Mouse pairs and Suslin cardinals

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Abstract

Building on the results of [19] and [18], we obtain optimal Suslin representations for mouse pairs. This leads us to an analysis of HOD^M, for any model M of $\mathsf{AD}_\mathbb{R}$ in which there are Wadge-cofinally many mouse pairs. We show also that if there is a least branch hod pair with a Woodin limit of Woodin cardinals, then there is a model of AD^+ in which not every set of reals is ordinal definable from a countable sequence of ordinals.

0 Introduction

The book [19] introduces *mouse pairs*, and develops their basic theory. It then uses one variety of mouse pair, the *least branch hod pairs*, to analyze HOD in certain models of the Axiom of Determinacy. Here we shall carry that work further, by obtaining optimal Suslin representations for mouse pairs. This enables us to show that the collection of models M of $\mathsf{AD}_\mathbb{R}$ to which the HOD-analysis of [19] applies is closed downward under Wadge reducibility. It also leads to a characterization of the Solovay sequence in such models M of $\mathsf{AD}_\mathbb{R}$ in terms of the Woodin cardinals of HOD^M.[1]

Unless otherwise stated, we shall assume AD^+ throughout this paper.

0.1 Mouse pairs

Let us recall, in outline, some of the definitions and results of [19].

A *pure extender premouse* is a transitive structure $M = (J_\alpha^{\vec{E}}, \in, \vec{E}, F)$, where $\vec{E}^\frown F$ is a coherent sequence of extenders. ($F = \emptyset$ is allowed.) We use Jensen indexing: if $E = E_\alpha^M$, then $\alpha = i_E(\mathrm{crit}(E)^{+,M})$. We also use the *projectum-free spaces* fine structure of [19], for reasons that are explained in [19].[2] So in this paper, *premouse* means *pfs premouse of type 1* in the sense of [19]. The new elements of pfs fine structure will rarely come up in this paper, however, so familiarity with the standard fine structure should suffice to follow all but a few details.

[1] [19] is a substantial revision and expansion of *Normalizing iteration trees and comparing iteration strategies*, which first appeared on the author's webpage in 2015. The revisions and expansions to the 2015 draft that were completed before August 2020 are currently available on the author's webpage. The remainder will appear in [19].

[2] The new element is that $\rho_{k+1}(M)$ and $\rho_k(M)$ are always put into the hull collapsing to the $k+1$-st core of M. One must do this, in one form or another, in order to obtain iteration strategies that normalize well.

A *least branch premouse* (lpm) is a transitive structure M that is like a pure extender premouse, except that at certain "branch active" stages, information about an iteration strategy for M is fed into M. When working in the AD^+ context, we deal primarily with countable premice, and iteration strategies for them that are defined on countable trees.

A *mouse pair* is a pair (P, Σ) such that P is a premouse of one of the two types, and Σ is a *complete iteration strategy* for P that normalizes well, is internally lift consistent, and has strong hull condensation. A complete strategy is one that is defined on countable stacks of countable[3] *plus trees*. In this paper it will suffice to consider the restriction of Σ to stacks of plus trees that are λ-*tight*, in the sense of [19], and these are just what are called *normal* iteration trees in the standard terminology. That is, the extenders used have increasing lengths, and they are applied to the largest possible initial segment of the earliest possible model.[4]

Normalizing well, internal lift consistency, and strong hull condensation are explained by

Definition 0.1. Suppose Σ is an iteration strategy for a premouse P.

(1) (Tail strategy) If s is a stack by Σ with last model Q, then Σ_s is the strategy for Q given by: $\Sigma_s(t) = \Sigma(s^\frown t)$.

(2) (Pullback strategy) If $\pi\colon N \to P$ is elementary, then Σ^π is the strategy for N given by: $\Sigma^\pi(s) = \Sigma(\pi s)$, where πs is the lift of s by π to a stack on P.

(3) (Normalizes well) Σ *normalizes well* iff whenever s is a stack of plus trees by Σ with last model Q, and $\mathcal{V} = V(s)$ is the quasi- normalization of s, with associated map $\pi\colon Q \to R$, then

  (i) $\mathcal{V}$ is by Σ,

  (ii) $\Sigma_s = (\Sigma_{\langle \mathcal{W} \rangle})^\pi$, and

  (iii) for any plus tree $\mathcal{T}$ on Q, $\mathcal{T}$ is by Σ_s iff its normal companion is by Σ_s.

(4) (Strong hull condensation) Σ has *strong hull condensation* iff whenever $\mathcal{U}$ is a plus tree by Σ, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is a tree embedding, then

  (i) $\mathcal{T}$ is by Σ, and

  (ii) if $\pi\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{u(\alpha)}$ is a map of Φ, then $\Sigma_{\mathcal{T}\restriction\alpha+1} = \Sigma^\pi_{U\restriction u(\alpha+1)}$.

[3] In [19], the requirement is that the strategy be defined on finite stacks. One can show that any such strategy extends to countable stacks, however. See [16] or [18].

[4] For E an extender on the M-sequence, E^+ is the extender representing E-then-D, where D is the order zero measure of $\mathrm{Ult}(M, E)$ on $\lambda(E)$. Plus trees are allowed to use extenders of the form E^+, where E is on the sequence of the current last model. $\mathcal{T}$ is λ-tight if it never makes use of this option. $\mathcal{T}$ is λ-*separated* if it always makes use of this option. [19] makes use of the good behavior of Σ of λ-separated trees to deduce that it behave well on arbitrary plus trees, and hence on the λ-tight plus trees. We are taking [19] as our starting point, so we can sweep the non-λ-tight plus trees under the rug, and we are left with the usual normal trees. [19] uses "normal" so as to allow extenders of the form E^+ to be used, but we shall not do that here.

(5) (Internal lift consistency) Σ is *internally lift consistent* iff whenever (Q, Ω) is an iterate of (P, Σ) via $\mathcal{T}$, $R \lhd Q$, and $\mathcal{U}$ is a plus tree on R, then $\mathcal{U}$ is by $\Sigma_{\mathcal{T},R}$ iff $\mathcal{U}^+$ is by $\Sigma_{\mathcal{T},Q}$, where $\mathcal{U}^+$ is the lift of $\mathcal{U}$ to a tree on Q.

The reader should see [19] for more complete definitions. Item (3) implies that Σ behaves in the same way with respect to embedding normalizations $s \mapsto W(s)$, because $W(s)$ is the normal companion of $V(s)$. We don't need to go into quasi-normalization in this paper.

We understand elementarity fine structurally, of course. Our convention is that every premouse P has a degree of soundness $k(P)$ attached to it, and elementarity means elementarity at that quantifier level. We shall say that $\mathcal{T}$ *is a pseudo-hull of* $\mathcal{U}$ iff there is a tree embedding of $\mathcal{T}$ into $\mathcal{U}$.[5] Internal lift consistency is related to strong hull condensation, and the two can be combined naturally into one property.

There is one additional property demanded of least branch hod pairs (lbr hod pairs) (P, Σ): if s is a stack on P by Σ with last model Q, then $\dot{\Sigma}^Q \subseteq \Sigma_s$. That is, the strategy predicate of Q must be consistent with the tail strategy for Q determined by Σ. This property is called *pushforward consistency* in [19]

If (M, Σ) is a mouse pair, and s is a stack by Σ with last model N, then we call (N, Σ_s) an *iterate of* (M, Σ). If the branch M-to-N of s does not drop, we call it a non-dropping iterate. In that case, we have an iteration map $i_s \colon M \to N$.

Definition 0.2. Let M be a premouse, then M is *projectum stable* iff for $k = k(M)$, the $r\Sigma_k$ cofinality of $\rho_k(M)$ is not measurable by the M-sequence. A pair (M, Σ) is projectum stable iff M is projectum stable.

The projectum-free spaces fine structure requires that $\rho_{k(M)}(M)$ not be measurable by the M-sequence, but 0.2 is a stronger requirement. Iterations of a projectum stable premouse M preserve all projecta $\rho_i(M)$, where $i \leq k(M)$. This implies that they preserve the projectum-free spaces notion of soundness, in a way that iterations of arbitrary premice may not. We shall be able to restrict our attention to projectum stable mouse pairs.[6]

Notice that if $k(M) = 0$, then M is projectum stable.

Theorem 0.3 (Comparison theorem, [19])**.** *Assume* AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be projectum stable mouse pairs of same type; then they have a common iterate* (R, Ω) *such that on at least one of the two sides, the iteration does not drop.*

The iteration trees on P and Q witnessing 0.3 can be taken to be normal, but they cannot always be obtained by iterating away least extender disagreements, as in the standard comparison process.[7][8]

[5]Tree embeddings, and much of the general machinery around embedding normalization, were developed by Farmer Schlutzenberg and the author. See [16] and [19]. See also [4]. Strong hull condensation is a stronger version of the hull condensation property isolated by Sargsyan in [14].

[6][19] uses "strongly stable" instead of "projectum stable".

[7][19] first proves that comparison via λ-separated trees is possible, then uses this to prove that comparison via λ-tight trees is possible.

[8]There is a comparison theorem for pairs that are not projectum stable, but its statement is more awkward.

Even for pure extender pairs, Theorem 0.3 goes beyond the usual comparison theorem for premice, because of the agreement between tail strategies it requires. In fact, it is no easier to prove the theorem for pure extender pairs than it is to prove it for least branch hod pairs. The proof in both cases is the same, and it makes use of the properties of the iteration strategies isolated in the definition of mouse pair.

Working in the category of mouse pairs enables one to state a general Dodd-Jensen lemma. Let us say $\pi\colon (P,\Sigma) \to (Q,\Psi)$ is elementary iff π is elementary from P to Q, and $\Sigma = \Psi^\pi$. The iteration maps associated to non-dropping iterations of a mouse pair are elementary.[9]

Theorem 0.4 (Dodd-Jensen lemma, [19]). *Let (P,Σ) be a mouse pair, and (Q,Ψ) be an iterate of (P,Σ) via the stack s. Suppose $\pi\colon (P,\Sigma) \to (Q,\Psi)$ is elementary; then s does not drop, and for all ordinals $\eta \in P$, $i_s(\eta) \leq \pi(\eta)$.*

The proof is just the usual Dodd-Jensen proof; the point is just that the language of mouse pairs enables us to formulate the theorem in its proper generality. There is no need to restrict to mice with unique iteration strategies, as is usually done.

Similarly, we can define the mouse order in its proper generality, without restricting to mice with unique iteration strategies. If (P,Σ) and (Q,Ψ) are projectum stable pairs of the same type, then $(P,\Sigma) \leq^* (Q,\Psi)$ iff (P,Σ) can be elementarily embedded into an iterate of (Q,Ψ). The Comparison and Dodd-Jensen theorems imply that $\leq^*$ is a prewellorder on each type.

If (P,Σ) is a mouse pair, then $\Sigma \subseteq \mathrm{HC}$. We shall often want to think of Σ as a set of reals, so let us fix a natural coding. For $x \in \mathbb{R} = \omega^\omega$, we say $\mathrm{Code}(x)$ iff $E_x =_{\mathrm{df}} \{\langle n,m\rangle \mid x(2^n 3^m) = 0\}$ is a wellfounded, extensional relation on ω. If $\mathrm{Code}(x)$, then

$$\pi_x\colon (\omega, E_x) \cong (M, \in)$$

is the transitive collapse map, and

$$\mathrm{set}(x) = M \text{ and } \mathrm{set}_0(x) = \pi_x(0).$$

So Code is Π^1_1, and set_0 maps Code onto HC. For $A \subseteq \mathrm{HC}$, we let

$$\mathrm{Code}(A) = \{x \in \mathrm{Code} \mid \mathrm{set}_0(x) \in A\}.$$

Definition 0.5. Let $A \subset \mathrm{HC}$ and $\kappa \in \mathrm{OR}$; then A is *κ-Suslin* iff $\mathrm{Code}(A)$ is κ-Suslin.

0.2 Hod pair capturing

We shall show that least branch hod pairs can be used to analyze HOD in models of $\mathsf{AD}_\mathbb{R}$, provided that there are enough such pairs.

Definition 0.6. (AD^+)

[9] This is actually not obvious; it is a property of the iteration strategy known as pullback consistency. It follows from strong hull condensation.

(a) *Hod Pair Capturing* (HPC) is the assertion: for every Suslin-co-Suslin set A, there is a least branch hod pair (P, Σ) such that A is definable from parameters over $(\mathrm{HC}, \in, \Sigma)$.

(b) $L[E]$ *capturing* (LEC) is the assertion: for every Suslin-co-Suslin set A, there is a pure extender pair (P, Σ) such that A is definable from parameters over $(\mathrm{HC}, \in, \Sigma)$.

An equivalent (under AD^+) formulation would be that the sets of reals coding strategies of the type in question, under some natural map of the reals onto HC, are Wadge cofinal in the Suslin-co-Suslin sets of reals. The restriction to Suslin-co-Suslin sets A is necessary, for AD^+ implies that if (P, Σ) is a pair of one of the two types, then Σ is Suslin and co-Suslin. This is one of the main results of the present paper. It is easy to show that HPC implies the capturing pairs (P, Σ) can be taken to satisfy $k(P) = 0$, and hence to be projectum stable. Similarly for LEC.

Another equivalent (under AD^+) to LEC or HPC, as the case may be, is the assertion that the mouse order on pairs of the given type has order type equal to the sup of the Suslin cardinals.

Remark. HPC is a cousin of Sargsyan's *Generation of Full Pointclasses*. See [13] and [14], §6.1.

Assuming AD^+, LEC follows from [10] the well known Mouse Capturing: for reals x and y, x is ordinal definable from y iff x is in a pure extender mouse over y. This equivalence is shown in [24]. (See especially Theorem 16.6.) [19] shows that under AD^+, LEC implies HPC. (See Theorem 5.58.) We do not know whether HPC implies LEC, or whether LEC implies Mouse Capturing.

The natural conjecture is that LEC and HPC hold in all models of AD^+ that have not reached an iteration strategy for a premouse with a long extender. Because our capturing mice have only short extenders on their sequences, LEC and HPC cannot hold in larger models of AD^+.

Definition 0.7. NLE ("No long extenders") is the assertion: there is no countable, $\omega_1 + 1$-iterable pure extender premouse M such that there is a long extender on the M-sequence.

Conjecture 0.7.1. Assume AD^+ and NLE; then LEC.

Conjecture 0.7.2. Assume AD^+ and NLE; then HPC.

As we remarked above, 0.7.1 implies 0.7.2. Conjecture 0.7.1 follows from[11] a slight strengthening of the usual Mouse Set Conjecture MSC. (The hypothesis of MSC is that there is no iteration strategy for a pure extender premouse with a superstrong, which is slightly stronger than NLE.) MSC has been a central target for inner model theorists for a long time.[12]

We shall prove here that both types of capturing localize to initial segments of the Wadge hierarchy, in the following sense.

Theorem 0.8. *Assume* AD^+, *and let* $M \models \mathsf{AD}^+$ *be such that* $\mathbb{R} \cup OR \subseteq M$; *then*

[10] Modulo the intertranslatability of the projectum-free spaces hierarchy with the standard one. This should be relatively straightforward, but it has not been carried out as yet.

[11] Modulo the translation just mentioned.

[12] We retract the claim to have proved LEC implies MSC made in [19].

(i) if there is a pure extender pair (P, Σ) such that $\Sigma \notin M$, then LEC *holds in M, and*

(ii) if there is an lbr hod pair (P, Σ) such that $\Sigma \notin M$, then HPC *holds in M.*

For LEC, this was done in [24], but our proof here is different even in that case. We shall also sketch a proof of

Theorem 0.9. *Assume AD^+, and that there is a countable ω_1+1-iterable pure extender mouse with a long extender on its sequence. Let $M \models \mathsf{AD}^+ + \mathsf{NLE}$ be such that $\mathbb{R} \cup OR \subseteq M$; then $M \models \mathsf{LEC}$, and hence $M \models \mathsf{HPC}$.*

So, stepping out to the full universe, where ZFC holds and, it seems quite likely, there is a Hom_∞ iteration strategy for a mouse with a long extender, we see that conjectures 0.7.1 and 0.7.2 hold in all Wadge cuts in Hom_∞. This makes it likely that the theory $\mathsf{AD}^+ + \mathsf{NLE}$ proves LEC and HPC. A counter-model would have to be "nonstandard".

0.3 HOD in models of $\mathsf{AD}_\mathbb{R}$

We shall show that, assuming HPC and $\mathsf{AD}_\mathbb{R}$, and letting θ be the least ordinal that is not the surjective image of $\mathbb{R}$, $V_\theta \cap \mathrm{HOD}$ is a direct limit of lbr hod mice under the iteration maps given by comparisons. This yields

Theorem 0.10. *Assume $\mathsf{AD}_\mathbb{R}$ and HPC; then $V_\theta \cap HOD$ is the universe of a least branch premouse.*

Under $\mathsf{AD}_\mathbb{R} + V = L(P(\mathbb{R})) + \mathsf{HPC}$, the full HOD is $L[M_\infty]$, where M_∞ is a direct limit of least branch hod mice. So HOD $\models \mathsf{GCH}$, and more generally, can be analyzed fine-structurally. In particular, by [28] and [29], the Schimmerling-Zeman characterization of $\square$ holds in HOD.

We believe Theorem 0.10 remains true if $\mathsf{AD}_\mathbb{R}$ is weakened to AD^+ in its hypothesis, but we do not have a proof.

Our proof of 0.10 gives a characterization of the Solovay sequence in terms of the Woodin cardinals in HOD.

Definition 0.11. (AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set

$$\begin{aligned} \theta_0 &= \theta(\emptyset), \\ \theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\ \theta_\lambda &= \bigcup_{\alpha<\lambda} \theta_\alpha, \text{ for } \lambda \text{ a limit ordinal..} \end{aligned}$$

$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$.[13] Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B. The sequence of θ_α's is called the *Solovay sequence*. It is an important feature of any model of AD^+, one tied to the pattern of pointclasses with the scale property.

[13] Θ is the least ordinal which is not the surjective image of $\mathbb{R}$.

Theorem 0.12. *Assume* $\mathsf{AD}_{\mathbb{R}} + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then the following are equivalent:*

(1) HOD $\models$ *"*δ *is Woodin, and there is no extender* E *on the HOD-sequence such that* $\mathrm{crit}(E) < \delta$ *and* $V_\delta \subseteq \mathrm{Ult}(V, E)$*"*[14]*,*

(2) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

In particular, θ_0 *is the least Woodin cardinal in HOD.*

That θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD is due to Woodin, cf. [9]. Woodin also proved an approximation to the statement that they are cutpoints in HOD (unpublished). The rest of (2) $\to$ (1), and all of (1) $\to$ (2), will be proved here.

We shall also prove a conjecture of G. Sargsyan that provides a similar characterization of arbitrary successor Woodin cardinals in HOD.

Definition 0.13. For any κ, we let $\theta(\kappa^\omega)$ be the least ordinal α such that there is no ordinal definable surjection from $[\kappa]^\omega$ onto α.

Theorem 0.14. *Assume* $\mathsf{AD}_{\mathbb{R}} + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then for any* $\kappa < \theta$ *such that* κ *is a successor cardinal of HOD,* $\theta(\kappa^\omega)$ *is the least* $\delta > \kappa$ *such that HOD* $\models \delta$ *is Woodin.*

We do not have any neat descriptive set theoretic characterization of those limits of Woodins in HOD that are themselves Woodin.

0.4 Organization

In §1 we review some material from [19] that did not find its way into this section, and we describe briefly the results on full normalization and very strong hull condensation for mouse pairs that are proved in [18].[15] In §2 we use very strong hull condensation and full normalization to obtain optimal Suslin representations for mouse pairs. This is the main technical tool used in the rest of the paper. §3 concerns those Suslin representations in the special case that our mouse pair is mouse-minimal among pairs not in some strongly closed pointclass. We prove 0.8 and 0.9 in this section. In §4 we prove 0.10, and in §5 we prove 0.12 and 0.14. Finally, in §6 we show how to obtain models of some strong determinacy theories from hypotheses on the existence of mouse pairs.

1 Preliminaries

Let us review some material from [19] and [18].

[14]That is, δ is a cutpoint Woodin of HOD. See 2.16.

[15]Part of the author's work on normalization was done, either independently by or jointly with, Farmer Schlutzenberg. See [16].

1.1 Background universes and constructions

Assume AD^+, and let Γ, Γ_1 be good (i.e. closed under $\exists^{\mathbb{R}}$) lightface pointclasses with the scale property such that $\Gamma \subseteq \Delta_1$. Let A be a universal Γ_1 set, and let $U \subset \mathbb{R}$ code $\{\langle \varphi, x \mid (V_{\omega+1}, \in, A) \models \varphi[x]\}$. Let S and T be trees on some $\omega \times \kappa$ that project to U and $\neg U$. Using his work in [9], Woodin has shown ([22, Lemma 3.13]) that there is a countable transitive N^*, a wellorder w of N^*, and an iteration strategy Σ such that for $\delta = o(N^*)$,

(a) (fullness) $N^* = V_\delta^{L(N^* \cup \{S,T,w\})}$,

(b) N^* is f-Woodin, for all $f \colon \delta \to \delta$ such that $f \in C_\Gamma(N^*, w)$,[16]

(c) for all $\eta \leq \delta$, there is an $f \colon \eta \to \eta$ such that $f \in C_{\Gamma_1}(V_\eta^{N^*}, \lhd \cap V_\eta^{N^*})$ and $V_\eta^{N^*}$ is not f-Woodin, and

(d) Σ is an (ω_1, ω_1)-iteration strategy for $L(N^*, S, T, w)$, with respect to nice[17] trees based on N^*.

Concerning item (d), recall that ω_1-iterability implies $\omega_1 + 1$-iterability, granted AD.

Definition 1.1. Assume AD^+, and let Γ be a good pointclass with the scale property, and let N^*, δ, S, T, w, and Σ be as in (a)-(d); then

(1) we call $\langle N^*, \delta, S, T, w, \Sigma\rangle$ a *coarse* Γ*-Woodin tuple*, and

(2) letting $M = (L[N^*, S, T, w], \in, S, T)$, we call (M, Σ) the associated *coarse* Γ*-Woodin pair.*

Of course, S and T determine U, and hence A and Γ_1. U is self-dual, so S is only there for notational convenience.

Let $M = L[N^*, S, T, w]$, where $\langle N^*, \delta, S, T, w, \Sigma\rangle$ is a coarse Γ-Woodin tuple. Let $p[T] = \mathrm{Th}(V_{\omega+1}, \in, A)$, and let Γ_1 be the good pointclass whose universal set is A. If P is a wellfounded iterate of M, and g is is P-generic over $\mathrm{Col}(\omega, i(\delta))$, then $P[g]$ is projectively-in-A correct.[18] Thus the C_Γ and C_{Γ_1} operators are correctly defined over $P[g]$. It follows that M and its iterates are C_{Γ_1}-full, and Σ is guided at $\mathcal{T}$ by a Q-structure in $C_{\Gamma_1}(\mathcal{M}(\mathcal{T}))$, where

$$\mathcal{M}(\mathcal{T}) = (\bigcup_{\alpha < lh(\mathcal{T})} V^{M_\alpha}_{\mathrm{lh}(E_\alpha)}, \bigcup_{\alpha < \mathrm{lh}(\mathcal{T})} i_{0,\alpha}(w) \cap V^{M_\alpha}_{\mathrm{lh}(E_\alpha)}).$$

(We have omitted some superscript $\mathcal{T}$'s here.) That is,

[16] For a countable and transitive, $C_\Gamma(a)$ is the largest countable $\Gamma(a \cup \{a\})$ subset of $P(a)$. Its theory (under determinacy hypotheses) was first developed by Kechris and Moschovakis. See [5], [6], [7], and the survey [20]. Harrington and Kechris showed in [1] that $C_\Gamma(a) = P(a) \cap L[T, a]$, for any tree T of a Γ scale on a universal Γ set. This is probably the most useful characterization of $C_\Gamma(a)$ is our context.

[17] Roughly, $\mathcal{T}$ is nice iff the extenders it uses have inaccessible lengths and strengths.

[18] $p[T] \subseteq p[i(T)]$ and $p[S] \subseteq p[i(S)]$, while $p[(i(T)] \cap p[i(S)] = \emptyset$ because P is wellfounded. So $p[T] = p[i(T)]$.

Lemma 1.2. *Assume* AD^+*, and let* (M,Σ) *be a coarse* Γ*-Woodin pair. Let* $\vec{T}$,$\mathcal{U}$ *be a stack of nice trees played by* Σ*; then the following are equivalent*

(1) $\Sigma_{\vec{T}}(\mathcal{U}) = b$,

(2) $C_{\Gamma_1}(\mathcal{M}(\mathcal{U})) \subseteq \mathcal{M}_b^{\mathcal{U}}$,

(3) $\mathcal{M}_b^{\mathcal{U}}$ *is wellfounded.*

Since Σ is guided by C_{Γ_1} Q-structures, $\mathrm{Code}(\Sigma)$ is projective in A, hence Wadge reducible to U via a recursive function. On the other hand, letting T_0 be the tree of a Γ scale on a universal Γ set A_0, there are club many $\eta < \delta$ such that $L(V_\eta, w \cap V_\eta, T_0) \models \eta$ is Woodin, and this is preserved in iterations by Σ. Moreover, $p[i(T_0)] = A_0$ for such iterations, so letting τ be a $\mathrm{Col}(\omega,\delta)$-term for $p[T_0]$, (M,Σ,τ) *captures* A_0.

Let (M,Σ) be the coarse Γ-Woodin pair associated to $\langle N^*, \delta, S, T, w, \Sigma\rangle$; then

$$\mathcal{F}^{N^*,w} = \{E \in V_\delta^M \mid M \models E \text{ is nice and } i_E(w) \cap V_{\mathrm{lh}(E)+1}^{Ult(V,E)} = w \cap V_{\mathrm{lh}(E)+1}^{\mathrm{Ult}(V,E)}\}.$$

We call $(w,\mathcal{F})$ the *maximal coherent pair* associated to $\langle N^*, \delta, S, T, w, \Sigma\rangle$. There are two types of background construction associated to the tuple $\langle N^*, \delta, S, T, w, \Sigma\rangle$:

(a) the *maximal pure extender pair construction* $\mathbb{C} = \langle (M_{\nu,k}, \Omega_{nu,k}) \mid \langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \delta, 0\rangle\rangle$, whose levels $(M_{\nu,k}, \Omega_{\nu,k})$ are pure extender pairs, and

(b) the *maximal hod pair construction* $\mathbb{C} = \langle (M_{\nu,k}, \Omega_{nu,k}) \mid \langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \delta, 0\rangle\rangle$, whose levels $(M_{\nu,k}, \Omega_{\nu,k})$ are lbr hod pairs.

Both constructions add an extender F whenever doing so produces a premouse, and there is a background extender $F^* \in \mathcal{F}^{N^*,w}$ such that $F \subseteq F^*$. There is at most one such F at any stage. We choose a background extender for F by minimizing first in the Mitchell order, and then in w. This choice of background extenders enters into the definition of $\Omega_{\nu,k}$, which is the iteration strategy obtained by lifting trees on $M_{\nu,k}$ to trees on N^*, and then using Σ to pick branches. The hod pair construction adds information about the $\Omega_{\nu,k}$ as well.

One of the main results of [19] is

Theorem 1.3. *[Comparison II, [19]] Assume* AD^+*, let* (P,Σ) *be a projectum stable mouse pair, and let* Γ *be an good pointclass such that* $\mathrm{Code}(\Sigma) \in \Gamma \cap \breve{\Gamma}$*. Let* $\langle N^*, \delta, S, T, w, \Sigma\rangle$ *be a coarse* Γ*-Woodin tuple such that* $P \in HC^{N^*}$*, and let* $\mathbb{C}$ *be the associated maximal construction of the same mouse pair type as* (P,Σ)*, with levels* $(M_{\nu,k}, \Omega_{\nu,k})$*; then there is a* $\langle \nu, k\rangle$ *such that*

(a) $\langle \nu, k\rangle <_{\mathrm{lex}} \langle \delta, 0\rangle$,

(b) (P,Σ) *iterates to* $(M_{\nu,k}, \Omega_{\nu,k})$ *via a normal tree, and*

(c) for all $\langle \eta, j\rangle <_{\mathrm{lex}} \langle \nu, k\rangle$, (P,Σ) *iterates strictly past* $(M_{\eta,j}, \Omega_{\eta,j})$ *via a normal tree.*

This easily implies Theorem 0.3. We do not know how to prove Theorem 0.3 without going through 1.3.

1.2 Very strong hull condensation

One of the main results of [18] is

Theorem 1.4 ([18])**.** *Assume* AD^+*, and let* (P, Σ) *be a projectum stable mouse pair. Let* (Q, Ψ) *be an iterate of* (P, Σ)*; then there is a normal tree* $\mathcal{T}$ *such that* (Q, Ψ) *is an iterate of* (P, Σ) *via* $\mathcal{T}$*.*

The normal tree $\mathcal{T}$ in the theorem is uniquely determined by P, Q, and Σ. So we get

Corollary 1.5. *Let* (P, Σ) *be a projectum stable mouse pair; then* Σ *is positional. That is, if* (Q, Ψ) *and* (Q, Λ) *are iterates of* (P, Σ)*, then* $\Psi = \Lambda$*.*

Definition 1.6. Let (P, Σ) be a projectum stable mouse pair, and let s be a normal stack by Σ with last model Q; then $X(s)$ is the unique normal tree by Σ with last model Q. If $s = \langle \mathcal{T}, \mathcal{U} \rangle$ is a stack of length 2, then we write $X(\mathcal{T}, \mathcal{U})$ for $X(s)$.

Remark. $X(s)$ depends on Σ as well as s. If t is another stack by Σ with last model Q, then $X(t) = X(s)$, so we could write $X(Q)$ or $X(Q, \Sigma)$ for the common value. By positionality, $\Sigma_s = \Sigma_t$, and we shall write Σ_Q for the common value.

The key to the proof of Theorem 1.4 is the definition of *weak tree embedding*, and a proof that if (P, Σ) is a projectum stable mouse pair, then Σ condenses to itself under weak tree embeddings. Let $\mathcal{T}$ and $\mathcal{U}$ be normal trees on the same premouse P. A weak tree embedding of $\mathcal{T}$ into $\mathcal{U}$ is a system of maps

$$\Phi = \langle u, \langle s_\xi \mid \xi < \mathrm{lh}(\mathcal{T}) \rangle, \langle t_\xi \mid \xi + 1 < \mathrm{lh}(\mathcal{T}) \rangle, p \rangle$$

having the properties of a tree embedding (see [19]), except that somewhat less is required of the map p sending exit extenders of $\mathcal{T}$ to exit extenders of $\mathcal{U}$. Namely, $p(E^{\mathcal{T}}_\alpha)$ may not be $t_\alpha(E^{\mathcal{T}}_\alpha)$, but instead connected to $t_\alpha(E^{\mathcal{T}}_\alpha)$ inside $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$ via a sequence of fine structural hulls. The Condensation Theorem for $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$ implies that $p(E^{\mathcal{T}}_\alpha)$ is on its extender sequence.

Definition 1.7. Let $\mathcal{T}$ and $\mathcal{U}$ be iteration trees on a premouse P; then $\mathcal{T}$ is a *weak hull of* $\mathcal{U}$ iff there is a weak tree embedding of $\mathcal{T}$ into $\mathcal{U}$.

The definition of weak tree embedding is natural, but it would take some space to lay it out here in full, because of the fine structure involved. So we ask the reader to see [18] or [21]. Fortunately, we don't need to make use of the details of the definition here. One elementary fact we use is

Proposition 1.8. *If* $\Phi \colon \mathcal{T} \to \mathcal{U}$ *is a weak tree embedding,* $\gamma \leq \mathrm{lh}(\mathcal{T})$*, and* $\mathcal{V}$ *extends* $\mathcal{U}$*, then* $\Phi \restriction \gamma$ *is a weak tree embedding of* $\mathcal{T} \restriction \gamma$ *into* $\mathcal{V}$*.*

Here $\Phi \restriction \gamma$ consists of the maps of Φ that act on $\mathcal{T} \restriction \gamma$. The deeper facts we use are

Theorem 1.9 ([18], [21])**.** *Let* (P, Σ) *be a projectum stable mouse pair, and suppose that* $\mathcal{U}$ *is by* Σ*, and* $\mathcal{T}$ *is a weak hull of* $\mathcal{U}$*; then* $\mathcal{T}$ *is by* Σ*.*

This is a strengthening of the strong hull condensation property that is part of the definition of mouse pair. It is called *very strong hull condensation* in [18] and [21].

Theorem 1.10 ([18], [21])**.** *Let (P, Σ) be a projectum stable mouse pair, and let $\langle \mathcal{T}_\alpha \mid \alpha < \beta \rangle$ be a stack of normal trees by Σ; then*

(a) $\mathcal{T}_0$ is a weak hull of $X(\vec{\mathcal{T}})$, and

(b) $X(\vec{\mathcal{T}})$ is a weak hull of $W(\vec{\mathcal{T}})$.

(Recall that $W(\vec{\mathcal{T}})$ is the embedding normalization of $\vec{\mathcal{T}}$; see [19] or [16].) The weak tree embeddings asserted to exist in 1.10 are constructed in an inductive process that is woven into the proof that $\mathcal{T}$ can be fully normalized. The process gives a concrete inductive construction of $X(\vec{\mathcal{T}})$, in which these weak tree embeddings play a central role. Theorems 1.9 and 1.10 show that the process produces Σ-iterates of P.

2 Optimal Suslin representations

Let (P, Σ) be a projectum stable mouse pair. By $\mathcal{F}(P, \Sigma)$ we mean the system of all non-dropping Σ-iterates (Q, Ψ) of (P, Σ), ordered by

$$(Q, \Psi) \prec (R, \Phi) \text{ iff } (R, \Phi) \text{ is a non-dropping iterate of } (Q, \Psi).$$

By the Comparison Theorem, $\mathcal{F}(P, \Sigma)$ is countably directed, and by Dodd-Jensen, the iteration map

$$\pi^{\Psi}_{Q,R} \colon (Q, \Psi) \to (R, \Phi)$$

is independent of Φ and the stack of trees used to get from (Q, Ψ) to (R, Φ). We let

$$M_\infty(P, \Sigma) = \text{ direct limit of } \mathcal{F}(P, \Sigma)$$

under the iteration maps. By countable directedness, $M_\infty(P, \Sigma)$ is wellfounded, so we assume it has been transitivized. Let

$$\pi^{\Sigma}_{P,\infty} \colon P \to M_\infty(P, \Sigma)$$

be the direct limit map.[19]

Proposition 2.1. *Let (P, Σ) be a projectum stable mouse pair; then*

(a) $M_\infty(P, \Sigma) \in HOD$,

(b) Σ is the unique Ψ such that $(P, \Psi) \equiv^* (P, \Sigma)$ and $\pi^{\Sigma}_{P,\infty} = \pi^{\Psi}_{P,\infty}$, and hence

(c) Σ is ordinal definable from $\pi^{\Sigma}_{P,\infty}$.

[19] When $(Q, \Psi) \in \mathcal{F}(P, \Sigma)$, then $\Psi = \Sigma_Q$, and we may sometimes write $\pi^{\Sigma}_{Q,R}$ for $\pi^{\Sigma_Q}_{Q,R}$. There is no ambiguity here, because Σ determines P, so Q and Σ determine Σ_Q.

Proof. For (a): If (P,Σ) and (Q,Ψ) have a common non-dropping iterate, then $M_\infty(P,\Sigma) = M_\infty(Q,\Psi)$. Thus $M_\infty(P,\Sigma)$ is definable from the ordinal rank of (P,Σ) in the mouse order. But $M_\infty(P,\Sigma)$ has a definable wellorder, so $M_\infty(P,\Sigma) \in \mathrm{HOD}$.

For (b): Let (R,Φ) be a common nondropping iterate of (P,Σ) and (P,Ψ), with $i = \pi^\Sigma_{P,R}$ and $j = \pi^\Psi_{P,R}$. Then

$$\begin{aligned}\pi^\Phi_{R,\infty} \circ i &= \pi^\Sigma_{P,\infty}\\ &= \pi^\Psi_{P,\infty}\\ &= \pi^\Phi_{R,\infty} \circ j.\end{aligned}$$

Thus $i = j$. But $\Sigma = \Phi^i$ and $\Psi = \Phi^j$ by pullback consistency, and so $\Sigma = \Psi$.

(c) follows at once from (b). □

If $\theta_0 \leq o(M_\infty(P,\Sigma))$, then Σ is not ordinal definable from a real. Thus under reasonable hypotheses, (c) cannot be improved. Similarly, if $\theta_0 \leq o(M_\infty(P,\Sigma))$, then there is a $\Psi \neq \Sigma$ such that (P,Ψ) is mouse equivalent to (P,Σ). (By the Basis Theorem, there is also a $\boldsymbol{\Delta}^2_1$ strategy Λ such that (P,Λ) is a mouse pair, so P can have mouse-inequivalent expansions.)

Proposition 2.2. *Let (P,Σ) and (Q,Ψ) be projectum stable mouse pairs; then $(P,\Sigma) \equiv^* (Q,\Psi)$ if and only if $M_\infty(P,\Sigma) = M_\infty(Q,\Psi)$.*

Proof. The "only if" part is obvious. Suppose now that $M_\infty(P,\Sigma) = M_\infty(Q,\Psi)$. Let A be a set of reals coding Σ and Ψ, let G be V-generic over $\mathrm{Col}(\omega_1,\mathbb{R})$, and let $N = L(A,\mathbb{R})[G]$, so that $N \models \mathsf{ZFC}$. In N, we have sequences (P_α,Σ_α) and (Q_α,Ψ_α) that are cofinal in $\mathcal{F}(P,\Sigma)$ and $\mathcal{F}(Q,\Psi)$ respectively, such that if $\pi_{\alpha,\beta} = \pi^\Sigma_{P_\alpha,P_\beta}$, and $\tau_{\alpha,\beta} = \pi^\Psi_{Q_\alpha,Q_\beta}$, then

(a) for λ a limit ordinal, $P_\lambda = \bigcup_{\alpha<\lambda} \mathrm{ran}(\pi_{\alpha,\lambda})$ and $Q_\lambda = \bigcup_{\alpha<\lambda} \mathrm{ran}(\tau_{\alpha,\lambda})$, and

(b) $M_\infty(P,\Sigma) = \bigcup_{\alpha<\omega_1} \mathrm{ran}(\pi_{\alpha,\infty})$ and $M_\infty(Q,\Psi) = \bigcup_{\alpha<\omega_1} \mathrm{ran}(\tau_{\alpha,\infty})$.

Since $M_\infty(P,\Sigma) = M_\infty(Q,\Psi)$, we can find a $\lambda < \omega_1$ such that $\mathrm{ran}(\pi_{\lambda,\infty}) = \mathrm{ran}(\tau_{\lambda,\infty})$. Clearly, this implies $P_\lambda = Q_\lambda$ and $\pi_{\lambda,\infty} = \tau_{\lambda,\infty}$. By Proposition 2.1(b), $\Sigma_\lambda = \Psi_\lambda$. Thus (P,Σ) and (Q,Ψ) have a common iterate (in $V[G]$, but therefore in V), as desired. □

The key to our Suslin representation of Σ is the fact that $M_\infty(P,\Sigma)$ is an iterate of P via a normal tree all of whose countable weak hulls are by Σ. We show this now.

Lemma 2.3. *[Strategy extension] Let (P,Σ) be a projectum stable mouse pair; then there is a unique extension Ψ of Σ such that*

(a) Ψ is defined on all normal trees $\mathcal{T}$ by Ψ such that $\mathcal{T}$ is the surjective image of $\mathbb{R}$, and

(b) if $\mathcal{T}$ is by Ψ, then all countable weak hulls of $\mathcal{T}$ are by Σ.

Proof. For uniqueness, suppose b and c are cofinal branches of $\mathcal{T}$, and that all countable weak hulls of $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are by Σ. If $b \neq c$, then taking a Skolem hull of the situation, we get countable weak hulls $\mathcal{S}^\frown d$ and $\mathcal{S}^\frown e$ of $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ such that $d \neq e$. But $d = \Sigma(\mathcal{S}) = e$, contradiction.

So it is enough to show that if $\mathcal{T}$ is a normal tree of limit length, $\mathcal{T}$ is the surjective image of $\mathbb{R}$, and all countable weak hulls of $\mathcal{T}$ are by Σ, then $\mathcal{T}$ has a cofinal branch b such that all countable weak hulls of $\mathcal{T}^\frown b$ are by Σ. Fix then such a $\mathcal{T}$.

Let μ be the supercompactness measure on $P_{\omega_1}(\mathcal{T})$. Since $\mathcal{T}$ is wellorderable in order type $< \theta$, μ exists. For μ-a.e. σ, let

$$\pi_\sigma \colon \mathcal{T}_\sigma \to \mathcal{T}$$

be the uncollapse map. It is clear that μ-a.e., π_σ generates a weak hull embedding of $\mathcal{T}_\sigma$ into $\mathcal{T}$. (In fact, it generates a full hull embedding.) So $\mathcal{T}_\sigma$ is by Σ, and we can set

$$b_\sigma = \Sigma(\mathcal{T}_\sigma).$$

Set

$$b = [\lambda\sigma.b_\sigma]_\mu.$$

We are taking the ultrapower of wellorderable structures, so there is enough elementarity that b is a cofinal branch of $\mathcal{T}$. Suppose that $\mathcal{S}^\frown d$ is a countable weak hull of $\mathcal{T}^\frown b$. It is easy to see that for μ a.e. σ, $\mathcal{S}^\frown d$ is a weak hull of $\mathcal{T}_\sigma{}^\frown b_\sigma$. Thus $\mathcal{S}^\frown d$ is by Σ, as desired. □

Definition 2.4. Let (P, Σ) be a projectum stable mouse pair; then for normal trees $\mathcal{T}$ on P such that $\mathcal{T}$ is the surjective image of $\mathbb{R}$, $\Sigma^+(\mathcal{T}) = b$ iff every countable weak hull of $\mathcal{T}^\frown b$ is by Σ.

Lemma 2.5. *Let (P, Σ) be a projectum stable mouse pair; then there is a normal tree $\mathcal{U}$ by Σ^+ such that*

(a) $\mathcal{U}$ has last model $\mathcal{M}^{\mathcal{U}}_\infty = M_\infty(P, \Sigma)$, and

(b) $i^{\mathcal{U}}_{0,\infty} = \pi^\Sigma_{P,\infty}$.

Proof. We build $\mathcal{U}$ by induction, iterating away extender disagreements with $M_\infty(P, \Sigma)$, and using Σ^+ to pick branches. Suppose that at some stage we have constructed $\mathcal{T}$ of length $\alpha + 1$.

Claim 1. If β is least such that $\mathcal{M}^{\mathcal{T}}_\alpha|\beta \neq M_\infty(P, \Sigma)|\beta$, then $\mathcal{M}^{\mathcal{T}}_\alpha|\beta$ is extender active, and $M_\infty(P, \Sigma)|\beta$ is passive.

Proof. Note that if P is an lpm, this says something about the agreement of the internal strategies. We aren't trying to line up external strategies, although it would be possible to say something sensible in that direction.

Let $A \subseteq \mathbb{R}$ be the codeset of Σ. It is easy to see that $\mathcal{T}$ and $M_\infty(P, \Sigma)$ are definable from A in $L_\xi(A, \mathbb{R})$, for ξ least such that ZF^- holds in $L_\xi(A, \mathbb{R})$. Using $\mathsf{DC}_\mathbb{R}$ we get a countable $\tau \subseteq \mathbb{R}$ and an embedding

$$\pi \colon L_\gamma(B, \tau) \to L_\xi(A, \mathbb{R}).$$

Let $\pi(\mathcal{S}) = \mathcal{T}$ and

$$\pi(N) = M_\infty(P, \Sigma).$$

$B = A \cap \tau$ codes $\Sigma \restriction L_\gamma(B, \tau)$, so

$$N = \text{ direct limit of } \mathcal{F}(P, \Sigma) \cap L_\gamma(B, \tau).$$

Working outside $L(B, \tau)$ we can pick a countable stack s cofinal in this direct limit system, and by 1.4, we have the normal tree

$$X = X(s)$$

by Σ which iterates (P, Σ) to (N, Σ_N). But note that $\mathcal{S}$ is also by Σ, because it is a weak hull of $\mathcal{T}$, and $\mathcal{T}$ is by Σ^+. Normal trees by a fixed strategy are determined by their last models, so $\mathcal{S}$ is an initial segment of X. Thus our claim is true about $\mathcal{S}$ in $L_\gamma(B, \mathbb{R})$. By elementarity, our claim is true. □

Claim 2. Neither of $\mathcal{M}_\alpha^\mathcal{T}$ and $M_\infty(P, \Sigma)$ is a proper initial segment of the other.

Proof. The reflection argument of Claim 1 shows this as well. □

Claims 1 and 2 yield Lemma 2.5 at once. □

Definition 2.6. Let (P, Σ) be a projectum stable mouse pair; then $\mathcal{U}(P, \Sigma)$ is the unique normal tree by Σ^+ whose last model is $M_\infty(P, \Sigma)$.

One might be tempted to think that $(P, \Sigma) <^* (Q, \Psi)$ iff $o(M_\infty(P, \Sigma)) < o(M_\infty(Q, \Psi))$, but this is trivially false. Let P be the least active mouse, and let Q be a mouse such that $P \lhd Q$ with $Q \models$ " all sets are countable". Let Σ and Ψ be the (in case, unique) strategies for the two. Then $o(M_\infty(P, \Sigma)) > \omega_1$, but $M_\infty(Q, \Psi) = Q$, because it is impossible to iterate Q without dropping.

Note also that in the example above, Ψ is not $o(M_\infty(Q, \Psi))$-Suslin. What we get directly is a Suslin representation for the $\mathcal{U}(Q, \Psi)$-certified part of Ψ.

Definition 2.7. Let (P, Σ) be a projectum stable mouse pair; then

(a) For $\mathcal{T}$ a countable normal iteration tree on P, $\mathcal{T}$ is $\mathcal{U}(P, \Sigma)$-*certified* iff $\mathcal{T}$ is a weak hull of $\mathcal{U}(P, \Sigma)$.

(b) For s a countable stack on P, s is $\mathcal{U}(P, \Sigma)$-certified iff $W(s)$ is a weak hull of $\mathcal{U}(P, \Sigma)$.

(c) Σ^{uc} is the set of all countable $\mathcal{U}(P, \Sigma)$-certified stacks.

Clearly, every $\mathcal{U}(P, \Sigma)$-certified normal tree or stack is by Σ. This includes all trees and stacks that do not drop on their main branches, by the following two lemmas.

Lemma 2.8. *Let (P, Σ) be a projectum stable mouse pair, and $\mathcal{T}$ a countable normal tree on P. Equivalent are*

(a) $\mathcal{T}$ *is* $\mathcal{U}(P,\Sigma)$*-certified,*

(b) *there is a countable, normal tree* $\mathcal{S}$ *by* Σ *whose main branch does not drop, and such that* $\mathcal{T}$ *is a weak hull of* $\mathcal{S}$.

Proof. That (a) implies (b) follows by a simple Skolem hull argument. Now let $\mathcal{S}$ be such that $\mathcal{S}$ is by Σ, and $\mathcal{S}$ has last model Q, P-to-Q does not drop, and $\mathcal{T}$ is a weak hull of $\mathcal{S}$. Let $\mathcal{V}$ be the normal tree by Σ_Q^+ on Q having last model $M_\infty(Q,\Sigma_Q) = M_\infty(P,\Sigma)$. A simple Skolem hull argument shows that

$$X(\mathcal{S},\mathcal{V}) = \mathcal{U},$$

and by using 1.10, that $\mathcal{S}$ is a weak hull of $\mathcal{U}$. It follows that $\mathcal{T}$ is a weak hull of $\mathcal{U}$. $\square$

For stacks, we have

Lemma 2.9. *Let* (P,Σ) *be a projectum stable mouse pair, and* s *a countable normal stack on* P *that is by* Σ*. Suppose that either*

(a) s *has limit length, and does not drop along its unique branch, or*

(b) s *has a last tree* $\mathcal{T}$*, with base model* $Q = \mathcal{M}_0^{\mathcal{T}}$ *such that* P*-to-*Q *does not drop, and* $\mathcal{T}$ *is* $\mathcal{U}(Q,\Sigma_Q)$*-certified.*

Then s *is* $\mathcal{U}(P,\Sigma)$*-certified.*

Proof. If (a) holds, then we can extend s by one model to t by Σ that has a last model, and does not drop going to it. But then $W(t)$ is by Σ, has a last model, and does not drop going to it. So $W(t)$ is $\mathcal{U}(P,\Sigma)$-certified by Lemma 2.8. But $W(s) = W(t)$.

So assume (b) holds, and let $s = t^\frown\langle\mathcal{T}\rangle$, where t is by Σ and doesn't drop, with last model Q, and $\mathcal{T}$ is $\mathcal{U}(Q,\Sigma_Q)$-certified. By Lemma 2.8, $\mathcal{T}$ is a weak hull of $\mathcal{V}$, for some $\mathcal{V}$ on Q by Σ_Q such that $\mathcal{V}$ has a last model, and does not drop going to it. Since $t^\frown\langle\mathcal{V}\rangle$ is by Σ, so is $W(t^\frown\langle\mathcal{V}\rangle)$, so by 2.8, $W(t^\frown\langle\mathcal{V}\rangle)$ is a weak hull of $\mathcal{U}(P,\Sigma)$ But by [18], $W(t^\frown\mathcal{T})$ is a weak hull of $W(t^\frown\mathcal{V})$. So $W(t^\frown\mathcal{T})$ is a weak hull of $\mathcal{U}(P,\Sigma)$, as desired. $\square$

Definition 2.10. Let (P,Σ) be a projectum stable mouse pair; then

(a) Σ^{nd} is the collection of countable normal trees by Σ having a last model, and such that the branch to that model does not drop.

(b) A countable normal $\mathcal{T}$ is $M_\infty(P,\Sigma)$*-relevant* iff $\mathcal{T}$ has some (perhaps improper) extension $\mathcal{S}$ such that $\mathcal{S}$ is in Σ^{nd}.

(c) A countable stack s is $M_\infty(P,\Sigma)$*-relevant* iff s is by Σ, and either has limit length and does not drop along its unique branch, or has a last tree $\mathcal{T}$ with base model Q such that P-to-Q does not drop, and $\mathcal{T}$ is $M_\infty(Q,\Sigma_Q)$-relevant.

(d) Σ^{rl} is the restriction of Σ to all $M_\infty(P,\Sigma)$-relevant stacks.

One can give a more concrete characterization of the M_∞-relevant normal trees, but we don't need it here.

Given sets of reals A and B, we say that A is *positive*-$\mathbf{\Sigma^1_n}(B)$ iff A is definable over $(V_{\omega+1}, \in, B)$ from parameters by a Σ_n formula in which $\dot{B}$ has only positive occurrences. Note that if A is positive-$\mathbf{\Sigma^1_1}(B)$ and B is κ-Suslin, then A is κ-Suslin. Note also that each of the codesets of Σ^{nd}, Σ^{rl}, and Σ^{uc} is positive-$\mathbf{\Sigma^1_1}$ in each of the others, by Lemmas 2.8 and 2.9 and some simple computations. So we have

Proposition 2.11. *Let (P, Σ) be a projectum stable mouse pair; then for any κ, Σ^{uc} is κ-Suslin iff Σ^{nd} is κ-Suslin iff Σ^{rl} is κ-Suslin.*

Σ^{rl} is the fragment of Σ that gets used in forming $M_\infty(P, \Sigma)$. It is contained in Σ^{uc}, but the containment may be proper. By 2.8, Σ^{uc} consists of all weak hulls of trees by Σ^{rl}.

Theorem 2.12. *Let (P, Σ) be a projectum stable mouse pair; then*

(a) Σ^{uc} is $|M_\infty(P, \Sigma)|$-Suslin, and

(b) Σ^{uc} is not α-Suslin for any $\alpha < |M_\infty(P, \Sigma)|$.

Proof. Part (b) follows easily from the Kunen-Martin theorem. For let

$$(Q, \eta) R (S, \gamma) \Leftrightarrow [(Q, \Sigma_Q) \prec (S, \Sigma_S) \wedge \pi^\Sigma_{Q,\infty}(\eta) < \pi^\Sigma_{S,\infty}(\gamma)].$$

$\mathrm{Code}(R)$ is positive-$\mathbf{\Sigma^1_1}$ in $\mathrm{Code}(\Sigma^{\mathrm{uc}})$, so if $\mathrm{Code}(\Sigma^{\mathrm{uc}})$ is α Suslin, then so is $\mathrm{Code}(R)$. But R is a prewellorder of order type $o(M_\infty(P, \Sigma))$, so by Kunen-Martin, $\alpha \geq |M_\infty(P, \Sigma)|$.

For (a), let $\mathcal{U} = \mathcal{U}(P, \Sigma)$, and $\kappa = |M_\infty(P, \Sigma)| = |\mathcal{U}(P, \Sigma)|$. We may assume $\Sigma^{\mathrm{uc}} \neq \emptyset$, as otherwise (a) is trivial. This implies $\kappa \geq \omega_1$. Let $C \subset \kappa$ code $\mathcal{U}$ and $M_\infty(P, \Sigma)$ in some simple way. (Note that they are both essentially sets of ordinals.) Let α be least such that $L_\alpha[C] \models \mathsf{ZFC}^-$.

Let T be the tree on $\omega \times L_\alpha[C]$ of all attempts to build an (x, π) such that $\pi \colon (\omega, E_x) \to L_\alpha[C]$ is elementary, and $\pi(0) = \mathcal{U}$. For $(x, \pi) \in [T]$, let

$$\sigma_x = \pi \circ \pi_x^{-1}$$

be the uncollapse map from $(\mathrm{set}(x), \in)$ to $(\mathrm{ran}(\pi), \in)$. Let $\mathcal{U}_x = \sigma_x^{-1}(\mathcal{U}) = \mathrm{set}_0(x)$. So if $x \in p[T]$, then $\mathcal{U}_x$ is a psuedo-hull of $\mathcal{U}$, and every countable weak hull of $\mathcal{U}$ is a weak hull of some $\mathcal{U}_x$, where $x \in p[T]$. Let

$$\begin{aligned} A(y, x) \Leftrightarrow\ & \mathrm{Code}(y) \wedge \mathrm{Code}(x) \wedge \\ & \mathrm{set}_0(y) \text{ is a stack of normal trees on } P \wedge \\ & W(\mathrm{set}_0(y)) \text{ is a weak hull of } \mathrm{set}_0(x). \end{aligned}$$

Clearly, A is Σ^1_2 in P, hence κ-Suslin. But

$$y \in \mathrm{Code}(\Sigma^{\mathrm{uc}}) \Leftrightarrow \exists x (A(y, x) \wedge x \in p[T]).$$

Thus $\mathrm{Code}(\Sigma^{\mathrm{uc}})$ is κ-Suslin. $\square$

In order to get a Suslin representation for the full Σ, and not just its M_∞-relevant part, we must amalgamate Suslin representations like that above.

Definition 2.13. H^e_α is the common value of $M_\infty(P,\Sigma)$, for all projectum stable pure extender pairs (P,Σ) of mouse rank α. H^h_α is the common value of $M_\infty(P,\Sigma)$, for all projectum stable lbr hod pairs (P,Σ).

When context permits, we shall simply write H_α, letting the superscript "e" or "h" be understood. Notice that H_α is essentially a set of ordinals, and thus $\langle H^e_\alpha \mid \alpha < \theta\rangle$ and $\langle H^h_\alpha \mid \alpha < \theta\rangle$ are in HOD. Assuming $\mathsf{AD}_\mathbb{R} + \mathsf{HPC}$, $\mathrm{HOD} = L[\langle H^h_\alpha \mid \alpha < \theta\rangle]$, as we shall see.

The next lemma strengthens 2.5.

Lemma 2.14. *Let (P,Σ) be a projectum stable mouse pair of mouse rank α; then for all $\beta \le \alpha$ there is a unique normal tree $\mathcal{W}_\beta$ on P such that $\mathcal{W}_\beta$ is by Σ^+, and has last model H_β.*

Remark. Of course, the branch P-to-H_β in $\mathcal{W}_\beta$ drops iff $\beta < \alpha$.

Proof. Fix β, and let $M_\infty(Q,\Psi) = H_\beta$. We may assume $\beta < \alpha$, as otherwise 2.5 applies. Comparing (P,Σ) with (Q,Ψ), we obtain a normal $\mathcal{T}$ on P by Σ with last model (R,Φ) such that (R,Φ) is a nondropping iterate of (Q,Ψ). Then $H_\beta = M_\infty(R,\Phi)$, so by 2.5 there is a normal tree $\mathcal{U}$ by Φ^+ on R whose last model is H_β. We set

$$\mathcal{W}_\beta = X(\mathcal{T},\mathcal{U}).$$

Then $\mathcal{W}_\beta$ is a normal tree on P by Σ^+ that has last model H_β. (This can be shown by passing to the appropriate countable elementary submodel of V.) Uniqueness is clear.

□

Corollary 2.15. *Let (P,Σ) be a mouse pair of mouse rank α, and let $\gamma = \sup_{\beta\le\alpha} o(H_\beta)$; then*

(1) Σ is γ-Suslin, and

(2) either

(a) Σ is not μ-Suslin for any $\mu < |\gamma|$, or

(b) γ is singular, and $\gamma = \nu^+$ for some cardinal ν such that Σ is ν-Suslin, but not μ-Suslin for any $\mu < \nu$.

Proof. (Sketch.) Let $\langle \mathcal{W}_\beta \mid \beta \le \alpha\rangle$ be the sequence of normal trees by Σ^+ given by the lemma. We verify that a countable stack s is by Σ by producing a $\beta \le \alpha$ and a weak tree embedding from $W(s)$ into $\mathcal{W}_\beta$. The details can be arranged as they were in the proof of Theorem 2.12. The result is a tree

$$T \subseteq \omega \times (\alpha+1) \times \gamma,$$

such that

$$p[T] = \mathrm{Code}(\Sigma).$$

We claim that $|\alpha| \leq \gamma$. For if not, then since each H_β is essentially a subset of $o(H_\beta)$, we get a sequence of γ^+ distinct subsets of γ, contrary to the boldface GCH. Thus Σ is γ-Suslin.

Suppose now that $\nu \leq |\gamma|$ is least such that $\mathrm{Code}(\Sigma)$ is ν-Suslin. For each $\beta < \alpha$, there is a dropping iterate (Q, Ψ) of (P, Σ) such that $M_\infty(Q, \Psi) = H_\beta$. But $\mathrm{Code}(\Psi)$ is positive-$\mathbf{\Sigma}^1_1$ in $\mathrm{Code}(\Sigma)$, so $\mathrm{Code}(\Psi)$ is ν-Suslin, so $|H_\beta| \leq \nu$ by 2.12.Thus $\gamma \leq \nu^+$. If $\gamma < \nu^+$ then we have conclusion (a).

Assume then $\gamma = \nu^+$. We have also $|\alpha| \leq \nu$, because the relation

$$\begin{aligned} Q < R \Leftrightarrow{} & \text{there are normal trees } \mathcal{T} \text{ by } \Sigma \text{ with last model } R \text{ and} \\ & \mathcal{S} \text{ by } \Sigma_R \text{ with last model } Q \text{ such that } R\text{-to-}Q \text{ drops} \end{aligned}$$

is positive-$\mathbf{\Sigma}^1_1$ in $\mathrm{Code}(\Sigma)$, and hence ν-Suslin. It follows that the $o(H_\beta)$ for $\beta \leq \alpha$ witness that ν^+ is singular. Thus we have (b).

□

Remark. We don't know whether clause (b) in the conclusion of Corollary 2.15 ever actually occurs.

We consider now a second way to get Suslin cardinals from mouse pairs.

Definition 2.16. If P is a premouse and $\kappa \in P$, then $o(\kappa)^P = \sup\{\eta \mid \mathrm{crit}(E^P_\eta) = \kappa\}$. (The sup is nonstrict.) δ is a *cutpoint* of P iff $\forall \alpha < \delta (o(\alpha)^P \leq \delta)$. δ is a *strong cutpoint* of P iff $\forall \alpha < \delta (o(\alpha)^P < \delta)$.

By this definition, there could be E on the P-sequence such that $\mathrm{lh}(E)$ is a cutpoint of P. However, we are usually interested in cutpoints that are cardinals of P, hence not of this form.

Definition 2.17. Let P be a premouse; then

(1) $\tau^P = \sup\{\kappa^{+,P} \mid \kappa < \rho_{k(P)}(P) \wedge \exists \eta > \kappa^{+,P}(\kappa = \mathrm{crit}(E^P_\eta))\}$.

(2) P has *limit type* iff τ^P is a limit of P-cardinals, and $\forall \alpha < \tau^P (o(\alpha)^P < \tau^P)$. *$P$ has a top block* iff P is not of limit type.

(3) If P has a top block, then β^P is the least $\alpha < \tau^P$ such that $o(\alpha)^P \geq \tau^P$. We say that β^P *begins the top block* of P.

By convention, $\tau^P = 0$ when there are no E on the P sequence with critical point $< \rho_{k(P)}(P)$. In that case, P has no non-dropping iterates, and we aren't interested in it at the moment.

Clearly, either τ^P is a cardinal of P, or $\tau^P = o(P)$ and P has no largest cardinal. If α is the largest P-cardinal $< \tau^P$, then $o(\alpha)^P > \tau^P$. Thus β^P does indeed exist whenever P is not of limit type. If P has limit type, then τ^P is a limit of successor cardinal strong cutpoints in P. If P has a top block, then β^P is a limit of successor cardinal strong cutpoints in P.[20]

[20] We prove the second statement. If $\alpha < \beta^P$, then $o(\alpha)^P < \beta^P$, as otherwise $o(\alpha)^P \geq \tau^P$ by coherence, contradiction. But β^P is a limit (in fact, measurable) cardinal of P, and it is easy to see by coherence that $o(\alpha)^{+,P}$ is a strong cutpoint of P.

Definition 2.18. Let (P, Σ) be a projectum stable mouse pair; then $\tau_\infty(P, \Sigma) = \pi^\Sigma_{P,\infty}(\tau^P)$. If P has a top block, then $\beta_\infty(P, \Sigma) = \pi^\Sigma_{P,\infty}(\beta^P)$.

Theorem 2.19. *Let (P, Σ) be a projectum stable mouse pair; then Σ^{uc} is $\tau_\infty(P, \Sigma)$-Suslin.*

Proof. Let $\tau_\infty = \tau_\infty(P, \Sigma)$, and let $\mathcal{U} = \mathcal{U}(P, \Sigma)$. We have $i^{\mathcal{U}}_{0,\alpha} = \pi_{(P,\Sigma),\infty}$, where $\alpha + 1 = \mathrm{lh}(\mathcal{U})$. $[0, \alpha)_U$ uses no extenders overlapping τ_∞, since this is in the range of the branch embedding. But $[0, \alpha)_U$ uses no extenders with critical point $\geq \tau_\infty$, because τ^P is defined so that any such extender would cause $[0, \alpha)_U$ to drop. Thus all extenders used in $[0, \alpha)_U$ have length $< \tau_\infty$. It follows that all extenders used in $\mathcal{U}$ have length $< \tau_\infty$.

Now we get a τ_∞ Suslin representation of Σ^{uc} from the proof of 2.12. □

It follows that $|o(M_\infty(P, \Sigma)| \leq \tau_\infty(P, \Sigma)$. That is also easy to see directly from the proof of 2.19.

It is possible that neither τ_∞ nor β_∞ are cardinals of V, where V is the model of AD^+ in which we are living. For example, let δ be the Woodin cardinal of M_1, κ the least strong to δ, and ν a cardinal of M_1 such that $\kappa < \nu < \delta$. Let $P = M_1|\nu$, and let Σ be its unique iteration strategy. Then $\omega_1 < \beta_\infty(P, \Sigma) < \tau_\infty(P, \Sigma) < \omega_2$. If we take $P = M_1|\delta$ instead, we still have $\omega_1 < \beta_\infty(P, \Sigma) < \omega_2$, but now $\tau_\infty(P, \Sigma)$ is a cardinal, namely ω_ω. (See [2].)[21]

Suppose (P, Σ) has a top block. Must Σ^{uc} be $\beta_\infty(P, \Sigma)$-Suslin? In general, no, as the example of M_1 and its strategy shows. What becomes Suslin at β_∞ is the fragment of Σ consisting of weak hulls of $\mathcal{U}(P, \Sigma) \upharpoonright (\beta_\infty(P, \Sigma) + 1)$.

Definition 2.20. Let (P, Σ) be a projectum stable mouse pair having a top block, let $\mathrm{lh}(\mathcal{U}(P, \Sigma)) = \gamma + 1$, and $\nu \in [0, \gamma]_U$ be least such that $\mathrm{crit}(i^{\mathcal{T}}_{\nu,\gamma}) > i_{0,\nu}(\beta^P)$; then we set $\mathcal{U}_0(P, \Sigma) = \mathcal{U}(P, \Sigma) \upharpoonright \nu + 1$. We write $M^0_\infty(P, \Sigma)$ for the last model of $\mathcal{U}_0(P, \Sigma)$.

Another way of saying it is, the main branch extender of $\mathcal{U}_0(P, \Sigma)$ is $E \upharpoonright \beta_\infty$, where E is the main branch extender of $\mathcal{U}(P, \Sigma)$. We have $|\mathcal{U}_0(P, \Sigma)| = |\beta_\infty(P, \Sigma)|$. In the cases we shall care about below, β_∞ is a cardinal of V, so $\mathrm{lh}(\mathcal{U}_0(P, \Sigma)) = \beta_\infty(P, \Sigma) + 1$.

Definition 2.21. Let (P, Σ) be a projectum stable mouse pair; then

(a) For $\mathcal{T}$ a countable normal iteration tree on P, $\mathcal{T}$ is $\mathcal{U}_0(P, \Sigma)$-*certified* iff $\mathcal{T}$ is a weak hull of $\mathcal{U}_0(P, \Sigma)$.

(b) For s a countable stack on P, s is $\mathcal{U}_0(P, \Sigma)$-*certified* iff $W(s)$ is a weak hull of $\mathcal{U}_0(P, \Sigma)$.

(c) Σ^{ucb} is the set of all countable $\mathcal{U}_0(P, \Sigma)$-certified stacks.

Definition 2.22. Let (P, Σ) be a projectum stable mouse pair, and $i\colon P \to Q$ an iteration map coming from a non-dropping iteration by Σ; then i is ν-*generated* iff for $k = k(P)$, $Q^k = \{i(f)(a) \mid f \in P^k \wedge a \in [\nu]^{<\omega}\}$.

[21] Note that τ_∞ is the next Suslin after β_∞ in this example. That is part of a more general pattern.

Of course, when $\nu = \rho_{k(P)}(P)$, every nondropping iteration map i is $i(\nu)$-generated. We say a branch b of $\mathcal{T}$ is ν-generated iff b does not drop, and $i_b^{\mathcal{T}}$ is ν-generated. We are mainly interested below in the case that P has a top block, and b is $i_b(\beta^P)$-generated. In this case, $\mathcal{T}^\frown b$ is $\mathcal{U}_0(P,\Sigma)$-certified:

Lemma 2.23. *Let (P,Σ) be a projectum stable mouse pair having a top block, and $\mathcal{T}$ a countable normal tree on P. Equivalent are*

(a) $\mathcal{T}$ is $\mathcal{U}_0(P,\Sigma)$-certified,

(b) $\mathcal{T}$ is a weak hull of some countable, normal tree $\mathcal{S}$ by Σ whose main branch b does not drop, and is $i_b(\beta^P)$-generated.

Proof. That (a) implies (b) follows by a simple Skolem hull argument.

Now let $\mathcal{S}$ be as in (b), with last model Q. As in the proof of 2.8, letting $\mathcal{V}$ be the normal tree by Σ_Q^+ on Q having last model $M_\infty(Q,\Sigma_Q) = M_\infty(P,\Sigma)$, we have

$$X(\mathcal{S},\mathcal{V}) = \mathcal{U},$$

where $\mathcal{U} = \mathcal{U}(P,\Sigma)$. So there is a weak tree embedding Φ of $\mathcal{S}$ into $\mathcal{U}$. Let $\alpha+1 = \mathrm{lh}(\mathcal{S})$, and

$$\Phi = \langle u, \langle s_\xi \mid \xi \le \alpha\rangle, \langle t_\xi \mid \xi < \alpha\rangle, p\rangle,$$

and let $v = v^\Phi$. We have that

$$s_\alpha \colon Q \to \mathcal{M}^{\mathcal{U}}_{v(\alpha)},$$

moreover $v(\alpha) \le_U \gamma$ where $\gamma + 1 = \mathrm{lh}(\mathcal{U})$,

$$i^{\mathcal{U}}_{0,v(\alpha)} = s_\alpha \circ i^{\mathcal{S}}_{0,\alpha},$$

and

$$\mathcal{M}^{\mathcal{U}}_{v(\alpha)} = \{ i^{\mathcal{U}}_{0,v(\alpha)}(f)(a) \mid a \in [\lambda]^{<\omega}\},$$

where $\lambda = \sup(\{s_\alpha(\lambda(E)) \mid E \text{ is used in } [0,\alpha)_S\})$. Since the branch extender of $[0,\alpha)_S$ has all measures concentrating on β^P,

$$\lambda \le s_\alpha(i^{\mathcal{S}}_{0,\alpha}(\beta^P)) = i^{\mathcal{U}}_{0,v(\alpha)}(\beta^P).$$

But this implies $v(\alpha)+1 \le \mathrm{lh}(\mathcal{U}_0)$, where $\mathcal{U}_0 = \mathcal{U}_0(P,\Sigma)$. (In fact, because $\mathcal{S}$ is countable, $v(\alpha)$ is somewhere on the main branch of $\mathcal{U}_0$ strictly below its last model). Thus Φ is a weak tree embedding of $\mathcal{S}$ into $\mathcal{U}_0$. So $\Phi \restriction \mathcal{T}^\frown b$ is a weak tree embedding of $\mathcal{T}^\frown b$ into $\mathcal{U}_0$.

□

The next two lemmas show that $M^0_\infty(P,\Sigma)$ can be viewed a a direct limit of countable mouse pairs.

Lemma 2.24. *Let (P, Σ) be a projectum stable mouse pair, let $\mu < \rho_{k(P)}(P)$, and let s and t be nondropping stacks on P with last models Q and R that are $i_s(\mu)$ and $i_t(\mu)$-generated, respectively. Then there are normal trees $\mathcal{T}$ on Q by Σ_Q and $\mathcal{V}$ on R by Σ_R with a common last model S, and such that for $j = i^{\mathcal{T}} \circ i_s = i^{\mathcal{V}} \circ i_t$, j is $j(\mu)$-generated.*

Proof. We can compare (Q, Σ_Q) with (R, Σ_R) by iterating away least extender disagreements. Because Σ is positional, no strategy disagreements show up as we do so, and eventually we get $\mathcal{T}$ and $\mathcal{V}$ whose last models have no disagreement at all. By Dodd-Jensen, $\mathcal{T}$ and $\mathcal{V}$ have a common last model (S, Σ_S), neither side drops, and $i^{\mathcal{T}} \circ i_s = i^{\mathcal{V}} \circ i_t$. Let $j = i^{\mathcal{T}} \circ i_s$; we must see j is $j(\mu)$-generated.

Suppose not, and let $\gamma+1 = \mathrm{lh}(\mathcal{T})$. Since i_s is $i_s(\mu)$-generated, the generator of j above $j(\mu)$ must have been introduced by $i^{\mathcal{T}}$. Thus we have a least $\alpha <_T \gamma$ such that $i^{\mathcal{T}}_{0,\alpha} \circ i_s(\mu) = j(\mu)$. Similarly, for $\delta + 1 = \mathrm{lh}(\mathcal{V})$, we have a least $\beta <_V \delta$ such that $i^{\mathcal{V}}_{0,\beta} \circ i_t(\mu) = j(\mu)$. But then

$$\begin{aligned}\mathcal{M}^{\mathcal{T}}_\alpha &= \mathrm{Ult}(P, E_j \restriction j(\mu)) \\ &= \mathcal{M}^{\mathcal{V}}_\beta.\end{aligned}$$

Since $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ and $\beta + 1 < \mathrm{lh}(\mathcal{V})$, and we were iterating away extender disagreements, this is a contradiction. $\square$

Remark. Lemma 2.24 describes another way in which the system of iterates of a mouse pair behaves like the system of iterates of a sound mouse projecting to ω.

By the lemma, for any $\mu \leq \rho_{k(P)}(P)$, we can form a directed system

$$\mathcal{F}_\mu(P, \Sigma) = \{(Q, \Sigma_s) \mid i_s \colon P \to Q \text{ is } i_s(\mu)\text{-generated }\}.$$

For $\mu = \rho_{k(P)}(P)$, this is just $\mathcal{F}(P, \Sigma)$. Let

$$M^\mu_\infty(P, \Sigma) = \text{ direct limit of } \mathcal{F}_\mu(P, \Sigma),$$

and $\pi^\mu_{P,\infty} \colon P \to M^\mu_\infty(P, \Sigma)$ be the canonical embedding. It is quite easy to see

Lemma 2.25. *Let (P, Σ) be a projectum stable mouse pair, and $\mathcal{U} = \mathcal{U}(P, \Sigma)$ have length $\gamma + 1$. Let $\mu < \rho_{k(P)}(P)$ and let $\alpha \leq \gamma$ be largest such that $i^{\mathcal{U}}_{0,\alpha}$ is $i^{\mathcal{U}}_{0,\alpha}(\mu)$-generated; then $\mathcal{M}^{\mathcal{U}}_\alpha = M^\mu_\infty(P, \Sigma)$, and $i^{\mathcal{U}}_{0,\alpha} = \pi^\mu_{P,\infty}$.*

In the special case $\mu = \beta^P$, we write $M^0_\infty(P, \Sigma)$ for $M^\mu_\infty(P, \Sigma)$, etc. The normal tree from P to $M^0_\infty(P, \Sigma)$ is then the initial segment of $\mathcal{U}(P, \Sigma)$ that we have called $\mathcal{U}_0(P, \Sigma)$

Definition 2.26. Let (P, Σ) be a projectum stable mouse pair with a top block; then

(a) Σ^{ndb} is the collection of countable normal trees $\mathcal{T}$ by Σ having a last model, and such that the branch b to that model does not drop, and is $i_b(\beta^P)$-generated.[22]

[22] The "b" in the superscript of Σ^{ndb}, etc., is for "beta" or "bottom".

(b) A countable normal $\mathcal{T}$ is $M_\infty^0(P,\Sigma)$-*relevant* iff $\mathcal{T}$ has some (perhaps improper) extension $\mathcal{S}$ such that $\mathcal{S}$ is in Σ^{ndb}.

(c) A countable stack s is $M_\infty^0(P,\Sigma)$-*relevant* iff s is by Σ, and either

(i) s has limit length, does not drop along its unique branch b, and b is $i_b(\beta^P)$-generated, or

(ii) $s = v^\frown\langle\mathcal{T}\rangle$, v does not drop, has last model Q, i_v is $i_v(\beta^Q)$-generated, and $\mathcal{T}$ is $M_\infty^0(Q,\Sigma_Q)$-relevant.

(d) Σ^{rlb} is the restriction of Σ to all $M_\infty^0(P,\Sigma)$-relevant stacks.

Σ^{rlb} is the fragment of Σ relevant to forming $M_\infty^0(P,\Sigma)$. It is contained in Σ^{ucb}, perhaps properly. By 2.23, Σ^{ucb} is the collection of weak hulls of trees by Σ^{rlb}. For any α, Σ^{ndb} is α-Suslin iff Σ^{rlb} is α-Suslin iff Σ^{ucb} is α-Suslin.

Theorem 2.27. *Let (P,Σ) be a projectum stable mouse pair having a top block; then Σ^{ucb} is $\beta_\infty(P,\Sigma)$- Suslin, but not α-Suslin for any $\alpha < |\beta_\infty(P,\Sigma)|$.*

Proof. We verify that a stack s is by Σ^{ucb} by producing a weak tree embedding from $W(s)$ into some countable Skolem hull of $\mathcal{U}_0(P,\Sigma)$. This implies that $\text{Code}(\Sigma^{\text{ucb}})$ is β_∞-Suslin. The details are the same as those in the proof of 2.12.

Again, the second part follows easily from Kunen-Martin. Let $\alpha < |\beta_\infty(P,\Sigma)|$, and let

$$(Q,\eta)R(S,\gamma) \Leftrightarrow [(Q,\Sigma_Q) \prec (S,\Sigma_S) \wedge \pi^\Sigma_{Q,\infty}(\eta) < \pi^\Sigma_{(S,\infty}(\gamma)].$$

$\text{Code}(R)$ is positive-$\mathbf{\Sigma}^1_1$ in $\text{Code}(\Sigma^{\text{rlb}})$, and R is a prewellorder of order type $o(M_\infty^0(P,\Sigma))$, so $\text{Code}(\Sigma^{\text{rlb}})$ cannot be α-Suslin. $\square$

So if (P,Σ) has a top block, there are two Suslin cardinals associated to it, $|\beta_\infty|$ and $|\tau_\infty|$. It is possible that they are equal; for example, this is true if P is a cardinal initial segment M_1 strictly below its Woodin cardinal δ but above the least strong to δ, and Σ is the strategy induced by the unique strategy for M_1. If $P = M_1$ and Σ is its strategy, then τ_∞ is the next Suslin cardinal after β_∞. We conjecture that these are the only two possibilities.

Conjecture 2.27.1. Let (P,Σ) be a mouse pair having a top block, and suppose that $\beta_\infty(P,\Sigma) < |\tau_\infty(P,\Sigma)|$; then $\tau_\infty(P,\Sigma)$ is the least Suslin cardinal strictly greater than $\beta_\infty(P,\Sigma)$.

We shall prove some approximations to this conjecture now, and use them in the later sections. A full proof of the conjecture would require a finer analysis than we have done, and perhaps involve new results on pointclass envelopes and self justifying systems.

The proof of 2.27 shows that in certain cases, the full Σ^{rl} is $\beta_\infty(P,\Sigma)$-Suslin.

Definition 2.28. Let P be a premouse. We say that P is *top block stable* iff P is projectum stable, P has a top block, and

(a) $o(\beta^P) = \tau^P$, and

(b) whenever E is on the P-sequence and $\mathrm{crit}(E) < \rho_{k(P)}(P)$, then i^P_E is continuous at τ^P.

We say that a mouse pair (P, Σ) is top block stable iff P is top block stable.

β^P is the least strong to τ^P, so when $\tau^P < o(P)$, clause (a) is equivalent to τ^P being a cutpoint. Clause (a) and coherence imply that τ^P is not the critical point of any extender from the P-sequence, so it is a strong cutpoint. Clause (b) can be put somewhat more concretely:

Proposition 2.29. *Let P have a top block, and suppose $o(\beta^P) = \tau^P$; then P is top block stable iff P is projectum stable, and either $\tau^P = \rho_{k(P)}(P)$, or $\tau^P < \rho_{k(P)}(P)$ and $\mathrm{cof}^P(\tau^P)$ is not the critical point of a total extender from the P-sequence.*

Lemma 2.30. *Let (P, Σ) be a mouse pair having a top block, and suppose that P is not top block stable; then*

(a) whenever s is by Σ^{rl}, then $W(s)$ is $\mathcal{U}_0(P, \Sigma)$ certified,

(b) Σ^{rl} is $\beta_\infty(P, \Sigma)$-Suslin, and

(c) Σ is γ-Suslin, where $\gamma = \sup(\beta_\infty(P, \Sigma) \cup \{o(H_\beta) \mid \beta <$ mouse rank of $(P, \Sigma)\}$.

Proof. We prove (a). Part (b) then follows at once, and we get (c) using the proof of Theorem 2.15.

Let $\mathcal{U} = \mathcal{U}(P, \Sigma)$ and $\mathcal{U}_0 = \mathcal{U}_0(P, \Sigma)$. Let $s = t^\frown\langle\mathcal{V}\rangle$ be a stack with last tree $\mathcal{V}$ by Σ^{rl}; then we can extend $\mathcal{V}$ normally to $\mathcal{V}_1$ by Σ^{nd}_t. $W(s)$ is a weak hull of $\mathcal{T}$, where $\mathcal{T} = W(t^\frown\langle\mathcal{V}_1\rangle)$. Thus it is enough to show that whenever $\mathcal{T}$ is a normal tree by Σ^{nd}, then $\mathcal{T}$ is a weak hull of $\mathcal{U}_0$.

So fix a normal $\mathcal{T}$ by Σ^{nd}, with last model $Q = \mathcal{M}^{\mathcal{T}}_\gamma$. Let

$$\begin{aligned}\delta &= \sup(\{\lambda(E^{\mathcal{T}}_\xi) \mid \xi + 1 < \mathrm{lh}(\mathcal{T})\}) \\ &= \sup(\{\lambda(E^{\mathcal{T}}_\xi) \mid \xi + 1 \leq_T \gamma)\}).\end{aligned}$$

Claim 1. $\delta \leq \tau^Q$.

Proof. Suppose not; then there must be an $\alpha <_T \gamma$ such that $i^{\mathcal{T}}_{0,\alpha}(\tau^P) \leq \mathrm{crit}(i^{\mathcal{T}}_{\alpha,\gamma}))$. Fix such an α, and let $R = \mathcal{M}^{\mathcal{T}}_\alpha$ and $E = E^{\mathcal{T}}_\xi$ where $\xi + 1 \leq_T \gamma$ and $\alpha = T\mathrm{pred}(\xi + 1)$. E is total over R and $\mathrm{crit}(E) < \rho_{k(R)}(R)$ because $[0, \gamma)_T$ does not drop in model or degree. By coherence and the initial segment condition, the normal measure of E is indexed on the R-sequence, so $\mathrm{crit}(E) < \tau^R$. But $\tau^R = i^{\mathcal{T}}_{0,\alpha}(\tau^P)$, contradiction. □

Claim 2. If $\delta < \tau^Q$, then $\mathcal{T}$ is $\mathcal{U}_0$-certified.

Proof. If $\beta^Q \geq \delta$, then $\mathcal{T}$ is $\mathcal{U}_0$ certified by Lemma 2.23. Otherwise, we have $\beta^Q < \delta < \tau^Q$, so we have E on the sequence of Q such that $\mathrm{crit}(E) = \beta^Q$ and $\delta < \mathrm{lh}(E)$. But then the normal extension $\mathcal{T}^\frown\langle E\rangle$ of $\mathcal{T}$ is $\mathcal{U}_0$-certified by Lemma 2.23, so $\mathcal{T}$ is $\mathcal{U}_0$-certified.

□

So we assume $\delta = \tau^Q$. Since τ^Q is in $\mathrm{ran}(i^{\mathcal{T}}_{0,\gamma})$, it is not $\lambda(E)$ for any E used in $\mathcal{T}$, so γ is a limit ordinal. Let $b = [0,\gamma)_T$and $i_b = i^{\mathcal{T}}_b$. We have that b does not drop,

$$i_b(\tau^P) = \delta(\mathcal{T}),$$

$Q = \mathcal{M}^{\mathcal{T}}_b$, and

$$i_b(\langle \beta^P, \tau^P \rangle) = \langle \beta^Q, \tau^Q \rangle$$

by the elementarity of i_b.

If $o(\beta^P)^P > \tau^P$, then $o(\beta^Q)^Q > \tau^Q$, so there is an E on the Q sequence with $\mathrm{crit}(E) = i_b(\beta^P)$ and $\mathrm{lh}(E) > \delta(\mathcal{T})$. Let

$$\mathcal{S} = \mathcal{T}^\frown \langle E \rangle,$$

where E is applied to $\mathcal{M}^{\mathcal{T}}_\nu$, for $\nu \in b$ least such that $\mathrm{crit}(i^{\mathcal{T}}_{\nu,b}) > i^{\mathcal{T}}_{0,\nu}(\beta^P)$. That is, $\mathcal{S}$ is normal. Then $\mathcal{S}$ is $\mathcal{U}_0$-certified by Lemma 2.23, so $\mathcal{T}$ is $\mathcal{U}_0$-certified.

So we may assume $o(\beta^P) = \tau^P$, and thus clause (b) of top block stability fails for P. By the proposition, letting η_0 be the $r\Sigma_k$ cofinality of τ^P we have that $\eta_0 < \tau^P$, and η_0 is measurable in P. Because P is projectum stable, there is an $f_0 \in P$ mapping η_0 cofinally into τ^P. Letting $\eta = i_b(\eta_0)$ and $f = i_b(f_0)$, f witnesses that τ^Q has measurable cofinality η in Q.

Now let D be a total normal measure on η from the Q sequence. Let

$$\mathcal{S}_0 = X(\mathcal{T}^\frown b, \langle D \rangle),$$

and

$$\Phi \colon \mathcal{T}^\frown b \to \mathcal{S}_0$$

be the associated weak hull embedding. The last model of $\mathcal{S}_0$ is $N = \mathrm{Ult}(Q, D)$, and i^Q_D is discontinuous at τ^Q, so we have E on the N sequence with $\mathrm{crit}(E) = \beta^N$ and $\mathrm{lh}(E) > \sup(i^Q_D``\tau^Q)$. The extenders used on $b$ have lengths $< \tau^Q$, so the extenders used on the main branch of $\mathcal{S}_0$ have lengths $< \sup(i^Q_D``\tau^Q) < \mathrm{lh}(E)$. Thus $\mathcal{S}_1$ is normal, where

$$\mathcal{S}_1 = {\mathcal{S}_0}^\frown \langle E \rangle$$

is obtained by applying E to the appropriate earlier model in $\mathcal{S}_0$. Letting W be the last model of $\mathcal{S}_1$, the branch extender P-to-W of $\mathcal{S}_1$ has all component measures on β^P. It follows by our previous arguments that there is a weak tree embedding

$$\Psi \colon \mathcal{S}_1 \to \mathcal{U}_0.$$

But then $\Psi \circ \Phi$ is a weak tree embedding of $\mathcal{T}^\frown b$ into $\mathcal{U}_0$, as desired.

□

The next lemma says that when dealing with top block stable (P, Σ), we may as well assume $\tau^P = o(P)$ and $k(P) = 0$.

Proposition 2.31. *Let (P, Σ) be top block stable, $\tau = \tau^P$, $\tau_\infty = \tau_\infty(P, \Sigma)$; then*

(1) $\Sigma^{rl} = \Sigma^{rl}_{P|\langle\tau,0\rangle}$*, and*

(2) $M_\infty(P,\Sigma)|\langle\tau_\infty,0\rangle = M_\infty(P|\langle\tau,0\rangle,\Sigma_{P|\langle\tau,0\rangle})$*.*

Proof. Let $\Psi = \Sigma_{P|\langle\tau,0\rangle}$. Suppose that $\mathcal{T}$ is by Σ^{rl}, with models $M_\alpha = \mathcal{M}^{\mathcal{T}}_\alpha$. Let

$$N_\alpha = M_\alpha|\langle \hat{i}^{\mathcal{T}}_{0,\alpha}(\tau),0\rangle$$

if $[0,\alpha)_T$ has not dropped so far that $\hat{i}^{\mathcal{T}}_{0,\alpha}(\tau)$ is undefined. Otherwise, let $N_\alpha = M_\alpha$. Thus $N_0 = P|\langle\tau,0\rangle$. Let $\mathcal{S}$ use the same extenders as $\mathcal{T}$, but have models N_α. We claim that $\mathcal{S}$ is an iteration tree by Ψ. (The main thing is that it is an iteration tree at all.)

For this, suppose $\mathcal{S} \restriction \alpha+1$ is a tree by Ψ. First, we must see $E = E^{\mathcal{T}}_\alpha$ is on the N_α sequence. If $\hat{i}^{\mathcal{T}}_{0,\alpha}(\tau)$ is undefined, this is clear. But if $\hat{i}^{\mathcal{T}}_{0,\alpha}(\tau)$ is defined, then $\mathrm{lh}(E) < \hat{i}^{\mathcal{T}}_{0,\alpha}(\tau^P)$, since otherwise, $\hat{i}^{\mathcal{T}}_{0,\alpha}(\tau^P) < \mathrm{crit}(E)$, so E is applied in $\mathcal{T}$ to M_α, forcing a drop that cannot be un-dropped, and $\mathcal{T}$ is irrelevant. So again, E is on the N_α sequence.

Second, we must see that $\mathcal{M}^{\mathcal{S}}_{\alpha+1} = N_{\alpha+1}$. Let $\beta = \mathrm{pd}_T(\alpha+1)$. Since the same extenders are used in $\mathcal{S}$ and $\mathcal{T}$, $\beta = \mathrm{pd}_S(\alpha+1)$. Let $Q = (\mathcal{M}^{\mathcal{T}}_{\alpha+1})^*$ be what E is applied to in $\mathcal{T}$. If $Q \lhd N_\beta$, then then $\mathcal{M}^{\mathcal{S}}_{\alpha+1} = \mathrm{Ult}(Q,E) = M_{\alpha+1} = N_{\alpha+1}$. If $N_\beta \lhd Q$, then $\mathrm{crit}(E) < o(N_\beta) = \hat{i}^{\mathcal{T}}_{0,\beta}(\tau)$, and $\hat{i}^{\mathcal{T}}_{\beta,\alpha+1}$ is continuous at $o(N_\beta)$ by our regularity assumption in stability. Thus $\mathcal{M}^{\mathcal{S}}_{\alpha+1} = i^Q_E(N_\beta) = N_{\alpha+1}$.

This proves (1) of 2.31. Part (2) follows easily.

□

Definition 2.32. Let (P,Σ) be top block stable; then we say (P,Σ) is *minimal* iff $\tau^P = o(P)$ and $k(P) = 0$.

The next definition isolates an important class of $\mathcal{U}_0$-certified trees.

Definition 2.33. Let (P,Σ) be a mouse pair having a top block, and let $\mathcal{T}$ be a normal tree of limit length by Σ; then we say $\mathcal{T}$ is (P,Σ)-*maximal* iff for $b = \Sigma(\mathcal{T})$,

(i) b does not drop, and

(ii) if $\tau^P < o(P)$, then $i^{\mathcal{T}}_b(\tau^P) = \delta(\mathcal{T})$, and

(iii) if $\tau^P = o(P)$, then $\delta(\mathcal{T}) = o(\mathcal{M}^{\mathcal{T}}_b)$.

Otherwise, we say that $\mathcal{T}$ is (P,Σ)-*short.*

So if τ^P is a cutpoint of P, $\mathcal{T}$ is (P,Σ)-maximal, and $b = \Sigma(\mathcal{T})$, then $\mathcal{T}^\frown b$ has no proper, normal, $M_\infty(P,\Sigma)$-relevant extension.

Proposition 2.34. *Let* (P,Σ) *be top block stable and minimal, and let* $\mathcal{T}$ *be a countable, normal tree of limit length by* Σ*; then equivalent are*

(a) $\mathcal{T}$ *is* $M^0_\infty(P,\Sigma)$*-relevant,*

(b) $\mathcal{T}$ is $M_\infty(P,\Sigma)$-relevant, and short.

Proof. Let $\mathcal{T}$ be $M^0_\infty(P,\Sigma)$ relevant, as witnessed by a normal $\mathcal{S}$ extending $\mathcal{T}$ that has a last model Q, and is such that P-to-Q does not drop, and is β^Q-generated. Suppose $\mathcal{T}$ is maximal, and let $b = \Sigma(\mathcal{T})$. Then $\mathcal{T}^\frown b$ has no proper normal extensions, so $\mathcal{S} = \mathcal{T}^\frown b$. But b is not $i_b(\beta^P)$-generated, by maximality. This is a contradiction.

Conversely, let $\mathcal{T}$ be $M_\infty(P,\Sigma)$-relevant, and short. Let $b = \Sigma(\mathcal{T})$. We shall show that there is a normal $\mathcal{S}$ extending $\mathcal{T}^\frown b$ by at most one model such that $\mathcal{S}$ has a last model Q, the branch P-to-Q does not drop, and P-to-Q is β^Q-generated.

For suppose first that b does not drop. If $i^{\mathcal{T}}_b(\beta^P) \geq \delta(\mathcal{T})$, then we can take $\mathcal{S} = \mathcal{T}$. If $i^{\mathcal{T}}_b(\beta^P) < \delta(\mathcal{T})$, then since $i^{\mathcal{T}}_b(\tau^P) > \delta(\mathcal{T})$, there is an E on the sequence of $M^{\mathcal{T}}_b$ such that $\mathrm{lh}(E) > \delta(\mathcal{T})$ and $\mathrm{crit}(E) = i^{\mathcal{T}}_b(\beta^P)$. We can then take $\mathcal{S} = \mathcal{T}^\frown b^\frown\langle E\rangle$.

Now suppose that b does drop. Since $\mathcal{T}$ is $M_\infty(P,\Sigma)$-relevant, there is an E on the $\mathcal{M}^{\mathcal{T}}_b$ sequence such that $\mathrm{lh}(E) > \delta(\mathcal{T})$ and $\mathrm{crit}(E) < \delta(\mathcal{T})$. (Note $\mathrm{lh}(E) = \delta(\mathcal{T})$ is impossible.) Let κ be the least possible critical point of such an E. κ is a strong cutpoint of $\mathcal{M}^{\mathcal{T}}_b$. Let ν be least such that $\lambda(E^{\mathcal{T}}_\nu) > \kappa$.

Claim 1.

(i) For all $\eta \geq \nu$, $\mathrm{crit}(E^{\mathcal{T}}_\eta) > \kappa$.

(ii) For all $\eta \geq \nu+1$, κ is a strong cutpoint in $\mathcal{M}^{\mathcal{T}}_\eta$, and $\mathrm{crit}(E^{\mathcal{T}}_\eta) > \kappa$.

(iii) $[0,\nu)_T \cap D^{\mathcal{T}} = \emptyset$.

(iv) $P(\kappa) \cap \mathcal{M}^{\mathcal{T}}_\nu = P(\kappa) \cap \mathcal{M}^{\mathcal{T}}_b$.

Proof For (i): Let $\lambda = \lambda(E_\eta)$. Since λ is a cardinal in $\mathcal{M}^{\mathcal{T}}_b$ and $o(\kappa) > \lambda$ in $\mathcal{M}^{\mathcal{T}}_b$, $\mathcal{M}^{\mathcal{T}}_b|\lambda \models o(\kappa) = \infty$. Letting $\mu = \mathrm{crit}(E_\eta)$, we get that $\mathcal{M}^{\mathcal{T}}_\eta|\mu \models \exists\xi(o(\xi) = \infty)$. Fix such a ξ; then $\mathcal{M}^{\mathcal{T}}_{\eta+1}|\lambda \models o(\xi) = \infty$. This implies $o(\xi) \geq \kappa$ in $\mathcal{M}^{\mathcal{T}}_b$, and hence $\kappa \leq \xi$ because κ is a strong cutpoint in $\mathcal{M}^{\mathcal{T}}_b$. But $\xi < \mu$, so $\kappa < \mu$.

For (ii): Let $\lambda = \lambda(E_\nu)$. If $\xi < \kappa$ and $o(\xi) \geq \kappa$ in $\mathcal{M}^{\mathcal{T}}_\eta$, then since λ is a cardinal, then $\mathcal{M}^{\mathcal{T}}_\eta|\lambda \models o(\xi) = \infty$. But $\mathcal{M}^{\mathcal{T}}_\eta|\lambda = \mathcal{M}^{\mathcal{T}}_b|\lambda$, so κ is not a strong cutpoint in $\mathcal{M}^{\mathcal{T}}_b$, contradiction.[23]

For (iii) and (iv): Since $\mathcal{T}$ is $M_\infty(P,\Sigma)$-relevant, we can fix a normal extension $\mathcal{S}$ of $\mathcal{T}^\frown b$ such that $\mathrm{lh}(\mathcal{S}) = \eta+1$ and $[0,\eta)_S \cap D^{\mathcal{S}} = \emptyset$. We may assume that $\eta = \xi+1$ for some $\xi \geq \mathrm{lh}(\mathcal{T})$. By the proof of (ii), for all γ such that $\nu+1 \leq \gamma$, κ is a strong cutpoint in $\mathcal{M}^{\mathcal{S}}_\gamma$, and $\kappa \leq \mathrm{crit}(E^{\mathcal{S}}_\gamma)$ if $\gamma \leq \xi$. Thus $\nu \leq_S \eta$, so if $[0,\nu)_T$ does not drop. Similarly, $\mathrm{crit}(i_{\nu,\eta}) \geq \kappa$, so $P(\kappa) \cap \mathcal{M}^{\mathcal{T}}_\nu = P(\kappa) \cap \mathcal{M}^{\mathcal{T}}_\eta$, which implies (iv).

$\square$

Now let E on the $M^{\mathcal{T}}_b$ sequence be such that $\mathrm{lh}(E) > \delta(\mathcal{T})$ and $\mathrm{crit}(E) = \kappa$. We claim that $\mathcal{S} = \mathcal{T}^\frown b^\frown\langle E\rangle$ witnesses that $\mathcal{T}$ is $M^0_\infty(P,\Sigma)$-relevant. It is enough to show that the measures of its main branch extender concentrate on β^P, and for that, it is enough to see that $\kappa \leq i^{\mathcal{T}}_{0,\nu}(\beta^P)$. But if not, then $i^{\mathcal{T}}_{0,\nu}(\beta^P) < \kappa < i^{\mathcal{T}}_{0,\nu}(\tau^P)$, so $i^{\mathcal{T}}_{0,\nu}(\beta^P)$ is strong to κ in $M^{\mathcal{T}}_\nu$, and hence in $M^{\mathcal{T}}_b$. By coherence then, $i^{\mathcal{T}}_{0,\nu}(\beta^P)$ is strong to $\delta(\mathcal{T})$ in $M^{\mathcal{T}}_b$, contrary to our choice of κ. $\square$

[23] The proof does not show that κ is a strong cutpoint in $\mathcal{M}^{\mathcal{T}}_\nu$, because λ might not be a cardinal there.

Corollary 2.35. *Let (P,Σ) be top block stable and minimal; then all short, $M_\infty(P,\Sigma)$-relevant normal trees are $\mathcal{U}_0(P,\Sigma)$-certified.*

One could extend the notions to stacks of normal trees in a natural way, but it turns out then that a stack s is (P,Σ)-maximal if and only if $W(s)$ (equivalently $X(s)$) is (P,Σ)-maximal. So we shall just take that as our definition. Notice that maximality and shortness involve Σ, as well as $\mathcal{T}$ and P, and that Q-structures are irrelevant.

Definition 2.36. Let (P,Σ) be a mouse pair having a top block; then

(i) Σ^{sh} is the restriction of Σ to (P,Σ)-short trees.

(ii) $\Sigma^{\mathrm{sh,rl}}$ is the restriction of Σ to $M_\infty(P,\Sigma)$ relevant, (P,Σ)-short trees.

Notice that Σ^{sh} can be partial, that is, undefined at trees played according to it. Σ itself cannot be partial this way. This leads to the possibility that Code(Σ^{sh}) could be complete in some nicely closed nonselfdual pointclass. That could not happen with the full Σ.

Corollary 2.37. *Let (P,Σ) be a projectum stable mouse pair having a top block; then*

(a) for any stack s by $\Sigma^{sh,rl}$, $W(s)$ is a weak hull of $\mathcal{U}_0(P,\Sigma)$.

(b) $\Sigma^{sh,rl}$ is $\beta_\infty(P,\Sigma)$-Suslin, but not α-Suslin for any $\alpha<|\beta_\infty(P,\Sigma)|$.

(c) Σ^{sh} is γ-Suslin, where $\gamma=\sup(\beta_\infty(P,\Sigma)\cup\{o(H_\beta)\mid\beta<$ mouse rank of $(P,\Sigma)\}$.

Proof. (a) and (b) follow at once from 2.30, 2.31, and 2.35. The proof of 2.15 then gives (c). □

The following lemma imples that $\beta_\infty(P,\Sigma)$ is a limit of Suslin cardinals in the cases we shall be interested in later.

Lemma 2.38. *Let (P,Σ) be a projectum stable mouse pair, and γ a strong cutpoint of P. Suppose that for all $\alpha<\gamma$, there is a $\beta<\gamma$ such that β is a strong cutpoint cardinal of P, and $\Sigma^{rl}_{P|\beta}$ is not Σ_1-definable from parameters over $(HC,\in,\Sigma^{rl}_{P|\alpha})$. Then $\sup(\pi^\Sigma_{P,\infty}``\gamma)$ is a limit of Suslin cardinals.*

Proof. Let $\alpha<\beta<\gamma$ be strong cutpoint cardinals of P such that $\Sigma^{\mathrm{rl}}_{P|\beta}$ is not Σ_1-definable from parameters over $(\mathrm{HC},\in,\Sigma^{\mathrm{rl}}_{P|\alpha})$. We may assume α and β are successor cardinals, so that

$$\alpha_\infty=_{\mathrm{df}}\pi^\Sigma_{P,\infty}(\alpha)=\tau_\infty(P|\alpha,\Sigma_{P|\alpha}),$$

and

$$\beta_\infty=_{\mathrm{df}}\pi^\Sigma_{P,\infty}(\beta)=\beta_\infty(P|\alpha,\Sigma_{P|\beta}).$$

By 2.12, $|\alpha_\infty|$ and $|\beta_\infty|$ are Suslin cardinals, so it is enough to show $|\alpha_\infty|<|\beta_\infty|$. But if not, then $\Sigma^{\mathrm{rl}}_{P|\beta}$ is $|\alpha_\infty|$-Suslin. Since $\Sigma^{\mathrm{rl}}_{P|\alpha}$ is not κ-Suslin for any $\kappa<|\alpha_\infty|$, every $|\alpha_\infty|$-Suslin set is Σ_1-definable from parameters over $(\mathrm{HC},\in,\Sigma^{\mathrm{rl}}_{P|\alpha})$. (See [3], §3.) Thus $\Sigma^{\mathrm{rl}}_{P|\beta}$ is Σ_1-definable from parameters over $(\mathrm{HC},\in,\Sigma^{\mathrm{rl}}_{P|\alpha})$, contradiction. □

Definition 2.39. Let (P,Σ) be a projectum stable mouse pair having a top block; then (P,Σ) is *projectively inaccessible* iff whenever (Q,Ψ) is a non-dropping iterate of (P,Σ), and $\alpha < \beta^Q$, there is a $\xi < \beta^Q$ such that ξ is a strong cutpoint cardinal of Q, and $\Psi^{\mathrm{rl}}_{Q|\xi}$ is not Σ_1-definable from parameters over $(\mathrm{HC}, \in, \Sigma^{\mathrm{rl}}_{Q|\alpha})$.

Corollary 2.40. *Let (P,Σ) have a top block, and be projectively inaccessible; then $\beta_\infty(P,\Sigma)$ is a limit of Suslin cardinals, moreover* $\mathrm{cof}(\beta_\infty(P,\Sigma)) > \omega$.

Proof. This follows at once from 2.38. Note that β^P is measurable in P and $< \rho_{k(P)}(P)$, so β_∞ has uncountable cofinality. □

The next theorem is our main result on the adjacency of $|\beta_\infty|$ and τ_∞ in the sequence of Suslin cardinals.

Theorem 2.41. *Let (P,Σ) be a projectum stable, projectively inaccessible mouse pair having a top block, let $\beta_\infty = \beta_\infty(P,\Sigma)$, $\tau_\infty = \tau_\infty(P,\Sigma)$, and suppose that Σ^{sh} is β_∞-Suslin. Let μ be the least Suslin cardinal $> \beta_\infty$, if one exists; then either*

(a) Σ^{rl} is β_∞-Suslin, or

(b) μ exists, and Σ^{rl} is μ-Suslin.

Remark. The hypothesis that Σ^{sh} is $\beta_\infty(P,\Sigma)$-Suslin is not redundant. We have shown that $\Sigma^{\mathrm{sh,rl}}$ is β_∞-Suslin, but the full Σ^{sh} may not be β_∞-Suslin.

Proof. If P is not top block stable, then (a) holds by Lemma 2.30, so we may assume P is top block stable. By Proposition 2.31, we may assume $o(P) = \tau^P$, and $k(P) = 0$. If $|\tau_\infty| = \beta_\infty$, then (a) holds by Theorem 2.19, so we assume $\beta_\infty < |\tau_\infty|$. It follows that μ exists.

By 2.39, β_∞ is a Suslin limit of Suslin cardinals, and of uncountable cofinality. By results of Kechris (see [3], §3), there is an ω-parametrized pointclass Γ_0 that is "Σ^1_2-like", in that it is closed under $\exists^{\mathbb{R}}$, number quantifiers, recursive substitution, and has the scale property, such that for any set of reals B, B is β_∞-Suslin iff B is $\Gamma_0(x)$ for some real x. We may assume that $\mathrm{Code}((P,\Sigma^{\mathrm{sh}}))$ is Γ_0. Let $A_0 \subseteq \omega \times \mathbb{R}$ be a universal Γ_0 set, and let T_0 be the tree of a Γ_0 scale on A_0.

Let $\langle N^*, \delta^*, w, S, T, \Sigma^*\rangle$ be a coarse Γ_0-Woodin tuple, and let $\mathbb{C} = \langle (M_{\nu,k}, \Omega_{\nu,k}) \mid \langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \delta^*, 0\rangle\rangle$ be the maximal construction associated to $\langle N^*, \delta^*, w, S, T, \Sigma^*\rangle$ that has the same mouse pair type as (P,Σ).

For $\eta \leq \delta^*$, let

$$N^*|\eta = (V^{N^*}_\eta, w \cap V^{N^*}_\eta).$$

Let δ be the least Γ_0-Woodin of N^*, that is, δ is least such that

$$L[T_0, N^*|\delta] \models \delta \text{ is Woodin.}$$

The Woodinness automatically witnessed by extenders in $\mathcal{F}^{N^*|\delta} = \mathcal{F} \cap V_\delta^{N^*}$. $\delta < \delta^*$ because a scale on $\neg A_0$ is coded into A_1. Let $\mathbb{C}$ be the maximal construction of N^* of the same mouse-type as P. The levels of $\mathbb{C}$ are mouse pairs $(M_{\nu,k}, \Omega_{\nu,k})$, and for $\langle \nu, k\rangle \leq_{\text{lex}} \langle \delta, 0\rangle$, these are just the levels of the maximal construction of $L[T_0, N^*|\delta]$. By the Comparison Theorem 1.3 either

(i) (P, Σ) iterates to $(M_{\nu,k}, \Omega_{\nu,k})$ for some $\langle \nu, k\rangle <_{\text{lex}} \langle \delta, 0\rangle$, or

(ii) (P, Σ) iterates past $(M_{\delta,0}, \Omega_{\delta,0})$ (perhaps not strictly).

Note here that Σ^* is defined on all countable trees in V, hence all the $\Omega_{\nu,k}$ are as well. N^* is sufficiently correct that whichever of (i) and (ii) holds in N^*, holds also in V of the extended strategies.

Suppose first that (i) holds, and let $(M, \Omega) = (M_{\nu,k}, \Omega_{\nu,k})$. Let $\eta < \delta$ be such that Ω is induced by Ψ, where Ψ is the restriction of Σ^* to trees based on $N^*|\eta$. Since δ is the least Γ_0 Woodin of (N^*, w), Ψ is guided by C_{Γ_0}, in that for $\mathcal{T}$ of limit length by Ψ,

$$\Psi(\mathcal{T}) = b \text{ iff } \exists f \in C_{\Gamma_0}(\mathcal{M}(\mathcal{T}))[\mathcal{M}(\mathcal{T}) \text{ is not } f\text{-Woodin}].$$

Thus Code(Ψ) is in $\Gamma_0(N^*|\eta)$, and hence Code(Ω) is in $\Gamma_0(N^*|\eta)$. But Σ is pullback consistent, so $\Sigma = \Sigma^i_M = \Omega^i$, where $i\colon P \to M$ is the iteration map. So Code(Σ) is $\Gamma_0(N^*|\eta, i)$, and therefore β_∞-Suslin.

Suppose next that (ii) holds, and let $(M, \Omega) = (M_{\delta,0}, \Omega_{\delta,0})$. Let $\mathcal{T}$ be the normal tree on P of limit length by which (P, Σ) iterates past (M, Ω), and let $b = \Sigma(\mathcal{T})$. Since $\mathcal{T}$ is normal, all its proper initial segments are short. $\mathcal{T}$ itself cannot be short, for otherwise $\mathcal{T}^\frown b \in L[T_0, N^*|\delta]$ because Σ^{sh} is coded into A_0, so $\Sigma^{\text{sh}} \cap L[T_0, N^*|\delta] \in L[T_0, N^*|\delta]$. But $\mathcal{T}^\frown b$ kills the Woodinness of δ. (Let $i_{\eta,b}(\gamma) = \delta$; then $i_{\eta,b}$ is continuous at γ, witnessing that δ is singular.)

So $\mathcal{T}$ is (P, Σ)-maximal. Looking from the point of view of the full N^*, we see then that (P, Σ) iterates to (M, Ω) via $\mathcal{T}^\frown b$. N^* is sufficiently correct that this is true in V. We have that $\Sigma = \Omega^{i_b}$, so Code$(\Sigma) = f^{-1}(\text{Code}(\Omega))$, where f is a $\boldsymbol{\Delta^1_2}$ function. So it will suffice to show that Ω is μ-Suslin. But Ω is obtained from $\Sigma^*_{N^*|\delta}$ via a conversion system ([19]), so Code$(\Omega) = g^{-1}(\text{Code}(\Sigma^*_{N^*|\delta}))$ for some $\boldsymbol{\Delta^1_2}$ function g. Thus it suffices to show that $\Sigma^*_{N^*|\delta}$ is μ-Suslin. For that, it is enough to consider the restriction of Σ^* to normal trees based in $N^*|\delta$.

Let S_μ be the collection of μ-Suslin sets. Second Periodicity, Martin's characterization of the next Suslin cardinal via pointclass envelopes, (see [3]), and results of Woodin on self-justifying systems (see [31]) show that there is a family $\mathcal{B}$ of sets of reals such that

(1) $\mathcal{B} \subseteq S_\mu \cap \check{S}_\mu$,

(2) $A_0 \in \mathcal{B}$, $\mathcal{B}$ is closed under complements, and

(3) for all $B \in \mathcal{B}$, there is a scale on B all of whose associated prewellorders are in $\mathcal{B}$.

Such a $\mathcal{B}$ is called a self-justifying system. In the case that $\Gamma_0 = S_{\beta_\infty}$ is not closed under $\forall^{\mathbb{R}}$, the Second Periodicity Theorem yields $\mathcal{B}$ as above. In this case, μ is the prewellordering ordinal of $\forall^{\mathbb{R}}\Gamma_0$, and $\mu = \beta^+_\infty$. We can then take $\mathcal{B} \subseteq \forall^{\mathbb{R}}\Gamma_0$. $S_\mu = \exists^{\mathbb{R}}\forall^R\Gamma_0$, so (1) is not the best possible upper bound on $\mathcal{B}$. In the case that Γ_0 is closed under $\forall^{\mathbb{R}}$, we get a self-justifying system $\mathcal{B}$ as above such that $\mathcal{B}$ is contained in the boldface envelope $\boldsymbol{\Lambda}(\boldsymbol{\Gamma_0}, \beta_\infty)$ of Γ_0. (See [3] and [31].) Again, this is a better upper bound than given in (1).

We may assume that $\mathcal{B}$ has been chosen such that for all $B \in \mathcal{B}$ there is an $x \in N^*$ such that $B = (A_1)_x$. This then gives us standard $\mathrm{Col}(\omega, \delta)$ terms τ_B in N^* such that $(\tau_B)_g = B \cap N^*[g]$ for all g on $\mathrm{Col}(\omega, \delta)$. We then get that $\Sigma^*_{N^*|\delta}$ is determined by the fact that it moves these terms correctly. More precisely, if $\mathcal{T}$ is a normal tree on $N^*|\delta$ of limit length, then $\Sigma^*(\mathcal{T})$ is the unique cofinal wellfounded branch b of $\mathcal{T}$ such that

$$C_{\Gamma_0}(\mathcal{M}(\mathcal{T})) \subseteq \mathcal{M}_b^{\mathcal{T}},$$

and either

(i) there is an $f\colon \delta(\mathcal{T}) \to \delta(\mathcal{T})$ witnessing that $\delta(\mathcal{T})$ is not Woodin with respect to extenders in $\mathcal{M}(\mathcal{T})$, with $f \in C_{\Gamma_0}(\mathcal{M}(\mathcal{T}))$, or

(ii) $i_b^{\mathcal{T}}(\delta) = \delta(\mathcal{T})$, and for all $B \in \mathcal{B}$, for all g on $\mathrm{Col}(\omega, i_b(\delta))$, $i_b(\tau_B)_g = B \cap \mathcal{M}_b^{\mathcal{T}}$.

The reader should see [22] for a full explanation here. The relation $R(\mathcal{T}, b) \Leftrightarrow C_{\Gamma_0}(\mathcal{M}(\mathcal{T})) \subseteq \mathcal{M}_b^{\mathcal{T}}$ is in Γ_0-dual, hence in S_μ. Clause (i) above is a Γ_0 condition. Clause (ii) is also an S_μ condition, because $\mathcal{B} \subseteq S_\mu \cap \breve{S}_\mu$, and the quantifier $\forall g$ can be replaced by "for comeager many g". Thus the restriction of Σ^* to normal trees based on $N^*|\delta$ is μ-Suslin, as desired.

□

Corollary 2.42. *Under the hypotheses of 2.41, either $|\tau_\infty| = \beta_\infty$, or $|\tau_\infty|$ is the least Suslin cardinal strictly greater than β_∞.*

We would guess that the hypothesis that (P, Σ) is projectively inaccessible is not needed in Theorem 2.41 and its corollary. We used projective inaccessibility to conclude that S_{β_∞} has the scale property. Probably it would suffice to assume that $|\beta_\infty|$ has uncountable cofinality.

We can improve Corollary 2.42 in the case that S_{β_∞} is inductive-like.

Theorem 2.43. *Let (P, Σ) be a projectum stable mouse pair having a top block, $\beta_\infty = \beta_\infty(P, \Sigma)$, and $\tau_\infty = \tau_\infty(P, \Sigma)$. Suppose that S_{β_∞} is closed under $\forall^{\mathbb{R}}$, and that $\Sigma^{sh} \in S_{\beta_\infty}$; then τ_∞ is the least Suslin cardinal strictly greater than β_∞.*

Proof. Let Γ_0 be an ω-parametrized pointclass that is is closed under both real quantifiers, recursive substitution, has the scale property, and is such that for any set of reals B, B is β_∞-Suslin iff B is $\Gamma_0(x)$ for some real x. We may assume that $\mathrm{Code}((P, \Sigma^{\mathrm{sh}}))$ is Γ_0. Let $A_0 \subseteq \omega \times \mathbb{R}$ be a universal Γ_0 set. The prewellordering ordinal of Γ_0 is β_∞.

If (Q, Ψ) is a nondropping iterate of (P, Σ), and $(R, \Phi) = (Q, \Psi)|\eta$ for some cardinal cutpoint $\eta < \beta^Q$, then both Φ and its complement are $\boldsymbol{\Gamma_0}$. (Φ is total, and coded into Σ^{sh}.) Such pairs (Q, Φ) are Wadge cofinal in $\boldsymbol{\Delta_0}$ because the ordinals of the form $o(M_\infty(Q, \Phi))$ are cofinal in β_∞. Thus (P, Σ) is projectively inaccessible.

Σ^{rl} cannot be in Γ_0, because it is (essentially) a total function, so it would then be in Δ_0. More precisely, a normal tree $\mathcal{T}$ is *not* by Σ^{rl} iff there is some limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{T} \upharpoonright \lambda$ is by Σ^{sh}, and a b such that $b \neq [0, \lambda)_T$, and either $\Sigma^{\mathrm{sh}}(\mathcal{T}) = b$ or $\Sigma^{\mathrm{rl}}(\mathcal{T}) = b$. So if Σ^{rl} is $\mathbf{\Gamma}_0$, it is $\mathbf{\Delta}_0$, contradiction.

Thus Σ^{rl} is not β_∞-Suslin. By 2.30, P is top block stable, and by 2.41, $|\tau_\infty| = \mu$, where μ is the least Suslin cardinal $> \beta_\infty$. We want to show $\tau_\infty = \mu$. By 2.31 we may assume that $\tau^P = o(P)$ and $k(P) = 0$.

Let Γ_1, N^*, δ, $\mathbb{C}$, and the self-justifying system $\mathcal{B}$ be as in the proof of the Theorem 2.41. We showed that if (P, Σ) iterates to some $(M_{\nu,k}, \Omega_{\nu,k})$ with $\nu < \delta$, then Σ is β_∞-Suslin, so we have that (P, Σ) iterates past $(M, \Omega) = (M_{\delta,0}, \Omega_{\delta,0})$. As we showed, this implies (P, Σ) iterates to (M, Ω), via some tree $\mathcal{T}^\frown b$ such that $i_b(\tau^P) = \delta$. So $M_\infty(P, \Sigma) = M_\infty(M, \Omega)$, and it is enough to show that for all $\gamma < \delta$, $i^{\Omega}_{M,\infty}(\gamma) < \mu$.

So fix $\gamma < \delta$. Because iterations of (M, Ω) convert to iterations of $N^*|\delta$ by Σ^* in the way they do, $\pi^{\Omega}_{M,\infty}(\gamma) \leq \pi^{\Sigma^*}_{N^*|\delta,\infty}(\gamma)$ so it would be enough to show $\pi_{(N^*|\delta,\Sigma^*),\infty}(\gamma) < \mu$. However, to do things this way, we need a comparison theorem for coarse iterates of $(N^*, \Sigma^*_{N^*|\delta})$. That can be proved, but we shall take a more direct route.

We recall briefly some well-known concepts. (See for example [30].) For any countable transitive N, let

$$P(N) = C_{\Gamma_0}(N) = L[T_0, N]|o(N)^+,$$

where $o(N)^+$ is computed in $L[T_0, N]$. Say that N is *suitable* iff $P(N) \models o(N)$ is Woodin, and N is a rank initial segment of $P(N)$. For $B \in \mathcal{B}$, and N suitable, let τ^N_B be the standard term capturing B over $L[T_0, N]$. (That is, $(\tau^N_B)_g \cap L[T_0, B][g] = B \cap L[T_0, N][g]$ for any g on $\mathrm{Col}(\omega, o(N))$, and τ^N_B is its own forcing relation.) Let

$$P_B(N) = (P(N), \tau^N_B),$$

and

$$\gamma_B(N) = \sup(\{\xi < o(N) \mid \xi \text{ is definable over } P(N)\{).$$

We assume that B has been chosen so that $\gamma < \gamma_B(N^*|\delta)$, and A_0 and $\neg A_0$ are Wadge reducible to B.

If N is suitable, and $\mathcal{T}$ is a normal tree on N, we say that $\mathcal{T}$ is Γ_0-*guided* iff for all limit $\lambda < \mathrm{lh}(\mathcal{T})$, there is an $f\colon \delta(\mathcal{T}) \to \delta(\mathcal{T})$ witnessing that $\delta(\mathcal{T})$ is not Woodin with respect to extenders in $\mathcal{M}(\mathcal{T})$, with $f \in C_{\Gamma_0}(\mathcal{M}(\mathcal{T})) \cap \mathcal{M}^{\mathcal{T}}_\lambda$. Say that $\mathcal{T}$ is Γ_0 *maximal* iff $\mathcal{T}$ is Γ_0 guided, $\mathcal{T}$ has limit length, and $M(\mathcal{T})$ is suitable. In this case, if b is a cofinal branch of $\mathcal{T}$, then b *respects* B iff, regarding $\mathcal{T}$ as a tree on $P(N)$, $M^{\mathcal{T}}_b = P(\mathcal{M}(\mathcal{T}))$, $i_b(o(N)) = \delta(\mathcal{T})$, and $i_b(\tau^N_B) = \tau^{\mathcal{M}(\mathcal{T})}_B$.

Now consider the tree W on HC, whose nodes are finite sequences

$$\langle (N_0, \gamma_0, \mathcal{T}_0), ..., (N_k, \mathcal{T}_k, \gamma_k) \rangle$$

such that

(a) $N_0 = N^*|\delta$ and $\gamma_0 = \gamma$,

(b) for $m < k$,N_m and N_{m+1} are suitable, and $\mathcal{T}_m$ is a normal tree on N_m by having last model N_{m+1} such that $i^{\mathcal{T}_m}_{0,\infty}(\gamma_m) > \gamma_{m+1}$, and

(c) for $m < k$, either $\mathcal{T}_m$ is Γ_0-guided, or $\mathcal{T}_m = \mathcal{S}^\frown b$ where $\mathcal{S}$ is Γ_0-maximal, and b respects B.

The order on W is of course sequence extension. To see that W is wellfounded, note that if $\mathcal{S}$ on N is Γ_0-maximal, b respects B, and $c = \Sigma^*(\mathcal{S})$, then $\mathcal{M}^{\mathcal{S}}_b = \mathcal{M}^{\mathcal{S}}_b$ and $i^{\mathcal{S}}_b$ agrees with $i^{\mathcal{S}}_c$ on γ^N_B. Thus an infinite path through W would give rise to an infinite stack of trees by Σ^* with illfounded direct limit model. Finally, the conversion of trees on (M, Ω) to trees on $N^*|\delta$ by Σ^* shows that

$$\pi^{\Omega}_{M,\infty}(\gamma) \leq |W|.$$

(Here $|W|$ is the ordinal rank.) So it is enough to show that $|W| < \mu$.

Because Γ_0 is inductive like, its envelope is closed under real quantifiers. W is clearly projective in B. There is a prewellorder Y such that $|Y| = |W|$ and Y is projective in W, and hence in the envelope of Γ_0. Let V be a tree on $\omega \times \mu$ such that $p[V] = \neg A_0$. If $\mu \leq |Y|$, then by the Coding Lemma, V is $\boldsymbol{\Sigma}^1_1(Y)$ in the codes. This implies that every Γ_0-dual relation has a uniformization in the envelope of Γ_0. That is not true ([3][§3]). So $|W| < \mu$, as desired.

□

Which of the Suslin cardinals are associated to mouse pairs? We would guess that all of them are.

Conjecture 2.43.1. Assume $\mathsf{AD}^+ + \mathsf{NLE}$, and let κ be a Suslin cardinal; then

(1) there is a pure extender pair (P, Σ) such that $\kappa = |\tau_\infty(P, \Sigma)|$, or (P, Σ) has a top block, and $\kappa = |\beta_\infty(P, \Sigma)|$, and

(2) there is a least branch pair (P, Σ) such that $\kappa = |\tau_\infty(P, \Sigma)|$, or (P, Σ) has a top block, and $\kappa = |\beta_\infty(P, \Sigma)|$.

Part (1) is known in the case κ is a projective ordinal, i.e., some δ^1_{2n+1} or its cardinal predeccessor. We believe that part (2) also holds for projective κ, but do not have a full proof. We have stated the conjecture so that one would need to show NLE implies LEC in order to prove it, because the statement is prettier that way. The more accessible version would be: LEC implies (1), and HPC implies (2).[24]

3 Pointclass generators

Here is some terminology related to local mouse capturing.

Definition 3.1. Let $\boldsymbol{\Delta}$ be a pointclass; then

[24]Those versions have recently been proved. See [17].

(a) $\mathbf{\Delta}$ is *strongly closed* iff $\mathbf{\Delta}$ is closed under complements and real quantification, and closed downward under $\leq_w$,

(b) $\mathbf{\Delta} \models$ "A is Suslin" iff A admits a scale $\vec{\varphi}$ such that $\{(n,x,y) \mid \varphi_n(x) \leq \varphi_n(y)\} \in \mathbf{\Delta}$,

(c) $\mathbf{\Delta} \models \mathsf{HPC}$ iff whenever $\mathbf{\Delta} \models$ "A is Suslin", then $A \leq_w \mathrm{Code}(\Sigma)$ for some lbr hod pair (P,Σ) such that $\mathrm{Code}(\Sigma) \in \mathbf{\Delta}$. We define $\mathbf{\Delta} \models \mathsf{LEC}$ in parallel fashion.

If $\mathbf{\Delta}$ is strongly closed, then its Wadge ordinal, its prewellordering ordinal, and the sup of the lengths of $\mathbf{\Delta}$ wellfounded relations are all the same. (Cf. [8].) We write $o(\mathbf{\Delta})$ for this ordinal. Then $\mathbf{\Delta} \models$ "A is Suslin" iff A is κ-Suslin for some $\kappa < o(\mathbf{\Delta})$, and $\mathbf{\Delta} \models$ " all sets are Suslin" iff $o(\mathbf{\Delta})$ is a limit of Suslin cardinals.

The following is Theorem 0.8 from the introduction.

Theorem 3.2. *Let $\mathbf{\Delta}$ be strongly closed; then*

(a) if there is a pure extender pair (P,Σ) such that $\mathrm{Code}(\Sigma) \notin \mathbf{\Delta}$, then $\mathbf{\Delta} \models \mathsf{LEC}$, and

(b) if there is an lbr hod pair (P,Σ) such that $\mathrm{Code}(\Sigma) \notin \mathbf{\Delta}$, then $\mathbf{\Delta} \models \mathsf{HPC}$.

Proof. We prove (a). The proof of (b) is the same, mutatis mutandis.

Let (P,Σ) be a pure extender pair of minimal mouse rank such that $\mathrm{Code}(\Sigma) \notin \mathbf{\Delta}$. We may as well assume $\mathbf{\Delta} \models$ " all sets are Suslin, since otherwise we can just replace $\mathbf{\Delta}$ with $\{A \mid \mathbf{\Delta} \models$ "A is Suslin and co-Suslin".$\}$.

Claim 1. If $\Sigma^{\mathrm{rl}} \in \mathbf{\Delta}$, then $\mathbf{\Delta} \models \mathsf{LEC}$.

Proof. Let α be the mouse rank of (P,Σ), and for $\beta \leq \alpha$, let $H_\beta = H^e_\beta$ be the common $M_\infty(Q,\Psi)$ for all (Q,Ψ) of mouse rank β. Since $\Sigma^{\mathrm{rl}} \in \mathbf{\Delta}$, H_α has a code in $\mathbf{\Delta}$. By the minimality of (P,Σ), each H_β for $\beta < \alpha$ has a code in $\mathbf{\Delta}$. The $o(H_\beta)$ for $\beta < \alpha$ must be unbounded in $o(\mathbf{\Delta})$, since otherwise Corollary 2.15 implies that Σ is γ-Suslin for some $\gamma < o(\mathbf{\Delta})$, so $\mathrm{Code}(\Sigma) \in \mathbf{\Delta}$. But then the Wadge ranks of mouse pairs $<^* (P,\Sigma)$ must be unbounded in $\mathbf{\Delta}$. □

Claim 2. If $\Sigma^{\mathrm{rl}} \notin \mathbf{\Delta}$ and P has limit type, then $\mathbf{\Delta} \models \mathsf{LEC}$.

Proof. We have that τ^P is a limit of successor cardinal cutpoints in P, and that $\tau_\infty(P,\Sigma) \geq o(\mathbf{\Delta})$. (Otherwise $\Sigma^{\mathrm{rl}} \in \mathbf{\Delta}$ by 2.19.) It follows that for each $\gamma < o(\mathbf{\Delta})$ there is a non-dropping iterate (Q,Σ_Q) of (P,Σ) and a strict cutpoint $(R,\Sigma_R) \lhd^* (Q,\Sigma_Q)$ such that $o(M_\infty(R,\Sigma_R)) \geq \gamma$. But then the $\mathrm{Code}(\Sigma_R)$ for such R must be Wadge cofinal in $\mathbf{\Delta}$. □

Claim 3. If $\Sigma^{\mathrm{rl}} \notin \mathbf{\Delta}$ and P has a top block, then $\mathbf{\Delta} \models \mathsf{LEC}$.

Proof. $o(\mathbf{\Delta})$ is a limit of Suslin cardinals, so if $\beta_\infty(P,\Sigma) < o(\mathbf{\Delta})$, then $\tau_\infty(P,\Sigma) < o(\mathbf{\Delta})$, by 2.42. This would imply $\Sigma^{\mathrm{rl}} \in \mathbf{\Delta}$. Thus $\beta_\infty(P,\Sigma) \geq o(\mathbf{\Delta})$. On the other hand,

$$\beta_\infty(P,\Sigma) = \sup(\{\tau_\infty(Q|\beta^Q, \Sigma_{Q|\beta^Q}) \mid (Q,\Sigma_Q) \text{ is an iterate of } (P,\Sigma)\}.$$

Thus $\beta_\infty(P,\Sigma) \leq o(\mathbf{\Delta})$ by the minimality of (P,Σ), and pairs of the form $(Q|\beta^Q, \Sigma_{Q|\beta^Q})$ are Wadge cofinal in $\mathbf{\Delta}$. $\square$

Claims 1-3 complete the proof. $\square$

Definition 3.3. Let $\mathbf{\Delta}$ be a selfdual pointclass closed downward under $\leq_w$, and (P,Σ) be a mouse pair; then we say (P,Σ) is a *generator of* $\mathbf{\Delta}$ iff $\Sigma \notin \mathbf{\Delta}$, and whenever $(Q,\Psi) <^* (P,\Sigma)$, then $\Psi \in \mathbf{\Delta}$. We say (P,Σ) is a *pointclass generator* iff it is a generator of some strongly closed $\mathbf{\Delta}$.

It would of course make sense to look at generators for pointclasses that are not strongly closed, but we shall not do that.

Clearly (P,Σ) generates $\mathbf{\Delta}$ iff it generates $\{A \mid \mathbf{\Delta} \models\ A$ and $\neg A$ are Suslin $\}$. So (P,Σ) is a pointclass generator iff it is a generator for some strongly closed $\mathbf{\Delta}$ such that $\mathbf{\Delta} \models$ " all sets are Suslin". Clearly, if (P,Σ) and (Q,Ψ) are generators of $\mathbf{\Delta}$ of the same mouse type, then they are mouse-equivalent.

If $\mathbf{\Delta}$ is the collection of projective sets, then it has a unique generators (P,Σ) and (Q,Ψ) in each hierarchy. P and Q are least in their respective hierarchies such that $\mathbb{R} \cap P$ and $\mathbb{R} \cap Q$ are the lightface projective reals. In this case, Σ^{rl} and Ψ^{rl} are trivial, in fact, empty. For $\mathbf{\Delta}$ with more closure, this sort of thing does not happen.[25] For example, in the pure extender hierarchy, the generator for $P(\mathbb{R}) \cap L(\mathbb{R})$ (or equivalently, for $\mathbf{\Delta}_1^{2\,L(\mathbb{R})}$) is (P,Σ), where $P = M_\omega|\delta$, for δ the least Woodin cardinal of M_ω, and Σ is the canonical strategy for P. In the hod pair hierarchy, the generator of $P(\mathbb{R}) \cap L(\mathbb{R})$ is something similar. In both cases, $\Sigma^{\mathrm{rl}} \notin L(\mathbb{R})$.

If (P,Σ) generates $\mathbf{\Delta}$, then $o(H_\beta) < o(\mathbf{\Delta})$ for all β strictly less than the mouse rank of (P,Σ). So our results on Suslin representations, when applied to pointclass generators, yield the following. Recall that whenever $\mathbf{\Delta}$ is strongly closed and $\mathrm{cof}(o(\mathbf{\Delta}) > \omega$, then $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$, for some nonselfdual $\mathbf{\Gamma}$ closed under $\forall^{\mathbb{R}}$. (See [27].)

Theorem 3.4. *Let $\mathbf{\Delta}$ be strongly closed, $\mathbf{\Delta} \models$ " all sets are Suslin", and $\delta = o(\mathbf{\Delta})$. Suppose $\mathrm{cof}(\delta) > \omega$, and let Γ be nonselfdual, closed under $\forall^{\mathbb{R}}$, and such that $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. Let (P,Σ) be a pointclass generator for $\mathbf{\Delta}$, $\tau_\infty = \tau_\infty(P,\Sigma)$, and $\beta_\infty = \beta_\infty(P,\Sigma)$ if P has a top block; then:*

(1) (P,Σ) has mouse rank δ.

(2) If (P,Σ) has limit type, then $\tau_\infty \leq \delta$, Σ is δ-Suslin, and $\mathrm{cof}(\delta) = \mathrm{cof}(\tau_\infty) < \delta$.

(3) If P has a top block, then $\beta_\infty \leq \delta$, and Σ^{sh} is δ-Suslin; moreover

(a) if P is not stable, then Σ is δ-Suslin, and $\mathbf{\Gamma}$ is not closed under $\exists^{\mathbb{R}}$,

(b) if P is stable and Σ is δ-Suslin, then $\mathbf{\Gamma}$ is not closed under $\exists^{\mathbb{R}}$, and

(c) if $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$, then $\beta_\infty = \delta$, and τ_∞ is the least Suslin cardinal $> \delta$, and Σ is τ_∞-Suslin.

[25] One could look instead at a $<^*$-minimal (P,Σ) such that $\Sigma^{\mathrm{rl}} \notin \mathbf{\Delta}$, or equivalently, $o(\mathbf{\Delta}) \leq o(M_\infty(P,\Sigma))$. In the pure extender hierarchy, the pointclass generator for the projective sets in this alternative sense is (P,Σ), where P is the minimal ladder mouse (in the sense of [12]), and Σ is its unique iteration strategy.

Proof. For (1): if $(Q, \Psi) <^* (P, \Sigma)$, then $\mathrm{Code}(\Psi) \in \boldsymbol{\Delta}$. The mouse order below (Q, Ψ) is represented by a wellfounded relation on $\mathbb{R}$ that is Δ^1_1 in $\mathrm{Code}(\Psi)$ (namely, $N < M$ iff M and N are Ψ-iterates of Q, and N is a Ψ_M-iterate of M via a tree that drops on its main branch). So (Q, Ψ) has mouse rank $< \delta$. Thus (P, Σ) has mouse rank $\leq \delta$.

To see that (P, Σ) has rank at least δ, let $\xi < \delta$. We claim there is a $(Q, \Psi) \in \boldsymbol{\Delta}$ such that setting

$$\gamma = \begin{cases} \tau_\infty(Q, \Psi) & \text{if } Q \text{ has limit type, and} \\ \beta_\infty(Q, \Psi) & \text{otherwise,} \end{cases}$$

we have

$$\xi \leq \gamma.$$

For otherwise, by 2.15, every (Q, Ψ) in $\boldsymbol{\Delta}$ is μ-Suslin, where μ least Suslin cardinal $> \xi$. (In applying 2.15, note that $o(M_\infty(Q, \Psi)) < \gamma^+$ by Kunen-Martin.) Let

$$C = \{\eta < \gamma \mid \eta \text{ is a strong cutpoint of } Q\}.$$

It is easy to see that C has order type γ, and that C is mapped order preservingly into the order type of the mouse order by sending $\eta \in C$ to the mouse rank of all (R, Φ) such that $M_\infty(R, \Phi) = M_\infty(Q, \Psi)|\eta$. So the mouse order below (Q, Ψ) has order type at least ξ, as desired.

For (2): Suppose (P, Σ) has limit type. Then τ_∞ is a limit of successor cardinal cutpoints η in $M_\infty(P, \Sigma)$, and for each such η there is a $(Q, \Psi) <^* (P, \Sigma)$ such that $M_\infty(Q, \Psi) = M_\infty(P, \Sigma)|\eta$. It follows that $\tau_\infty \leq \delta$. It is easy to see that such pairs (Q, Ψ) are cofinal in the mouse order below (P, Σ), so $\mathrm{cof}(\tau_\infty) = \mathrm{cof}(\delta)$. Thus $\mathrm{cof}(\tau_\infty) > \omega$, so $\pi_{P,\infty}$ is discontinuous at τ^P, so there is a $\mu < \tau^P$ so that $\mu < \rho_{k(P)}(P)$ and μ is measurable in P and some $r\Sigma_{k(P)}(P)$ map of μ cofinally into τ^P. But then $\mathrm{cof}(\tau_\infty) = \mathrm{cof}(\pi_{P,\infty}(\mu))$, so τ_∞ and δ are singular. Finally, for each $\beta < \delta$, $o(H_\beta) < \delta$, and thus by 2.15, Σ is δ-Suslin.

We turn to (3). Suppose P has a top block. The proof in (2) that $\tau_\infty \leq \delta$ shows that $\beta_\infty \leq \delta$ in this case. Since $o(H_\alpha) < \delta$ for all $\alpha < \delta$, we then have, by Theorem 2.37 (c), that Σ^{sh} is δ-Suslin.

If P is not stable, then Σ is δ-Suslin by Lemma 2.30(c). In this case, $\boldsymbol{\Gamma}$ cannot be closed under $\exists^{\mathbb{R}}$, for otherwise, $\boldsymbol{\Gamma}$ is the class of all δ-Suslin sets, so $\mathrm{Code}(\Sigma) \in \boldsymbol{\Gamma}$. But Σ is a complete strategy, so s is not by Σ iff there is a t diverging from s at some limit step such that t is by Σ. So $\neg\,\mathrm{Code}(\Sigma)$ is positive-$\boldsymbol{\Sigma^1_1}$ in $\mathrm{Code}(\Sigma)$, and hence $\neg\,\mathrm{Code}(\Sigma)$ is δ-Suslin. This implies $\mathrm{Code}(\Sigma) \in \boldsymbol{\Delta}$, contradiction. This proves (3)(a).

For (3)(b), suppose P is stable. Clearly, P is minimal. The argument of the last paragraph shows that if Σ is δ-Suslin, then $\boldsymbol{\Gamma}$ is not closed under $\exists^{\mathbb{R}}$.

For (3)(c), let $\boldsymbol{\Gamma}$ be closed under $\exists^{\mathbb{R}}$, so that $\boldsymbol{\Gamma}$ is the class of δ-Suslin sets.

Claim 1. $\Sigma^{\mathrm{rl}} \notin \boldsymbol{\Gamma}$

Proof. Otherwise Σ^{rl} is δ-Suslin, so $o(H_\delta) < \delta^+$ by Kunen-Martin. But $o(H_\alpha) < \delta$ for all $\alpha < \delta$, so $\mathrm{Code}(\Sigma)$ is δ-Suslin by 2.15. But Σ is a complete strategy, so $\neg\,\mathrm{Code}(\Sigma)$ is positive-$\boldsymbol{\Sigma^1_1}$ in $\mathrm{Code}(\Sigma)$, and hence $\neg\,\mathrm{Code}(\Sigma)$ is δ-Suslin. Thus $\mathrm{Code}(\Sigma) \in \boldsymbol{\Delta}$, contradiction. □

Claim 2. $\beta_\infty = \delta$.

Proof. β_∞ is the sup of all $\tau_\infty(Q|\beta^Q, \Sigma_{Q|\beta^Q})$ such that (Q, Σ_Q) is a nondropping iterate of (P, Σ), and thus $\beta_\infty \leq \delta$. δ is a limit of Suslin cardinals, so if $\beta_\infty < \delta$ then $\tau_\infty < \delta$ by 2.42. But that would imply $\Sigma^{\mathrm{rl}} \in \mathbf{\Delta}$, contrary to claim 1. Thus $\beta_\infty = \delta$. □

The claims together with Theorem 2.43 complete the proof of (3)(c).

□

Remark. We do not know whether the converse of (3)(b) holds. If P is top block stable and $\mathbf{\Gamma}$ is not inductive-like, must Σ be δ-Suslin? A related question is: what is the pointclass generator for the sets Kleene-recursive in the real quantifier, the "Kleene pointclass"?

It is worth noting that our results show that assuming HPC, the class of Suslin cardinals is closed strictly below its sup.

Corollary 3.5. *Assume* HPC*, and let* δ *be a limit of Suslin cardinals such that there are Suslin cardinals* $> \delta$*; then* δ *is a Suslin cardinal.*

Proof. Let $\mathbf{\Delta} = \{A \mid \exists\gamma < \delta(A \text{ is } \gamma\text{-Suslin }\}$, and let (P, Σ) be a pointclass generator for $\mathbf{\Delta}$. If Σ is δ-Suslin, then Σ witnesses that δ is a Suslin cardinal. If Σ is not δ-Suslin, then P has a top block and is stable, and the proofs of claims (1) and (2) just given show that $\beta_\infty(P, \Sigma) = \delta$. But then Σ^{sh} witnesses that δ is a Suslin cardinal. □

Of course, AD^+ implies the class of Suslin cardinals is closed, period. We have developed the theory of mouse pairs assuming AD^+, so it is a tacit hypothesis in Corollary 3.5. However, we would guess that $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ is sufficient to develop the theory, perhaps without much difficulty. Further, we conjecture

Conjecture 3.5.1. $\mathsf{AD} + \mathsf{HPC}$ implies AD^+.

This may be significantly easier to prove than the well-known conjecture that AD implies AD^+.

Finally, we sketch a proof of Theorem 0.9 from the introduction. We shall rely on the fact that the theory of mouse pairs under AD^+ can be extended so that it applies to pure extender pairs (P, Σ) such that P is active, with its last extender $\dot{F}^P$ being long, and all other extenders on the P-sequence being short. This seems to be a completely routine matter of putting together [19] and [10], but as of this writing, it has not yet been done. That is why this is a sketch, rather than a full proof.

Theorem 3.6. *Assume that there is a countable* $\omega_1 + 1$*-iterable pure extender mouse with a long extender on its sequence. Let* $M \models \mathsf{AD}^+ + \mathsf{NLE}$ *be such that* $\mathbb{R} \cup OR \subseteq M$*; then* $M \models \mathsf{LEC}$.

Proof. (*Sketch.* Let (P, Σ) be a minimal pure extender pair such that $\dot{F}^P$ is long. Our hypothesis implies that there is such a pair. Because P is active, it has a top block, and because $\dot{F}^P$ is long, $\beta(P, \Sigma) < \mathrm{crit}(\dot{F}^P)$. We have that $(Q, \Psi) <^* (P, \Sigma)$ iff (Q, Ψ) is a pure extender pair with only short extenders on its sequence.

Let $M \models \mathsf{AD}^+ + \mathsf{NLE}$ with $\mathbb{R} \cup \mathrm{OR} \subseteq M$, and let

$$\mathbf{\Delta} = \{A \mid \exists (Q, \Psi) \in M (A \leq_w \mathrm{Code}(\Psi))\}.$$

Suppose toward contradiction that $M \not\models \mathsf{LEC}$, so that we have a Suslin cardinal μ such that $o(\mathbf{\Delta}) < \mu < \theta^M$. It is enough to show that there is a $(Q, \Psi) <^* (P, \Sigma)$ such that $(Q, \Psi) \notin \mathbf{\Delta}$, for then the mouse-least such (Q, Ψ) is μ-Suslin by Theorem 3.4, and hence in M, contradiction.

So suppose all $(Q, \Psi) <^* (P, \Sigma)$ are in $\mathbf{\Delta}$. Then

$$\beta_\infty(P, \Sigma) \leq \sup(\{o(M_\infty(Q, \Psi)) \mid (Q, \Psi) <^* (P, \Sigma)\}) = o(\mathbf{\Delta}).$$

But (P, Σ) is not top block stable, because it has a last extender, and so Σ is $o(\mathbf{\Delta})$-Suslin. This implies $\Sigma \in M$, contrary to $M \models \mathsf{NLE}$. □

4 Local HOD computation

We assume $\mathsf{AD}_\mathbb{R} + \mathsf{HPC} + V = L(P(\mathbb{R}))$ throughout this section. Our goal is to show that $\mathrm{HOD}|\theta$ is a least branch premouse.

Some preliminary definitions:

Definition 4.1. If (Q, Ψ) and (R, Ω) are mouse pairs, then $(Q, \Psi))$ is a *strong initial segment* of (R, Ω) iff $Q = R|\eta$ for some cutpoint η of R such that η is not the critical point or length of any (even partial) extender on the R-sequence, and $\Psi = \Omega_Q$. We write $(Q, \Psi) \trianglelefteq^* (R, \Omega)$ in this case.

Note that if Q is passive and $(Q, \Psi) \trianglelefteq^* (R, \Omega)$, then $E^R_{o(Q)} = \emptyset$.

Definition 4.2. Let (P, Σ) and (R, Ω) be mouse pairs of the same type such that $(P, \Sigma) \leq^* (R, \Omega)$. We call $\langle \mathcal{T}, \mathcal{U} \rangle$ a *minimal comparison* of (P, Σ) with (R, Ω) iff

(a) $\mathcal{T}$ is a normal tree on (P, Σ) (by Σ) with last model (Q, Ψ), and $\mathcal{U}$ is a normal tree on (R, Ω) (by Ω) with last model (S, Φ),

(b) P-to-Q does not drop, and $(Q, \Psi) \trianglelefteq^* (S, \Phi)$, and

(c) for all α, $\mathrm{crit}(E^\mathcal{U}_\alpha) \leq o(Q)$.

It is clear that if $(P, \Sigma) \leq^* (R, \Omega)$, then there is a minimal comparison of the two.[26]

Now let

$$\bar{\theta} \in \{\theta_\xi \mid \xi = 0 \text{ or } \exists \alpha (\xi = \alpha + 1)\},$$

and recalling that $P_\gamma(\mathbb{R}) = \{A \mid |A|_w < \gamma\}$, let

$$\mathbf{\Gamma} = \{A \mid P_{\bar{\theta}}(\mathbb{R}) \models \ A \text{ is Suslin }\},$$

[26] This comparison may not proceed by iterating away least extender disagreements, however.

and

$$\mathbf{\Delta} = \mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}.$$

So $\mathbf{\Gamma}$ is closed under both real quantifiers and has the scale property, $o(\mathbf{\Delta})$ is the largest Suslin cardinal $< \bar{\theta}$, and $\bar{\theta}$ is itself a Suslin cardinal. Fix an lbr hod pair (P, Σ) that is a pointclass generator for $\mathbf{\Delta}$. By Theorem 3.4, (P, Σ) has a top block, is top block stable and minimal, and

$$\beta_\infty(P, \Sigma) = o(\mathbf{\Delta}),$$

and

$$\tau_\infty(P, \Sigma) = \bar{\theta}.$$

We shall show that $M_\infty(P, \Sigma) = \mathrm{HOD}|\bar{\theta}$. The first step is to prove that (P, Σ) is mouse-fullness preserving below images of β^P.

Lemma 4.3. *Let (Q, Ψ) be an iterate of (P, Σ) via an iteration such that P-to-Q does not drop, let κ be a strong cutpoint cardinal of Q, and let $\eta = \kappa^{+,Q}$; then there is no mouse pair (S, Φ) such that $(Q|\eta, \Psi_{Q|\eta}) \trianglelefteq (S, \Phi)$ and $\rho(S) \leq \beta^Q$.*

Proof. Suppose $(Q|\eta, \Psi_{Q|\eta}) \trianglelefteq (S, \Phi)$ and $\rho(S) \leq \kappa$, and no proper initial segment of (S, Φ) has this property. Let $k = k(S)$, so that $\kappa < \rho_k(S)$. We may assume that κ is not measurable by the S-sequence, for otherwise we can replace S with $R = \mathrm{Ult}_k(S, D)$, where D is the order zero total measure of S on κ. By strategy coherence, $(Q|\eta, \Psi_{Q|\eta}) \trianglelefteq (R, \Phi_{\langle D})$, so the replacement is valid. Similarly, we may assume that S is $\kappa + 1$-sound, in that

$$S = \mathrm{Hull}^S_{k+1}(\kappa + 1 \cup p_{k+1}(S)).$$

It follows from [19] that (S, Φ) is ordinal definable from $(Q|\eta, \Psi_{Q|\eta})$.[27] Since $(Q|\eta, \Psi_{Q|\eta}) <^* (P, \Sigma)$, $(Q|\eta, \Psi_{Q|\eta}) \in \mathbf{\Delta}$, so $(S, \Phi) \in P_{\bar{\theta}}(\mathbb{R})$, so $(S, \Phi) <^* (Q, \Psi)$. This leads to a contradiction, the argument being more awkward than usual because we do not have comparison by least extender disagreement available.

Let $\mathcal{T}$ on S and $\mathcal{U}$ on Q be a minimal comparison witnessing $(S, \Phi) <^* (Q, \Psi)$, with models $S_\xi = \mathcal{M}^\mathcal{T}_\xi$ and $Q_\xi = \mathcal{M}^\mathcal{U}_\xi$. Let $\mathcal{T} \restriction \gamma + 1$ be the initial segment of $\mathcal{T}$ that is on $S|\eta = Q|\eta$; then $\mathcal{T} \restriction \gamma + 1$ and $\mathcal{U} \restriction \gamma + 1$ use the same extenders and pick the same branches because $\Psi_{Q|\eta} = \Phi_{S|\eta}$. Let

$$i = i^\mathcal{T}_{0,\gamma} \restriction (S|\eta) = i^\mathcal{U}_{0,\gamma} \restriction (Q|\eta).$$

Letting S_θ and Q_δ be the last models, we have that $\gamma \leq_T \theta$ and $\gamma \leq_U \delta$ because $i(\kappa)$ is a strong cutpoint in S_γ and Q_γ.

Suppose $S_\theta \in Q_\delta$; then $i(\eta)$ is not a cardinal in Q_δ because $S_\gamma = \mathrm{Hull}^{S_\theta}(i(\kappa) + 1 \cup r)$ for some finite r, and $i(\eta) \subseteq S_\gamma$. But $i(\eta)$ is a cardinal of Q_γ, and $\mathrm{lh}(E^\mathcal{U}_\gamma) > i(\eta)$, so $i(\eta)$ is a cardinal of Q_δ, contradiction.

So S_θ and Q_δ are the same as bare premice. If $[0, \delta)_U$ does not drop, then $S_\theta \triangleleft Q_\delta$, and since $k(Q_\delta) = 0$, $S_\theta \in Q_\delta$, contradiction. Thus $[0, \delta)_U$ drops, and $k = k(Q_\delta)$ as well. But $[0, \gamma)_U$ does not drop, and $Q_\gamma|i(\eta) \trianglelefteq Q_\delta$, so $i(\eta) \leq \rho_{k+1}(Q_\delta)$, contrary to $\rho_{k+1}(S_\theta) \leq i(\kappa)$. □

[27] See Claim 3 in the proof of Theorem 11.3.2 in [19].

Now we show (P, Σ) has a kind of mouse-fullness at images of $o(P)$.

Lemma 4.4. *Let (R, Ω) be a projectum stable lbr hod pair such that $(P, \Sigma) \leq^* (R, \Omega)$, and let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a minimal comparison of (P, Σ) with (R, Ω), with $(Q, \Psi) \trianglelefteq^* (S, \Phi)$ being the last models; then*

(a) R-to-S does not drop (in model or degree), and

(b) $o(Q) \leq \rho_{k(S)}(S)$, $o(Q)$ is $r\Sigma_{k(S)}(S)$-regular

Proof. Fix $\mathcal{T}$ and $\mathcal{U}$ with last models (Q, Ψ) and (S, Φ) such that $(Q, \Psi) \trianglelefteq^* (S, \Phi)$.

Claim 1. There is no (N, Λ) such that $(Q, \Psi) \trianglelefteq (N, \Lambda) \trianglelefteq (S, \Phi)$, and for some $\eta < o(Q)$ and $r \in N$, $\mathrm{Hull}^N_{k(N)+1}(\eta \cup r)$ is cofinal in $o(Q)$.

Proof. Assume not, and let (N, Λ) be the first such initial segment of (S, Φ). It follows that $o(Q)$ is $r\Sigma^N_{k(N)}$-regular, and a cutpoint cardinal of N. But then

$$M_\infty(Q, \Psi) \trianglelefteq M_\infty(N, \Lambda),$$

and

$$\pi^\Lambda_{N,\infty} \restriction o(Q) = \pi^\Psi_{Q,\infty}.$$

Letting η_∞ and r_∞ be the images of η and r under $\pi^\Lambda_{N,\infty}$, we have that

$$\mathrm{Hull}^{M_\infty(N,\Lambda)}_{k+1}(\eta_\infty \cup r_\infty) \text{ is cofinal in } o(M_\infty(Q, \Psi)).$$

But $M_\infty(N, \Lambda) \in \mathrm{HOD}$, and $\bar{\theta} = o(M_\infty(P, \Sigma)) = o(M_\infty(Q, \Psi))$, so $\bar{\theta}$ is singular in HOD. However, successor points in the Solovay sequence are regular in HOD.[28] □

Claim 2. For all $\alpha + 1 < \mathrm{lh}(\mathcal{U})$, $\mathrm{crit}(E^{\mathcal{U}}_\alpha) < o(Q)$.

Proof. $\mathrm{crit}(E^{\mathcal{U}}_\alpha) \leq o(Q)$ by minimality of the comparison. Suppose $\mathrm{crit}(E_\alpha)^{\mathcal{U}} = o(Q)$. Then $Q \trianglelefteq \mathcal{M}^{\mathcal{U}}_\alpha$ because $\mathcal{U}$ is normal, so β^Q is strong to $\mathrm{crit}(E^{\mathcal{U}}_\alpha)$ in $\mathcal{M}^{\mathcal{U}}_\alpha$, so by coherence, $o(\beta^Q) > o(Q)$ in $\mathcal{M}^{\mathcal{U}}_\alpha$. So in fact, $E^{\mathcal{U}}_\alpha$, which by normality is the first extender F on the $\mathcal{M}^{\mathcal{U}}_\alpha$ sequence with $\mathrm{lh}(F) \geq o(Q)$, must have critical point β^Q. □

Claim 3. The branch R-to-S in $\mathcal{U}$ does not drop.

Proof. Suppose toward contradiction that R-to-S drops.

Suppose first that there is a last extender F used on the branch R-to-S. If $\lambda(F) < o(Q)$, then because the branch dropped, $S = \mathrm{Hull}\,^S_{k(S)+1}(\lambda(F) \cup r)$ for $r = p_{k+1}(S)$. This contradicts Claim 1, so $\lambda(F) \geq o(Q)$. The proof of Claim 2 then yields that $\kappa = \mathrm{crit}(F)$ is a strong cutpoint of Q, and a cardinal of Q. (Thus $\kappa \leq \beta_Q$). This implies that $\rho(S) \leq \kappa$, which contradicts Lemma 4.3. Thus there is no last extender used in R-to-S.

[28] In fact, they are Woodin in HOD, but regualrity is much easier to prove.

But then we can find some model W far enough out on R-to-S that there is no further dropping, and $i^{\mathcal{U}}_{W,S}(\delta) = o(Q)$ for some δ. Letting $k = k(W)$, we must have that $\text{Hull}^W_{k+1}(\eta \cup r)$ is cofinal in δ, for η the sup of the generators of R-to-W, and some r. But $\eta < \delta$, so using $i^{\mathcal{U}}_{W,S}$ to move this situation up to S, we have a contradiction to Claim 1. (Note here that $i^{\mathcal{U}}_{V,S}$ is continuous at δ, because δ is $r\Sigma_{k(W)}$-regular and not measurable in W, because $o(Q)$ has those properties in S.) □

So we have (a) of the lemma, and (b) follows at once from Claim 1.

□

Corollary 4.5. *If (P, Σ) is a generator of $P_{\bar{\theta}}(\mathbb{R})$, where $\bar{\theta} = \theta_0$ or $\bar{\theta} = \theta_{\alpha+1}$ for some α, then whenever $(P, \Sigma) \leq^* (R, \Omega)$, then $M_\infty(P, \Sigma) \trianglelefteq^* M_\infty(R, \Omega)$.*

Proof. Let $(Q, \Psi) \trianglelefteq^* (S, \Phi)$ be the last pairs in a minimal comparison of (P, Σ) with (R, Ω). By (b) of the lemma, $M_\infty(Q, \Psi) \trianglelefteq^* M_\infty(S, \Phi)$. But $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$, and by part (a) of the lemma, $M_\infty(R, \Omega) = M_\infty(S, \Phi)$. □

We are assuming $\mathsf{AD}_{\mathbb{R}}$, so the successor points in the Solovay sequence are cofinal in θ.

Definition 4.6. $\mathcal{H}$ is the unique least branch premouse N such that $o(N) = \theta$, and whenever (P, Σ) is a pointclass generator for $P_{\theta_{\alpha+1}}(\mathbb{R})$ for some α, then $M_\infty(P, \Sigma) \trianglelefteq^* N$.

The following is Theorem 0.10 from the introduction.

Theorem 4.7. *($\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC} + V = L(P(\mathbb{R}))$) The universe of $\mathcal{H}$ is V_θ^{HOD}, and $HOD = L[\mathcal{H}]$.*

Proof. It is clear that $\mathcal{H} \in$ HOD. It is known that under $\mathsf{AD}_{\mathbb{R}} + V = L(P(\mathbb{R}))$, HOD $= L(V_\theta^{\text{HOD}})$ (see [23]), and that Θ is a strong limit cardinal in HOD, so it is enough to show that every bounded subset of Θ that is ordinal definable belongs to $\mathcal{H}$. So fix A bounded in Θ such that A is ordinal definable.

Lemma 4.8. *There are $\xi < \gamma < \Theta$ and a formula $\varphi(u, v)$ of the language of set theory such that $A \subseteq \xi$, and whenever $X \subseteq \mathbb{R}$ has Wadge rank at least γ, then for all α,*

$$\alpha \in A \Leftrightarrow L(X, \mathbb{R}) \models \varphi[\alpha, \xi].$$

Proof. We can define A from the stage at which it becomes ordinal definable. More precisely, let $A \subseteq \mu$, and let Y be a prewellorder of $\mathbb{R}$ of length μ. Woodin's results show that A is OD in $L(X, \mathbb{R})$ for some X of Wadge rank $< \theta(Y)$. We wellorder the OD subsets of μ: for any OD set $B \subseteq \mu$, let $f(B)$ be the lexicographically least triple $\langle \nu, \eta, \psi \rangle$ such that whenever Z has Wadge rank ν, then B is the unique t such that $L(Z, \mathbb{R}) \models \psi[\eta, t]$. f is definable over $L(X, \mathbb{R})$ from μ whenever X has Wadge rank at least $\theta(Y)$, via a formula that does not depend on X. Let ξ be an ordinal that codes $(\xi)_0 = \mu$ and $(\xi)_1 = f(A)$ in some simple way, and let

$$\varphi(u, v) = \text{“}\exists B \subseteq (v)_0 (f(B) = (v)_1 \wedge u \in B)\text{”}.$$

It is easy to see that this works. □

Fix ξ, γ, and φ as in 4.8. Let $\gamma < \theta_{\alpha+1}$, and let (P, Σ) be a pointclass generator for $P_{\theta_{\alpha+1}}(\mathbb{R})$. We can take (P, Σ) so that ξ and γ are in $\operatorname{ran}(\pi^{\Sigma}_{P,\infty})$, say

$$\pi_{P,\infty}(\langle\bar{\xi}, \bar{\gamma}\rangle) = \langle\xi, \gamma\rangle.$$

For any iterate (Q, Ψ) of (P, Σ) put

$$\alpha \in A^Q \Leftrightarrow \pi_{Q,\infty}(\alpha) \in A.$$

Lemma 4.9. *There is an iterate (R, Λ) of (P, Σ) such that whenever (Q, Ψ) is an iterate of (R, Λ), then $A^Q \in Q$.*

Lemma 4.9 suffices for the theorem. For let (R, Λ) be as in 4.9. There is no infinite chain $(Q_n, \Psi_n) \prec^{\mathcal{F}(P,\Sigma)} (Q_{n+1}, \Psi_{n+1})$ such that $(Q_0, \Psi_0) = (R, \Lambda)$ and $\pi_{Q_n,Q_{n+1}}(A^{Q_n}) \neq A^{Q_{n+1}}$ for all n. (Consider the preimage of A^{Q_ω}.) So there is a (Q, Ψ) such that whenever (S, Ω) is an iterate of (Q, Ψ), then $\pi_{Q,S}(A^Q) = A^S$. This implies $A = \pi_{Q,\infty}(A^Q)$, so $A \in \mathcal{H}$.

So let us prove 4.9. This is precisely where we use that (P, Σ) is a hod pair, and not a pure extender pair, and that it is mouse full in the sense of Lemma 4.4. Suppose we could find an iterate (R, Λ) of (P, Σ) and an (S, Φ) such that

(i) $(R, \Lambda) \trianglelefteq^* (S, \Phi)$, and

(ii) $S \models$ "$\lambda > o(R)$ and λ is a limit of Woodin cardinals".

By the generic interpretability theorem of [19], $\operatorname{Code}(\Lambda)$ is in the derived model of S at λ, and this enables S to compute the theory of $L(\operatorname{Code}(\Lambda), \mathbb{R})$, as well as the map $\pi_{R,\infty} \colon R \to M_\infty(R, \Lambda)$.[29] Thus $A^R \in S$, and hence $A^R \in R$ by Lemma 4.4. Moreover, if (Q, Ψ) is an iterate of (R, Λ), then the iteration is by Φ and carries S along, so $A^Q \in Q$ by the same proof.

The problem with this sketch is that S is a strategy mouse with Woodin cardinals, and such mice are difficult to construct. With more care, we could make do with an S that has a small finite number of Woodin cardinals above $o(R)$, but that doesn't help much at this stage. Once we have shown that $\mathcal{H} = \mathrm{HOD}$, the fact that Θ is a limit of Woodins in HOD implies the existence of (R, Λ) and (S, Φ) as in (i) and (ii).[30] But at this stage we are trying to show $\mathcal{H} = \mathrm{HOD}$, so this doesn't help.

We solve our problem by letting S be a hybrid, an extension of R that continues to insert the strategy information in Λ, but is a pure extender premouse otherwise. That is, $S = M_\omega^{\Lambda,\sharp}$, the minimal active such hybrid with ω Woodin cardinals. $M_\omega^{\Lambda,\sharp}$ can compute the theory of $L(\operatorname{Code}(\Lambda), \mathbb{R})$ by consulting its derived model, and it can be constructed using AD^+ in $L(\operatorname{Code}(\Lambda)^\sharp, \mathbb{R})$. Let us proceed to the details.

[29] Via an $\mathbb{R}$-genericity iteration of S above $o(R)$ we can arrange that the reals of the derived model are the reals of V.

[30] With S satisfying some reasonable fragment of ZFC. One could not ask for a measurable cardinal in S above λ, because we could be living in the minimal model of $\mathsf{AD}_{\mathbb{R}}$.

Proof of Lemma 4.9. Let Γ be a good pointclass closed under $\forall^{\mathbb{R}}$ with the scale property, and such that $\mathrm{Code}(\Sigma) \in \Gamma \cap \check{\Gamma}$. Let $\langle N^*, \delta, S, T, w, \Sigma^* \rangle$ be a coarse Γ-Woodin tuple, and let $\mathbb{C}$ be the associated maximal hod pair construction. By Theorem 1.3, we may fix $\nu < \delta$ such that $(M^{\mathbb{C}}_{\nu,0}, \Omega^{\mathbb{C}}_{\nu,0})$ is an iterate of (P, Σ), and (P, Σ) iterates strictly past all earlier levels of $\mathbb{C}$. Set

$$(R, \Lambda) = (M^{\mathbb{C}}_{\nu,0}, \Omega^{\mathbb{C}}_{\nu,0}).$$

Claim 1. For all $\langle \alpha, l \rangle$ such that $\langle \nu, 0 \rangle \leq_{\text{lex}} \langle \alpha, l \rangle \leq_{\text{lex}} \langle \delta, 0 \rangle$,

(a) $o(R)$ is a cutpoint of $M^{\mathbb{C}}_{\alpha,l}$, and $o(R)$ is not the critical point of an extender on the $M^{\mathbb{C}}_{\alpha,l}$ sequence,

(b) $o(R) \leq \rho(M^{\mathbb{C}}_{\alpha,l})$, and

(c) $o(R)$ is regular in $M^{\mathbb{C}}_{\alpha,l}$.

Proof. By induction on $\langle \alpha, l \rangle$. For (a), it is enough to see that if $F = \dot{F}^{M_{\alpha,0}}$ and $F \neq \emptyset$, then $o(R) < \mathrm{crit}(F)$. But if not, then letting $F^* = B^{\mathbb{C}}(F)$ be the background extender for F,

$$(R, \Lambda) = (M_{\nu,0}, \Omega_{\nu,0})^{i_{F^*}(\mathbb{C})}$$

and

$$\nu < i_{F^*}(\nu).$$

$i_{F^*}(P) = P$ and $i_{F^*}(\Sigma) \subseteq \Sigma$, so (P, Σ) iterates to (R, Λ) in $\mathrm{Ult}(N^*, F^*)$, but (P, Σ) iterates strictly past (R, Λ) in $\mathrm{Ult}(N^*, F^*)$ because $\nu < i_{F^*}(\nu)$, contradiction.

For (b), suppose that

$$\rho = \rho_{l+1}(M_{\alpha,l}) < o(R).$$

Let

$$\begin{aligned} (M, \Omega) &= (M_{\alpha,l}, \Omega_{\alpha,l}), \\ (H, \Phi) &= (M_{\alpha,l+1}, \Omega_{\alpha,l+1}), \\ \pi &= \text{ anticore map from } H \text{ to } M. \end{aligned}$$

As in the proof of 4.4, we reach a contradiction by showing that $o(M_\infty(R, \Lambda))$ is singular in HOD. The new element here is that we have not reached $(M_{\alpha,l}, \Omega_{\alpha,l})$ in an iteration. Instead, let us consider the comparison of (M, H, ρ) with M, done in some larger Γ_1-Woodin universe N^{**} where M is countable and $\mathrm{Code}(\Omega)$ has been captured, as in the proof in [19] that (M, Ω) is parameter solid.

Assume first that M is projectum stable. By [19, §10.6] there are (S, Ψ) and iteration maps

$$j \colon (M, \Omega) \to (S, \Psi)$$

and

$$i\colon (H,\Phi) \to (S,\Psi)$$

such that $\mathrm{crit}(i) > \rho$, $\mathrm{crit}(j) > \rho$, and $i = j \circ \pi$. Letting $p = p(M) = p_{l+1}(M)$, we claim that there is a $\tau < j(o(R))$ such that $\mathrm{Hull}_{l+1}(\tau \cup j(p))$ is cofinal in $j(o(R))$. For let $\mathcal{T}$ be the iteration tree on (M,H,ρ) giving rise to i, $S = \mathcal{M}^{\mathcal{T}}_{\gamma}$, and $\xi <_T \gamma$ be least such that $j(o(R)) \in \mathrm{ran}(i^{\mathcal{T}}_{\xi,\gamma})$. Let

$$i^{\mathcal{T}}_{\xi,\gamma}(\mu) = j(o(R)).$$

Either $\xi = 1$ and $\mathcal{M}^{\mathcal{T}}_{\xi} = H$, or $\xi = \tau + 1$ for some $\tau \geq 1$, and in either case there is a $\beta < \mu$ such that

$$\mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_{\beta}}_{l+1}(\beta \cup i_{1,\xi} \circ \pi^{-1}(p)) \text{ is cofinal in } \mu.$$

(If $\xi = 1$, then $\beta = \rho$, and otherwise $\beta = \lambda(E^{\mathcal{T}}_{\xi-1})$.) By induction, $j(o(R))$ is $r\Sigma_l$ regular and not measurable in S, so μ is $r\Sigma_l$ regular and not measurable in $\mathcal{M}^{\mathcal{T}}_{\xi}$, and hence $i^{\mathcal{T}}_{\xi,\gamma}$ is continuous at μ. Thus letting $\tau = i^{\mathcal{T}}_{\xi,\gamma}(\beta)$ and noting that $i_{1,\gamma} \circ \pi^{-1}(p) = j(p)$, we get

$$\tau < j(o(R))$$

and

$$\mathrm{Hull}^{S}_{l+1}(\tau \cup j(p)) \text{ is cofinal in } j(o(R)).$$

But then, since $o(R)$ is an $r\Sigma_l$ regular cutpoint in M, $(j(R), \Psi_{j(R)})$ is an iterate of (R,Λ) via an initial segment of the tree from M to S, so

$$M_\infty(R,\Lambda) = M_\infty(j(R), \Psi_{j(R)}) \trianglelefteq M_\infty(S,\Psi).$$

Since $M_\infty(S,\Psi) \in \mathrm{HOD}$, $o(M_\infty(R,\Lambda))$ is singular in HOD, a contradiction.

If M is not projectum stable, then let D be the order zero measure of M on η, where η is the $r\Sigma^M_l$ cofinality of $\rho_l(M)$. M is a pfs premouse, so $\eta < \rho_{l+1}(M)$. Let N be the strong core of M, $\pi\colon N \to M$ be the anticore map (cf. [19]). π is the identity on $\rho_l(M)$. Let

$$(M_1, \Omega_1) = (\mathrm{Ult}_l(N,D), \Omega^{\sigma}_{\langle D\rangle}),$$

where $\sigma\colon \mathrm{Ult}_l(N,D) \to \mathrm{Ult}_l(M,D)$ is the copy map. (M_1, D_1) is strongly stable[31]. Let

$$(R_1,\Lambda_1) = (\mathrm{Ult}_0((R,\Lambda),D), \Lambda_{\langle D\rangle}).$$

$R_1 \triangleleft M_1$ and $\sigma \restriction R_1 = \mathrm{id}$, so $\Lambda_1 = (\Omega_1)_{R_1}$. Moreover $\rho_{l+1}(M_1) \leq i_D(\rho_{l+1}(M) < i_D(o(R)) = o(R_1)$. Since $M_\infty(R,\Lambda) = M_\infty(R_1,\Lambda_1)$, we now have the bad situation from above, but with M_1 being projectum stable, a contradiction.

For (c), suppose that $o(R)$ is $r\Sigma^M_l$ regular and $r\Sigma^M_{l+1}$ singular, and not measurable by the M-sequence. Let $\pi\colon M \to M_\infty(M,\Omega)$; then the relevant ultrapower maps are all continuous

[31] $\rho_l(M_1) = \sup i_D``\rho_l(M)$, and therefore has $r\Sigma_l$ cofinality η is M_1. But η is not measurable by the M_1-sequence.

at the current image of $o(R)$, so π is continuous at $o(R)$, so again $o(M_\infty(R,\Lambda))$ is singular in HOD, contradiction. $\square$

Now let

$$(K,\Upsilon) = (M^{\mathbb{C}}_{\delta,0}, \Omega^{\mathbb{C}}_{\delta,0}).$$

$(R,\Lambda) \trianglelefteq^* (K,\Upsilon)$, and in K, $o(R)$ is a limit cardinal, a cutpoint, and not measurable. Working in K, we do the maximal pure extender construction $\mathbb{D}$ relative to (R,Λ). That is,

$$(M^{\mathbb{D}}_{0,0}, \Omega^{\mathbb{D}}_{0,0}) = (R,\Lambda),$$

and we continue by adding extenders F whenever $o(R) < \mathrm{crit}(F)$, F has a nice background extender F^* that is on the K-sequence, and we get a hybrid Λ-premouse by doing so. We add branch information about Λ at branch active stages just as in [19]. No branch information beyond that in Λ gets added. Because we are working in K, no level projects below $o(R)$, so

$$(R,\Lambda) \trianglelefteq (M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$$

for all $\langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \delta, 0\rangle$. The well known arguments for pure extender constructions show that $\mathbb{D}$ does not break down.

For any lbr hod pair (S,Φ), let $M^\sharp_\omega(S,\Phi)$ be the minimal active pure extender (S,Φ)-mouse N with ω Woodin cardinals such that $S \trianglelefteq N$. Let $\Psi(S,\Phi)$ be its canonical iteration strategy. Here $\Psi(S,\Phi)$ only acts on iteration trees that are above $o(S)$, and these iterations move the predicate for Λ correctly. In this general case, some level of N may project across $o(S)$; nevertheless $S \trianglelefteq N$ because all cores include $S \cup \{S\}$ by fiat.

Claim 2. There is a $\nu < \delta$ such that $M^\sharp_\omega(R,\Lambda)$ is the core of $(M^{\mathbb{D}}_{\nu,0}, \Omega^{\mathbb{D}}_{\nu,0})$.

Proof. Let $Q = M^\sharp_\omega(R,\Lambda)$ and $\Psi = \Psi(R,\Lambda)$. In the comparison of Q with $M^{\mathbb{D}}_{\delta,0}$ only Q moves, so if the claim is false, then Q iterates past $M^{\mathbb{D}}_{\delta,0}$ by Ψ. Let $\mathcal{U}$ be the tree on Q of length δ whereby it does so, and $b = \Psi(\mathcal{U})$. Since Ψ is projective in $\mathrm{Code}(\Sigma)^\sharp$ and $\mathrm{Code}(\Sigma)^\sharp \in \Gamma \cap \check{\Gamma}$, $\mathcal{U}$ and b belong to $L[N^*, T_0]$, where T_0 is the tree of a Γ scale on a universal Γ set. But then δ is singular in $L[N^*, T_0]$, contrary to its being Woodin there. $\square$

Now let

$$(Q,\Psi) = (M^{\mathbb{D}}_{\nu,0}, \Omega^{\mathbb{D}}_{\nu,0}),$$

where ν is as in Claim 2. It is enough to show that $A^R \in Q$. For this it will be enough to show that (Q,Ψ) can compute truth in $L(\mathbb{R}, \mathrm{Code}(\Lambda))$ by consulting its derived model. The difficulty here is that Q has only been directly given $\Lambda \cap Q$, and not the action of Λ on trees in its derived model.

For $n < \omega$ and S a non-dropping iterate of (Q,Ψ), let δ^S_n be the n-th Woodin cardinal of S above $o(R)$, and let $\lambda^S = \sup_{n<\omega} \delta^S_n$. Let $\mathcal{A}(n,S)$ be the set of normal trees $\mathcal{U}$ on S such that for some cutpoint cardinal $\eta < \delta^S_{n+1}$, $\mathcal{U}$ is based on $S|\eta$ and $\delta^S_n < \mathrm{crit}(E^{\mathcal{U}}_\alpha)$ for all $\alpha + 1 < \mathrm{lh}(\mathcal{U})$.

Claim 3. Let (S,Φ) be a non-dropping iterate of (Q,Ψ) and $n < \omega$; then $\Phi(\mathcal{U}) \in S$ for all $\mathcal{U} \in \mathcal{A}(n,S) \cap S|\lambda^S$, and $\Phi \restriction \mathcal{A}(n,S) \cap S|\lambda^S$ is definable over S. Moreover, the definition is uniform in S and n.

Proof. We use Q-structures to compute Φ. Given $\mathcal{U} \in \mathcal{A}(n, S)$ of limit length that is by Φ, S uses its own extender sequence (restricted to critical points $> \delta(\mathcal{U})$) and $\Lambda \cap S$ to rebuild a Λ hybrid premouse extending $\mathcal{M}(\mathcal{U})$. It must reach a first structure $Q(\mathcal{U})$ projecting across or killing the Woodinness of $\delta(\mathcal{U})$. Then $\Phi(\mathcal{U})$ is the unique b such that $Q(\mathcal{U}) \trianglelefteq \mathcal{M}_b^{\mathcal{U}}$. $\square$

Claim 4. Let (S, Φ) be a non-dropping iterate of (Q, Ψ) and $n < k < \omega$; then there is a term

$$\tau = \tau_{n,k}^S$$

in S such that whenever g is $\mathrm{Col}(\omega, \delta_k^S)$-generic over S then $\Phi(\mathcal{U}) \in S[g]$ for all $\mathcal{U} \in \mathcal{A}(n, S) \cap (S|\lambda^S)[g]$, and

$$\Phi \restriction \mathcal{A}(n, S) \cap (S|\lambda^S)[g] = \tau_g.$$

Moreover, the definition of $\tau_{n,k}^S$ is uniform in S,n, and k.

Proof. (Sketch.) We use the Boolean-valued comparison method to compute Φ on $S[g]$. We work in $S[g]$. Let σ be a term such that $\mathcal{U} = \sigma_g$, and suppose that the empty condition forces that σ is a tree of limit length in $\mathcal{A}(n, S)$ on $S|\eta$, and σ is according to the procedure we are defining now. For each $p \in \mathrm{Col}(\omega, \delta_k^S)$ let $g_p(i) = p(i)$ if $i \in \mathrm{dom}(p)$ and $g_p(i) = g(i)$ otherwise. Let $\mathcal{U}_p = \sigma_{g_p}$, and

$$N_p = \mathcal{M}(\mathcal{U}_p).$$

We assume that our procedure is correct so far, that is, that each $\mathcal{U}_p$ is by Φ. We must find branches b_p such that $\mathcal{U}_p{}^\frown b_p$ is by Φ. For this, it is enough to find the Q-structures for them, that is to find Λ-hybrids

$$N_p \trianglelefteq Q_p$$

that are iterable above $\delta(\mathcal{U}_p)$ in a way that moves the predicate for Λ correctly. Such Q_p do exist, but we must show that they belong to $S[g]$, and identify them.

We do this by simultaneously comparing $S|\eta$ with all the N_p. This results in trees $\mathcal{T}$ on $S|\eta$ and $\mathcal{T}_p$ on N_p. By Claim 3 and the fact that we have symmetrized the situation by considering all N_p simultaneously, $\mathcal{T} \in S$. This allows us to identify the correct branches for $\mathcal{T}_p$; given $\mathcal{T}_p \restriction \gamma$ for γ a limit ordinal, the comparison will have produced some $\mathcal{M}_\xi^{\mathcal{T}}$ such that

$$\mathcal{M}(\mathcal{T}_p \restriction \gamma) \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}.$$

We then extend $\mathcal{T}_p \restriction \gamma$ by choosing the unique b such that $Q(b, \mathcal{T}_p \restriction \gamma) \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}$.

For any p we must reach a stage γ such that

$$\mathcal{M}_\gamma^{\mathcal{T}_p} \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}$$

for some ξ, and $[0, \gamma)_{T_p}$ does not drop, so that we have $i = i_{0,\gamma}^{\mathcal{T}} \colon N_p \to M$, where $M \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}$. Let $Q(M)$ be the first level of $\mathcal{M}_\xi^{\mathcal{T}}$ that projects across $o(M)$ or kills its Woodinness, and let r be the standard parameter of $Q(M)$. Since $\mathcal{T}_p$ is by Φ, i can be extended to a map from Q_p to $Q(M)$, and hence

$$Q_p = \text{ transitive collapse of } \mathrm{Hull}^{Q(M)}(\mathrm{ran}(i) \cup r).$$

This lets us identify Q_p, and hence b_p, in $S[g]$. $\square$

Claim 5. Let (S, Φ) be a non-dropping iterate of (Q, Ψ), $k < \omega$, and g be $\mathrm{Col}(\omega, \delta_k^S)$-generic over S; then

(a) for all $\mathcal{U} \in S[g]$ such that $\mathrm{lh}(\mathcal{U}) < \lambda^S$ and $\mathcal{U}$ is by Λ, $\Lambda(\mathcal{U}) \in S[g]$, and

(b) there is a term $\tau = \tau_k^S$ in S such that

$$\Lambda \cap (S|\lambda^S)[g] = \tau_g.$$

Moreover, τ_k^S is independent of g, and definable over S, uniformly in S and k.

Proof. Let $\delta = \delta_0^S$. Working in S, let $\mathcal{F}$ be the collection of nice extenders E such that $\mathrm{crit}(E) > o(R)$ and $E = E_\xi^S \restriction \mathrm{lh}(E)$ for some $\xi < \delta$, and let w be the canonical wellorder of $S|\delta$. Let $\mathbb{G}$ be the maximal $(w, \mathcal{F})$ hod pair construction[32] of length δ. By Claim 3, S can compute Φ on all $\mathcal{F}$-trees in $S|\lambda^S$, so the construction $\mathbb{G}$ can be done in S. The arguments of [19] show that it does not break down.

Still in S, we can apply Theorem 1.3 to (R, Λ) and $\mathbb{G}$. Since δ is Woodin, (R, Λ) cannot iterate past $(M_{\delta,0}, \Omega_{\delta,0})^{\mathbb{G}}$. Thus we can fix $\xi < \delta$ and a normal tree $\mathcal{W}$ such that letting

$$(R_1, \Lambda_1) = (M_{\xi,0}, \Omega_{\xi,0})^{\mathbb{G}}$$

we have

$$S \models (R, \Lambda) \text{ iterates to } (R_1, \Lambda_1) \text{ via } \mathcal{W}.$$

Let $i = i^{\mathcal{W}} \colon R \to R_1$ be the iteration map.

The iteration strategy Σ^* acts on all trees in V, so the strategy $\Upsilon = \Omega(K, \mathbb{C}, \Sigma^*)$ that it induces can be extended so as to act on all trees in V. Thus the strategy $\Psi = \Omega(Q, \mathbb{D}, \Upsilon)$ for Q extends so as to act on all trees in V. Letting $j \colon (Q, \Psi) \to (S, \Phi)$ be the iteration map, this means that the $j(\mathbb{D})$-induced strategy $\Lambda_1 = \Omega(R_1, \mathbb{G}, \Phi)$ extends so as to act on all trees in V. Let

$$\Lambda_1^* = \text{ extension of } \Lambda_1 \text{ to } V \text{ determined by } \mathbb{G}, \mathbb{D}, \mathbb{C}, \text{ and } \Sigma^*.$$

By Claim 4, $S[g]$ has a term for $\Lambda_1^* \cap S[g]$.

We claim that $\Lambda_1^* = \Lambda_{\mathcal{W}, R_1}$ holds in V, not just in S. This follows from the proof of Theorem 1.3. For by [19], if $\mathcal{U}$ is of limit length and is by both strategies, and $b = \Lambda_1^*(\mathcal{U})$, then

$$W(\mathcal{W}, \mathcal{U}^\frown b) \text{ is a pseudo-hull of } i_b^*(\mathcal{W}),$$

and

$$\mathcal{M}_b^{\mathcal{U}^*} \models i_b^*(\mathcal{W}) \text{ is by } i_b^*(\Lambda).$$

So it is enough to see that Φ-iterations move the internal predicate of S for Λ correctly. But Σ^* iterations with critical point $> o(R)$ move $\Lambda \cap N^*$ correctly (because they move Σ^* correctly),

[32] Over nothing, like $\mathbb{C}$, not beginning with (R, Λ) as did $\mathbb{D}$.

so by strong hull condensation for the relevant lifting maps, Υ-iterations above $o(R)$ move the predicate of K for Λ correctly. Similarly, Ψ iterations by Q (which perforce are above $o(R)$) move the predicate of Q for Λ correctly. $\mathcal{M}_b^{\mathcal{U}^*}$ comes from such an iteration, so indeed its internal Λ-predicate is correct.

So S has a term for $\Lambda_{\mathcal{W},R^1}$. But S has i in it, and $\Lambda = (\Lambda_{\mathcal{W},R_1})^i$ by pullback consistency. So S has a term for $\Lambda \cap S[g]$, as desired. $\square$

We can now finish the proof of Lemma 4.9. Let S be an $\mathbb{R}$-genericity iterate of Q by Ψ, with $i\colon Q \to S$ the iteration map. We have g $\mathrm{Col}(\omega, < \lambda^S)$-generic over S such that $\mathbb{R} = \mathbb{R}^*_g$, where $\mathbb{R}^*_g$ are the reals of the symmetric collapse. Using $i(\langle \tau_k^Q \mid k < \omega\rangle)$ and Claim 5, we see that $\Lambda \in S(\mathbb{R}^*_g)$, and is definable in $S(\mathbb{R}^*_g)$ from parameters in S. But then $M_\infty(R, \Lambda)$ and ithe direct limit map $\pi\colon R \to M_\infty(R,\Lambda)$ are also in $S(\mathbb{R}^*_g)$ and definable there from parameters in S. But let $\pi(\langle \xi_{,0}, \gamma_0\rangle) = \langle \xi, \gamma\rangle$; then

$$\xi \in A^R \text{ iff } L(\mathrm{Code}(\Lambda, \mathbb{R}) \models \varphi[\alpha, \pi(\xi_0), \pi(\gamma_0)].$$

So $A^R \in S$ by the homogeneity of $\mathrm{Col}(\omega, < \lambda^S)$, and hence $A^R \in R$.

The proof just given shows that whenever (R_1, Λ_1) is an iterate of (R, Λ), then $A^{R_1} \in R_1$. This is what we need. $\square$

The proof of Lemma 4.9 completes the proof of Theorem 4.7. $\square$

Theorem 4.7 characterizes HOD in models of $\mathsf{AD}_\mathbb{R} + \mathsf{HPC}$. It leaves open

Question. Let $M \models \mathsf{AD}^+ + \mathsf{HPC}$, and suppose that M has a largest Suslin cardinal. What is HOD^M, as a mouse?

We believe that the arguments above show that $\mathrm{HOD}|\theta$ is an lpm, but there are difficulties in analyzing the full HOD. Even when $M = L(\mathbb{R})$, we have no natural characterization of HOD^M in the least branch hierarchy. One seems to need to shift over to an extender-biased hierarchy at θ. We do not know how to do this properly for larger M.

5 Woodins in HOD and the Solovay sequence

By pushing the proof of Theorem 4.7 further, we get our characterization of the successor points in the Solovay sequence as the cutpoint Woodins in HOD. We can also identifty the successor Woodins in HOD as the successor points in a certain refinement of the Solovay sequence.

The following is Theorem 0.12 from the introduction.

Theorem 5.1. *Assume $\mathsf{AD}_\mathbb{R} + \mathsf{HPC}$; then for any $\eta < \theta$, the following are equivalent:*

(a) $\eta = \theta_0$, or $\eta = \theta_{\alpha+1}$ for some α,

(b) η is a cutpoint Woodin cardinal of HOD.

Proof. Let us prove (a) ⇒ (b). Let η be as in (a). Of course, Woodin showed ([9]) that η is Woodin in HOD. Let (P, Σ) be a pointclass generator of $P_\eta(\mathbb{R})$. By Corollary 4.5, $M_\infty(P, \Sigma)$ is a strong initial segment of $\mathcal{H}$. Since $\eta = o(M_\infty(P, \Sigma))$, this means that η is a cutpoint of $\mathcal{H}$.

Now let us prove (b) ⇒ (a). Let η_0 be a cutpoint Woodin cardinal of HOD. Let

$$\eta_0 < \eta_1 < \ldots < \eta_5 < \bar{\theta}$$

where $\bar{\theta}$ and the η_i for $i \geq 1$ are successor points in the Solovay sequence, and thus cutpoint Woodins in HOD, as we have just shown. Let (P, Σ) be a pointclass generator for $P_{\bar{\theta}}(\mathbb{R})$,

$$\pi^{\Sigma}_{P,\infty}(\delta_i) = \eta_i,$$

and

$$N = P|\delta_0.$$

Thus $M_\infty(N, \Sigma_N) \trianglelefteq^* M_\infty(P, \Sigma) \trianglelefteq^* \mathcal{H}$, where $\mathcal{H}$ is HOD viewed as an lpm, and the initial segments are cardinal cutpoint initial segments in both cases. N is top block stable, and

$$\eta_0 = \tau_\infty(N, \Sigma_N).$$

Let

$$\Psi = \Sigma_N^{\mathrm{rl}}$$

and

$$\Psi_0 = \Sigma_N^{\mathrm{sh,rl}}.$$

We write

$$\beta_\infty = \beta_\infty(N, \Sigma_N).$$

By our results on optimal Suslin representations, Ψ is η_0-Suslin, but not α-Suslin for any $\alpha < |\tau_\infty|$, while Ψ_0 is β_∞-Suslin, but not α-Suslin for any $\alpha < |\beta_\infty|$.

Recall that for $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal not the image of $\mathbb{R}$ under a map f that is $\mathrm{OD}(A)$. Equivalently, $\theta(A) = \theta_{\alpha+1}$, where α is least such that $A \in P_{\theta_{\alpha+1}}(\mathbb{R})$. The main step in our proof is:

Claim 1. $\beta_\infty < \theta(\mathrm{Code}(\Psi_0)) \leq \eta_0$.

Proof. The direct limit system $\mathcal{F}^0(N, \Sigma_N)$ is definable from Ψ_0, and it yields a prewellorder of order type $o(M^0_\infty(N, \Sigma))$. So $\beta_\infty < \theta(\mathrm{Code}(\Psi_0))$.

Our proof that $\theta(\mathrm{Code}(\Psi_0)) \leq \eta_0$ follows the outline of an argument due to Hjorth and/or Woodin. (See [2], and the proof of Lemma 3.28 in [30].) Let $f \colon \mathbb{R} \to \eta_0$ be ordinal definable from $\mathrm{Code}(\Psi_0)$. We shall show that $\mathrm{ran}(f)$ is bounded in η_0.

By AD^+, f is OD from $\mathrm{Code}(\Psi_0)$ in $L(X, \mathbb{R})$, whenever X has Wadge rank $\geq \theta(\mathrm{Code}(\Psi_0))$. $\mathrm{Code}(\Sigma_{P|\delta_1})$ is one such X, so we have a formula $\varphi(u, v, w, z)$ and an ordinal $\nu_0 < \eta_1$ such that for all reals x and $\alpha < \eta_0$

$$f(x) = \alpha \Leftrightarrow L(\Sigma_{P|\delta_1}, \mathbb{R}) \models \varphi[x, \alpha, \nu_0, \mathrm{Code}(\Psi_0)].$$

We may assume

$$\nu_0 = \pi^{\Sigma}_{P,\infty}(\bar{\nu}).$$

Let (Q, Σ_Q) be a nondropping iterate of (P, Σ), with iteration map i. We write

$$\delta^Q_k = i(\delta_k), \text{ and } \Sigma^Q_k = \Sigma_{Q|i(\delta_k)}.$$

We want to look at what happens when we iterate P below δ_0 in such a way that some real x is extender-algebra-generic over Q at δ^Q_0. In particular, we want to see that if the generic g adds enough information, then $Q[g]$ can identify some $\bar{\alpha}$ such that $\pi^{\Sigma}_{Q,\infty}(\bar{\alpha}) = f(x)$, from x and the image of $\bar{\nu}$. (Then we use the fact that the extender algebra of Q is δ^Q_0-c.c. to find a bound $\xi < \delta^Q_0$ on all the possible $\bar{\alpha}$, and we get that $\pi^{\Sigma}_{Q,\infty}(\xi)$ is a bound on $\mathrm{ran}(f)$.)

The generic interpretability theorem of [19] says that each iterate Q of P has terms $\dot{\Sigma}_i$ such that for all h set generic over Q, $(\dot{\Sigma}_i)_h = \Sigma^Q_i \cap Q[h]$. Going one step further, we have

Subclaim A. There is a term $\tau \in P$ such that whenever $i\colon (P, \Sigma) \to (Q, \Sigma_Q)$ is an iteration map, and g is is $\mathrm{Col}(\omega, \delta^Q_2)$-generic over Q, and $\mathbb{R}^*_g = \mathbb{R} \cap Q[g]$, then

$$\begin{aligned} i(\tau)_g &= \mathrm{Code}(\Sigma^Q_1)^\sharp \cap Q[g] \\ &= (\mathrm{Code}(\Sigma^Q_1) \cap Q[g])^\sharp. \end{aligned}$$

Proof. The proofs of Claims 3-5 in the proof of 4.7 show that $\mathrm{Code}(\Sigma_1)^\sharp$ is Wadge reducible to $\mathrm{Code}(\Sigma_2)$ via a recursive function.[33] By Lemma 7.3 of [19], there is a term $\sigma \in P$ such that $\sigma_h = \mathrm{Code}(\Sigma_2) \cap P[h]$ for all h set generic over P. This gives us τ such that $\tau_h = \mathrm{Code}(\Sigma_1)^\sharp \cap P[h]$ for all h set generic over P.

We get the second equality from the fact that τ^g satisfies the witness condition for sharps of sets of reals. That is because the witness condition for interpretations of τ is a Π^1_2 fact about the interpretation, and we have Woodin cardinals δ_3 and δ_4 in P, and in $P[g]$, we have a UB code for $\mathrm{Code}(\Sigma_1)^\sharp$.[34]

The same proof shows that if $i\colon (P, \Sigma) \to (Q, \Sigma_Q)$ is an iteration map, then $i(\tau)$ captures $\mathrm{Code}(\Sigma^Q_1)^\sharp$. □

Note that if (Q, Σ_Q) is as in the subclaim, then $L(\Sigma^Q_1, \mathbb{R}) = L(\Sigma^P_1, \mathbb{R})$, so $i(\tau)$ can be used in the evaluation of $f(x)$. The problem lies with the parameter $\mathrm{Code}(\Psi_0)$ that occurs in the definition of $f(x)$. We cannot replace Ψ_0 with $(\Sigma^Q_0)^{\mathrm{sh}}$ when we move from P to Q, for then our formula φ will define a different function. We solve this problem by showing that for (Q, Σ_Q) arbitrarily far out in $\mathcal{F}(P, \Sigma)$, we can add a UB code for $\mathrm{Code}(\Psi_0)$ to Q via the δ^Q_0-generator

[33] We reconstruct an iterate Q of $M_\omega^{(P|\delta_1, \Sigma_1)}$, together with the iteration strategy Ψ for Q, inside $P|\delta_2$. Ψ acts on all trees in V, and Q has a term for Σ_1 on its derived model that is moved correctly by Ψ. So from $\mathrm{Code}(\Psi)$ we can recover $\mathrm{Code}(\Sigma_1)^\sharp$. We can recover Ψ from Σ_2.

[34] Note $\pi^{-1}(\sigma)_k = \mathrm{Code}(\Sigma_2) \cap \pi^{-1}(P|\eta)[g][k]$ for $\pi\colon N \to P|\eta$ with $\mathrm{crit}(\pi) > \delta_2$, and genericity iterations of N inside $P[h]$ also move $\pi^{-1}(\sigma)$ correctly.

version of the extender algebra at δ_0^Q. It is crucial here that we are only trying to add a UB code for the short tree component of $\Sigma_{P|\delta_0}$. The full $\Sigma_{P|\delta_0}$ lets us compute the iteration map $i\colon P|\delta_0 \to Q|i(\delta_0)$, so it will in general collapse δ_0^Q.

Given our UB code for Ψ_0 in $Q[g]$, where g has not collapsed δ_0^Q, we then consider homogeneous collapse extensions $Q[g][h]$ in our definition of $f(x)$ over $Q[g]$. But it is important that the parameters in the definition were added by δ_0^Q-c.c. forcing. The further collapse extensions are just a tool whereby $Q[g]$ computes what is true of these parameters in $L(\Sigma_1^Q, \mathbb{R})$.

Remark. The argument below can be simplified if $\theta(\Psi_0) = \theta(\Sigma_{P|\alpha})$ for some $\alpha < \delta_0$. In that case, we can make $\mathrm{Code}(\Sigma_{P|\alpha})$ our parameter in the definition of f, and add a real coding it relative to $\mathrm{Code}(\Sigma_1^Q)$ via the ω-generator extender algebra at δ_0^Q. This simpler case is exactly the case when $L(P_{\theta(\Psi_0)}(\mathbb{R}))$ does not satisfy LSA. See the next section.

The next two subclaims execute our plan. For any premouse Q and any $\delta < o(Q)$, let $\mathbb{B}_\delta^Q$ be the δ-generator extender algebra of Q at δ. [35]

Subclaim B. There is a $\mathbb{B}_{\delta_0}^P$ term ρ such that whenever $(S, \Sigma_S) \in \mathcal{F}(P, \Sigma)$, and $x \in \mathbb{R}$; then there is a $(Q, \Sigma_Q) \in \mathcal{F}(S, \Sigma_S)$ such that letting $\mathcal{U}$ be the unique normal tree on P by Σ that has last model Q, $i = i^{\mathcal{U}}$, and $\delta = i(\delta_0)$, we have

(1) $i^{\mathcal{U}}_{0,\delta}(\delta_0) = \delta$, and

(2) $\langle P, \mathcal{U} \restriction \delta, x\rangle = i(\rho)_g$, for some $\mathbb{B}_\delta^Q$-generic g.

Proof. Let (S, Σ_S) and x be given. We show how to iterate it to a (Q, Σ_Q) such that for the corresponding $\mathcal{U}$, the $A \subset \delta$ naturally coding $\langle P, \mathcal{U}, x\rangle$ is $\mathbb{B}_\delta^Q$ generic. The desired term ρ is then just the decoding method.

Let Γ be a good pointclass closed under $\forall^{\mathbb{R}}$ with the scale property, and such that $\mathrm{Code}(\Sigma) \in \Gamma \cap \breve{\Gamma}$. Let $\langle N^*, \delta^*, S, T, w, \Sigma^*\rangle$ be a coarse Γ-Woodin tuple such that $x, S, P \in N^*$, and let $\mathbb{C}$ be the associated maximal hod pair construction. By Theorem 1.3, we may fix $\nu < \delta^*$ such that $(M_{\nu,0}^{\mathbb{C}}, \Omega_{\nu,0}^{\mathbb{C}})$ is an iterate of (P, Σ), and (P, Σ) iterates strictly past all earlier levels of $\mathbb{C}$. Set

$$(Q, \Sigma_Q) = (M_{\nu,0}^{\mathbb{C}}, \Omega_{\nu,0}^{\mathbb{C}}),$$

and let $\mathcal{U}$ be the normal tree by Σ from P to Q.

By 1.3, there is also a level of $\mathbb{C}$ that is a normal iterate of (S, Σ_S), and such that (S, Σ_S) iterates strictly past all earlier levels. By Dodd-Jensen, this level must be $(M_{\nu,0}, \Omega_{\nu,0})$ as well.

[35] For the reader's convenience: letting $\mathcal{L}$ be the propositional language with sentence symbols $\dot{A}_\alpha$ for $\alpha < \delta$, $\mathbb{B}_\delta^Q$ is the Lindenbaum algebra of an $\mathcal{L}_{\delta,0}$ theory $T(Q, \delta)$. The axioms of T are induced by E on the $Q||\delta$ sequence. If $\kappa = \mathrm{crit}(E)$, then

$$\bigvee_{\alpha<\kappa} \varphi_\alpha \leftrightarrow i_E(\bigvee_{\alpha_\kappa} \varphi_\alpha) \restriction \nu$$

is an axiom of $T(Q, \delta)$, whenever $i_E(\bigvee_{\alpha_\kappa} \varphi_\alpha) \restriction \nu \in Q|\eta$, for some cardinal η of Q such that $\eta < i_E(\kappa)$. If δ is Woodin in Q, then $\mathbb{B}_\delta^Q$ is δ-c.c. in Q. Moreover, if $A \subset \delta$ and $A \models T(Q, \delta)$ (where $A \models \dot{A}_\alpha$ iff $\alpha \in A$), then $G_A = \{[\varphi] \mid A \models \varphi\}$ is $\mathbb{B}_\delta^Q$ generic over Q.

So $(S,\Sigma_S) \prec (Q,\Sigma_Q)$ in $\mathcal{F}(P,\Sigma)$. Letting $\delta = \delta_0^Q = i^{\mathcal{U}}_{P,Q}(\delta_0)$, we have that δ is Woodin in Q, and hence a limit of measurable cardinals in N^*. But P is countable in N^*. This implies $\mathcal{U} \restriction \delta$ is a tree on $P|\delta_0$, and $i^{\mathcal{U}}_{0,\delta}(\delta_0) = \delta$.

We are left to verify part (2). Let $A(P,\mathcal{U},x)$ be a subset of δ that codes $\langle P,\mathcal{U} \restriction \delta, x\rangle$ in some fixed, natural way. We must see that $A(P,\mathcal{U},x) \models T(Q,\delta)$[36]. In order to suppress the coding a little, let us write $\langle P,\mathcal{U},x\rangle \models \varphi$ instead of $A(P,\mathcal{U},x) \models \varphi$ below. Let

$$\bigvee_{\alpha<\kappa} \varphi_\alpha \leftrightarrow i_E(\bigvee_{\alpha_\kappa} \varphi_\alpha) \restriction \gamma$$

be an axiom induced by E. This implies that $i_E(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma \in Q|\mu$, where $\mu < \lambda(E)$ and μ is a cardinal of Q. Our natural coding is such that $A(P,\mathcal{U},x) \cap \mu$ is determined by P, x, and $\mathcal{U} \restriction \mu$, and thus only $\langle P,\mathcal{U} \restriction \mu, x\rangle$ is relevant to whether $\langle P,\mathcal{U},x\rangle \models i_E(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma$.

E occurs on the $M^{\mathbb{C}}_{\nu,k}$ sequence. Let $\sigma\colon M_{\nu,k}|\operatorname{lh}(E) \to M_{\eta,0}$ be the resurrection map of $\mathbb{C}$, and E^* the background extender for $\sigma(E) = \dot{F}^{M_{\eta,0}}$ given by $\mathbb{C}$. Note $\sigma \restriction (M_{\nu,k}|\mu) = $ identity. It follows that

$$\begin{aligned} i_E(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma &= i_{\sigma(E)}(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma \\ &= i_{E^*}(\bigvee_{\alpha_\kappa} \varphi_\alpha) \restriction \gamma. \end{aligned}$$

Now assume that

$$\langle P,\mathcal{U},x\rangle \models i_E(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma,$$

where again $\langle P,\mathcal{U},x\rangle$ should be replaced by its code $A(P,\mathcal{U},x)$ as a subset of δ. We must show that $\langle P,\mathcal{U},x\rangle \models \bigvee_{\alpha<\kappa} \varphi_\alpha$. We have that

$$\langle P,\mathcal{U},x\rangle \models i_{E^*}(\bigvee_{\alpha<\kappa} \varphi_\alpha) \restriction \gamma.$$

Moreover,

$$i_{E^*}(\mathcal{U}) \restriction \mu = \mathcal{U} \restriction \mu.$$

This is because $i_{E^*}(\Sigma) \subseteq \Sigma$, since N^* has captured Σ, and $i_{E^*}(M^{\mathbb{C}}_{\nu,0})|\mu = M^{\mathbb{C}}_{\nu,0}|\mu$ by coherence. So $A(P,\mathcal{U},x) \cap \mu = i_{E^*}(A(P,\mathcal{U},x)) \cap \mu$, so

$$i_{E^*}(\langle P,\mathcal{U},x\rangle) \models i_{E^*}(\bigvee_{\alpha<\kappa} \varphi_\alpha),$$

and thus

$$\langle P,\mathcal{U},x\rangle \models \bigvee_{\alpha<\kappa} \varphi_\alpha,$$

[36] $T(Q,\delta)$ is defined in a previous footnote.

as desired. This proves Subclaim B. $\square$

Now we show that if Q and g are as in Subclaim B, then $Q[g]$ has a UB code for $\text{Code}(\Psi_0)$. Fix terms $\tau, \rho \in P$ as above.

Subclaim C. There is a term $\sigma \in P$ such that whenever $Q, \mathcal{U}, x, i, \delta$, and g are such that

(a) $(Q, \Sigma_Q) \in \mathcal{F}(P, \Sigma)$, $x \in \mathbb{R}$, $\mathcal{U}$ is the unique normal tree on P by Σ with last model Q, and $i = i^{\mathcal{U}}$,

(b) $\delta = i(\delta_0) = i^{\mathcal{U}}_{0,\delta}(\delta_0)$, and

(c) $\langle P, \mathcal{U} \restriction \delta, x \rangle = i(\rho)_g$,

then $i(\sigma)_g$ is a $\text{Col}(\omega, \delta_2^Q)$ term such that for any $Q[g]$-generic h on $\text{Col}(\omega, \delta_2^Q)$,

$$\text{Code}(\Psi_0) \cap Q[g][h] = (i(\sigma)_g)_h.$$

Proof. We show how to define $\Psi_0 \cap Q[g][h]$ inside $Q[g][h]$ using parameters from Q in $\text{ran}(i)$, together with $\langle P, \mathcal{U} \restriction \delta, x \rangle$. The procedure is uniform, so we get the fixed term $\sigma \in P$.

Let $S = Q|\delta$. If k is $Q[g][h]$-generic over $\text{Col}(\omega, \delta_3^Q)$, then inside $Q[g][h][k]$ we can form $M_\infty(S, \Sigma_S)$ and the normal tree $\mathcal{U}(S, \Sigma_S)$ from S to $M_\infty(S, \Sigma_S)$. Here the trees entering into the direct limit are those that are countable in $Q[g][h][k]$. Σ_S has a UB code in Q, and the forcing is homogeneous, so $M_\infty(S, \Sigma_S), \mathcal{U}(S, \Sigma_S) \in Q$. Let us write

$$\mathcal{T} = \mathcal{U}(S, \Sigma_S)^{Q[g][h][k]}$$

for this tree. From $\mathcal{T}$ and $\mathcal{U} \restriction \delta$, we can construct

$$\mathcal{X} = X(\mathcal{U} \restriction \delta, \mathcal{T}).$$

Here we are thinking of $\mathcal{U} \restriction \delta$ and $\mathcal{X}$ as trees on $N = P|\delta_0$. Some care is needed, because $\mathcal{U} \restriction \delta$ is a tree of limit length, and its good branch $b = [0, \delta]_U$ is not available in $Q[g]$. However, $\delta(\mathcal{U} \restriction \delta) = \delta$ is a cutpoint in $\mathcal{M}^{\mathcal{U}}_b$, and $\mathcal{T}$ is a tree on the common part model $M(\mathcal{U} \restriction \delta)$, so $X((\mathcal{U} \restriction \delta)^\frown b), \mathcal{T})$ has the form $\mathcal{X}^\frown c$, where only c, and not $\mathcal{X}$, depends on b. So we may write $X(\mathcal{U} \restriction \delta, \mathcal{T}) = \mathcal{X}$. We have

$$\mathcal{X} \in Q[g],$$

although b and c are not in $Q[g]$, since they collapse δ.

All weak hulls of $\mathcal{X}$ are by Ψ, so it is enough to show that whenever $\mathcal{W} \in Q[g][h]$ is a short, normal tree of countable limit length by Ψ_0 (hence relevant), then in $Q[g][h]$ there is a proper extension of $\mathcal{W}$ that is a weak hull of $\mathcal{X}$. By a simple absoluteness argument, it is enough to show that

$$Q[g][h] \models \text{Col}(\omega, \delta_3^Q) \Vdash \exists \mathcal{V}(\mathcal{V} \text{ properly extends } \mathcal{W} \text{ and } \mathcal{V} \text{ is a weak hull of } \mathcal{X}).$$

For this, let $\mathcal{V}$ be a proper normal extension of $\mathcal{W}^\frown c$ with last model M such that the branch $P|\delta_0$-to-M does not drop, and is generated by the image of $\beta(P|\delta_0)$. Let us iterate Q above δ_2^Q to make $\mathcal{V}$ generic at the image of δ_3^Q, at the same time making $[0,\delta]_U$ generic. This gives us

$$j\colon Q[g][h] \to R[g][h]$$

with $\mathrm{crit}(j) > \delta_2^Q$, and k on $\mathrm{Col}(\omega, \delta_3^R)$ such that

$$\mathcal{V}, [0,\delta]_U \in R[g][h][k].$$

Since j is elementary, it is enough to see that in $R[g][h][k]$ there is a weak hull embedding of $\mathcal{V}$ into $j(\mathcal{X})$. In $R[g][h][k]$ we have $\pi = i^{\mathcal{U}}_{0,\delta} \restriction P|\delta_0$, and we have Σ_S on all countable trees, so we can compute $\Sigma_{P|\delta_0} = \Sigma_S^\pi$ on all countable trees. Thus we can compute $\Sigma_M = (\Sigma_{P|\delta_0})_{\mathcal{V},M}$, and we have the normal tree

$$\mathcal{S} = \mathcal{U}(M, \Sigma_M)^{R[g][h][k]}$$

from M to

$$M_\infty(M, \Sigma_M)^{R[g][h][k]} = j(M_\infty(S, \Sigma_S)^{Q[g][h]}).$$

From this we obtain

$$\begin{aligned} X(\mathcal{V}, \mathcal{S}) &= \mathcal{U}(P|\delta_0, \Sigma_{P|\delta_0})^{R[g][h][k]} \\ &= X(\mathcal{U} \restriction (\delta+1), \mathcal{U}(Q|\delta, \Sigma_{Q|\delta})^{R[g][h][k]}). \end{aligned}$$

Thus there is a weak hull embedding Φ of $\mathcal{V}$ into $X(\mathcal{U} \restriction (\delta+1), \mathcal{U}(Q|\delta, \Sigma_{Q|\delta})^{R[g][h][k]})$. But this tree is just $j(\mathcal{X})^\frown d$, for some d. Since the generators of $P|\delta_0$-to-M are contained in the image of $\beta(P|\delta_0)$, Φ is actually a weak hull embedding into $j(\mathcal{X})$, as desired. $\square$

Let $\sigma \in P$ be a term witnessing Subclaim C.

Subclaim D. There is a term $\dot{z} \in P$ and a Σ^1_2 formula γ such that whenever $Q, \mathcal{U}, x, i, \delta$, and g are as in Subclaim C, and h is $\mathrm{Col}(\omega, \delta_2^Q)$-generic over $Q[g]$, then $\dot{z}_{g,h} = z$ is a real such that

$$\mathrm{Code}(\Psi_0) \cap Q[g][h] = \{y \mid Q[g][h] \models \gamma[y, \mathrm{Code}(\Sigma_1^Q), z]\}.$$

Proof. It is enough to see that $Q[g] \models \gamma_0[i(\sigma)_g, i(\tau)]$, where $\gamma_0(u,v)$ is the formula saying that for collapse generics h, $(i(\sigma)g)_h$ is Σ^1_2 in $(\dot{\Sigma}_1^Q)_{g,h}$ and some real. (Then we can just take $\dot{z}$ to be a name such that $i(\dot{z})$ interprets as such a real whenever one exists.) But let

$$j\colon Q[g] \to R[g]$$

come from a genericity iteration with all critical points $> \delta$ such that there is a collapse δ_2^R generic h with

$$[0,\delta]_U \in R[g][h].$$

In $R[g][h]$ we have $i^{\mathcal{U}}_{0,\delta}\colon P|\delta_0 \to Q|\delta = R|\delta$, and we have $\Sigma_{R|\delta}$, so we have the full $\Psi = \Sigma^{i_{0,\delta}}_{R|\delta}$. Moreover, $\mathrm{Code}(\Psi)$ is Σ^1_2 in $\mathrm{Code}(\Sigma_1)$ and a real coding $i_{0,\delta}$. Thus

$$R[g][h] \models j(i(\sigma)_g)_h \in L(\Sigma^R_1, \mathbb{R}).$$

So $R[g] \models \gamma[j(i(\sigma)_g), j(i(\tau))]$, so $Q[g] \models \gamma_0[i(\sigma)_g, i(\tau)]$. □

Now let (Q, Σ_Q) be any nondropping iterate of (P, Σ), and $i\colon P \to Q$ the iteration map. We define an ordinal $\xi < \delta = \delta^Q_0$ by considering generics g on $\mathbb{B}^Q_\delta$. Let such a g be given. We define an ordinal $\xi(g) < \delta$ in $Q[g]$ from g as follows:

(i) if $i(\rho)_g$ is not a triple whose third coordinate is a real y, let $\xi(g) = 0$. Otherwise, let y be that real, and go on.

(ii) Let h be $\mathrm{Col}(\omega, \delta^Q_2)$ generic, and $z = \dot{z}_{g,h}$. If it is not the case that z is a real, and in $Q[g][h]$, $(i(\sigma_g)_h)$ is defined by γ from z and $\mathrm{Code}(\Sigma^Q_1)$, then set $\xi(g) = 0$.

(iii) Otherwise, working in $Q[g][h]$, consider $L(\Sigma^Q_1, \mathbb{R})$. Let $\pi\colon Q|\delta \to M_\infty(Q|\delta, \Sigma^Q_0)$ be the direct limit map of the model $L(\Sigma^Q_1, \mathbb{R})$.

(iv) For each x recursive in y, using z and $i(\tau)_{g,h} = (\Sigma^Q_1)^\sharp$ as oracles, find the least $\mu < \delta$ such that $L(\Sigma^Q_1, \mathbb{R}) \models \varphi[\pi(\mu), \pi(\bar{\nu}), x, Z]$, where $Z = \{u \mid \gamma(u, z, \mathrm{Code}(\Sigma^Q_1))\}$, if such a μ exists. Let $\bar{\alpha}(x) = \mu$ if μ exists, and $\bar{\alpha}(x) = 0$ otherwise.

(v) Finally, let $\xi(g)$ be the sup of all $\bar{\alpha}(x)$ for x recursive in y.

Since $\mathbb{B}$ is δ-cc, $\xi(g) < \delta$ for all g, and in fact, there is a $\xi < \delta$ such that for all g, $\xi(g) < \xi$. We let ξ^Q be the least such ξ.

It is clear from the definition of ξ^Q that whenever (R, Σ_R) is an iterate of (Q, Σ_Q), then

$$\pi^\Sigma_{Q,R}(\xi^Q) = \xi^R.$$

But our Subclaims show that for any real x, there is an iterate (Q, Σ_Q) of (P, Σ) and a g that is $\mathbb{B}^Q_\delta$ generic such that $i(\rho)_g$ has the form $\langle P, \mathcal{U}, x\rangle$, and $\pi^\Sigma_{Q,\infty}(\bar{\alpha}(x)) = f(x)$. So in fact

$$\mathrm{ran}(f) \subseteq \pi^\Sigma_{P,\infty}(\xi^P) < \eta_0.$$

This proves Claim 1. □

Claim 2. $\eta_0 = \theta(\Psi_0)$.

Proof. We must show $\eta_0 \leq \theta(\Psi_0)$. We have that β_∞ is strong to η_0 in HOD, so if $\theta(\Psi_0) < \eta_0$, then $\theta(\Psi_0)$ is not a cutpoint in HOD. However, $\theta(\Psi_0)$ is in the Solovay sequence, so by the first part of the theorem, it is a cutpoint in HOD. □

Claim 2 clearly finishes the proof of the theorem. □

We now characterize the successor Woodins in HOD in terms of a modified Solovay sequence. The characterization was suggested to the author by Grigor Sargsyan.

Definition 5.2. For any ordinal κ, $\theta({}^\omega\kappa)$ is the least ordinal α such that there is no ordinal definable map of ${}^\omega\kappa$ onto α.

Notice that there is no $f\colon {}^\omega\kappa \overset{\text{onto}}{\longrightarrow} \theta({}^\omega\kappa)$ such that for some $s\colon \omega \to \kappa$ f is OD(s).

Theorem 5.3. *Assume* $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$, *let HOD* $\models$ *"η is a Woodin cardinal but not a limit of Woodin cardinals". Let* $\kappa = \sup(\{\gamma < \eta \mid HOD \models \text{“}\gamma \text{ is Woodin”}\})^{+,HOD}$; *then* $\eta = \theta({}^\omega\kappa)$.

Proof. We show first that $\eta \leq \theta({}^\omega\kappa)$. Let $\alpha < \eta$; we want to define a surjection f from ${}^\omega\kappa$ onto α. Let $\gamma_0 > \eta$ be a cardinal cutpoint of HOD.

Claim 1. Let (P, Σ) and (P, Φ) be lbr hod pairs, $\pi = \pi^\Sigma_{P,\infty}$, and $\sigma = \pi^\Phi_{P,\infty}$. Suppose

(a) $M_\infty(P, \Sigma) = M_\infty(P, \Phi) = \mathrm{HOD}|\gamma_0$,

(b) $\pi(\langle\bar\kappa, \bar\alpha, \bar\eta\rangle) = \sigma(\langle\bar\kappa, \bar\alpha, \bar\eta\rangle) = \langle\kappa, \alpha, \eta\rangle$, and

(c) $\pi \restriction \bar\kappa = \sigma \restriction \bar\kappa$;

then $\pi \restriction \bar\alpha = \sigma \restriction \bar\alpha$.

Proof. Let $\pi = \pi^\Sigma_{P,\infty}$ and $\sigma = \pi^\Phi_{P,\infty}$. Let $\pi(\bar\kappa) = \kappa$. We are given that $\pi \restriction \bar\kappa + 1 = \sigma \restriction \bar\kappa + 1$.

Let

$$\mathcal{T} = \mathcal{U}(P, \Sigma)$$

and

$$\mathcal{U} = \mathcal{U}(P, \Phi)$$

be the normal trees from P to $\mathrm{HOD}|\gamma$ by Σ^+ and Φ^+ respectively. The main branch maps of $\mathcal{T}$ and $\mathcal{U}$ are π and σ respectively. Let ξ be largest on the main branch of $\mathcal{T}$ such that whenever E is used in $[0, \xi)_T$, then $\lambda(E) \subseteq \kappa$, and let β be largest on the main branch of $\mathcal{U}$ such that whenever E is used in $[0, \beta)_U$, then $\lambda(E) \subseteq \kappa$. (Equivalently, ξ is least on the main branch such that $\kappa < \mathrm{crit}(i^{\mathcal{T}}_{\xi,\infty}$, and least anywhere in $\mathcal{T}$ such that $\kappa < \lambda(E^{\mathcal{T}}_\xi)$. Similarly for β and $\mathcal{U}$.) The branch extender of $[0, \xi)_T$ is $E_\pi \restriction \kappa$, and that of $[0, \beta)_U$ is $E_\sigma \restriction \kappa$, so the two branch extenders are the same.[37] The base model is P in each case, so

$$\mathcal{M}^{\mathcal{T}}_\xi = \mathcal{M}^{\mathcal{U}}_\beta.$$

(This does not imply $\xi = \beta$, however.)

Now let ξ_1 be largest on the main branch of $\mathcal{T}$ such that whenever E is used in $[0, \xi_1)_T$ then $\lambda(E) \subseteq \alpha$, and β_1 be largest on the main branch of $\mathcal{U}$ such that whenever E is used in $[0, \beta_1)_T$, then $\lambda(E) \subseteq \alpha$. We claim that the part of $\mathcal{T}$ between ξ and ξ_1 is a tree on $\mathcal{M}^{\mathcal{T}}_\xi$, and the same

[37] At this point we use that κ is a successor cardinal of HOD. Since $\bar\kappa$ is a successor cardinal of P, and π is continuous at κ, $E_\pi \restriction \kappa$ is determined by $\pi \restriction (P|\bar\kappa)$.

as the part of $\mathcal{U}$ between β and β_1. More precisely, we show by induction on $\xi \leq \gamma \leq \xi_1$ that there is a tree $\mathcal{W}_\gamma$ on $\mathcal{M}_\xi^\mathcal{T} = \mathcal{M}_\beta^\mathcal{U}$ such that, setting $\gamma^* = \beta + (\gamma - \xi)$,

$$\mathcal{T} \restriction \gamma + 1 = \mathcal{T} \restriction (\xi + 1)^\frown \mathcal{W}_\gamma$$

and

$$\mathcal{U} \restriction \gamma^* + 1 = \mathcal{U} \restriction (\beta + 1)^\frown \mathcal{W}_\gamma.$$

The main thing is that in this interval, we are comparing $\mathcal{M}_\xi^\mathcal{T} = \mathcal{M}_\beta^\mathcal{U}$ with HOD$|\alpha$, and HOD has no Woodin cardinals between κ and η. So if we have $\mathcal{W}_\gamma$ for γ a limit ordinal, then $\mathcal{W}_{\gamma+1}$ is determined by the Q-structure for $M(\mathcal{W}_\gamma)$ that HOD provides. At successor steps, granted $\mathcal{M}_\gamma^\mathcal{T} = \mathcal{M}_{\gamma^*}^\mathcal{U}$ by induction, $E_\gamma^\mathcal{T} = E_{\gamma^*}^\mathcal{U}$ because we are iterating away the least extender disagreement with HOD in both cases. Further, $\kappa < \text{crit}(E_\gamma^\mathcal{T})$, because otherwise crit$(E_\gamma)$ is a limit of Woodin cardinals of HOD, and hence in $\mathcal{M}_\xi^\mathcal{T}$, which implies there are arbitrarliy large Woodins in $\mathcal{M}_\gamma^\mathcal{T}|\lambda(E_\gamma^\mathcal{T})$, and since $\lambda(E_\gamma^\mathcal{T})$ is a cardinal of HOD, that there are Woodins of HOD between κ and α.[38] So $E_\gamma^\mathcal{T}$ and $E_{\gamma^*}^\mathcal{T}$ are applied to the same model in $\mathcal{T}$ and $\mathcal{U}$ (though it is indexed differently), and we have $\mathcal{M}_{\gamma+1}^\mathcal{T} = \mathcal{M}_{\gamma^*+1}^\mathcal{U}$, and we have $\mathcal{W}_{\gamma+1}$.

We thus have $\mathcal{M}_{\xi_1}^\mathcal{T} = \mathcal{M}_{\xi_1^*}^\mathcal{U}$. $\xi_1^* \leq \beta_1$, because the extenders E in $\mathcal{W}_{\xi_1}$ have $\lambda(E) \subseteq \alpha$. If $\xi_1^* < \beta_1$, then $\mathcal{M}_\xi^\mathcal{T} = \mathcal{M}_{\xi_1^*}^\mathcal{U}$ disagrees with HOD below α, contradiction. Thus $\xi_1^* = \beta_1$, so $\mathcal{M}_{\xi_1}^\mathcal{T} = \mathcal{M}_{\beta_1}^\mathcal{U}$. Moreover, $[\xi, \xi_1)_T$ and $[\beta, \beta_1)_U$ have the same branch extender, namely the extender of the main branch of $\mathcal{W}_{\xi_1}$.

It follows that $[0, \xi_1)_T$ and $[0, \beta_1)_U$ have the same branch extender, and thus $\pi \restriction \bar{\alpha} = i_{0,\xi_1}^\mathcal{T} \restriction \bar{\alpha} = i_{0,\beta_1}^\mathcal{U} \restriction \bar{\alpha} = \sigma \restriction \bar{\alpha}$, as desired. □

We can now define the desired surjection of ${}^\omega\kappa$ onto α. For $s \in {}^\omega\kappa$, we let $f(s) = 0$ unless s decodes as a tuple $\langle t, P, \bar{\kappa}, \bar{\alpha}, \bar{\eta}, \xi \rangle$ such that

(i) P is a countable lpm, $\bar{\kappa} < \xi < \bar{\alpha} < \bar{\eta}$ are ordinals of P,

(ii) $t \colon \bar{\kappa} \to \kappa$,

(iii) there is a Σ such that (P, Σ) is an lbr hod pair, $M_\infty(P, \Sigma) = \text{HOD}|\gamma_0$, $\pi_{P,\infty}^\Sigma(\langle \bar{\kappa}, \bar{\alpha}, \bar{\eta} \rangle) = \langle \kappa, \alpha, \eta \rangle$, and $\pi_{P,\infty}^\Sigma \restriction \bar{\kappa} = t$.

If s does decode to a tuple $\langle t, P, \bar{\kappa}, \bar{\alpha}, \bar{\eta}, \xi \rangle$ as in (i)-(iii), then we let $f(s)$ be the common value of $\pi_{P,\infty}^\Sigma(\xi)$, for all Σ as in (iii). By the claim, there is a common value. Clearly, f is surjective.

This shows $\eta \leq \theta({}^\omega\kappa)$. The proof that $\theta({}^\omega\kappa) \leq \eta$ follows the outline of the proof of Claim 1 in the proof of 5.1, but it is easier. We just give a sketch.

Let $\eta_0 = \eta$, and let

$$\eta_0 < \eta_1 < \ldots < \eta_5 < \bar{\theta}$$

[38] Let $\sigma = \sup(\{\xi < \eta \mid \text{HOD} \models \xi \text{ is Woodin}\})$. If crit$(E_\gamma) = \sigma$, then κ is a limit of Woodins in HOD. If crit$(E_\gamma) < \sigma$, then E_γ overlaps a Woodin of HOD, hence a Woodin of $\mathcal{M}_\xi^\mathcal{T}$, so again crit$(E_\gamma)$ is a limit of Woodins in HOD.

where $\bar{\theta}$ and the η_i for $i \geq 1$ are successor points in the Solovay sequence, and thus cutpoint Woodins in HOD. Let

$$\mathrm{HOD}|\bar{\theta} = M_\infty(P, \Sigma),$$

and $\pi_{P,\infty}(\bar{\kappa}) = \kappa$, and

$$\pi_{P,\infty}(\delta_i) = \eta_i, \text{ for } 0 \leq i \leq 5.$$

We omit the superscript Σ in these formulae, because from now on all strategies will be tails of Σ. Suppose that $f\colon {}^\omega\kappa \to \eta_0$ is ordinal definable; we want to show that its range is bounded. We may assume that we have a formula $\varphi(u, v, w)$ and an ordinal $\nu_0 < \eta_1$ such that for all $s \in^\omega \kappa$, $\alpha < \eta_0$,

$$f(s) = \alpha \Leftrightarrow L(\Sigma_{P|\delta_1}, \mathbb{R}) \models \varphi[s, \alpha, \nu_0].$$

We may also assume

$$\nu_0 = \pi^\Sigma_{P,\infty}(\bar{\nu}).$$

We show that there is a $\xi < \delta_0$ such that $\mathrm{ran}(f) \subseteq \pi_{P,\infty}(\xi)$, so $\mathrm{ran}(f)$ is bounded in η_0, as desired.

ξ is defined as follows. Let $\mathbb{B}$ be the $\bar{\kappa}$-generator extender algebra of P at δ_0, where the axioms are obtained using only extenders with critical points $> \bar{\kappa}$. Via some simple decoding term ρ, a $\mathbb{B}$-generic g over P yields $t = \rho_g$ such that $t \in^\omega \bar{\kappa}$. Let

$$s(n) = \pi_{P,\infty}(t(n)).$$

$P[g]$ now consults its term for $(\Sigma_{P|\delta_1})^\sharp$ (cf. Subclaim A in the proof of 5.1) to determine whether there is a $\bar{\alpha} < \delta_0$ such that

$$L(\Sigma_{P|\delta_1}, \mathbb{R}) \models \varphi[s, \pi_{P|\delta_1,\infty}(\bar{\alpha}), \pi_{P|\delta_1,\infty}(\bar{\nu})]$$

If so, then $\bar{\alpha}(g)$ is the least such $\bar{\alpha}$, otherwise $\bar{\alpha}(g) = 0$. By the δ_0-cc, we have $\xi < \delta_0$ such that for all such g, $\bar{\alpha}(g) < \xi$. We write $\xi = \xi^P$.

If $i\colon (P, \Sigma) \to (Q, \Sigma_Q)$ is an iteration map, then $\xi^Q = i(\xi^P)$ is defined from $i(\mathbb{B})$ and $i(\bar{\nu})$ over Q in the same way. To see that $\mathrm{ran}(f)$ is bounded by $\pi_{P,\infty}(\xi^P)$, let $s \in^\omega \kappa$ be given. We can iterate (P, Σ) to (R, Σ_R) so that

$$s \cup \{f(s)\} \subseteq \mathrm{ran}(\pi_{R,\infty}).$$

Let $j\colon P \to R$ be the iteration map. Let

$$\pi_{R,\infty}(t(n)) = s(n).$$

Now we can genericity iterate (R, Σ_R) to (Q, Σ_Q) with all critical points $> j(\bar{\kappa})$ so that, letting $i\colon P \to Q$ be the iteration map, there is a g on $i(\mathbb{B})$ generic over Q such that

$$t = \rho_g.$$

But then $\pi_{Q,\infty}(t(n)) = \pi_{R,\infty}(t(n)) = s(n)$, so $\pi^{-1}_{Q,\infty}(f(s)) < \bar{\alpha}(g)^Q < \xi^Q$, as desired. □

The following definition is due to Sargsyan.[39]

Definition 5.4. Assume AD^+. We set

$$\begin{aligned}\eta_0 &= \theta({}^\omega\omega) = \theta_0,\\ \eta_{\alpha+1} &= \theta({}^\omega\kappa), \text{ where } \kappa = (\eta_\alpha)^{+,\mathrm{HOD}},\\ \eta_\lambda &= \bigcup_{\alpha<\lambda} \eta_\alpha.\end{aligned}$$

We have shown

Corollary 5.5. *Assume $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$; then for any $\delta < \theta$, δ is a successor Woodin cardinal of HOD iff $\delta = \eta_0$ or $\delta = \eta_{\alpha+1}$ for some α.*

6 LSA from least branch hod pairs

We go back to AD^+ as our background theory.

Recall that LSA is the theory AD^+ + "there is an α such that θ_α is the largest Suslin cardinal". That is, the largest Suslin cardinal exists, and belongs to the Solovay sequence. LSA stands for "Largest Suslin Axiom". The members of the Solovay sequence are all Suslin cardinals, so if θ_α is the largest Suslin, then $\theta = \theta_{\alpha+1}$.

LSA is stronger than many familiar determinacy theories. For example, if θ_α is the largest Suslin, then $L(P_{\theta_\alpha}(\mathbb{R})) \models \mathsf{AD}_{\mathbb{R}}$ + "θ is regular". A lower bound for LSA in terms of hod pair existence follows from the fact that if θ_α is the largest Suslin, then in HOD, θ_α is a limit of Woodin cardinals, and θ_α is strong to θ, which is itself a Woodin cardinal. In [15], Sargsyan showed that a hypothesis on the existence of hod pairs in the rigidly layered hierarchy that is close to this lower bound implies the existence of models of LSA. Here we shall obtain something close to Sargsyan's upper bound, but in the least branch hierarchy.

Here is a theory still stronger than LSA.

Definition 6.1. LSA^+ is the theory: $\mathsf{AD}^+ + \exists A \subseteq \mathbb{R} \forall \alpha < \theta \forall s \colon \omega \to \alpha (A \notin \mathrm{OD}(s))$.

Proposition 6.2. LSA^+ *implies* LSA.

Proof. If B is Suslin and co-Suslin, then B is homogeneously Suslin (cf. [11]), so B is definable from a homogeneity system $\bar{\mu}$ such that $B = \{x \mid \bar{\mu}_x \text{ is wellfounded }\}$. Since any measure on an ordinal $< \theta$ is ordinal definable, B is $\mathrm{OD}(s)$ for some $s \in \bigcup_{\alpha<\theta} {}^\omega\alpha$.

So LSA^+ implies that not all sets are Suslin, and thus there is a largest Suslin cardinal κ. Let A be a complete κ-Suslin set. It is enough to see that A is not $\mathrm{OD}(B)$, for all Suslin-co-Suslin sets B. But if it were, A would be $\mathrm{OD}(s)$ for some $s \colon \omega \to \kappa$. By results of Woodin, if $\theta(A) < \theta$, then $\theta(A)$ is a Suslin cardinal. So $\theta(A) = \theta$; that is, every set of reals is OD from A and a real. So every set of reals is OD from s and a real, contrary to LSA^+. $\square$

[39] One might call this the *Sargsyan sequence*.

It seems very likely that LSA^+ is strictly stronger than LSA, but we do not have a full proof. The most natural proof would proceed by showing that HOD in the minimal model of LSA is a weaker mouse than HOD in the minimal model of LSA^+. This involves proving HPC in those models, which is work in progress.[40]

The next theorem gives hod mouse upper bounds on LSA and LSA^+.

Theorem 6.3. *Suppose that there is an lbr hod pair (P, Σ) such that for some δ, λ, the following hold in P:*

(a) ZFC + λ *is a limit of Woodin cardinals,*

(b) $\delta < \lambda$, *and* δ *is Woodin, and*

(c) letting κ be the least $< \delta$-strong cardinal, κ is a limit of Woodins.

Then there is a pointclass Γ such that $L(\Gamma, \mathbb{R}) \models \mathsf{LSA}$. If in addition

(d) δ is a limit of Woodin cardinals in P,

then there is a pointclass Γ such that $L(\Gamma, \mathbb{R}) \models \mathsf{LSA}^+$.

Remark. One could obtain what is essentially equivalent to Sargsyan's mouse by taking P as in the theorem, letting $Q = P|\delta$, letting $\Psi = \Sigma_Q^{\mathrm{sh}}$ be the short-tree component of Σ_Q, and considering the mouse $N = M_\omega^\Psi(Q)$. Our proof of Theorem 6.3 should work assuming only the existence of this weaker mouse. Unfortunately, N is not in the least branch hierarchy, because above δ we are only inserting Σ_Q^{sh} and not the full Σ. Objects like N show up in an important way in the theory of [15], but not at all in the theory of [19]. We therefore do not have at this moment a reasonable theory of N, and cannot use its existence as our hypothesis in Theorem 6.3. Developing such a theory is closely related to the problem of characterizing HOD assuming only $\mathsf{AD}^+ + \mathsf{HPC}$; "short-tree-strategy mice" like N should be part of the characterization.

Proof. Let $(P, \Sigma), \delta$, and λ be as in the hypotheses. We may assume $\lambda = \sup_{i<\omega} \delta_i$, where δ_i is the i^{th} Woodin cardinal above $\delta = \delta_0$. Let g be $\mathrm{Col}(\omega, <\lambda)$ generic over P, and

$$L(\mathbb{R}_g^*, \mathrm{Hom}_g^*) = D(P, <\lambda)$$

be the associated derived model. By [19][§7], $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*) \models \mathsf{AD}_{\mathbb{R}} + P(\mathbb{R}) = \mathrm{Hom}_g^*$, and the $\mathrm{Code}(\Sigma_{P|\delta_i}) \cap \mathbb{R}_g^*$, for $i < \omega$, are Wadge cofinal in Hom_g^*. Thus $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*) \models \mathsf{HPC}$. It is also shown in [19] that the lbr hod pairs $(P|\delta_i, \Sigma_{P|\delta_i})$ are fullness preserving in $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)$, so that

$$\mathcal{H} = (\mathrm{HOD}|\theta)^{L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)} = \bigcup_{i<\omega} M_\infty(P|\delta_i, \Sigma_{P|\delta_i}).$$

We will be working in $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)$ now, so let us write $\mathbb{R} = \mathbb{R}_g^*$ and $P(\mathbb{R}) = \mathrm{Hom}_g^*$. Let

$$\eta_i = o(M_\infty(P|\delta_i, \Sigma_{P|\delta_i})) = \tau_\infty(P|\delta_i, \Sigma_{P|\delta_i}).$$

[40] [15] proves that the analog of HPC in a rigidly layered hierarchy holds in the minimal model of LSA.

The η_i are cutpoint Woodins of HOD, and thus successor points in the Solovay sequence. Moreover,

$$\beta_\infty = \beta_\infty(P|\delta_0, \Sigma_{P|\delta_0})$$

is a limit of cutpoint Woodins of HOD, and hence a limit point in the Solovay sequence. Since there are no Suslin cardinals strictly between $\beta_\infty(P|\delta_0, \Sigma_{P|\delta_0})$ and $|\tau_\infty(P|\delta_0, \Sigma_{P|\delta_0})| = \eta_0$,

$$L(P_{\eta_0}(\mathbb{R})) \models \beta_\infty \text{ is the largest Suslin cardinal.}$$

Thus $L(P_{\eta_0}(\mathbb{R})) \models \mathsf{LSA}$.

Now suppose that δ_0 is a limit of Woodins in P. We shall show that $L(P_{\eta_0}(\mathbb{R})) \models \mathsf{LSA}^+$. Let us write

$$\Psi_0 = (\Sigma_{P|\delta_0})^{\text{sh}}$$

as before. Code(Ψ_0) is β_∞-Suslin, and hence belongs to $L(P_{\eta_0}(\mathbb{R}))$. Let $s\colon \omega \to \alpha$, where $\alpha < \eta_0$. It will be enough to show that Ψ_0 is not ordinal definable from s in $L(P_{\eta_0}(\mathbb{R}))$. The ordinal parameter in such a definition could be taken $< \eta_0$, and then absorbed in s. Moreover, η_0 is definable in $L(P(\mathbb{R}))$. So assume toward contradiction that we have a formula φ such that

$$\Psi_0(\mathcal{T}) = b \Leftrightarrow L(P(\mathbb{R})) \models \varphi[s, \mathcal{T}, b].$$

$L(P(\mathbb{R}))$ is a derived model of P, and also of any Σ-iterate Q of P such that the iteration is given by a tree coded in $\mathbb{R}$. Thus P has a $\text{Col}(\omega, <\lambda)$-term $\dot{X}$ such that whenever $k\colon P \to N$ is an iteration map via a tree coded in $\mathbb{R}$, and h is $\text{Col}(\omega, < k(\lambda^N))$ generic over N with $\mathbb{R}^*_h = \mathbb{R}$, then

$$k(\dot{X})_h = \{\langle s, \mathcal{T}, b\rangle \mid L(P(\mathbb{R})) \models \varphi[s, \mathcal{T}, b]\}.$$

Let (Q, Σ_Q) be an iterate of (P, Σ) via a tree coded in $\mathbb{R}$ such that $\{\alpha\} \cup s \subseteq \text{ran}(\pi_{Q,\infty})$, and $i\colon P \to Q$ the iteration map. (All strategies now are tails of Σ, so we have dropped the superscript.) Let t be such that for all n,

$$\pi_{Q,\infty}(t(n)) = s(n),$$

and let $\pi_{Q,\infty}(\bar{\alpha}) = \alpha$. Let γ be a Woodin cardinal of Q such that $\bar{\alpha} < \gamma < i(\delta_0)$. (This is precisely where we use that δ_0 is a limit of Woodin cardinals in P!) Now let

$$j\colon Q \to R$$

come from a genericity iteration with critical points above $\bar{\alpha}$ and be such that

$$t \in R[g]$$

for some g that is $\text{Col}(\omega, j(\gamma))$-generic over R. In $R[g]$ we can compute Ψ_0 on all trees of size $< j(i(\lambda))$, for

$$\Psi_0(\mathcal{T}) = b \Leftrightarrow \text{Col}(\omega, < j(i(\lambda))) \Vdash \langle s, \mathcal{T}, b\rangle \in j(i(\dot{X})).$$

In particular, we can compute $\mathcal{U} \restriction j(i(\delta_0))$, where $\mathcal{U}$ is the normal tree from P to R. But $\mathcal{U} \restriction j(i(\delta_0))$ collapses all cardinals of $R[g]$ in the interval $(j(\gamma), j(i(\delta_0)))$, contradiction.

$\square$

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