) such that A is in the N-dropdown sequence of Q_α and $\kappa \leq o(P) < \operatorname{lh}(H)$, but we are not iterating it away, that is, $H \neq \psi_\alpha(E_\alpha^{\mathcal{T}})$. Then $\operatorname{res}_\alpha(P) \neq \operatorname{Res}_{Q_\alpha, \operatorname{Res}_{Q_\alpha}[N]}[P]$ is possible, and it causes trouble in showing $W(\mathcal{T}, \mathcal{U})^*$ is an initial segment of $W(\mathcal{T}^*, \mathcal{U}^*)$ in general. We shall see in the next section that we can afford to restrict ourselves to trees $\mathcal{T}$ such that his troublesome case will never occur in the proof that $W(\mathcal{T}, \mathcal{U})^* = W(\mathcal{T}^*, \mathcal{U}^*)$.

5 ρ-separated iteration trees

We now isolate a class of plus trees for which the troublesome case in lifting to a background that we just described cannot occur.

Definition 5.1 *Let $\mathcal{T}$ be a normal plus tree on a premouse, with models M_α and exit extenders E_α. We say that $\mathcal{T}$ is* ρ-separated *iff whenever $\alpha + 1 < \operatorname{lh}(\mathcal{T})$, then there is no pfs-violation (N, κ, D) such that*

(a) D is on the sequence of M_α strictly before E_α^-, and

(b) $M_\alpha|\operatorname{lh}(E_\alpha) \trianglelefteq N \trianglelefteq M_\alpha$.

In a ρ-separated tree, whenever we encounter a pfs-violation (N, κ, D), we must either iterate by D, or skip past N. We are not allowed to iterate by an extender that is in N, but past D. In particular, if E_α^- is part of a pfs violation, then E_α^- must have order zero. (Otherwise, letting D to be the order zero measure on $\text{crit}(E_\alpha^-)$, D is on the $\mathcal{M}_\alpha$-sequence before E_α^-.)

Recall that a λ-separated tree is one in which every $E_\alpha^{\mathcal{T}}$ has plus type.

Definition 5.2 *A normal plus tree is* well separated *iff it is both λ-separated and ρ-separated.*

Iterations into a backgrounded construction can be organized as well-separated plus trees.

Lemma 5.3 *Let P be a countable premouse, and Σ a UB iteration strategy for P defined on normal plus trees. Let $\mathbb{C}$ be a w-construction, suppose that (P, Σ) iterates strictly past each $M_{\eta,j}^{\mathbb{C}}$ for $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$. Let $\mathcal{T}$ be the unique λ-separated plus tree whereby (P, Σ) iterates past $M_{\nu,k}^{\mathbb{C}}$; then $\mathcal{T}$ is ρ-separated.*

Proof. Suppose that $\mathcal{T}$ fails to be ρ-separated at α. Let N, κ, D be a pfs-violation such that for $E = E_\alpha^{\mathcal{T}}$ and $M = \mathcal{M}_\alpha^{\mathcal{T}}$, we have $M|\text{lh}(E) \trianglelefteq N \trianglelefteq M$. D is on the sequence of M before E, so it is also on the $M_{\nu,k}$-sequence. D is total on N, so $\kappa < \lambda_E$. Since $M_{\nu,k} \trianglelefteq \mathcal{M}_\infty^{\mathcal{T}}$ and E was used in $\mathcal{T}$, $\text{lh}(E)$ is a cardinal in $M_{\nu,k}$. Also, $\rho_j(M_{\nu,k}) \geq \lambda(E)$ for all $j < k$.[12] This implies that D resurrects to itself in $\mathbb{C}$, and hence has a background extender D^* such that $D \subseteq D^*$. But then $M_{\nu,k}||\text{lh}(E)$ is amenably closed at κ, whereas $\text{Th}_{k(N)+1}^N(\kappa \cup p(N))$ witnesses that $M_{\nu,k}||\text{lh}(E) = M||\text{lh}(E)$ is not amenably closed at κ. ☐

We believe that one can show that if Σ for P has strong hull condensation, then its action on well-separated trees determines its action on arbitrary normal plus trees. Here is a sketch. Given an arbitrary normal $\mathcal{T}$ on P, we define a well-separated tree $\mathcal{T}^{\text{ws}}$ on P, and a tree embedding of $\mathcal{T}$ into $\mathcal{T}^{\text{ws}}$, by induction. Suppose we have determined $\mathcal{U}_\alpha = \mathcal{T}^{\text{ws}}\restriction v(\alpha) + 1$, and a tree embedding $\Phi_\alpha \colon \mathcal{T}\restriction(\alpha + 1) \to \mathcal{U}_\alpha$. Let $s_\alpha \colon \mathcal{M}_\alpha^{\mathcal{T}} \to Q = \mathcal{M}_{v(\alpha)}^{\mathcal{U}_\alpha}$ be given by Φ_α. Let $E = E_\alpha^{\mathcal{T}}$ and $F = s_\alpha(E^-)$. Let $N = Q|\text{lh}(F)$.

We first iterate away all the order zero pfs violations corresponding to points $A_i(Q, N)$ in the N-dropdown sequence of Q, We start with the largest such i, that

[12]Suppose not. Since $\text{lh}(E)$ is a cardinal in $S = \mathcal{M}_\infty^{\mathcal{T}}$, S and $M_{\nu,k}$ are the same, except possibly that $k(S) > k$. S is j-sound, so all extenders F used in $\mathcal{T}$ on the branch to S satisfy $\lambda(F) \leq \rho_j(S)$. On the other hand, this branch must use an F such that $\lambda(F) \leq \lambda(E)$, contradiction

is the smallest $\rho^-(A_i)$. If new violations appear as we iterate, we iterate them away too. The result is a finite (possibly empty) normal iteration $\langle D_k \mid k < n\rangle$ of N by order zero measures. Let $G = i_{\vec{D}}(F)$. Note that none of the $\rho^-(A_i)$ are strictly below $\lambda^0(E_\gamma^{\mathcal{U}_\alpha})$ for some $\gamma < v(\alpha)$. Hence $\mathcal{S} = \mathcal{U}_\alpha^\frown \vec{D}$ is a normal tree with last model indexed at $v(\alpha)+k$, and $v(\alpha) <_S v(\alpha+1 <_S \ldots <_S v(\alpha)+k$. We set $u(\alpha) = v(\alpha+k)$, $v(\alpha+1) = u(\alpha)+1$, $t_\alpha = i_{\vec{D}} \circ s_\alpha$, and $\mathcal{U}_{\alpha+1} = \mathcal{S}^\frown\langle G^+\rangle$. One must then check that we have extended our tree embedding Φ_α to $\Phi_{\alpha+1}\colon \mathcal{T}\restriction(\alpha+2) \to \mathcal{U}_{\alpha+1}$.

6 $W(\mathcal{T},\mathcal{U})^* = W(\mathcal{T}^*,\mathcal{U}^*)$ when $\mathcal{T}$ is well-separated

Finally, we get

Theorem 6.1 *Let $P_0 = M_{\nu,k}^{\mathbb{C}}$, where $\mathbb{C}$ is a w-construction. Let $\langle\mathcal{T},\mathcal{U}\rangle$ be a maximal stack on P_0 such that $\mathcal{T}$ is well separated, and $\mathcal{U}$ is normal; then $W(\mathcal{T},\mathcal{U})^*$ is an initial segment of $W(\mathcal{T}^*,\mathcal{U}^*)$.*

Proof. We shall cut-and-paste the repair from [3], modifying it in the appropriate places.

Let $\mathcal{T}$ be on $M_{\nu_0,k_0}^{\mathbb{C}}$, and

$$\mathrm{lift}(\mathcal{T}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi^{\mathcal{T}}, l_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle, \langle \psi_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle\rangle.$$

Let

$$\mathrm{lift}(\psi_{\xi_0}^{\mathcal{T}}\mathcal{U}, M_{\eta_{\xi_0}^{\mathcal{T}}, l_{\xi_0}^{\mathcal{T}}}^{i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})}, i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})) = \langle \mathcal{U}^*, \langle\langle \eta_\xi^{\mathcal{U}}, l_\xi^{\mathcal{U}}\rangle \mid \xi < \mathrm{lh}\,\mathcal{U}\rangle, \langle \rho_\xi \mid \xi < \mathrm{lh}\,\mathcal{U}\rangle\rangle.$$

Let $\tau_\xi : \mathcal{M}_\xi^{\mathcal{U}} \to \mathcal{M}_\xi^{(\psi_{\xi_0}^{\mathcal{T}})\mathcal{U}}$ be the copy map, and

$$\psi_\xi^{\mathcal{U}} = \rho_\xi \circ \tau_\xi,$$

so that

$$\psi_\xi^{\mathcal{U}} : \mathcal{M}_\xi^{\mathcal{U}} \to Q_\xi,$$

where Q_ξ is the appropriate model in $i_{0,\xi}^{\mathcal{U}^*}(\mathbb{C})$. We show that $W(\mathcal{T},\mathcal{U})$ lifts to an initial segment of $W(\mathcal{T}^*,\mathcal{U}^*)$. This is done by induction: set $\mathcal{W}_\gamma = W(\mathcal{T},\mathcal{U} \restriction \gamma+1)$ and $\mathcal{W}_\gamma^* = W(\mathcal{T}^*,\mathcal{U}^* \restriction \gamma+1)$, and

$$\mathrm{lift}(\mathcal{W}_\gamma, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{S}_\gamma^*, \langle\langle \eta_\xi^\gamma, l_\xi^\gamma\rangle \mid \xi < \mathrm{lh}\,\mathcal{W}_\gamma\rangle, \langle \psi_\xi^\gamma \mid \xi \leq z(\gamma)\rangle\rangle.$$

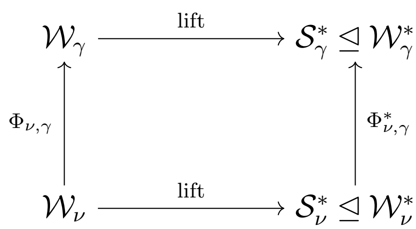
Here $z(\gamma) = \mathrm{lh}(\mathcal{W}_\gamma) - 1$. Let also $z^*(\gamma) = \mathrm{lh}(\mathcal{W}^*_\gamma) - 1$. Because we have full normalization at the background level,

$$\mathcal{M}^{\mathcal{U}^*}_\gamma = \mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}.$$

Let also $\mathbb{C}^\gamma_\xi = i^{\mathcal{S}^*_\gamma}_{0,\xi}(\mathbb{C})$ and $Q^\gamma_\xi = M^{\mathbb{C}^\gamma_\xi}_{\eta^\gamma_\xi, l^\gamma_\xi}$, so that

$$\psi^\gamma_\xi \colon \mathcal{M}^{\mathcal{W}_\gamma}_\xi \to Q^\gamma_\xi.$$

We shall show that $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction z(\gamma) + 1$. Thus $\mathcal{S}^*_\gamma$ is by Σ^*, so $\mathcal{W}_\gamma$ is by Σ. With $\gamma + 1 = \mathrm{lh}(\mathcal{U})$, this is what we want. The overall plan is summarized in the diagram:



Here $\Phi_{\nu,\gamma}$ and $\Phi^*_{\nu,\gamma}$ are the tree embeddings we get from the two embedding normalization processes.

We have by induction that the diagram holds at all $\xi \leq \gamma$, and that

$$\psi^\xi_{z(\xi)} \circ \sigma_\xi = \psi^{\mathcal{U}}_\xi$$

for all $\xi \leq \gamma$.[13] Part of this is that $\langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma \rangle = \langle \eta^\gamma_{z(\gamma)}, l^\gamma_{z(\gamma)} \rangle$. Moreover, the construction of $\mathcal{M}^{\mathcal{U}^*}_\gamma = \mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}$ agrees with $\mathbb{C}^\gamma_{z(\gamma)}$ up to this point, and the background universes $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z(\gamma)}$ and $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}$ agree to a rank past where all background extenders used for $\mathbb{C}^\gamma_{z(\gamma)}$ live. In effect, it won't hurt to assume $z^*(\gamma) = z(\gamma)$.[14] So

$$Q_\gamma = Q^\gamma_{z(\gamma)}.$$

[13] $\sigma_\xi \colon \mathcal{M}^{\mathcal{U}}_\xi \to \mathcal{M}^{\mathcal{W}_\xi}_{z(\xi)}$ comes from embedding normalization.

[14] This can only fail if $[0, \gamma]_U$ drops.

We define

$$
\begin{aligned}
F &= \sigma_\gamma(E_\gamma^{\mathcal{U}}), \\
H &= \psi_{z(\gamma)}^{\gamma}(F) \\
&= \psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}}), \\
G &= \mathrm{res}_{Q_\gamma|\,\mathrm{lh}(H)}^{\mathbb{C}_{z(\gamma)}^{\gamma}}(H), \text{ and} \\
G^* &= B^{\mathbb{C}_{z(\gamma)}^{\gamma}}(G).
\end{aligned}
$$

Here we are writing $B^{\mathbb{D}}(K)$ for the background extender associated to a completely resurrected K by a construction $\mathbb{D}$.

Let

$$\alpha = \alpha(\mathcal{W}_\gamma, F).$$

The main thing we must show is that the background extender associated by $\mathbb{C}_\alpha^\gamma$ to $\psi_\alpha^\gamma(F)$ is G^*. That was Claim 4.44 of [2], where the λ-error occurred, and Claim 6.4 of [3], where the ρ-error occurred. Our assumption that $\mathcal{T}$ is well separated helps us avoid those errors.

Let

$$
\begin{aligned}
K &= \psi_\alpha^\gamma(F), \\
P &= Q_\alpha^\gamma|\,\mathrm{lh}(K), \\
E &= \psi_\alpha^\gamma(E_\alpha^{\mathcal{W}_\gamma}), \\
N &= Q_\alpha^\gamma|\,\mathrm{lh}(E).
\end{aligned}
$$

The following is what we need

Claim 6.2 *(a)* $G = \mathrm{res}_{Q_\alpha^\gamma}[P]^{\mathbb{C}_\alpha^\gamma}(K)$,

(b) $G^* = B^{\mathbb{C}_\alpha^\gamma}(G)$, *and*

(c) $\alpha = \alpha(\mathcal{W}_\gamma^*, G^*)$.

Proof.. If $\alpha = z(\gamma)$, then (a) and (b) are clear. That gives $\alpha(\mathcal{W}_\gamma^*, G^*) \le \alpha$. But if G^* is on the $\vec{F}^{\mathbb{C}}$-sequence of $\mathcal{M}_\xi^{\mathcal{W}_\gamma^*}$, where $\xi < \alpha$, then since it is also on the $\mathcal{M}_\alpha^{\mathcal{W}_\gamma^*}$ sequence, the two sequences agree up to where G^* sits by coherence of the background sequences. But $E_\xi^{\mathcal{W}_\gamma^*}$ is on the first sequence before G^*, and $E_\xi^{\mathcal{W}_\gamma^*} \notin \mathcal{M}_\alpha^{\mathcal{W}_\gamma^*}$.

So we may assume $\alpha < z(\gamma)$. $\mathcal{W}_\gamma$ was formed by inserting the F_ξ for $\xi < \gamma$ into images of $\mathcal{W}_0$, moreover $\lambda^0(F_\xi) < \lambda^0(F_\gamma)$ for all $\xi < \gamma$. It follows from the way that embedding normalization works that

$$\alpha \in \mathrm{ran}(u^{\Phi_{0,\gamma}}).$$

Since the plus case occurred everywhere in $\mathcal{W}_0$, the plus case occurs at α in $\mathcal{W}_\gamma$. It follows that that

$$\psi^\gamma_{z(\gamma)} \restriction \mathrm{lh}(E) = \mathrm{res}^\gamma_\alpha \circ \psi^\gamma_\alpha \restriction \mathrm{lh}(E)$$

Here $\mathrm{res}^\gamma_\alpha$ is the map we defined above when lifting $\mathcal{W}_\gamma$ at stage α, mapping the generators of E into the generators of $E_\alpha^{\mathcal{W}^*_\gamma}$. The displayed equality requires λ-separation in the non-anomalous case. Without it the agreement would only be up to $\lambda(E)$, and the argument would break down when $\lambda(E) < \mathrm{lh}(K) < \mathrm{lh}(E)$. In the anomalous case, we don't actually need λ-separation here, because of the way we defined res_α is that case.

We claim that one of the two agreement lemmas 4.1 and 4.2 applies to $\mathrm{res}^\gamma_\alpha$. For suppose Lemma 4.2 does not apply; then we have a projectum-critical A in the Q_α-dropdown sequence of N. Let A be the first one, and let (A, κ, D) be an order zero pfs-violation. Since $N \in \mathrm{ran}(\psi_\alpha)$, we have (B, μ, C) such that $\psi_\alpha((B, \mu, C)) = (N, \kappa, D)$. Now let $\alpha = u^{\Phi_{0,\gamma}}(\beta)$, $t = t_\beta^{\Phi_{0,\gamma}}$, and $M = \mathcal{M}^\mathcal{T}_\beta | \mathrm{lh}(E^\mathcal{T}_\beta)$. t is only partial, but $E_\alpha^{\mathcal{W}_\gamma} \in \mathrm{ran}(t)$, and Σ_1 facts about $E_\alpha^{\mathcal{W}_\gamma}$ have witnesses in $\mathrm{ran}(t)$. It follows that we have (I, ν, J) an order zero pfs violation such that I is in the $\mathcal{M}^\mathcal{T}_\beta$-dropdown sequence of $\mathcal{M}^\mathcal{T}_\beta$.

But $\mathcal{T}$ is ρ-separated! Thus $J = (E^\mathcal{T}_\beta)^-$, and moving back up under $\psi_\alpha \circ t$, we get that $K^- = D$. This means that Lemma 4.1 applies.

Let $S = \mathrm{Res}_{Q^\gamma_\alpha}[N]^{\mathbb{C}^\gamma_\alpha}$ be the complete resurrection of N in $\mathbb{C}^\gamma_\alpha$. Since $P \lhd N$, 4.1 and 4.2 give

$$\begin{aligned} H &= \psi^\gamma_{z(\gamma)}(F) \\ &= \mathrm{res}^\gamma_\alpha \circ \psi^\gamma_\alpha(F) \\ &= \mathrm{res}^\gamma_\alpha(K) \\ &= \mathrm{res}_{Q^\gamma_\alpha, S}[P](K). \end{aligned}$$

Put another way, $Q^\gamma_{z(\gamma)} | \mathrm{lh}(H) = \mathrm{Res}_{Q^\gamma_\alpha, S}[P]^{\mathbb{C}^\gamma_\alpha}$.

It is enough now to see that H is resurrected the same way in $\mathbb{C}^\gamma_{z(\gamma)}$ and $\mathbb{C}^\gamma_\alpha$, that is, letting $P_1 = \mathrm{Res}_{Q^\gamma_\alpha, S}[P]^{\mathbb{C}^\gamma_\alpha}$,

$$\mathrm{Res}_{Q^\gamma_{z(\gamma)}}[P_1]^{\mathbb{C}^\gamma_{z(\gamma)}} = \mathrm{Res}_S[P_1]^{\mathbb{C}^\gamma_\alpha}.$$

But let E_1 be the top extender of S, and let S_0 be S with E_1 removed. Let $E_1^* = B^{\mathbb{C}_\alpha^\gamma}(E_1) = E_\alpha^{\mathcal{W}_\gamma^*}$. Note that $\mathbb{C}_\alpha^\gamma \restriction \theta = \mathbb{C}_{z(\gamma)}^\gamma \restriction \theta$, where θ is such that $E_1^* = F_\theta^{\mathbb{C}_\alpha^\gamma}$ and $S = M_{\theta,0}^{\mathbb{C}_\alpha^\gamma}$, by coherence at the background level. Thus both $\mathbb{C}_{z(\gamma)}^\gamma$ and $\mathbb{C}_\alpha^\gamma$ reach S_0 in the same way, so since $P_1 \lhd S_0$,

$$\begin{aligned}\mathrm{Res}_S[P_1]^{\mathbb{C}_\alpha^\gamma} &= \mathrm{Res}_{S_0}[P_1]^{\mathbb{C}_\alpha^\gamma}\\ &= \mathrm{Res}_{S_0}[P_1]^{\mathbb{C}_{z(\gamma)}^\gamma}\\ &= \mathrm{Res}_{Q_{z(\gamma)}^\gamma}[P_1]^{\mathbb{C}_{z(\gamma)}^\gamma}.\end{aligned}$$

The last equality holds because in $\mathbb{C}_{z(\gamma)}^\gamma$, no level between S_0 and $Q_{z(\gamma)}^\gamma$ can project across $\mathrm{lh}(E_1)$.

This proves (a) and (b) of 6.2 in the case $\alpha < z(\gamma)$. We get (c) as in the case $\alpha = z(\gamma)$.

□

If γ is not $\mathrm{lift}(\psi_{\xi_0}^{\mathcal{T}}\mathcal{U}, M_{\eta_{\xi_0}^{\mathcal{T}}, l_{\xi_0}^{\mathcal{T}}}^{i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})}, i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C}))$ anomalous, then the rest of the proof of Theorem 6.1 is the same as that in [2].

So suppose γ is $\mathrm{lift}(\psi_{\xi_0}^{\mathcal{T}}\mathcal{U}, M_{\eta_{\xi_0}^{\mathcal{T}}, l_{\xi_0}^{\mathcal{T}}}^{i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})}, i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C}))$ anomalous. This case is actually less involved. Suppose first $E_\gamma^{\mathcal{U}}$ is not of plus type. We get then

$$\begin{aligned}&\mathcal{M}_{\gamma+1}^{\mathcal{U}^*} = \mathcal{M}_\gamma^{\mathcal{U}^*},\\ &\mathcal{W}_{\gamma+1}^* = \mathcal{W}_\gamma^*, \text{ and}\\ &\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma \restriction (\alpha+1)^\frown \langle F \rangle.\end{aligned}$$

So $z(\gamma+1) = \alpha+1$. What we need to see is that $Q_{\gamma+1} = Q_{\alpha+1}^{\gamma+1}$ and $\psi_{\gamma+1}^{\mathcal{U}} \circ \sigma_{\gamma+1} = \psi_{\alpha+1}^{\gamma+1}$. Let us assume $\alpha < z(\gamma)$, as the other case is easier.

Let us adopt the notation from the proof of 6.2. $Q_{\gamma+1}$ is obtained by working in $\mathcal{M}_{\gamma+1}^{\mathcal{U}^*} = \mathcal{M}_{z^*(\gamma)}^{\mathcal{W}_\gamma^*}$ with its construction, up to the point where that construction uses G^* as a background extender for G. This is the same as working in $\mathcal{M}_{z(\gamma)}^{\mathcal{W}_\gamma^*}$ with $\mathbb{C}_{z(\gamma)}^\gamma$ up to the point G^* is used. Say $G^* = F_\eta^{\mathbb{C}_{z(\gamma)}^\gamma}$, and let $\mathbb{D} = \mathbb{C}_{z(\gamma)}^\gamma \restriction \eta + 1$. What we do is resurrect H to G inside $\mathbb{D}$, see that the last or second-to-last uncoring $\pi\colon \mathrm{Res}_{Q_{z(\gamma)}^\gamma}[A] \to \mathrm{Res}_{Q_{z(\gamma)}^\gamma}[A^-]$ is projectum-critical[15], and set

$$Q_{\gamma+1} = \mathrm{Res}_{Q_{z(\gamma)}^\gamma}[A^-]^{\mathbb{D}}.$$

[15] A is either $Q_{z(\gamma)}^\gamma | \mathrm{lh}(H)$ or $A_1(Q_{z(\gamma)}^\gamma, Q_{z(\gamma)}^\gamma | \mathrm{lh}(H))$.

To obtain $Q^{\gamma+1}_{\alpha+1}$, we resurrect K (i.e. P) inside $\mathbb{C}^\gamma_\alpha$. Recall that in $\mathbb{C}^\gamma_\alpha$, $S = \mathrm{Res}_{Q^\gamma_\alpha}[N]$, and $H = \mathrm{res}_{Q^\gamma_\alpha}[N](K) = \mathrm{res}_{Q^\gamma_\alpha, S}[P](K)$. Moreover, $\mathbb{C}^\gamma_\alpha \restriction \theta = \mathbb{C}^\gamma_{z(\gamma)} \restriction \theta = \mathbb{D} \restriction \theta$, where $S = M^{\mathbb{C}^\gamma_\alpha}_{\theta,0}$ and $E^*_1 = F_\theta^{\mathbb{C}^\gamma_{z(\gamma)}}$. Note $\mathrm{lh}(\mathbb{D}) < \theta$. So in $\mathbb{C}^\gamma_\alpha$, the resurrection of H is a tail end of the resurrection of K, and it uses only $\mathbb{D}$. It follows that the same projectum-critical uncoring is encountered in resurrecting K, and

$$Q^{\gamma+1}_{\alpha+1} = Q_{\gamma+1}.$$

One can also check that $\psi^{\mathcal{U}}_{\gamma+1} = \psi^{\gamma+1}_{\alpha+1} \circ \sigma_{\gamma+1}$, and that things work out when $E^{\mathcal{U}}_\gamma$ has plus type, too. The main points is just that the resurrrection of K in $\mathbb{C}^\gamma_\alpha$ factors into resurrecting it to H, and then resurrecting H in $\mathbb{C}^\gamma_\alpha \restriction \theta = \mathbb{C}^\gamma_{z(\gamma)} \restriction \theta$. □

7 Semi-soundness and pfs-premice

Here we sketch a different approach to the resurrection-consistency issues above. Namely, we change our constructions so that $\rho(M_{\nu,k})$ is always put into the hull collapsing to $M_{\nu,k+1}$. This makes the resurrection maps more natural, but it leads to a notion of premouse that is more complicated. Our purpose is not to advocate this approach, but just to look briefly at how it might work. The approach resembles the long extender fine structure theory of [1], in which one sometimes does not core down "all the way".

We stick with λ-indexing, and retain the notion of potential premouse from [5]. Cores, projecta, and elementarity of maps are defined as before. We want to axiomatize the properties of the levels $M^{\mathbb{C}}_{\nu,k}$ of a construction $\mathbb{C}$ in which we always let $M_{\nu,k+1}$ be the transitive collapse of $h_{M_{\nu,k}}``((\rho(M_{\nu,k}) + 1) \cup p(M_{\nu,k}))$.

One important property is a form of *projectum free spaces*. The naive statement here would be that no projectum is the critical point of a total extender. Unfortunately, the naive form is not generally true.

The amenable closure argument, together with our modified construction, lets us show that $\rho_{k+1}(M_{\nu,k+1})$ is never measurable in $M_{\nu,k+1}$. However, it is possible that $\rho_k(M_{\nu,k+1})$ is measurable in $M_{\nu,k+1}$. If this happens, then letting $\pi\colon M^-_{\nu,k+1} \to M_{\nu,k}$ be the uncoring map, it must be that π is discontinuous at $\rho_k(M_{\nu,k+1})$.[16] One can show that the discontinuity of π at $\rho_k(M_{\nu,k+1})$ is equivalent to $M_{\nu,k+1}$ satisfying that ρ_k has Σ_k-cofinality some measurable cardinal $\mu \in (\rho_{k+1}, \rho_k)$. This gives us a first order characterization of the projecta that are allowed to be spaces.

[16] $\rho_k(M_{\nu,k}) = \sup \pi``\rho_k(M_{\nu,k+1})$, so if π is continuous at $\rho_k(M_{\nu,k+1})$, then $\rho_k(M_{\nu,k})$ is measurable in $M_{\nu,k}$, which, as we said, cannot happen.

Definition 7.1 *Let (N, D) be a pfs-violation, $k = k(N)$, $\rho_k = \rho_k(N) = \mathrm{crit}(D)$, and $\rho_{k+1} = \rho_{k+1} = \rho_{k+1}(N)$. Note $\rho_k < o(N)$, so $k > 0$. We say that (N, D) is* tame *iff*

(a) $\rho_{k+1} < \rho_k < \rho_{k-1}(N)$,

(b) letting μ be the Σ_k^N cofinality of ρ_k, $\rho_{k+1} < \mu < \rho_k$, and there is a total measure on the N-sequence with critical point μ.

Here "μ is the Σ_k cofinality of ρ_k" means that μ is regular, and there is a boldface Σ_k^N[17] partial function f with domain $\subseteq \mu$ and range cofinal in ρ_k. Since $\mu < \rho_k$, we can assume that f is total and increasing. The usual elementary argument shows that there is at most one such μ, so "the Σ_k cofinality" is justified.

Remark 7.2 One can show that the tame order zero pfs violations are precisely the order zero pfs violations that do not give rise to lift anomalies.

Definition 7.3 *Let M be a potential premouse; then M has* projectum free spaces *iff whenever (N, D) is a pfs-violation such that $N \lhd M$, then (N, D) is tame.*

If M has projectum-free spaces, and E is an extender on the M-sequence that is not total on M, then $\kappa = \mathrm{crit}(E)$ is not a cardinal of M. For let $M|\mathrm{lh}(E) \trianglelefteq N \lhd M$ be least such that some $A \subset \kappa$ is N-definable, but not in N. We have $o(N) < o(M)$, so we can choose $k = k(N)$ so that $\rho_k(N) \leq \kappa$. If $\rho_k(N) < \kappa$, then κ is not a cardinal of M. If $\rho_k(N) = \kappa$, then (N, E) is a pfs-violation. It is tame, so $\rho_{k+1}(N) < \kappa$, so again, κ is not a cardinal of M.

The other important property is semi-soundness.

Definition 7.4 *Let M be a potential premouse; then M is* semi-sound *iff letting $\rho = \rho(M)$ and N be the collapse of h_M"$(\rho \cup p(M))$, either*

(a) $M = N$, *or*

(b) $M = \mathrm{Ult}(N, U)$, *where (N^+, U) is a nontame order zero pfs violation.*

Definition 7.5 *M is a* pfs-premouse *iff M has projectum-free spaces, and every proper initial segment of M is semi-sound.*

[17] That is, Σ_1 over N^{k-1}.

Iterable pfs-premice can be compared, moreover, if one side comes out strictly shorter, then it cannot drop. The characterization of just when we are allowed to depart from soundness is needed here; otherwise, we might be allowing constructions which sometimes core at ρ and sometimes core at $\rho + 1$ in a random way, and that would lead to incomparable premice.

To see how this plays out, let M and N be iterable pfs-premice. We get $\mathcal{T}$ on M and $\mathcal{U}$ on N with comparable last models P and Q by iterating away the least disagreement, as usual. This much doesn't use semi-soundness or projectum free spaces. Now suppose toward contradiction that $P \lhd Q$ and M-to-P dropped. Let $R = \mathcal{M}^{*,\mathcal{T}}_{\alpha+1}$ be what we dropped to last. So R is semi-sound, and we have $j \colon R \to P$ the branch tail embedding. Since we dropped, $\mathrm{crit}(j) \geq \rho(R)$, and since P is semi-sound, $\mathrm{crit}(j) = \rho(R)$, j has unique generator $\rho(R)$, and R is fully sound. This implies $P = \mathrm{Ult}(R, D)$, where D is a normal measure in R on $\rho(R)$.

Since $\mathrm{Ult}(R, D)$ is a proper initial segment of Q, it is semi-sound. $\mathrm{Ult}(R, D)$ is not fully sound, so it must be the ultrapower of its core R by an order zero measure on $\rho(R)$, which must then be D. Moreover, the core constituted a nontame pfs violation; that is, (R^+, D) is nontame. But $R \lhd \mathcal{M}^{\mathcal{T}}_{\beta}$, where $\beta = T\text{-pred}(\alpha + 1)$, and $\mathcal{M}^{\mathcal{T}}_{\beta}$ has projectum free spaces. This is a contradiction.

Definition 7.6 *A PFS-construction is a sequence $\langle (M_{\nu,k}, F_\nu) \rangle | \langle \nu, k \rangle <_{\mathrm{lex}} \langle \theta, n \rangle \rangle$ satisfying the properties of listed in [2], except that*

(i) each $M_{\nu,k}$ is a pfs-premouse, and

(ii) $M_{\nu,k+1} =$ transitive collapse of $h_{M_{\nu,k}}$“$(\rho + 1 \cup r)$,, where $\rho = \rho(M_{\nu,k})$ and $r = p(M_{\nu,k})$.

We believe that we can show that PFS-constructions produce pfs-premice. Part of this is the proof of Theorem 2.1, with an additional argument to the effect that in 2.1(2)(c), the pfs violation is nontame. There are some fine structural definability calculations involved in showing that all projectum-critical initial segments correspond to tame pfs-violations that we have not checked carefully.

The point of the new constructions is that there is no case split in the definition of the resurrection maps. Letting

$$\pi \colon M_{\nu,k+1} \to M_{\nu,k}$$

be the uncoring map, and $N \lhd M_{\nu,k+1}$,

$$\mathrm{Res}_{M_{\nu,k+1},M_{\nu,k}}[N] = \pi(N), \text{ and}$$
$$\mathrm{res}_{M_{\nu,k+1},M_{\nu,k}}[N] = \pi.$$

The definitions are uniform in N. We then get the consistency-of-resurrections statement: let Q be a level of $\mathbb{C}$ and $N \lhd Q$. Suppose $\text{Res}_Q[N] \leq_{\mathbb{C}} Y \leq_{\mathbb{C}} Q$. Then for $P \lhd N$,

(a) $\text{Res}_{Q,Y}[P] = \text{res}_{Q,Y}[N](P)$, and

(b) $\text{res}_{Q,Y}[P] = \text{res}_{Q,Y}[N] \restriction P$.

Of course, if one wishes to pursue this approach, there is a little more work left to do.

References

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