Some λ-errors in NITCIS

John R. Steel

May 2020

Abstract

We correct some errors in *Normalizing iteration trees and comparing iteration strategies*. The corrections preserve the overall structure of the paper, but they do involve nontrivial changes at various points.

1 Introduction

NITCIS purports to show that background-induced iteration strategies normalize well and have strong hull condensation, and that those two properties suffice to compare iteration strategies. There are (related) problems in both parts of the proof. Both have to do with what happens near λ_E in an iteration that uses E. They stem from the fact that if E on the M-sequence is backgrounded by E^*, then the factor map from $\mathrm{Ult}(M, E)$ to $i_{E^*}(M)$ is only the identity up to λ_E, and not at λ_E.

Our solution to these problems involves replacing the class of normal iteration trees by a slightly different class of iteration trees. We call trees in the new class λ-*separated*. There is an accompanying strengthening to the requirements on the background extenders in a background construction. With these two changes, the main arguments of NITCIS become fully correct.

In section 1, we discuss an error in the NITCIS proof that background-induced strategies normalize well. What the proof actually shows is that, granted the strengthened background condition, such strategies normalize well on stacks of λ-separated trees, in a sense that is appropriate for such trees. We show in [1] that background-induced strategies do actually normalize well on arbitrary stacks of normal trees, but the proof involves a strategy comparison, so the result is not available while the basic theory that goes into strategy comparison is being developed.

In section 2 we discuss an error in the NITCIS proof of $(*)(P, \Sigma)$.[1] We then show that the NITCIS proof yields a variant of $(*)(P, \Sigma)$ in which the strategy Σ acts on stacks of λ-separated trees.

We assume throughout that the reader is familiar with NITCIS.[2]

2 Background-induced strategies may not normalize well

There is a λ-error in the NITCIS proof that background-induced strategies normalize well. Let $M = M^{\mathbb{C}}_{\nu,k}$ and

$$\Sigma = \Omega(M, \mathbb{C}, \Sigma^*)$$

be the strategy for M induced by $\mathbb{C}$ and Σ^*.[3] Let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a stack by Σ, and let $\langle \mathcal{T}^*, (\pi\mathcal{U})^* \rangle$ be the stack by Σ^* you get from lifting. NITCIS claims that $W(\mathcal{T}^*, (\pi\mathcal{U})^*)$ is the lift of $W(\mathcal{T}, \mathcal{U})$. But the proof it gives has an error, which stems from the fact that the factor map from $\mathrm{Ult}(M, E)$ to $i_{E^*}(M)$ is not the identity at $\lambda(E)$.[4]

To see this in a simple case, let E be on the M-sequence, and $\mathcal{T} = \langle E \rangle$, and let $\mathcal{U}$ be a tree on $M || \mathrm{lh}(E)$ such that the stack $\langle \mathcal{T}, \mathcal{U} \rangle$ is by Σ. The claim in NITCIS is that $W(\mathcal{T}, \mathcal{U})$ is by Σ. In this special case, $W(\mathcal{T}, \mathcal{U})$ starts out with $\mathcal{U}$, then finishes with an image of E, so part of the claim is that $\mathcal{U}$ is by Σ. Let us suppose that E needs no resurrecting, and E^* is its background. Let

$$\pi\colon\ \mathrm{Ult}(M, E) \to i_{E^*}(M)$$

be the natural copy map. $\Sigma_{\mathcal{T}}(\mathcal{U})$ is defined by looking at how $\pi\mathcal{U}$ is lifted via $i_{E^*}(\mathbb{C})$ in $\mathrm{Ult}(V, E^*)$, and following $\Sigma^*_{\mathcal{T}^*}$ there. So we need to see that the lift $\mathcal{U}^*$ of $\mathcal{U}$ via $\mathbb{C}$ to a tree on V is by Σ^*, and what we know is that the lift $(\pi\mathcal{U})^*$ of $\pi\mathcal{U}$ using $i_{E^*}(\mathbb{C})$ to a tree on S is by the tail $\Sigma^*_{\mathcal{T}^*}$. But for F such that

$$\lambda(E) < \mathrm{lh}(F) < \mathrm{lh}(E),$$

[1]Here $(*)(P, \Sigma)$ is (essentially) the statement that no strategy disagreements, and no extender disagreements involving extenders on the backgrounded side, show up when you compare (P, Σ) with a level $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ of a construction $\mathbb{C}$ done in a universe where P is countable and Σ is universally Baire.

[2]References below are to the online version dated Oct. 2019.

[3]We are assuming V is strongly uniquely iterable via Σ^*.

[4]Precisely, the error is located on p. 164, in the middle of the proof of Claim 4.44, in the statement "the part of the lifting and resurrecting maps acting on F does not change from α to $z(\gamma)$". That is fine if $lh(F) < \lambda(E^{\mathcal{W}_\gamma}_\alpha)$, but it is not the case when $\lambda(E^{\mathcal{W}_\gamma}_\alpha) < \mathrm{lh}(F) < \mathrm{lh}(E^{\mathcal{W}_\gamma}_\alpha$.

there is no connection between the $\mathbb{C}$-background of F and the $i_{E^*}(\mathbb{C})$-background of $\pi(F)$, so $\mathcal{U}^*$ and $(\pi\mathcal{U})^*$ may have no connection.

In this very simple case, the Condensation Theorem of NITCIS (see also[2]) shows that $\mathcal{U}$ is by Σ.[5] But if $\mathcal{U}$ is on M rather than $M||\,\mathrm{lh}(E)$, and M has extenders overlapping $\mathrm{lh}(E)$, then Condensation does not apply.

There are various ways one might attempt to repair this error. One is to use the strategy-extension theorem of Schlutzenberg and the author, according to which every iteration strategy Σ acting on normal trees that has strong hull condensation can be extended uniquely to a strategy acting on stacks of normal trees that has strong hull condensation and normalizes well.[6] Here one *defines* $\Sigma_{\mathcal{T}}(\mathcal{U})$ by choosing the unique branch that is consistent with $\Sigma(W(\mathcal{T},\mathcal{U}))$. The problem here is that one can no longer show that Σ moves itself to its tails, which is essential for a theory of strategy mice.[7]

There are other ways to attempt a repair that look good for a while, but fall short. Before we describe a repair that seems to work, let us look at the second major λ-error in NITCIS.

3 Comparing iteration strategies incorrectly

The proof of $(*)(P,\Sigma)$ in NITCIS also has a problem. We now outline enough of the proof that the problem will become visible.

For each $\langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \nu_0, k_0\rangle$, we have have a normal tree $\mathcal{W}^*_{\nu,k}$ on P whose last model extends $M^{\mathbb{C}}_{\nu,k}$. We have a normal tree $\mathcal{U}$ on $M = M_{\nu_0,k_0}$ by $\Omega = \Omega(\mathbb{C}, M, \Sigma^*)$, and we must see that $\mathcal{U}$ is by $\Sigma_{\mathcal{T},M}$.[8] We let $\mathcal{W}^*_0 = \mathcal{W}^*_{\nu_0,k_0}$. For $\gamma < \mathrm{lh}(\mathcal{U})$, we let $S^*_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma$. We have the lifting map

$$\psi^{\mathcal{U}}_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to M^{\mathbb{C}_\gamma}_{\eta_\gamma,l_\gamma},$$

[5]For each $\alpha < \mathrm{lh}(E)$ such that $\rho(M|\alpha) = \lambda(E)$, we have apply the Condensation Theorem to $\pi\colon M|\alpha \to \mathrm{Ult}(M|\alpha, E_\pi \restriction \lambda(E^*))$. Of course, we need a comparison process to prove Condensation for the levels of $\mathbb{C}$, but the proof is part of an induction, so this is not actually a problem.

[6]The extension to infinite stacks is due exclusively to Schlutzenberg.

[7]For example, suppose Σ is for M, $\mathcal{S} \in M$ is a tree by Σ, and $i^{\mathcal{T}}\colon M \to N$ is an iteration map by Σ. Why would $i^{\mathcal{T}}(\mathcal{S})$ be by $\Sigma_{\mathcal{T}}$? If $\Sigma_{\mathcal{T}}(i(\mathcal{S}))$ has been defined by the lifting procedure, then this works, because the background embedding i^* on V moves the process whereby $\mathcal{S}$ was justified as being by Σ in the appropriate way. If $\Sigma_{\mathcal{T}}(i(\mathcal{S}))$ is being justified by the normalization $W(\mathcal{T}, i(\mathcal{S}))$, then this justification seems to have nothing to do with how $\mathcal{S}$ was justified.

[8]We are ignoring some minor points.

where $\mathbb{C}_\gamma$ is the construction of S^*_γ. We set

$$\mathcal{W}^*_\gamma = \mathcal{W}^{S^*_\gamma}_{\eta_\gamma, l_\gamma}.$$

We have also the embedding normalizations $\mathcal{W}_\gamma = W(\mathcal{W}^*_0, \mathcal{U})$. The burden of the proof is to construct inductively tree embeddings

$$\Phi_\gamma \colon \mathcal{W}_\gamma \to \mathcal{W}^*_\gamma.$$

There is a λ-error in the step from Φ_γ to $\Phi_{\gamma+1}$.

Again, we are given $E^{\mathcal{U}}_\gamma$, and set $F = F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, where $\sigma_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma = \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)}$.[9] We set

$$\begin{aligned} H_\gamma &= \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma) \\ &= t^\gamma(F). \end{aligned}$$

Here $t^\gamma = t^{\Phi_\gamma}_{z(\gamma)}$ is the last t-map of Φ_γ. We have by induction that $\psi^{\mathcal{U}}_\gamma = t^\gamma \circ \sigma_\gamma$, which justifies the displayed equalities. We let

$$G = \mathrm{res}_\gamma(H_\gamma)$$

be the resurrection of H_γ in the construction of S^*_γ, and G^* be the background extender in S^*_γ for G. Thus

$$G^* = E^{\mathcal{U}^*}_\gamma.$$

Letting $\nu = U\text{-pred}(\gamma+1)$, and assuming for simplicity that $\mathcal{U}$ does not drop here, we get

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\nu).$$

We can also show that $\mathcal{W}^*_{\gamma+1}$ uses G. This is our avenue toward $\Phi_{\gamma+1}$, which will associate F to G.

Let

$$\alpha = \alpha(\mathcal{W}_\gamma, F),$$

so that $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$.[10] We have the problem inherited from the last section, that Σ may not normalize well in the W-sense, so showing that $\mathcal{W}_{\gamma+1}$ is by Σ doesn't do any good. But let's assume we have somehow shown that Σ normalizes well; there is still a problem.

[9] $z(\gamma) = \mathrm{lh}(\mathcal{W}_\gamma) - 1$. σ_γ comes from the embedding normalization process.

[10] $\alpha(\mathcal{S}, F)$ is the least ξ such that F is on the sequence of $\mathcal{M}^{\mathcal{S}}_\xi$. Equivalently, it is the least ξ such that $\mathrm{lh}(F) < \mathrm{lh}(E^{\mathcal{S}}_\xi)$, or $\xi + 1 = \mathrm{lh}(\mathcal{S})$.

For simplicity, assume there is no resurrection, so that $G = H_\gamma$. We need to be able to set

$$\Phi_{\gamma+1} \upharpoonright \alpha + 1 = \Phi_\gamma \upharpoonright \alpha + 1,$$

and

$$t_\alpha^{\gamma+1}(F) = G.$$

Let

$$\tau = \alpha(\mathcal{W}_\gamma^*, G).$$

We must have $G = E_\tau^{\mathcal{W}_{\gamma+1}^*}$, $u^{\gamma+1}(\alpha) = \tau$, and $v^{\gamma+1}(\alpha) = v^\gamma(\alpha)$. So in order for $\Phi_{\gamma+1}$ to be a tree embedding, we must have

$$v^\gamma(\alpha) \leq_{W_\gamma^*} \tau.$$

Claim 5.22 of NITCIS purports to show this, but what the proof really shows is that either $v^\gamma(\alpha) \leq_{\mathcal{W}_\gamma^*} \tau$, or $v^\gamma(\alpha) \leq_{W_\gamma^*} \tau + 1$. [11]

In this no-resurrection case, if τ is not above $v^\gamma(\alpha)$ in $\mathcal{W}_\gamma^*$, then Proposition 5.2 of NITCIS shows that we must have $\lambda(E_\tau^{\mathcal{W}_\gamma^*}) = \lambda(G) < \mathrm{lh}(G) < \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma^*})$. This implies we are past superstrongs a little. But there is a fine-structural variant of the problem in which $\lambda(E_\tau^{\mathcal{W}_\gamma^*}) < \lambda(H)$.[12] In Jensen indexing, the fine-structural variant seems to occur just past strong cardinals.

4 λ-separated trees

The normalization problem stems from the fact that if $\sigma\colon \mathrm{Ult}(M, E) \to i_{E^*}(M)$ is the factor map, then σ is not the identity on all extenders indexed in M before E, because $\lambda_E < \sigma(\lambda_E)$.[13] One might think that ms-indexing would avoid the problem if we are working below superstrongs, but it does not. If $\nu(E)$ is a limit ordinal and a generator of E^*, then $\nu(E) < \sigma(\nu(E))$, so σ is not the identity on the extenders that are ms-indexed before E.

[11] The author of NITCIS had some awareness of this problem. Proposition 5.2 and Lemma 5.6 make some reference to it. But then he seems to have just hand-waved his way past it at the the crucial points in the proofs of 5.6 and 5.22.

[12] The problem occurs in the proof of Lemma 5.6, in the proof of Claim 5.8.1 at the bottom of page 196. The proof neglects the possibility that it is $\alpha_1 + 1$ that is on the branch that realizes the uncoring map.

[13] One cannot strengthen the background extender demand to $\lambda_E = \lambda_{E^*}$ in general, for then not all whole initial segments of E will be on the M-sequence.

We do not face the problem if we are working in ms-indexing, and E has a largest generator. This may seem like a very special case, but it turns out that we can always compare premice iterating only by extenders having a largest generator, and we don't have to move to ms-indexing to do it.

Definition 4.1 *Let M be a premouse, and E be an extender on the M-sequence; then*

(1) E^+ is the extender with generators $\lambda_E \cup \{\lambda_E\}$ that represents $i_F^{\mathrm{Ult}(M,E)} \circ i_E^M$, where F is the order zero total measure on λ_E in $\mathrm{Ult}(M,E)$,

(2) $\nu(E^+) = \lambda_E$,

(3) $\gamma(E^+) = \mathrm{lh}(E)$, and

(4) $o(E^+) = (\mathrm{lh}(E)^+)^{\mathrm{Ult}(M,E^+)}$.

Definition 4.2 *G is* of plus type *iff $G = E^+$, for some extender E that is on the sequence of a Jensen premouse. In this case, we let $G^- = E$.*

$o(E^+)$ is where the order zero measure on λ_E of $\mathrm{Ult}(M,E)$ would be ms-indexed.

It is easy to code E^+ as an amenable subset of $\mathrm{lh}(E)$ that is Σ_0 over $M|\,\mathrm{lh}(E)$, and of course, E is Σ_0 over $(M||\,\mathrm{lh}(E), E^+)$. So E and E^+ have the same information, and it would be pointless to change definitions so that it is $(M||\,\mathrm{lh}(E), E^+)$ that is a premouse. We won't do that. However, we will strengthen the background extender requirement, by a condition that implies E^* backgrounds not just E, but E^+.

Definition 4.3 *A* good w-construction *is a sequence $\mathbb{C} = \langle (M_{\nu,k}, F_\nu^*) \rangle$ satisfying the properties of Definition 2.41 of NITCIS, except that*

(a) whenever $M_{\nu,0} = (M, F)$ is active, and $F^* = F_\nu^*$ is the associated background extender for F, then $i_{F^*}(M) \models \lambda_F$ is not measurable , and

(b) F_ν^* is minimized, in Mitchell order and then w, among extenders F^* satisfying (a).

We say a background extender F^* for F is λ-minimal *iff (a) holds, that is, $i_{F^*}(M) \models \lambda_F$ is not measurable .*

A *maximal* good w-construction is one that is active whenever possible. A good w-construction may not actually be a w-construction in the sense of NITCIS, since in (b) we are minimizing with respect to a more restrictive requirement on F^*. But this is an extremely minor difference. The notions related to background constructions from NITCIS still apply.

Notice that any background construction $\mathbb{C} = \langle M_{\nu,k}, F^*_\nu \rangle$ can be transformed to a good one $\mathbb{D} = \langle M_{\nu,k}, G^*_\nu \rangle$ with the same models. For if F^* is the background extender in $\mathbb{C}$ for F, which is being added to M, and $i_{F^*}(M) \models \lambda_F$ is measurable , then in $i_{F^*}(V)$ we have a background E^* for the order zero total measure on λ_F of $i_{F^*}(M)$. For η the least inaccessible above λ_F, $V_\eta \subseteq i_{F^*}(V)$, so η is still the least inaccessible above λ_F in $i_{F^*}(V)$. So $V_\eta \subseteq \mathrm{Ult}(V, E^*)$ holds in $i_{F^*}(V)$, and hence in V. But then we can take $G^* = i_{E^*}(F^*)$. It is easy to see that G^* still backgrounds F, and is λ-minimal.

λ-minimality for F^* implies that it backgrounds F^+:

Theorem 4.4 *Let $F^* = F^*_\nu$ be a λ-minimal background for F in the construction $\mathbb{C}$, where F is the last extender of $M = M^{\mathbb{C}}_{\nu,0}$; then $F^* \cap ([\lambda_F + 1]^{<\omega} \times M) = F^+$.*

We defer the proof of this theorem to the last section. It involves a phalanx comparison argument, one similar to the proof of "closure under initial segment" from FSIT.

We wish to consider iteration trees that are allowed to use extenders of the form E^+, where E is on the coherent sequence of the current model. To unify notation, if E is an extender on the sequence of some premouse, let us set

(i) $\nu(E) = \lambda(E) = \nu(E^+)$,

(ii) $\gamma(E) = \mathrm{lh}(E) = \gamma(E^+)$,

(iii) $E^- = E$, and

(iv) $o(E) = (\mathrm{lh}(E)^+)^{\mathrm{Ult}(M,E)} = o(E^+)$.

Definition 4.5 *Let M be a premouse, then a* plus-tree *on M is a system $\langle T, \langle E_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T})\rangle, \langle M_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle\rangle$ with the usual properties of an iteration tree, except:*

(a) either E_α is on the M_α-sequence, or $E_\alpha = F^+$ for some F on the M_α-sequence,

(b) if $\alpha < \beta$, then $\gamma(E_\alpha) < \gamma(E_\beta)$, and

(c) T-pred$(\alpha+1)$ *is the least* β *such that* $\text{crit}(E_\alpha^{\mathcal{T}}) < \nu(E_\beta)$ *or* $\beta = \alpha+1$*, and*

(d) $\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \text{Ult}(P, E_\alpha^{\mathcal{T}})$*, where* β *is as in (c), and* $P \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}}$ *is as long as possible.*

We say the "plus case" occurs at α if E_α is of plus type. Thus an ordinary normal iteration tree on M is a plus-tree in which the plus case never occurs. At the other extreme, we have

Definition 4.6 *A* λ-separated *iteration tree on* M *is a plus-tree* $\mathcal{T}$ *on* M *such that every* $E_\alpha^{\mathcal{T}}$ *has plus type.*

It may seem that clause (c) of Definition allows generators to move along branches of $\mathcal{T}$. The worry would be the case that $\beta = \xi + 1$, where $E_\xi = F^+$ for some F. But in this case, the only important generators of E_ξ are in $\lambda_F \cup \{\lambda_F\}$. Generators below $\lambda_F = \nu(E_\xi)$ are not moved by (c). λ_F itself has no total measures in M_β, and hence in M_α. The partial measures on λ_F are all indexed below $\text{lh}(F) = \gamma(E_\xi) \leq \gamma(E_\alpha)$. Thus E_α is not moving any important generators of E_ξ. It is quite possible that $\text{crit}(E_\alpha) < \lambda(E_\xi)$, however.

Remark 4.7 It should be possible to show that if $\mathcal{T}$ is a plus tree, then there is an "equivalent" Jensen normal tree $\mathcal{S}$. The models of $\mathcal{S}$ would be coded versions of the models of $\mathcal{T}$. This should be essentially some small fragment of Fuchs' work on translating between ms and Jensen indexing.

Remark 4.8 Regardless of the last remark, we don't want to think of plus trees as Jensen normal trees, because of the way we are using a good background construction to induce an iteration strategy. For our purposes below, hitting E-then-F, where F is the order zero measure on λ_E of $\text{Ult}(M, E)$, is lifted to hitting just the background E^* for E. If we then want to hit some extender G on $\text{Ult}(M, E)$ with $\lambda_E < \text{crit}(G) < i_F(\lambda_E)$, then the background G^* for G has to come from $\text{Ult}(V, E^*)$. We can't demand that E^* be strong enough that G^* is an extender over V.

The agreement of models in a plus tree is given by

Proposition 4.9 *Let* $\mathcal{U}$ *be a plus tree; then for* $\alpha < \beta < \text{lh}(\mathcal{U})$*,* $E = E_\alpha^{\mathcal{U}}$ *and* $F = E_\beta^{\mathcal{U}}$*,*

(a) $\mathcal{M}_\alpha^{\mathcal{U}}$ *agrees with* $\mathcal{M}_\beta^{\mathcal{U}}$ *below* $\gamma(E)$*,*

(b) E^- *is indexed at* $\gamma(E)$ *on the* $\mathcal{M}_\alpha^{\mathcal{U}}$ *sequence, but* $\gamma(E)$ *is a cardinal of* $\mathcal{M}_\beta^{\mathcal{U}}$*,*

(c) $\gamma(E) < \nu(F)$,

(d) *if* $o(E) \leq \nu(E^{\mathcal{U}}_{\alpha+1})$, *then* $\mathcal{M}^{\mathcal{U}}_\beta$ *agrees with* $\mathrm{Ult}(\mathcal{M}^{\mathcal{U}}_\alpha, E^{\mathcal{U}}_\alpha)$ *below* $o(E)$, *and*

(e) *if* $\nu(E^{\mathcal{U}}_{\alpha+1}) < o(E)$, *then* $\gamma(E) < \mathrm{crit}(E^{\mathcal{U}}_{\alpha+1})$, *and* $\gamma(E)$ *is a cutpoint of* $\mathcal{M}^{\mathcal{U}}_{\alpha+1}$, *and* $\mathcal{U} = \mathcal{U} \restriction (\alpha+1)^\frown \mathcal{W}$, *where* $\mathcal{W}$ *is a tree above* $\gamma(E)$ *on some level of* $\mathcal{M}^{\mathcal{U}}_{\alpha+1}$ *that projects to* $\gamma(E)$.

We omit the elementary proof. For the most part, what we need is that the $\gamma(E^{\mathcal{U}}_\alpha)$'s are strictly increasing, and measure the agreement between successive extender sequences. The $o(E^{\mathcal{U}}_\alpha)$ may strictly decrease at points, but only in the limited way described in (e).

Let us also record the agreement between maps in a copying construction.

Proposition 4.10 *Let* M *and* N *be premice,* $\pi\colon M \to N$ *be elementary, and* $\mathcal{U}$ *on* M *be a plus tree. Let* $\pi_\alpha\colon \mathcal{M}^{\mathcal{U}}_\alpha \to \mathcal{M}^{\pi\mathcal{U}}_\alpha$ *be the copy map; then for* $\alpha < \beta$,

(1) $\pi_\alpha \restriction \gamma(E^{\mathcal{U}}_\alpha) + 1 = \pi_\beta \restriction \gamma(E^{\mathcal{U}}_\alpha) + 1$, *and*

(2) *if the plus case occurs at* α, *then either*

 (i) $\pi_\alpha \restriction o(E^{\mathcal{U}}_\alpha) + 1 = \pi_\beta \restriction o(E^{\mathcal{U}}_\alpha) + 1$, *or*

 (ii) $\gamma(E^{\mathcal{U}}_\alpha) < \mathrm{crit}(E^{\mathcal{U}}_{\alpha+1}) < \nu(E^{\mathcal{U}}_{\alpha+1}) < o(E^{\mathcal{U}}_\alpha)$.

One can compare with the levels of a background construction via a λ-separated trees.

Lemma 4.11 *Let* P *be a countable pure extender premouse, and let* Σ *be a universally Baire iteration strategy for* P *defined on* λ*-separated trees. Let* $\mathbb{C}$ *be a background construction, and suppose that* P *iterates strictly past* $M^{\mathbb{C}}_{\nu,k}$ *via a* λ*-separated tree by* Σ, *for all* $\langle \nu, k\rangle <_{\mathrm{lex}} \langle \eta, n\rangle$; *then* P *iterates past* $M^{\mathbb{C}}_{\eta,n}$ *via a* λ*-separated tree that is by* Σ.

Proof. The standard proof works pretty much word-for-word. First we reduce to the case $n = 0$ and $M = M_{\eta,0}$ is active, with last extender E. Then we define the unique λ-separated $\mathcal{U} = \mathcal{T}^+$ we can hope works by induction: $E^{\mathcal{U}}_\alpha = G^+$, where G is the least disagreement between $\mathcal{M}^{\mathcal{U}}_\alpha$ and M, and Σ is used to extend $\mathcal{U}$ at limit steps. We have to see that we never reach an α such that $M || \mathrm{lh}(E) \trianglelefteq \mathcal{M}^{\mathcal{U}}_\alpha$, but E is not on the $\mathcal{M}^{\mathcal{U}}_\alpha$-sequence.

Suppose this happens at α. Let $j = i_{E^*}$ be the background embedding for E, and $\kappa = \mathrm{crit}(E)$. Then $\kappa <_{j(U)} j(\kappa)$, and $i^{j(\mathcal{U})}_{\kappa, j(\kappa)} = j \restriction \mathcal{M}^{\mathcal{U}}_\kappa$. Let $G = E^{j(\mathcal{U})}_\xi$, where $\xi + 1$ is the immediate successor of κ on the branch $(\kappa, j(\kappa)]_{j(U)}$. Both G and E are initial segments of E_j, so G is compatible with E. The first initial segment of G that is not in $\mathcal{M}^{j(\mathcal{U})}_{j(\kappa)}$ is G^-. The first initial segment of E that is missing is E. Thus $E = G^-$, and hence E is on the sequence of $\mathcal{M}^{j(\mathcal{U})}_{\xi}$. But $\mathcal{U} \restriction \alpha{+}1 = j(\mathcal{U}) \restriction \alpha{+}1$ uses only extenders H such that $\gamma(H) < \mathrm{lh}(E)$, so $\alpha \leq \xi$. Moreover, $\mathrm{lh}(E) \leq \gamma(E^{j(\mathcal{U})}_{\alpha}) \leq \gamma(E^{\mathcal{U}}_{\xi}) = \mathrm{lh}(E)$, so $\xi = \alpha$, and E is on the $\mathcal{M}^{\mathcal{U}}_{\alpha}$-sequence, a contradiction. □

Remark 4.12 We can compare two iterable premice by λ-separated trees, but it is not exactly comparison by least disagreement. For example, let $N = \mathrm{Ult}(M, E)$, and let D be the order zero total measure of N on λ_E. On the M-side, we would first hit E^+, that is, E-then-D. Then on the N-side we would hit D^+. Then on the M-side we hit $i_{E^+}(D)^+$, etc. The models line up after ω steps in this process.

We also have universality at a Woodin.

Lemma 4.13 *Let δ be Woodin, let P be a countable pure extender premouse, and let Σ be a δ^+-universally Baire iteration strategy for P defined on λ-separated trees. Let $\mathbb{C}$ be a maximal background construction below δ of either the ordinary or the good variety; then there is a ν, k such that $\mathcal{M}^{\mathbb{C}}_{\nu,k}$ exists (i.e. $\mathbb{C}$ has not broken down yet), and $M^{\mathbb{C}}_{\nu,k}$ is a Σ-iterate of P via a λ-separated tree.*

Proof. Again, the standard proof works. Let us take the case $\mathbb{C}$ is required to be good. If the lemma fails, we have a λ-separated tree $\mathcal{U}$ on P by Σ of length $\delta + 1$, with last model Q extending $M^{\mathbb{C}}_{\delta,0}$. Let $b = [0, \delta)_U$. Because δ is Woodin, we have $j \colon V \to N$ with $\mathrm{crit}(j) = \kappa$, and for $\xi{+}1 \in b$ with U-pred$(\xi{+}1) = \kappa$, $\xi{+}1 \in j(b)$, and $V_\eta \subseteq N$ for η the least inaccessible above $\xi + 1$. Let $G = E^{\mathcal{U}}_{\xi}$. We can also arrange that $j(M_{\delta,0})$ agrees with $M_{\delta,0}$ past $\gamma(G)$. But then $E_j \restriction \eta$ is a good background extender for G^- (i.e., it is a background extender for the full G). By maximality and the Bicephalus Lemma, G^- is on the $M_{\delta,0}$ sequence, contradiction. □

5 Tree embeddings and strong hull condensation

Tree embeddings and extended tree embeddings of plus trees are defined in exactly the way they were in the special case that the trees are normal, except that we allow extenders in $\mathcal{T}$ that are not of plus-type to be mapped to extenders that are. That is, for $\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T})\rangle\rangle$ a tree embedding of $\mathcal{T}$ into $\mathcal{U}$, we allow (but do not demand)

$$E^{\mathcal{U}}_{u(\alpha)} = t_\alpha(E^{\mathcal{T}}_{\alpha})^+,$$

to hold when $E_\alpha^{\mathcal{T}}$ is not of plus type. If the plus case does occur at α in $\mathcal{T}$, then our demand is

$$E_\alpha^{\mathcal{T}} = F^+ \Rightarrow E_{u(\alpha)}^{\mathcal{U}} = t_\alpha(F)^+.$$

Definition 5.1 *$\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$ iff there is a tree embedding from $\mathcal{T}$ into $\mathcal{U}$.*

For future reference, let us record the agreement properties of the maps in a tree embedding.

Definition 5.2 *Let $\mathcal{T}$ be a plus tree, and $\alpha < \mathrm{lh}(\mathcal{T})$; then*

$$\nu_\alpha^{\mathcal{T}} = \sup\{\tau_\xi \mid \xi < \alpha\},$$

where

$$\tau_\xi = \begin{cases} \gamma(E_\xi^{\mathcal{T}}) + 1 & \text{if the plus case occurs in } \mathcal{T} \text{ at } \xi \\ \lambda(E_\xi^{\mathcal{T}}) & \text{otherwise.} \end{cases}$$

Since the $\nu(E_\xi^{\mathcal{T}})$ increase with ξ, $\nu_\alpha^{\mathcal{T}} = \sup\{\nu(E_\xi^{\mathcal{T}}) + 1 \mid \xi <_T \alpha\}$. $\mathcal{M}_\alpha^{\mathcal{T}}$ is generated from ordinals strictly less than $\nu_\alpha^{\mathcal{T}}$ and points in the range of the branch embedding from the last drop on $[0, \alpha]_T$.

Proposition 5.3 *Let $\mathcal{T}$ and $\mathcal{U}$ be plus trees on M, and let $\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T})\rangle\rangle$ be a tree embedding from $\mathcal{T}$ into $\mathcal{U}$; then*

(1) If $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then s_α agrees with t_α on $\nu_\alpha^{\mathcal{T}}$.

(2) If $\alpha < \beta < \mathrm{lh}(\mathcal{T})$, then s_β agrees with t_α on $\gamma(E_\alpha^{\mathcal{T}}) + 1$.

(3) If $\alpha < \beta$, then s_α agrees with s_β on $\nu_\alpha^{\mathcal{T}}$.

(4) If $\alpha < \beta$ and $\beta + 1 < \mathrm{lh}(\mathcal{T})$, then t_β agrees with t_α on $\nu_{\alpha+1}^{\mathcal{T}}$.

It follows from (4) that if the plus case occurs at α, then t_α agrees with all later t_β on $\gamma(E_\alpha^{\mathcal{T}})$. In fact, they agree on $o(E_\alpha^{\mathcal{T}})$, except in the irrevocable-dropping case $\gamma(E_\alpha^{\mathcal{T}}) < \mathrm{crit}(E_{\alpha+1}^{\mathcal{T}}) < o(E_\alpha^{\mathcal{T}})$. If the plus case does not occur at α, then we might have t_α disagreeing at $\lambda(E_\alpha^{\mathcal{T}}) = \nu(E_\alpha^{\mathcal{T}})$ with $t_{\alpha+1}$.

Definition 5.4 *A* complete strategy *for a premouse M is a strategy defined on finite stacks of plus trees on M.*

We care most about the part of a complete strategy that acts on finite stacks of λ-separated trees.

Definition 5.5 *Let Σ be a complete iteration strategy for a premouse M. Then Σ has* strong hull condensation *iff whenever s is a stack of plus trees by Σ with last model N, and $\mathcal{U}$ is a plus tree on N by $\Sigma_{s,N}$, then for any plus tree $\mathcal{T}$ on N,*

(a) if $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, then $\mathcal{T}$ is by $\Sigma_{s,N}$, and

(b) if $\Phi\colon \mathcal{T} \to \mathcal{U}$ is an extended tree embedding, with last t-map $\pi\colon Q \to R \trianglelefteq \mathcal{M}^{\mathcal{U}}_\alpha$ then $\Sigma_{s^\frown\langle\mathcal{T}\rangle,Q} = (\Sigma_{s^\frown\langle\mathcal{U}\restriction(\alpha+1)\rangle,R})^\pi$.

Theorem 5.6 *Let $N^* \models \mathsf{ZFC}$ + "$\mathbb{C}$ is a good background construction". Let Σ^* be (ω,θ)-iteration strategy for $(N^*, \vec{F}^{\mathbb{C}})$. Suppose that $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$, and Σ is the complete iteration strategy for $M^{\mathbb{C}}_{\nu,k}$ induced by Σ^*. Suppose finally that Σ^* has strong hull condensation; then Σ has strong hull condensation.*

The proof is the same as that in NITCIS. Given that $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, we show that $\mathcal{T}^*$ is a psuedo-hull of $\mathcal{U}^*$, where $\mathcal{T}^*$ and $\mathcal{U}^*$ are the lifts of $\mathcal{T}$ and $\mathcal{U}$ to trees on N^*. The reader might worry at this point that allowing $E^{\mathcal{U}}_{u(\alpha)} = t_\alpha(E_\alpha)^+$ weakens the connection between exit extenders required for the proof. However, it does not, basically because when G is put on the sequence of $M^{\mathbb{C}}_{<\nu}$ by a good construction, G and G^+ have the same background extender G^*.

We can easily λ-separate any plus tree as follows.

Definition 5.7 *Let $\mathcal{T}$ be a plus tree on M. We define a λ-separated tree $\mathcal{U} = \mathcal{T}^s$, along with maps*

$$\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_\alpha$$

by: $\pi_0 =$ id , and

$$E^{\mathcal{U}}_\alpha = \begin{cases} \pi_\alpha(E^{\mathcal{T}}_\alpha) & \text{if } E^{\mathcal{T}}_\alpha \text{ is of plus type,} \\ \pi_\alpha(E^{\mathcal{T}}_\alpha)^+ & \text{otherwise,} \end{cases}$$

and $\pi_{\alpha+1}$ is the natural copy map from $\mathrm{Ult}(P, E^{\mathcal{T}}_\alpha)$ to $\mathrm{Ult}(\pi_\beta(P), E^{\mathcal{U}}_\alpha)$, where $\beta = T\text{-pred}(\alpha+1) = U\text{-pred}(\alpha+1)$, and $P \trianglelefteq \mathcal{M}^{\mathcal{T}}_\beta$ is what $E^{\mathcal{T}}_\alpha$ is applied to.

$\mathcal{T}$ is a psuedo-hull of $\mathcal{T}^s$, via the maps $u = v = \mathrm{id}$, and $s_\alpha = t_\alpha = \pi_\alpha$. So if Σ is induced by a good background construction, then $\mathcal{T}$ is by Σ iff $\mathcal{T}^s$ is by Σ. We don't need 5.6 for this, however, because the lifts of $\mathcal{T}$ and $\mathcal{T}^s$ under a good background construction are essentially the same. For let $\mathcal{U} = \mathcal{T}^s$, and let $\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_\alpha$ come from the λ-separation process. Let $\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi, l_\xi \mid \xi \leq \xi_0\rangle, \langle \psi_\xi \mid \xi \leq \xi_0\rangle\rangle$

and $\mathrm{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \nu_\xi, k_\xi \mid \xi \leq \xi_0 \rangle, \langle \phi_\xi \mid \xi \leq \xi_0 \rangle \rangle$. An routine induction shows that $E_\alpha^{\mathcal{T}^*} = E_\alpha^{\mathcal{U}^*}$, and $\psi_\alpha = \phi_\alpha \circ \pi_\alpha$ for all α. Thus

$$\mathcal{T}^* = (\mathcal{T}^s)^*,$$

and

$$\mathcal{T} \text{ is by } \Omega(\mathbb{C}, M, \Sigma^*) \text{ iff } \mathcal{T}^s \text{ is by } \Omega(\mathbb{C}, M, \Sigma^*).$$

Remark 5.8 If $\mathcal{T}$ is a Jensen-normal plus tree[14], then $\mathcal{T}^s$ can be re-arranged as a Jensen normal tree $\mathcal{U}$. When $\mathcal{T}^s$ uses F^+ in one step, $\mathcal{U}$ uses F, and then in the next step the order zero measure on λ_F in the ultrapower.

6 Background-induced strategies normalize reasonably well

Let $\mathbb{C}$ be a good background construction, and let $\mathcal{T}$ be a plus tree on $M = M_{\nu,k}^{\mathbb{C}}$. We can lift $\mathcal{T}$ to a normal tree $\mathcal{T}^*$ on V in the usual. The resulting system is

$$\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi, l_\xi \mid \xi \leq \xi_0 \rangle, \langle \psi_\xi \mid \xi \leq \xi_0 \rangle \rangle.$$

Here $\langle \eta_0, l_0 \rangle = \langle \nu, k \rangle$ and $\psi_0 = \mathrm{id}$. The successor step in the lifting process is given by: we have

$$\psi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{T}} \to M_{\eta_\alpha, l_\alpha}^{\mathbb{C}^\alpha},$$

where $\mathbb{C}^\alpha = i_{0,\alpha}^{\mathcal{T}^*}(\mathbb{C})$. Let E

$$E = (E_\alpha^{\mathcal{T}})^-.$$

So E is on the $\mathcal{M}_\alpha^{\mathcal{T}}$ sequence. Let res_α be the map that resurrects $\psi_\alpha(E)$ in $\mathbb{C}^\alpha$. We then set

$$E_\alpha^{\mathcal{T}^*} = \mathbb{C}^\alpha\text{-background extender for } \mathrm{res}_\alpha \circ \psi_\alpha(E).$$

Theorem 4.4 tells us that, because $\mathbb{C}^\alpha$ is good, $E_\alpha^{\mathcal{T}^*}$ is actually a background extender for $(\mathrm{res}_\alpha \circ \psi_\alpha(E))^+$. This lets us define $\psi_{\alpha+1}$ in the usual way.

The agreement properties of the lifting maps are given by

Proposition 6.1 *For $\alpha < \beta$,*

(1) $\psi_\beta \restriction \nu(E_\alpha^{\mathcal{T}}) = \mathrm{res}_\alpha \circ \psi_\alpha \restriction \nu(E_\alpha^{\mathcal{T}})$, *and*

[14] Meaning that if E is used before F along a branch, then $\lambda_E \leq \mathrm{crit}(F)$. Plus trees as defined here must be ms-normal. In [1] we relax slightly the length-increasing requirement in ms-normality.

(2) *if the plus case occurs at* α, *then*

(a) $\psi_\beta \restriction \gamma(E_\alpha^{\mathcal{T}}) + 1 = \mathrm{res}_\alpha \circ \psi_\alpha \restriction \gamma(E_\alpha^{\mathcal{T}}) + 1$, *and*

(b) *either*

(i) $\psi_\beta \restriction o(E_\alpha^{\mathcal{T}}) + 1 = \mathrm{res}_\alpha \circ \psi_\alpha \restriction o(E_\alpha^{\mathcal{T}}) + 1$, *or*

(ii) $\gamma(E_\alpha^{\mathcal{U}}) < \mathrm{crit}(E_{\alpha+1}^{\mathcal{U}}) < \nu(E_{\alpha+1}^{\mathcal{U}}) < o(E_\alpha^{\mathcal{U}})$.

The fact that the plus case leads to increased agreement is what motivates our focus on good background constructions and λ-separated trees.

The embedding normalization $W(\mathcal{T},\mathcal{U})$ of a stack of plus trees is defined just as it was for stacks of normal trees. Letting $\mathcal{W}_\gamma = W(\mathcal{T},\mathcal{U} \restriction \gamma + 1)$ and $F = F_\gamma$, one has

$$\begin{aligned}\mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)\\ &= (\mathcal{W} \restriction \alpha + 1)^\frown \langle F \rangle^\frown i_F \text{“}\mathcal{W}_\nu^{\geq \beta}\end{aligned}$$

where $\nu = U\text{-pred}(\gamma + 1)$, and

$$\begin{aligned}\alpha &= \alpha(\mathcal{W}_\gamma, F)\\ &= \text{ least } \xi \text{ such that } F^- \text{ is on the } \mathcal{M}_\xi^{\mathcal{W}_\gamma}\text{-sequence,}\end{aligned}$$

and

$$\begin{aligned}\beta &= \beta(\mathcal{W}_\gamma, F)\\ &= \text{ least } \xi \text{ such that } \mathrm{crit}(F) < \nu(E_\xi^{\mathcal{W}_\gamma}) \text{ or } \xi = \alpha(\mathcal{W}_\gamma, F).\end{aligned}$$

One has by induction that $\mathcal{W}_\nu \restriction \beta + 1 = \mathcal{W}_\gamma \restriction \gamma + 1$. Moreover, if $\beta + 1 < \mathrm{lh}(\mathcal{W}_\nu)$, then $\beta + 1 < \mathrm{lh}(\mathcal{W}_\gamma)$ and $\gamma(E_\beta^{\mathcal{W}_\nu}) \geq \gamma(E_\beta^{\mathcal{W}_\gamma})$, so that $\nu(E_\beta^{\mathcal{W}_\nu}) \geq \nu(E_\beta^{\mathcal{W}_\gamma})$. This means that $\beta = \beta(\mathcal{W}_\nu, F)$; that is, F gets applied to an initial segment of $\mathcal{M}_\beta^{\mathcal{W}_\nu}$ in $\mathcal{W}_{\gamma+1}$, and $\beta = W_{\gamma+1}\text{-pred}(\alpha + 1)$.

The construction of $W(\mathcal{T},\mathcal{U})$ gives us partial tree embeddings

$$\Phi_{\nu,\eta} \colon \mathcal{W}_\nu \to \mathcal{W}_\eta$$

for $\nu <_U \eta$, defined on an appropriate initial segment of $\mathcal{W}_\nu$.

What the “proof” of Theorem 4.41 of NITCIS actually shows is that the iteration strategy induced by a good background construction normalizes well on stacks of the form $\langle \mathcal{T},\mathcal{U} \rangle$, where $\mathcal{T}$ is λ-separated and $\mathcal{U}$ is a plus tree.

Theorem 6.2 *Let Σ^* witness that V is strongly uniquely iterable, and let $\mathbb{C}$ be a good background construction. Let $M = M^{\mathbb{C}}_{\nu,k}$ and $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$ be the strategy for M induced by $\mathbb{C}$ and Σ^*. Let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a stack by Σ such that*

(a) $\mathcal{T}$ is λ-separated, and

(b) $\mathcal{U}$ is a plus tree;

then $W(\mathcal{T}, \mathcal{U})$ is by Σ.

Proof. We just take things far enough in the NITCIS proof to see that the error in the proof of Claim 4.44 has been fixed. Our notation matches that of NITCIS.

Let $\mathcal{T}$ be on $M^{\mathbb{C}}_{\nu_0,k_0}$, and

$$\text{lift}(\mathcal{T}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta^{\mathcal{T}}_\xi, l^{\mathcal{T}}_\xi \mid \xi \leq \xi_0 \rangle, \langle \psi^{\mathcal{T}}_\xi \mid \xi \leq \xi_0 \rangle \rangle.$$

Let

$$\text{lift}(\psi^{\mathcal{T}}_{\xi_0} \mathcal{U}, M^{i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})}_{\eta^{\mathcal{T}}_{\xi_0}, l^{\mathcal{T}}_{\xi_0}}, i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})) = \langle \mathcal{U}^*, \langle \langle \eta^{\mathcal{U}}_\xi, l^{\mathcal{U}}_\xi \rangle \mid \xi < \text{lh}\,\mathcal{U} \rangle, \langle \rho_\xi \mid \xi < \text{lh}\,\mathcal{U} \rangle \rangle.$$

Let $\tau_\xi : \mathcal{M}^{\mathcal{U}}_\xi \to \mathcal{M}^{(\psi^{\mathcal{T}}_{\xi_0})\mathcal{U}}_\xi$ be the copy map, and

$$\psi^{\mathcal{U}}_\xi = \rho_\xi \circ \tau_\xi,$$

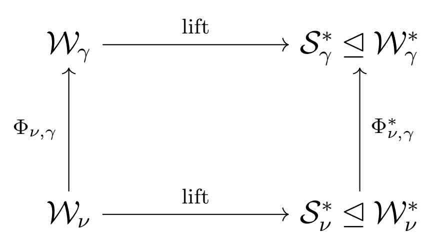
so that

$$\psi^{\mathcal{U}}_\xi : \mathcal{M}^{\mathcal{U}}_\xi \to Q_\xi,$$

where Q_ξ is the appropriate model in $i^{\mathcal{U}^*}_{0,\xi}(\mathbb{C})$. We show that $W(\mathcal{T}, \mathcal{U})$ lifts to an initial segment of $W(\mathcal{T}^*, \mathcal{U}^*)$. This is done by induction: setting $\mathcal{W}_\gamma = W(\mathcal{T}, \mathcal{U} \restriction \gamma + 1)$ and $\mathcal{W}^*_\gamma = W(\mathcal{T}^*, \mathcal{U}^* \restriction \gamma + 1)$, and

$$\text{lift}(\mathcal{W}_\gamma, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{S}^*_\gamma, \langle \langle \eta^\gamma_\xi, l^\gamma_\xi \rangle \mid \xi < \text{lh}\,\mathcal{W}_\gamma \rangle, \langle \psi^\gamma_\xi \mid \xi < \text{lh}\,\mathcal{W}_\gamma \rangle \rangle.$$

we show that $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction \text{lh}\,\mathcal{W}_\gamma$. Thus $\mathcal{S}^*_\gamma$ is by Σ^*, so $\mathcal{W}_\gamma$ is by Σ. With $\gamma + 1 = \text{lh}(\mathcal{U})$, this is what we want. The overall plan is summarized in the diagram:



Here $\Phi_{\nu,\gamma}$ and $\Phi^*_{\nu,\gamma}$ are the tree embeddings we get from the two embedding normalization processes.

We have by induction that the diagram holds at all $\xi \leq \gamma$, and that

$$\psi^{\xi}_{z(\xi)} \circ \sigma_\xi = \psi^{\mathcal{U}}_{\xi}$$

for all $\xi \leq \gamma$.[15] Part of this is that $\langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma \rangle = \langle \eta^\gamma_{z(\gamma)}, l^\gamma_{z(\gamma)} \rangle$. We define

$$\begin{aligned}
\langle \eta, l \rangle &= \langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma \rangle = \langle \eta^\gamma_{z(\gamma)}, l^\gamma_{z(\gamma)} \rangle, \\
F &= \sigma_\gamma(E^{\mathcal{U}}_\gamma), \\
H &= \psi^\gamma_{z(\gamma)}(F) \\
&= \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma), \\
G &= \mathrm{res}^{\mathbb{C}^\gamma_{z(\gamma)}}_H(H), \text{ and} \\
G^* &= B^{\mathbb{C}^\gamma_{z(\gamma)}}(G).
\end{aligned}$$

Here we are writing $B^{\mathbb{D}}(K)$ for the background extender associated to a completely resurrected K by a construction $\mathbb{D}$. We are also letting

$$\mathrm{res}^{\mathbb{D}}_{\eta,l,E} = \sigma^{\mathbb{D}}_{\eta,l}[M^{\mathbb{D}}_{\eta,l}|\gamma(E)]$$

be the map that resurrects E^- from stage $\langle \eta, l \rangle$ inside $\mathbb{D}$. (This map also resurrects all of E in the plus case.) Note here

Remark 6.3 If $M^{\mathbb{D}}_{\eta,l}|\,\mathrm{lh}(E) = M^{\mathbb{D}}_{\theta,n}|\,\mathrm{lh}(E)$, then $\mathrm{res}^{\mathbb{D}}_{\eta,l,E} = \mathrm{res}^{\mathbb{D}}_{\theta,n,E}$. (Since between $\langle \eta, l \rangle$ and $\langle \theta, n \rangle$ there can be no level projecting $< \mathrm{lh}(E)$.) So we may write $\mathrm{res}^{\mathbb{D}}_E$ for the common value.

We have

$$G^* = E^{\mathcal{U}^*}_\gamma.$$

So $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, F)$ and $\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, G^*)$, where $\nu = U\text{-pred}(\gamma+1) = \mathcal{U}^*\text{-pred}(\gamma+1)$. We must see that $\mathcal{W}_{\gamma+1}$ lifts to $\mathcal{W}^*_{\gamma+1}$, in a way that the diagram above (with γ replaced by $\gamma+1$) commutes. The main thing here is to show that the background extender associated to the lift of F is G^*. That was Claim 4.44 of NITCIS, where the λ-error occurred. Our assumption that $\mathcal{T}$ is λ-separated enters into the correct proof.

[15] $\sigma_\xi \colon \mathcal{M}^{\mathcal{U}}_\xi \to \mathcal{M}^{\mathcal{W}_\xi}_{z(\xi)}$ comes from embedding normalization.

More precisely, let

$$\alpha = \alpha(\mathcal{W}_\gamma, F),$$

and

$$K = \psi^\gamma_\alpha(F).$$

Claim 6.4 *(a)* $G = \mathrm{res}^{\mathbb{C}^\gamma_\alpha}_K(K)$,

(b) $G^* = B^{\mathbb{C}^\gamma_\alpha}(G)$, *and*

(c) $\alpha = \alpha(\mathcal{W}^*_\gamma, G^*)$.

Proof.. Let us assume $\alpha + 1 < \mathrm{lh}(\mathcal{W}_\gamma)$. The case they are equal is similar. $\mathcal{W}_\gamma$ was formed by inserting the F_ξ for $\xi < \gamma$ into images of $\mathcal{W}_0$, moreover $\gamma(F_\xi) < \gamma(F_\gamma)$ for all $\xi < \gamma$. It follows from the way that embedding normalization works that

$$\alpha \in \mathrm{ran}(u^{\Phi_{0,\gamma}}).$$

Since the plus case occurred everywhere in $\mathcal{W}_0$, the plus case occurs at α in $\mathcal{W}_\gamma$. It follows from 6.1(2) that

$$\psi^\gamma_{z(\gamma)} \restriction \gamma(E) = \mathrm{res}^{\mathbb{C}^\gamma_\alpha}_E \circ \psi^\gamma_\alpha \restriction \gamma(E),$$

where $E = E^{\mathcal{W}_\gamma}_\alpha$.[16] But $\gamma(F) < \gamma(E)$, so

$$\begin{aligned} H &= \psi^\gamma_{z(\gamma)}(F) \\ &= \mathrm{res}_\alpha \circ \psi_\alpha(F) \\ &= \mathrm{res}^{\mathbb{C}^\gamma_\alpha}_E(K). \end{aligned}$$

The second line here uses 6.1(2)(a), which applies because the plus case occurs at α in $\mathcal{W}_\gamma$.

Let $E_1 = \mathrm{res}^{\mathbb{C}^\gamma_\alpha}_E(E)$, θ be such that E_1 is the top extender of $M^{\mathbb{C}^\gamma_\alpha}_{\theta,0}$. To prove (a), it is enough to see that

$$\mathrm{res}^{\mathbb{C}^\gamma_\alpha}_{\theta,0,H}(H) = G.$$

But let $E^*_1 = B^{\mathbb{C}^\gamma_\alpha}(E_1) = E^{\mathcal{W}^*_\gamma}_\alpha$, and fix ξ such that $E^*_1 = F^{\mathbb{C}^\gamma_\alpha}_\xi$. By coherence of the background sequences, $\mathbb{C}^\gamma_\alpha$ and $\mathbb{C}^\gamma_{z(\gamma)}$ have the same background extenders with index $\eta < \xi$. This implies

$$M^{\mathbb{C}^\gamma_\alpha}_{\mu,k} = M^{\mathbb{C}^\gamma_{z(\gamma)}}_{\mu,k}$$

[16] This is where we use λ-separation. Without it the agreement would only be up to $\lambda(E)$, and the argument would break down when $\lambda(E) < \mathrm{lh}(F) < \mathrm{lh}(E)$.

for all $\langle \mu, k\rangle <_{\text{lex}} \langle \theta, 0\rangle$. That implies in turn that

$$\text{res}^{\mathbb{C}^\gamma_\alpha}_{\theta,0,H} = \text{res}^{\mathbb{C}^\gamma_{z(\gamma)}}_{\theta,0,H}.$$

Finally,

$$\text{res}^{\mathbb{C}^\gamma_{z(\gamma)}}_{\theta,0,H} = \text{res}^{\mathbb{C}^\gamma_{z(\gamma)}}_{\eta,l,H},$$

so we are done with (a) of 6.4. Our proof also showed (b).

The proof of (c) in NITCIS works here too. □

□

7 Comparing iteration strategies correctly

The problem in going from Φ_γ to $\Phi_{\gamma+1}$ that we identified in Section 3 stemmed from the fact that the conclusion of Proposition 5.2 in NITCIS is not quite strong enough for the purpose. The unwanted disjunct in this conclusion can be eliminated if we are starting with a λ-separated tree.

Proposition 7.1 *Let $\mathcal{S}$ be a λ-separated tree on some premouse, let $\delta \leq_S \eta$, and suppose that $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\eta$, but $P \ntrianglelefteq \mathcal{M}^{\mathcal{S}}_\sigma$ whenever $\sigma <_S \delta$. Suppose also that $P \in \text{ran}(\hat{\imath}^{\mathcal{S}}_{\delta,\eta})$. Let*

$$\begin{aligned} \alpha = &\ \textit{least } \xi \textit{ such that } P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\xi \\ = &\ \textit{least } \xi \textit{ such that } o(P) < \gamma(E^{\mathcal{S}}_\gamma) \textit{ or } \xi = \eta, \end{aligned}$$

and

$$\beta = \textit{least } \xi \in [0,\eta]_S \textit{ such that } o(P) < \text{crit}(\hat{\imath}^{\mathcal{S}}_{\xi,\eta}) \textit{ or } \xi = \eta.$$

Then $\beta \in [\delta, \eta]_S$, and $\alpha = \beta$. (We allow $\delta = \eta$, with the understanding $\hat{\imath}_{\delta,\delta}$ is the identity.)

Proof. By Proposition 4.9, for any $\xi < \eta$, $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\xi$ iff $\gamma(E^{\mathcal{S}}_\xi) > o(P)$. So the two characterizations of α are equivalent. Clearly, $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\beta$, and thus $\alpha \leq \beta$. We have that $o(P) \geq \gamma(E^{\mathcal{S}}_\sigma)$ for all $\sigma <_S \delta$, and hence for all $\sigma <_S \delta$ whatsoever. So $\delta \leq \alpha$, and $\beta \in [\delta, \eta]_S$.

Suppose $\alpha < \beta$; then $o(P) < \gamma(E^{\mathcal{S}}_\alpha)$, so $o(P) < \gamma(E^{\mathcal{S}}_\sigma)$ where σ is least such that $\alpha \leq \sigma$ and $\sigma + 1 \leq_S \beta$. But $\mathcal{S}$ is λ-separated, so

$$o(P) < \gamma(E^{\mathcal{S}}_\sigma) < i_{E_\sigma}(\text{crit}(E_\sigma)).$$

Since $\delta \leq \sigma$ and $P \in \operatorname{ran}(\hat{\imath}^{\mathcal{S}}_{\delta,\eta})$, we have $o(P) < \operatorname{crit}(E^{\mathcal{S}}_\sigma)$, which contradicts our definition of β.

Thus $\alpha = \beta$, and we are done. □

Remark 7.2 If $\mathcal{S}$ is merely normal, it can happen that $\beta = \alpha + 1$ and $\lambda(E^{\mathcal{S}}_\alpha) < o(P) < \operatorname{lh}(E^{\mathcal{S}}_\alpha)$. This was the problem discussed in Section 3.

Let us return now to the situation in Section 3, in order to see that the problem has been fixed.

Let (P, Σ) be a pure extender pair, in the sense that Σ has strong hull condensation, and normalizes well for stacks of the form $\langle \mathcal{T}, \mathcal{U} \rangle$, where $\mathcal{T}$ is λ-separated and $\mathcal{U}$ is an arbitrary plus tree. We assume P is countable and Σ is UB. Let $\mathbb{C}$ be a good background construction.

By Lemma 4.11, we may assume that for each $\langle \nu, k \rangle \leq_{\text{lex}} \langle \nu_0, k_0 \rangle$, we have a λ-separated tree $\mathcal{W}^*_{\nu,k}$ on P whose last model extends $M^{\mathbb{C}}_{\nu,k}$. We have a plus tree $\mathcal{U}$ on $M = M_{\nu_0,k_0}$ by $\Omega = \Omega(\mathbb{C}, M, \Sigma^*)$, and we must see that $\mathcal{U}$ is by $\Sigma_{\mathcal{T},M}$.[17] We let $\mathcal{W}^*_0 = \mathcal{W}^*_{\nu_0,k_0}$. For $\gamma < \operatorname{lh}(\mathcal{U})$, we let $S^*_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma$. We have the lifting map

$$\psi^{\mathcal{U}}_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to M^{\mathbb{C}_\gamma}_{\eta_\gamma, l_\gamma},$$

where $\mathbb{C}_\gamma$ is the construction of S^*_γ. We set

$$\mathcal{W}^*_\gamma = (\mathcal{W}^*_{\eta_\gamma, l_\gamma})^{S^*_\gamma}.$$

We have also the embedding normalizations $\mathcal{W}_\gamma = W(\mathcal{W}^*_0, \mathcal{U})$. The burden of the proof is to construct inductively extended tree embeddings

$$\Phi_\gamma \colon \mathcal{W}_\gamma \to \mathcal{W}^*_\gamma.$$

By strong hull condensation, we get that $\mathcal{W}_\gamma$ is by Σ, for all γ. Since Σ normalizes reasonably well, $\mathcal{U}$ is by Σ, as desired.

We are given $E^{\mathcal{U}}_\gamma$, and set $F = F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, where $\sigma_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma = \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)}$. We set

$$\begin{aligned} H &= \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma) \\ &= t^\gamma(F). \end{aligned}$$

[17]Again, we ignore relatively minor points, such as the possibility that $\mathcal{U}$ is on a proper initial segment of M.

Here $t^\gamma = t^{\Phi_\gamma}_{z(\gamma)}$. We have by induction that $\psi^{\mathcal{U}}_\gamma = t^\gamma \circ \sigma_\gamma$, which justifies the displayed equalities. We let

$$G = \mathrm{res}^{S^*_\gamma}_H(H)$$

be the resurrection of H in the construction of S^*_γ, and G^* be the background extender in S^*_γ for G. Thus

$$G^* = E^{\mathcal{U}^*}_\gamma.$$

Letting $\nu = U\text{-pred}(\gamma+1)$, and assuming for simplicity that $\mathcal{U}$ does not drop here, we get

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\nu).$$

We can also show that $\mathcal{W}^*_{\gamma+1}$ uses $(G^-)^+$; that is G if G is of plus-type, and G^+ if it is not.

Let

$$\alpha = \alpha(\mathcal{W}_\gamma, F),$$

so that $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$. Let us assume $\alpha + 1 < \mathrm{lh}(\mathcal{W}_\gamma)$. The case $\alpha + 1 = \mathrm{lh}(\mathcal{W}_\gamma)$ is similar, but $u^\gamma(\alpha)$ must be replaced by $z^*(\gamma)$, which is fine because Φ_γ is an extended tree embedding. As in our normalizing well proof, we have that the plus case occurs at α in $\mathcal{W}_\gamma$.[18]

Let

$$\tau = \alpha(\mathcal{W}^*_\gamma, H).$$

Claim. $\tau \in [v^\gamma(\alpha), u^\gamma(\alpha)]_{\mathcal{W}^*_\gamma}$.

Proof. Let

$$P = \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)} | \mathrm{lh}(F) = \mathcal{M}^{\mathcal{W}_\gamma}_\alpha | \mathrm{lh}(F^-),$$

and

$$Q = t^\gamma_{z(\gamma)}(P) = M^{\mathbb{C}_\gamma}_{\eta_\gamma, l_\gamma} | \mathrm{lh}(H^-).$$

We have $o(P) < \gamma(E^{\mathcal{W}_\gamma}_\alpha)$, so because the plus case occurred at α in $\mathcal{W}_\gamma$, 5.3(4) implies

$$Q = t^\gamma_\alpha(P).$$

But this means

$$Q = \hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha), u^\gamma(\alpha)} \circ s^\gamma_\alpha(P),$$

so $Q \in \mathrm{ran}(\hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha), u^\gamma(\alpha)}$. By Proposition 7.1, we get $\tau \in [v^\gamma(\alpha), u^\gamma(\alpha)]$, as desired. □

[18] $\mathcal{W}_0$ is λ-separated, and the F_ξ for $\xi < \gamma$ satisfy $\gamma(F_\xi) < \gamma(F)$.

If $G = H$, then enables us to set $u^{\gamma+1}(\alpha) = \tau$. The rest of the definition of $\Phi_{\gamma+1}$ is as in NITCIS.

If $G \neq H$, so that the resurrection map res_H of S^*_γ is nontrivial, then we proceed as in NITCIS. Let θ be such that G^- is the last extender predicate of $M^{\mathbb{C}_\gamma}_{\theta,0}$. We set

$$\mathcal{W}^{**}_\gamma = (\mathcal{W}^*_{\theta,0})^{S^*_\gamma}.$$

The proof of Lemma 5.6 of NITCIS shows that

$$\mathcal{W}^{**}_\gamma \restriction \tau + 1 = \mathcal{W}^*_\gamma \restriction \tau + 1,$$

and

$$\text{res}_H = \hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau,\xi},$$

where $\xi + 1 = \text{lh}(\mathcal{W}^{**}_\gamma)$. The λ-error in the proof of 5.6 is no longer present, because all the $\mathcal{W}^*_{\eta,n}$ are λ-separated.[19]

We go on from this point as in NITCIS.

8 Proof of Theorem 4.4

In this section we prove 4.4. It might seem at first that one could avoid doing that by strengthening the λ-minimality requirement on a background extender E^* for E. Why not just make it part of the definition that E^* must background E^+? The problem is of course that such background constructions may not produce enough mouse pairs. The NITCIS proof that there are strategy mice with subcompacts, granted the existence of a Hom_∞ iteration strategy for a pure extender premouse with a long extender on its sequence, or granted large cardinals in V and strong unique iterability, would have a gap. The NITCIS proof that LEC implies HPC would have a gap. We would have a comparison theorem for the mouse pairs reached by such constructions, but no strong evidence that they reach many lbr hod pairs.

We have observed above that the additional λ-minimality requirement on E^* does not restrict the possible E at all. Theorem 4.4 says that the requirement that E^* background E^+ is also not restrictive, because in fact it follows from λ-minimality.

Proof of 4.4. The proof resembles the proof of closure under initial segment in section 10 of FSIT. Let $F^* = F^*_\nu$ be a λ-minimal background for F in the construction $\mathbb{C}$, where F is the last extender of $M = M^{\mathbb{C}}_{\nu,0}$. background for F. We assume that M is a pure extender premouse. This simplifies the phalanx comparison.

[19]The error in the NICIS proof of 5.6 occurs in the proof of Claim 5.8.1. It could be that $\alpha_1 + 1$ is in the interval in question.

Let $\kappa = \text{crit}(F)$, and

$$G = \{(a, X) \mid a \in [\lambda_F + 1]^{<\omega} \wedge X \in M \cap P([\kappa]^{|a|}) \wedge a \in i^V_{F^*}(X)\}.$$

Our goal is to show that $G = F^+$. Assume toward contradiction that $G \neq F^+$. Let

$$N = \text{Ult}(M, G)|(\lambda_F^+)^{\text{Ult}(M,G)}.$$

If $G = F^+$, then $o(N) = \text{lh}(F)$, but for all we know now, $\text{lh}(F) < o(N)$ is possible. The factor embedding from $\text{Ult}(M, G)$ to $i_{F^*}(M)$ has critical point $\geq o(N)$, so $N \trianglelefteq i_{F^*}(M)$. Note also that $\text{lh}(F)$ is a passive stage in $i_{F^*}(M)$, because F^* was a Mitchell-minimal background for F.

For $\eta < \kappa^{+,M}$, the fragment $G_\eta = G \cap ([\lambda_F + 1]^{<\omega} \times M|\eta)$ belongs to N, by the usual Kunen argument. The G_η are constructed cofinally in $o(N)$, so we can code G by a predicate $\hat{G}$ that is amenable to N. The following are simple first order facts in the theory of $(N, \hat{G})$:

(*)] There is a largest cardinal ν, moreover

(1) ν is a cutpoint and a generator of G,[20]

(2) $\text{Ult}(N, G) \models \nu$ is not measurable,

(3) $(\nu^+)^{\text{Ult}(N, G|\nu)}$ is not active in $\text{Ult}(N, G)$, and

(4) G is not of plus type.

The structure $(N, \hat{G})$ has an iteration strategy Σ that we get from $\mathbb{C}$. We can replace $(N, \hat{G})$ by a countable elementary submodel of itself, and Σ by an $(\omega, \omega_1 + 1)$ iteration strategy for that model that has the weak Dodd-Jensen propety relative to some enumeration $\vec{e}$. We assume this has been done, and call the new objects N, G, $\hat{G}$, and Σ.

Let ν be the largest cardinal of N, and

$$P_0 = Q_0 = (N, \hat{G}),$$

and

$$P_1 = \text{Ult}(P_0, G \restriction \nu).$$

We are going to compare the phalanx (P_0, P_1, ν) with Q_0. The resulting tree on the phalanx we call $\mathcal{T}$, with models $P_\xi = \mathcal{M}_\xi^\mathcal{T}$, and the tree on Q_0 we call $\mathcal{U}$, with models

[20] That is, $\nu(G) \neq [a, f]^N_G$ when $a \subset \nu(G)$ and $f \in N$. This follows from the fact that $\lambda_F < \lambda_{F^*}$.

$Q_\xi = \mathcal{M}^{\mathcal{U}}_\xi$. At the same time, we lift $\mathcal{T}$ to a tree $\mathcal{T}^*$ with models P^*_ξ, and embeddings $\pi_\xi \colon P_\xi \to P^*_\xi$. Here $\pi_0 =$ id , and

$$P^*_1 = \mathrm{Ult}(P_0, G),$$

with π_1 being the natural factor map. The trees $\mathcal{T}^*$ and $\mathcal{U}$ are according to Σ.

$\mathcal{T}$ is not literally an iteration tree on P_0, since $G \restriction \nu$ is not on the P_0-sequence, but we will use iteration tree notation for it. In particular, $0 <_T 1$, and $i^{\mathcal{T}}_{0,1} = i_{G\restriction\nu}$.

The non-dropping iterates of P_0 in the trees $\mathcal{T}, \mathcal{T}^*$, and $\mathcal{U}$ all satisfy the elementary property $(*)$. P_0 also satisfies the "weak initial segment condition", in that whenever H is a whole proper initial segment of $G \restriction \nu(G)$, then (the Jensen completion of) H is indexed on the P_0 sequence. We have to deal with the possibility that this could fail for iterates of P_0.

The comparison proceeds by iterating away least disagreements. In this connection, if $P_\xi = (R, \hat{H})$ is such that $[0,\xi]_T$ has not dropped, and Q is the current model on the $\mathcal{U}$-side, and $R \trianglelefteq Q$, then $E^{\mathcal{T}}_\xi = H$ unless $Q = (R, \hat{H})$. In the latter case, the comparison is over.

If $E^{\mathcal{T}}_\xi$ exists, T-pred$(\xi+1) =$ least β such that crit$(E^{\mathcal{T}}_\xi) < \nu(E^{\mathcal{T}}_\beta)$. Here $\nu(E^{\mathcal{T}}_\beta) = \lambda(E^{\mathcal{T}}_\beta)$, or $\nu(E^{\mathcal{T}}_\beta)$ is the largest cardinal of $\mathcal{M}^{\mathcal{T}}_\beta$, and $E^{\mathcal{T}}_\beta$ is coded by the image of $\hat{G}$ along $[0,\beta]_T$. In this latter case, $\nu(E^{\mathcal{T}}_\beta)$ cannot be equal to crit$(E^{\mathcal{T}}_\alpha)$, because clause (2) above was preserved along $[0,\beta]_T$.

Claim 1. The comparison terminates.

Proof. This is not completely routine, because the weak initial segment condition may fail for iterates of $(N, \hat{G})$. Important generators are not moved along branches, so the usual proof gives us some countable α and $\eta+1, \xi+1$ such that

$$\alpha = T\text{-pred}(\eta+1) = U\text{-pred}(\xi+1),$$

and for $H = E^{\mathcal{T}}_\eta$ and $K = E^{\mathcal{T}}_\xi$, dom$(H) =$ dom(K) and H and K are compatible. This is impossible unless one of H and K is the image of G along the branch to its model.

Case 1. $[0,\eta]_T$ does not drop, and H is coded by the top predicate of P_η.

Let μ be the largest cardinal of H. We have that μ is a cutpoint of H, a generator of H, and for $\gamma = (\mu^+)^{\mathrm{Ult}(P_\eta, H\restriction\mu)}$, γ is not active in (Ult(P_η, H). It follows that the Jensen completion of H could not appear on the sequence of a Jensen premouse, because the Jensen initial segment condition fails for it.

Subcase 1A. $[0,\xi]_U$ does not drop, and K is coded by the top predicate of Q_ξ.

$H \neq K$, because otherwise the comparison was finished before we used them. Suppose K is a proper initial segment if H. We then have that $\pi_\eta \circ i^{\mathcal{U}}_{0,\xi}$ is a Σ_0 elementary, cardinal preserving map from $Q_0 = P^*_0$ into P^*_η. (It is Σ_0 in the language with a predicate for $\hat{G}$.) But letting ν be the largest cardinal of N, $\pi_\eta \circ i^{\mathcal{U}}_{0,\xi}(\nu) < i^{\mathcal{T}^*}_{0,\xi}(\nu)$. This contradicts the $\vec{e}$-minimality of Σ.[21] If H is a proper initial segment of K, then $i^{\mathcal{T}}_{0,\eta}$ is Σ_0 and cardinal preserving from $P_0 = Q_0$ into Q_ξ, and $i^{\mathcal{T}}_{0,\eta}(\nu) < i^{\mathcal{U}}_{0,\xi}(\nu)$, where ν is the largest cardinal of N. Again, this contradicts the $\vec{e}$-minimality of Σ.

Subcase 1B. Subcase 1A does not hold.

We then have that $Q_\xi | \operatorname{lh}(K)$ is a Jensen premouse. By the initial segment condition, if $H = K \restriction \nu(H) + 1$, then H^- is a whole proper initial segment of K, so H^- is indexed on the Q_ξ-sequence at $\gamma(H)$, and $\gamma(H) < \operatorname{lh}(K)$. This implies H^- is on the sequence of Q_{ω_1}, and hence on the sequence of P_{ω_1}, contrary to the fact that H was used in $\mathcal{T}$. Thus K must be a proper initial segment of H.

Suppose first that K is a proper initial segment of $H \restriction \mu$. Let ν be the largest cardinal of N, and $i = i^{\mathcal{T}}_{0,\eta}$, so that $i(\nu) = \mu$. For any $\tau < \nu$ such that $G \restriction \tau$ is whole, the Jensen completion of $G \restriction \tau$ is on the N sequence. It follows that for any $\tau < \sup i``\nu$ such tht $H \restriction \tau$ is whole, the Jensen completion of $H \restriction \tau$ is on the P_η sequence. Since K is not on the P_η sequence, we must have

$$\sup i``\nu \leq \lambda_K < i(\nu).$$

Thus ν is singular in N, and letting

$$\gamma = (\nu^+)^{\operatorname{Ult}(N, G\restriction\nu)} = \sup i^N_{G\restriction\nu}``\kappa^{+,N},$$

we have that $\gamma < o(N)$. Let S be the first level of N above γ that projects to ν. Letting $X \subset \kappa$ and $X \in N$, we have that

$$i_{G\restriction\nu}(X) = h_S(\beta, p(S)),$$

where h_S is the canonical Skolem function. This fact is preserved by i, so $i_{H\restriction\mu}(i(X)) = h_{i(S)}(i(\beta), p(i(S)))$. But this means

$$i_K(i(X)) = h_{i(S)}(i(\beta), p(i(S))) \cap \lambda_K.$$

Noting that $i(\beta) < \lambda_K$ and $\operatorname{ran}(i)$ is cofinal in $\operatorname{dom}(K)$, we see that $\operatorname{lh}(K)$ has cardinality λ_K in P_η. But K was used in $\mathcal{U}$ before we reached P_η, so $\operatorname{lh}(K)$ is a cardinal in the lined up part of P_η, and hence in P_η. This is a contradiction.

[21] We may assume $e_0 = \nu$.

Thus we must have $\lambda_K = \mu$. Let $i = i^{\mathcal{T}}_{\eta+1,\omega_1}$ and $j = i^{\mathcal{U}}_{\xi+1,\omega_1}$ be the branch tails. $\mu < \text{crit}(i)$, and μ is not measurable in $P_{\eta+1}$, so μ is not measurable in P_{ω_1}, so μ is not measurable in Q_{ω_1}. But $\mu = \lambda_K$ is measurable in $Q_{\xi+1}$. It follows that $E^{\mathcal{U}}_{\xi+1}$ is the order zero measure on λ_K, and $\xi + 1 <_U \xi + 2 <_U \omega_1$, so that K-then-$E^{\mathcal{T}}_{\xi+1}$ is the initial segment of the extender of $i^{\mathcal{T}}_{\alpha,\omega_1}$ with generators $\mu + 1$. This implies that

$$H = K\text{-then-}E^{\mathcal{T}}_{\xi+1},$$

so H is of plus type. But G is not of plus type, and this propagates under $i^{\mathcal{T}}_{0,\eta}$.

Case 2. $[0, \xi]_U$ does not drop, and K is coded by the top predicate of $\mathcal{M}^{\mathcal{U}}_{\xi}$.

This case is completely parallel to Case 1.

This proves Claim 1. □

Now let $\theta + 1 = \text{lh}(\mathcal{T})$ and $\tau + 1 = \text{lh}(\mathcal{U})$.

Claim 2. $P_\theta = Q_\tau$, neither $[0, \theta]_T$ nor $[0, \tau]_U$ drops, and $i^{\mathcal{T}}_{0,\theta} = i^{\mathcal{U}}_{0,\tau}$.

Proof. By standard Dodd-Jensen arguments, using of course $\pi_\theta \colon P_\theta \to P^*_\theta$ at various points. □

Claim 3. $1 \leq_T \theta$.

Proof. Suppose not. Let $\eta + 1 \leq_T \theta$ with T-pred$(\eta + 1) = 0$, and $\xi + 1 \leq_U \tau$ with U-pred$(\xi + 1) = 0$. Let $H = E^{\mathcal{T}}_{\eta}$ and $K = E^{\mathcal{U}}_{\xi}$. We reach the same contradictions we reached in the proof that the comparison process terminates. □

Now let $\xi + 1 \leq_U \tau$ and U-pred$(\xi + 1) = 0$, and let $K = E^{\mathcal{U}}_{\xi}$. By claims 2 and 3, $G \restriction \nu = K \restriction \nu$. (It is easy to see $\nu \leq \lambda_K$.) $G \restriction \nu$ is not in P_1, hence not in P_θ, hence not in $Q_{\xi+1}$. So the Jensen completion of $K \restriction \nu$ is not in $\text{Ult}(Q_\xi, K)$. K cannot itself be the Jensen completion of $K \restriction \nu$, since this is not in Q_0 by hypothesis, hence not in any later Q_ξ. It follows that K is coded by $i^{\mathcal{U}}_{0,\xi}(\hat{G})$. Since $\text{crit}(K) = \kappa = \text{crit}(G)$, and G had the weak initial segment condition below ν, we must have $\xi = 0$ and $K = G$.

It follows that ν is not measurable in Q_τ, hence not measurable in P_θ. Since ν is measurable in P_1, $E^{\mathcal{T}}_1$ must be the order zero measure on ν, and $G \restriction \nu$-then-$E^{\mathcal{T}}_1$ is an initial segment of the extender of $i^{\mathcal{T}}_{0,\theta}$. But then

$$G = G \restriction \nu\text{-then-}E^{\mathcal{T}}_1,$$

so G is of plus type, contradiction.

This completes the proof of Theorem 4.4. □

9 Mouse pairs

The repairs above mean that we need to change the basic definitions of pure extender pairs and least branch hod pairs a bit. The definitions need to axiomatize properties that we get directly from a background construction, and axiomatize enough properties that we can prove a comparison theorem.[22]

The iteration strategies need to be defined on at least the λ-separated trees. They have strong hull condensation and normalize well for stacks of λ-separated trees. (Presumably for countable stacks.) As we saw above, that determines their action on plus trees, and in fact on countable stacks of plus trees. The strategy on λ-separated trees is enough for comparison. This suggests

Definition 9.1 (M, Ω) *is a* pure extender pair *with scope* H_δ *iff*

(1) M *is a pure extender premouse, and* $M \in H_\delta$*,*

(2) Ω *is a* (δ, δ)*- iteration strategy for* M*, defined on stacks of plus trees,*

(3) Ω *has strong hull condensation, and*

(4) if s *is a stack of* λ*-separated trees by* Ω *with last model* N*, and* $\langle \mathcal{T}, \mathcal{U} \rangle$ *is a stack by* Ω_s *such that* $\mathcal{T}$ *is* λ*-separated, then* $W(\mathcal{T}, \mathcal{U})$ *is by* $\Omega_{s,N}$*, and* $\Sigma_{s^\frown\langle \mathcal{T},\mathcal{U}\rangle} = \Omega^\pi_{s^\frown\langle W(\mathcal{T},itU)\rangle}$*, for* π *the embedding normalization map.*

Good background constructions yield such pairs, by the corrected NITCIS. Ω is determined by its action on single λ-separated trees.[23] We care most about $\delta = \omega_1$ under AD^+, and under those hypotheses, and two such pairs can be compared.

Conjecture. Assume AD^+ and let (M, Ω) be a pure extender pair with scope HC; then Ω normalizes well.

At the moment, we believe we can prove this, by combining a strengthening of the direct-from-backgrounds proof with phalanx comparison argument. It's not clear how much hangs on the conjecture. For example, it seems likely that normalizing

[22] Definitions 9.1 and **??** below do not quite capture enough of what we get directly from a background construction. They are modified in [1].

[23] For example, let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a stack of plus trees. We saw above that $\mathcal{T}$ is by Ω iff $\mathcal{T}^s$ is by Ω. Let P be the last model of $\mathcal{T}$ and Q the last model of $\mathcal{T}^s$, and $\pi \colon P \to Q$ the natural map. By strong hull condensation again, $\Omega_{\mathcal{T},P} = \Omega^\pi_{\mathcal{T}^s,Q}$. So we just need to know whether $\pi\mathcal{U}$ is by $\Omega_{\mathcal{T}^s,Q}$. But $\mathcal{T}^s$ is λ-separated, so this holds iff $W(\mathcal{T}^s, \pi\mathcal{U})$ is a plus tree by Ω, i.e. iff $W(\mathcal{T}^s, \mathcal{U})^s$ is a λ-separated tree by Ω.

well for stacks of λ-separated trees is enough for the construction of optimal Suslin representations.

Least branch premice must be defined so that the strategy on λ-separated trees is inserted in the strategy predicate. One could go further and add the strategy on plus trees, or stacks of plus trees, but this is determined by the strategy on single λ-separated trees in a way the model can unravel. This leads to

Definition 9.2 (M, Ω) *is a* least branch hod pair *with scope* H_δ *iff*

(1) M *is a least branch premouse, and* $M \in H_\delta$*,*

(2) Ω *is a* (δ, δ)*- iteration strategy for* M*, defined on stacks of plus trees,*

(3) Ω *has strong hull condensation,*

(4) if s *is a stack of* λ*-separated trees by* Ω *with last model* N*, and* $\langle \mathcal{T}, \mathcal{U} \rangle$ *is a stack by* Ω_s *such that* $\mathcal{T}$ *is* λ*-separated, then* $W(\mathcal{T}, \mathcal{U})$ *is by* $\Omega_{s,N}$*, and* $\Sigma_{s^\frown\langle \mathcal{T}, \mathcal{U} \rangle} = \Omega^{\pi}_{s^\frown\langle W(\mathcal{T}, itU)\rangle}$*, for* π *the embedding normalization map, and*

(5) if s *is a stack of plus trees by* Ω *with last model* N*, then* $\dot{\Sigma}^N \subseteq \Omega_{s,N}$*.*

Again, good background constructions yield such pairs, by the corrected NITCIS.

In this strategy mouse context, the proof that the levels of a good background construction are least branch hod pairs, which is done by induction, has to fold in a proof of Theorem 4.4. This is somewhat similar to the way solidity, universality, and condensation were all proved for $(M^{\mathbb{C}}_{\nu,k}, \Omega_{\nu,k})^{\mathbb{C}})$ as part of an induction.

The results of NITCIS go through with these changes.[24]

References

[1] J. Steel, *Embedding normalization for mouse pairs*, avaliable at www.math.berkeley.edu/∼steel

[2] J. Steel and Nam Trang, *Condensation for mouse pairs*, avaliable at www.math.berkeley.edu/∼steel

[24]Definitions 9.1 and **??** are modified in [1], so as to record slightly more about what one gets directly for background-induced strategies.