§3. Construction of $\mathcal{H}_0$: closing under a strategy, fine structure, and condensation.

We are now ready to construct $(\mathcal{N}_{\xi+2}, \dot{\Psi}_{\xi+2})$. We note first

Lemma 3.1 Let h be $Col(\omega, <\kappa_1)$ generic, and suppose $\dot{\Omega}_h \notin \hat{J}(\Gamma)$; then $\dot{\Omega}_h$ is a pointclass generator for $\hat{J}(\Gamma)$.

Proof. Let $\Omega = \dot{\Omega}_h$. To see that $\hat{J}(\Gamma) \subseteq \Gamma((\mathcal{N}, E), \Omega)$, it suffices to show that if $(\mathcal{P}, \Lambda)$ is a $\hat{J}(\Gamma)$-hod-pair, then Λ is projective in the tail $\Omega_{\mathcal{W}}$, for some Ω-iterate $(\mathcal{W}, F)$ of $(\mathcal{N}, E)$. This follows from comparing $(\mathcal{P}, \Lambda)$ with $((\mathcal{N}, E), \Omega)$. The other inclusion, that $\Gamma((\mathcal{N}, E), \Omega) \subseteq \hat{J}(\Gamma)$, is essentially (i) of 2.4.

$\boxtimes$

We are seeking a pointclass generator for $\hat{j}(\Gamma)$, so for the remainder of this section we assume that $\mathring{\varphi}_{\xi+1} \in j(\mathring{\Gamma})$ holds in all $col(\omega, <\kappa_1)$ extensions of M. We are in the case that $(\mathcal{N}_{\xi+1}) = (\mathcal{N}, E)$, and we are using $\mathring{\Omega}$ for $\mathring{\varphi}_{\xi+1}$.

Letting h be $col(\omega, <\kappa_1)$-generic over M and $\Omega = \mathring{\Omega}_h$, we want to show that for any level $\mathcal{M}$ of $Lp^{\Omega}((\mathcal{N}, E))^{\hat{j}(\Gamma)}$ and any $k < \omega$, $(\mathcal{M}, k)$ is "$\hat{j}(\Gamma)$-tame" in the following sense.

<u>Definition 3.1.1</u> Let $\mathcal{M}$ be a level of $Lp^{\Omega}((\mathcal{N}, E))^{\hat{j}(\Gamma)}$ and let $k < \omega$; then $(\mathcal{M}, k)$ is <u>$\hat{j}(\Gamma)$-tame</u> iff

(1) $\mathcal{M}$ is k-sound, and the j-realization strategy for $(\mathcal{M}, k)$ exists, and is in $\hat{j}(\Gamma)$,

(2) $\rho_{k+1}(\mathcal{M}) > \theta^0$, and

(3) $\rho_{k+1}(\mathcal{M})$ is solid and universal.

Let us explain clause (1). We need to consider more than iterations of $\mathcal{M}$ that are above $\theta(\mathcal{N})$. In order to prove (2) (in the case we haven't reached a pointclass generator) we must consider iterations of $(\mathcal{N}, E)$ that carry $\mathcal{M}$ along on top. However, Ω doesn't tell us how to do that, because $\mathcal{M}$ may construct new functions $f : \theta^\circ \to \mathcal{N}$. So we must use the same method by which we constructed Ω to get a "j-realization strategy for $(\mathcal{M}, k)$". This is basically an iteration strategy for $(\mathcal{N}, E)$ that allows for ultrapowers of $(\mathcal{N}, E)$ and its images that use the appropriate functions, enough that the Σ_k theory of $\mathcal{M}$ gets moved properly. Call it $\Omega(\mathcal{M}, k)$.

We construct $\Omega(\mathcal{M}, k)$ in the same way that we constructed Ω. We show that it has the same good behavior, and in particular, that $\Omega(\mathcal{M},k) \in \mathrm{Hom}_h^*$. Repeating the proof of 3.1, we then get that if $\Omega(\mathcal{M},k) \notin \hat{\mathcal{J}}^a(\Gamma)$, it is our desired pointclass generator. If $\Omega(\mathcal{M},k) \in \hat{\mathcal{J}}(\Gamma)$, we then have (1) of definition 3.1.1, and we are ready to prove (2) and (3).

<u>Remark</u> It doesn't seem to matter that the strategy information fed into $\mathcal{M}$ comes from just Ω, not the $\Omega(\mathcal{M}|\eta, i)$'s. As soon as we finish with $L_p^{\Omega}((\mathcal{M},E))$, we'll start feeding in the $\Omega(\mathcal{M}|\eta, i)$ - information. It seems to be ok to wait that long.

What we need then to show is

<u>Lemma 3.2</u> Let $\mathcal{M}$ be a level of $Lp^{\Omega}((\mathbb{R},E))^{\hat{J}(\Gamma)}$ and $k<\omega$; then either $\hat{J}(\Gamma)$ has a pointclass generator in Hom_h^*, or $(\mathcal{M},k)$ is $\hat{J}(\Gamma)$-tame.

The proof will be by induction on levels $\mathcal{M}$, with a subinduction on k. Before we go to the proof, we ~~describe~~ prove a ~~general~~ condensation result that will be used heavily.

Lemma 3.2 will be true for hod pair constructions in general. Below O_h^P we get it from a strong form of condensation that does not hold in general. We state this strong condensation under the "$j \restriction \mathcal{N} \in M$" hypothesis. Later we shall prove a version that avoids "$j \restriction \mathcal{N} \in M$", and use it to avoid this hypothesis in our other arguments as well.

<u>Lemma 3.3</u> (Condensation below O_h^P, version I.)
Suppose that in $M[h]$ we have, with $(\mathcal{M}, \Psi) = (\mathcal{N}_\alpha, (\dot{\Psi}_\alpha)_h)$ a level of our ~~hod-type~~ $\hat{j}(\Gamma)$-hod-pair construction, the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "j"] \arrow[dr, "i"'] & & j(\mathcal{M}) \\
& Q \arrow[ur, "\tau"'] & \\
& Q|\eta \arrow[u, phantom, "\trianglelefteq" rotate=90] & \\
& P \arrow[u, "\pi"'] &
\end{tikzcd}$$

where Q is countable. (The diagram exists in M, so in particular, $j \restriction \mathcal{M} \in \mathcal{M}$.) Suppose that in $\mathcal{Q}$, η is either the index of a $\kappa^{\mathcal{Q}}$-relevant extender, or a limit of such indices. Suppose that π is either cofinal Σ_0 or else Σ_2 elementary, and that

$$\pi \restriction ((\kappa^{\mathcal{P}})^{+\mathcal{P}} + 1) = \text{identity}.$$

Let

$$\Sigma = \hat{j}(\Psi)^{\eta}$$

and

$$\Phi = (\Sigma_{\mathcal{Q}|\eta})^{\pi},$$

and suppose that Σ is ~~...~~ locally ν-consistent. Then there is an α such that

(1) $\mathcal{P} = \mathcal{Q}|\alpha$, and

(2) $\Phi = \Sigma_{\mathcal{Q}|\alpha}$.

<u>Proof</u> We show by induction on the full levels $\mathcal{W}$ of $\mathcal{P}$ that

(i) $\mathcal{W}$ is a full level of $\mathcal{Q}$, and

(ii) $\Phi_{\mathcal{W}} = \Sigma_{\mathcal{W}}$.

The base case for this induction is $\mathcal{W} = \mathcal{P} \restriction \kappa^{+\mathcal{P}}$, which is obvious.

We handle first the successor case of the induction in which $\mathcal{N}$ is ~~acti~~ e-active via an extender with $\mathrm{crit} = \kappa^{\mathcal{P}}$.
Let

$$\mathcal{N} = (R, E)$$

and

$$\pi : (R, E) \longrightarrow (Q \| \gamma, F).$$

We have that $R \trianglelefteq Q$ by induction, and $\Phi_R = \Sigma_R$ by induction. Write $F = \pi(E)$.

<u>Claim 1</u> E is τ-certified over (R, Σ_R).

<u>Proof</u> Because $\pi \restriction (\kappa^R)^{+R} + 1$ is the identity, E is an extender over Q. Let $k : Q \to \mathrm{Ult}(Q, E)$ and $\sigma : \mathrm{Ult}(Q, E) \longrightarrow \mathrm{Ult}(Q, \pi(E))$ be the natural maps. Note that $\pi(E)$ is τ-certified over $(Q\|\gamma, \Sigma_{Q\|\gamma})$ by 2.11 or 2.12. Let $t : Q \longrightarrow \mathrm{Ult}(Q, \pi(E))$, be the natural map, and $u : \mathrm{Ult}(Q, \pi(E)) \to j(\mathcal{M})$

be given by certification, i.e.

$$u(t(f)(a)) = r(f)\left(\pi^{\Sigma_{Q\|\gamma}}_{Q\|\gamma,\infty}(a)\right).$$

Thus $r = u \circ t$, and $\hat{j}(\Psi)^u$ is locally u-consistent, by 2.10. We have the diagram

$$\begin{tikzcd}
M \arrow[rrr, "j"] \arrow[dr, "i"'] & & & j(M) \\
 & Q \arrow[urr, "r"] \arrow[r, "t"] \arrow[dr, "k"'] & Ult(Q,\pi(E)) \arrow[ur, "u"'] & \\
 & & Ult(Q,E) \arrow[u, "\sigma"'] &
\end{tikzcd}$$

Note that R is a cutpoint in $Ult(Q,E)$, and $Q\|\gamma$ is a cutpoint in $Ult(Q,\pi(E))$, by the way we index extenders. We also have

$$\hat{j}(\Psi)^u_{Q\|\gamma} = \Sigma_{Q\|\gamma},$$

and

$$\hat{j}(\Psi)^{u\circ\sigma}_R = \Phi_R = \Sigma_R.$$

The first identity holds because $u \restriction (Q\|\gamma)$ is the iteration map of $\Sigma_{Q\|\gamma}$, which is pullback consistent. The second follows at once for Φ_R, and $\Phi_R = \Sigma_R$

is our induction hypothesis.

To see that E is τ-certified over (R, Σ_R), let $X \in R|(\kappa^R)^{+R} = Q|(\kappa^Q)^{+Q}$. Because j is continuous at $(\kappa^m)^{+m}$, $= o(H_0^+)$, i is continuous at $(\kappa^m)^{+m}$), so we can write

$$X = i(f)(a)$$

where $a \in [\kappa^Q]^{<\omega}$ and $f \in H_0^+$.

~~We must see~~ Let us write $\pi^{\Sigma_{Q\|\gamma}} : Q\|\gamma \to H_1$ and $\pi^{\Sigma_R} : R \to H_1$ for the maps of the direct limit system for $HOD^{L(\mathfrak{g}(\Gamma), \mathbb{R}^*_h)}$. We then have

$$\begin{aligned}
(b, X) \in E \ &\text{iff}\ (b, i(f)(a)) \in E \\
&\text{iff}\ (\sigma(b), i(f)(a)) \in \pi(E)
\end{aligned}$$

(since $\pi \restriction hE = \sigma \restriction hE$)

$$\begin{aligned}
&\text{iff}\ \sigma(b) \in t(i(f)(a)) \\
&\text{iff}\ \pi^{\Sigma_{Q\|\gamma}}(\sigma(b)) \in \tau(i(f)(a))
\end{aligned}$$

(since $\pi(E)$ is τ-certified over $(Q\|\gamma, \Sigma_{Q\|\gamma})$)

$$\begin{aligned}
&\text{iff}\ \pi^{\Sigma_{Q\|\gamma}}(\sigma(b)) \in j(f)(\pi^{\Sigma_{Q\|\gamma}}(a)) \\
&\text{iff}\ \pi^{\Sigma_R}(b) \in j(f)(\pi^{\Sigma_R}(a))
\end{aligned}$$

(by j-condensation, since $k(a) = a$, and $\tau(a) = \pi^{\Sigma_{Q\|\gamma}}(a)$)

$$\text{iff}\ \pi^{\Sigma_R}(b) \in \tau(X).$$

This proves claim 1.

☒

Claim 2. E is on the Q-sequence.

Proof. Let G be the first extender on the Q-sequence with length $\geq lh(E) = o(R)$, and critical point κ^Q. Since $\pi(E)$ exists, there is such a G. Let ν be the least cutpoint of $Ult(Q, G) > \kappa^Q$, so $lh\, G = (\nu^+)^{Ult(Q,G)}$. If H has critical point κ^Q and $lh(H) < lh(G)$, then H is on the Q-sequence iff H is on the R-sequence iff H is on the $Ult(Q,G)$-sequence. That is, $Ult(Q,G) | lh(G) = R$. By 2.11, G is τ-certified by $(Q \| lh G, \Sigma_{Q\|lh G}) = (R, \Sigma_R)$. Thus $G = E$.

☒

That finishes the proof that $\mathcal{W} = (R, E)$ is a full level of Q. We now show $\Phi_{\mathcal{W}} = \Sigma_{\mathcal{W}}$.

If not, then we have a stack $\vec{\mathcal{I}}$ on $\mathcal{W}$ by both strategies, with last model $\mathcal{S}$, and such that the tails

$$\Lambda = (\Phi_{\mathcal{W}})_{\vec{\mathcal{I}}, \mathcal{S} | \kappa^{\mathcal{S}+}}$$

and

$$\Omega = (\Sigma_{\mathcal{W}})_{\vec{\mathcal{I}}, \mathcal{S} | \kappa^{\mathcal{S}+}}$$

disagree. We have the diagram

$$\begin{tikzcd}
 & & j(\mathcal{M}) & \\
\mathcal{M} \arrow[r, "i"] & Q \arrow[ur, "\tau"] \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \mathcal{V} \arrow[u, "\rho"'] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \\
 & Q|\gamma \arrow[r] & \mathcal{U} & \\
 & \mathcal{W} \arrow[u, "\pi"] \arrow[r, "i^{\vec{\mathcal{I}}}"] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \mathcal{S} \arrow[u, "\lambda"'] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \\
 & Q \arrow[r] & \mathcal{A} \arrow[dr, "\theta"] & \\
 & & & j(\mathcal{M})
\end{tikzcd}$$

λ and ρ exist because $\vec{\mathcal{T}}$ was played by $\Phi_{\mathcal{W}}$, and θ exists because $\vec{\mathcal{T}}$ was played by $\Sigma_{\mathcal{W}}$. We have that

$$\hat{j}(\psi)^{\rho\circ\lambda}_{\mathcal{S}|k^{\mathcal{S}+}} \neq \hat{j}(\psi)^{\theta}_{\mathcal{S}|k^{\mathcal{S}+}}.$$

But now, let us restrict the maps in our diagram to H_0^+ and its images. We get

$$\begin{tikzcd}
 & & H_1^+ \\
H_0^+ \arrow[urr, "j"] \arrow[r, "i"] \arrow[drr, "j"'] & \mathcal{W}|k^{\mathcal{W}+} \arrow[r, "i^{\vec{\mathcal{T}}}"] & \mathcal{S}|k^{\mathcal{S}+} \arrow[u, "\rho\circ\lambda"'] \arrow[d, "\theta"] \\
 & & H_1^+
\end{tikzcd}$$

Our uniqueness-of-pullbacks lemma, 2.17.1, now gives a contradiction.

$\boxtimes$

If $\mathcal{P} = \mathcal{W}$, we are done with the proof of 3.3. So suppose not. Let β be the next index on the $\mathcal{P}$-sequence of an extender G such that $\mathrm{crit}(G) = \kappa^{\mathcal{P}}$. β must exist because $\pi : \mathcal{P} \to \mathcal{Q}|\eta$, where η is either such an index, or a limit of such indices, on the $\mathcal{Q}$-sequence. Let $\langle W_\xi \mid \xi \leq \alpha \rangle$ enumerate the full levels of $\mathcal{P}$ between $\mathcal{W}_0 = (R, E)$ and $\mathcal{W}_\alpha = \mathcal{P}||\beta$ in increasing order. We show by induction on ξ that

(i) $W_\xi \trianglelefteq \mathcal{Q}$,

and

(ii) $\Phi_{W_\xi} = \Sigma_{W_\xi}$.

We know it for $\xi = 0$, and the case ξ is a limit is trivial because then $\Phi_{W_\xi} = \bigoplus_{\nu<\xi} \Phi_{W_\nu} = \bigoplus_{\nu<\xi} \Sigma_{W_\nu} = \Sigma_{W_\xi}$. Now suppose we have (i) and (ii) for ξ, and $\xi < \alpha$. We need to see that

$W_{\xi+1}$ is the next full level of Q.

~~th~~ Now $W_{\xi+1}$ is an initial segment of $L_p^{\Lambda}(W_\xi)^{\hat{\jmath}(r)}$, where $\Lambda = \Phi_{W_\xi} = \Sigma_{W_\xi}$.

This is because the embedding π guarantees the required iterability for the Λ-mice over W_ξ that are stacked in $W_{\xi+1}$.

~~Thus $W_{\xi+1}$ is an initial segment of Q, provided that $W_{\xi+1}$ is marked "full" in Q.~~

<u>Claim 4.</u> $L_p^{\Lambda}(W_\xi)^{\hat{\jmath}(r)}$ is the next full level of Q after W_ξ.

<u>Proof</u> First, $o(W_\xi)$ is not e-active in Q. For otherwise, let F on the Q-sequence have $crit(F) = \kappa^Q$ and $lh(F) = o(W_\xi)$. Using $\hat{\jmath}$-condensation, we see that $L_p^{\Lambda}(W_\xi)^{\hat{\jmath}(r)} \trianglelefteq Ult(Q, F)$.

(The $\hat{j}(\Gamma)$ - fullness of $Ult(Q,F)$ comes from lemma 2.4, j-condensation.) Let $\mathcal{S} \triangleleft W_{\xi+1}$ be the collapsing structure for $o(W_\xi)$, which exists since $\xi \neq \alpha$. Then $\mathcal{S} \triangleleft Ult(Q,F)$, so $lh(F)$ is not a cardinal in $Ult(Q,F)$, contradiction.

Thus the next full level of Q is an initial segment of $Lp^{\Lambda}(W_\xi)^{\hat{j}(\Gamma)}$. It is all of $Lp^{\Lambda}(W_\xi)^{\hat{j}(\Gamma)}$ by j-condensation.

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<u>Claim 2</u> $W_{\xi+1} = Lp^{\Lambda}(W_\xi)^{\hat{j}(\Gamma)}$.

<u>Pf.</u> Let $G = E_\beta^{\mathcal{P}}$. In $Ult(\mathcal{P}, G)$, $W_{\xi+1}$ is the next full level after W_ξ. So in $Ult(Q, G)$, $W_{\xi+1}$ is the next full level after W_ξ. The claim now follows from j-condensation, using the factoring

$$\mathcal{M} \longrightarrow Q \longrightarrow Ult(Q,G) \longrightarrow Ult(Q,\pi(G)) \longrightarrow j(\mathcal{M}).$$

Claim 3 $\Phi_{W_{\xi+1}} = \Sigma_{W_{\xi+1}}$.

Proof This is just like the proof that $\Phi_{W_0} = \Sigma_{W_0}$. The difference is just that the map $i^{\vec{\mathcal{T}}}: W_{\xi+1} \to \mathcal{S}$ by both strategies can involve ~~more complicated~~ ultrapowers using more functions. We still get that the tail strategies disagree on $\mathcal{S}|(\kappa^{\mathcal{S}})^{+\mathcal{S}}$, and this contradicts 2.17.1.

$\boxtimes$

We have shown that $(\mathcal{P}\|\beta, \Phi_{\mathcal{P}\|\beta}) = (\mathcal{Q}\|\beta, \Sigma_{\mathcal{Q}\|\beta})$, where β is the next e-active, $\kappa^{\mathcal{P}}$-reduced level of $\mathcal{P}$. We are back to where we started.

We must also consider the case that ν is a limit of indices of extenders on the $\mathcal{P}$-sequence with critical point $\kappa^{\mathcal{P}}$, and

$\mathcal{W}$ is ~~the~~ a next full level of $\mathcal{P}$ between $\mathcal{P}|\beta = \mathcal{P}\|\beta$ and the next index of such an extender on the $\mathcal{P}$-sequence. This is done by induction on such full levels, just as in claims 1–3 above. We never actually used there that $\mathcal{W}_0$ is e-active. We get $\Phi_{W_0} = \Sigma_{W_0}$ in the current case because both are joins-of-strategies.

This finishes the proof of 3.3.

$\boxtimes$

From 3.3 we get the basic fine structure lemma 3.2.

<u>Proof of 3.2 (assuming $j\restriction\eta \in M$)</u>

The proof is by induction on levels $\mathcal{M}$ of $Lp^{\Omega}((\eta, E))^{\hat{j}(\Gamma)}$, with a subinduction on k, We are assuming that there is no pointclass generator in Hom_k^* for $\hat{j}(\Gamma)$.

Suppose $\mathcal{M} \trianglelefteq Lp^{\Omega}((\mathcal{N}, E))^{\hat{j}(\Gamma)}$, and that for all $\alpha < o(\mathcal{M})$ and all $k < \omega$, $(\mathcal{M}|\alpha, k)$ is $\hat{j}(\Gamma)$-tame. We show by induction on k that $(\mathcal{M}, k)$ is $\hat{j}(\Gamma)$-tame. The base case $k=0$ is typical.

<u>$k=0$</u> Clearly $\mathcal{M}$ is 0-sound. We construct $\Omega(\mathcal{M}, 0)$ just as we constructed Ω. Very roughly: given $i^{\vec{\mathcal{T}}} : \mathcal{M} \to Q$ by $\Omega(\mathcal{M}, 0)$, we'll have

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "j"] \arrow[dr, "i^{\vec{\mathcal{T}}}"'] & & j(\mathcal{M}) \\ & Q \arrow[ur, "\pi"'] & \end{tikzcd}$$

such that $j(\Sigma)^{\pi}$ is τ-consistent

for some uniquely-determined-by-$\vec{\mathcal{T}}$ map π!!. The $\vec{\mathcal{T}}$-tail of $\Omega(\mathcal{M}, 0)$ is then given by

(i) if you are iterating Q with critical points $> \kappa^Q$, then use τ to pull back a strategy from $j(\mathcal{M})$. Because we are below o_n^P, this iteration has dropped and cannot ~~and~~ un-drop, so no more realizing into $j(\mathcal{M})$ is needed.

(ii) If G is on the Q-sequence with $\mathrm{crit}(G) = \kappa^Q$, then G is τ-certified over $(Q \| \mathrm{lh}\, G, \Psi_{Q\|\mathrm{lh}\, G})$, where $\Psi = j(\Sigma)^\tau$.

(Recall Σ was the strategy of $\mathcal{N}$.) This follows literally from 2.11 and 2.12, because G must come from $i^{\vec{T}}((\mathcal{N}, E))$, so that we can literally apply 2.11 or 2.12. We then have

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "j"] \arrow[dr, "i"'] & & j(\mathcal{M}) \\ & Q \arrow[ur, "\tau"] \arrow[r, "i_G"'] & \mathrm{Ult}(Q, G) \arrow[u, "\sigma"'] \end{tikzcd}$$

by 2.10, and we are back to where we were.

(iii) If $\vec{u}$ is on $Q|K^Q$, then we use $j(Z)^T$. If this gives rise to $k : Q \to R$, then applying 2.13, we get

$$\begin{tikzcd}
\mathcal{M} \arrow[rr] \arrow[dr, "i"'] & & j(\mathcal{M}) \\
& Q \arrow[ur, "\varphi"] \arrow[r, "k"'] & R \arrow[u, "\sigma"']
\end{tikzcd}$$

and are back to our starting point.

This argument gives $\Omega(\mathcal{M}, k)$, for arbitrary k.

Remark | The way this differs from what we had in 2.10 – 2.13 before is that ~~the~~ in (ii) and (iii), the realizing maps ~~are~~ φ and σ are weak k-embeddings. This matches the degrees of the ultrapowers taken in $\mathcal{M} \xrightarrow{i} Q \xrightarrow{i_G} Ult(Q, G)$ and in $\mathcal{M} \xrightarrow{i} Q \xrightarrow{k} R$. That is because those two iterations are given by extenders with all their measures on Θ^0, and we have assumed $\rho_k(\mathcal{M}) > \Theta^0$ as an induction hypothesis. Thus the iteration maps $1_G \circ i$ and $k \circ i$ in (ii) and (iii) will be k-embeddings (and on the image of $\mathcal{M}$, not just the image of $\mathcal{N}$.) ~~Cor~~ The realization maps will then be weak k-embeddings.

This ends our sketch of the construction of $\Omega(\mathcal{M}, 0)$, and thus the verification of clause (1) of 3.1.1. We go on to (2).

<u>Claim 4</u> $\rho_1(\mathcal{M}) > \Theta^0$

<u>Proof</u> Consider $Q = Ult_0(\mathcal{M}, \nu)$, where ν is the order zero total measure on Θ^0. Let $q = i_\nu(p)$, where p is the standard parameter of Q. We have

(1) H_0^+ is a cutpoint in Q

(2) Q has the iteration strategy: the Q-tail of $\Omega(\mathcal{M}, 0)$. We ~~can~~ have

$\Sigma_{H_0} \subseteq \Omega(m,0)$. So we can regard Q as a hod-premouse over (H_0^+, Σ_{H_0}), with strategy $\Lambda \in \hat{j}(\Gamma)$.

(3) there is an $A \subseteq \theta^0$ such that A is Σ_1 definable over Q from q, but $A \notin H_0^+$.

Call a pair (Q, q) with properties (1) - (3) <u>minimal</u> if there is no Ψ-iteration $\pi : Q \xrightarrow{\vec{\mathcal{T}}} S$ and parameter $r < \pi(q)$ such that (S, r) has the properties (1) - (3), with respect to the $\vec{\mathcal{T}}$-tail of Ψ. Clearly, there is a minimal pair; fix (Q_0, q_0) such a pair, with strategy $\Psi_0 \in \hat{j}(\Gamma)$.

Let (R_0, r_0) with strategy $\Lambda \in \hat{j}(\Gamma)$ be any other minimal pair. We compare (Q_0, Ψ) and (R_0, Λ). ~~as~~ They are both Σ_{H_0}-hod-pairs, so the comparison is above θ^0. It is easy then to see that we get $i : Q_0 \to S$ and $t : R_0 \to S$ with $i(q_0) = t(r_0)$. It follows that

$Th_1^{Q_0}(q_0 \cup \theta^0) = Th_1^{R_0}(r_0 \cup \theta^0)$. But then this implies that $Th_1^{Q_0}(q_0 \cup \theta^0)$ is $OD_{\Sigma_{H_0}}(H_0)^{\hat{\delta}(\Gamma)}$. By Σ_{H_0}-mouse-capturing,

$Th_1^{Q_0}(q_0 \cup \theta^0) \in Lp^{\Sigma_{H_0}}(H_0)^{\hat{\delta}(\Gamma)} = H_0^+$.

This is a contradiction, yielding Claim 1.

$\boxtimes$

<u>Claim 2</u> $\rho_1(\mathcal{M})$ is 1-solid and 1-universal.

<u>Proof</u> Let $\rho = \rho_1(\mathcal{M})$, so that $\rho \geq \theta^{0+}$ by claim 1. Let H be the ~~$\pi : H_1^{\mathcal{M}}(\rho \cup$~~ transitive collapse

of $Hull_{\Sigma_1}^{\mathcal{M}}(\rho \cup p_1(\mathcal{M}))$, and $\pi : H \to_{\Sigma_1} \mathcal{M}$ the collapse embedding. By 3.3, H is an initial segment of $\mathcal{M}$, so in fact, $H = \mathcal{M}$. So $Hull_{\Sigma_1}^{\mathcal{M}}(\rho \cup p_1(\mathcal{M})) = \mathcal{M}$. That shows that $p_1(\mathcal{M})$ is 1-universal.

For solidity, let $p_1(\mathcal{M}) = \langle \alpha_0 \cdots \alpha_n \rangle$ and let H be the transitive collapse of $Hull_{\Sigma_1}^{\mathcal{M}}(\alpha_i \cup \{\alpha_0 \ldots \alpha_{i-1}\})$. Let $\pi : H \to \mathcal{M}$ be the collapse. Again, H is an initial segment of $\mathcal{M}$ by 3.3. We can't have $H = \mathcal{M}$, as otherwise $\mathcal{M} = Hull_{\Sigma_1}^{\mathcal{M}}(\alpha_i \cup \{\pi^{-1}(\alpha_0) \ldots \pi^{-1}(\alpha_{i-1})\})$, so $p_1(\mathcal{M})$ is Σ_1 definable over $\mathcal{M}$ from some parameter $<_{lex} p_1(\mathcal{M})$, which is impossible. Thus $H \in \mathcal{M}$, so $Th^{\mathcal{M}}_{\Sigma_1}(\alpha_i \cup \{\alpha_0 \dots \alpha_{i-1}\}) \in \mathcal{M}$, as desired.

This finishes the case $k=0$.

<u>$k>0$</u> The proof is the same as in the $k=0$ case.

This finishes the proof of 3.2.

$\boxtimes$

In view of 3.2, we may assume that for all levels $\mathcal{M}$ of $L_p^{\Omega}((\mathcal{N},E))^{\hat{\jmath}(\Gamma)}$, and all $k<\omega$, $(\mathcal{M},k)$ is $\hat{\jmath}(\Gamma)$ tame. We then set

$$\mathcal{N}_{\xi+2} = L_p^{\Omega}((\mathcal{N},E))^{\hat{\jmath}(\Gamma)}.$$

To get the desired "Σ_0-iteration strategy" for ~~$\mathcal{N}_{\xi+2}$, let $\rho = \inf\{\rho_k(\mathcal{M}) \mid \mathcal{M}$ is a~~

$\mathcal{N}_{\xi+2}$, we use the same method by which we constructed Ω and the $\Omega(\mathcal{M},k)$'s. We omit further detail here. Call that strategy $\Psi_{\xi+2}$. (See footnote.)

This defines $(\mathcal{N}_{\xi+2}, \dot{\Psi}_{\xi+2})$. We must verify $(\dagger)_{\xi+2}$ and $(*)_{\xi+2}$. We skip that for now, and go on to $(\mathcal{N}_{\xi+3}, \mathring{\Psi}_{\xi+3})$:

<u>Definition 3.5</u> Let $(\mathcal{M}, \Psi)$ be a $\hat{\jmath}(T)$-lod-pair having a top block. We say that ~~this~~ $(\mathcal{M}, \Psi)$ is <u>$\kappa^{\mathcal{M}}$-extender-ready</u> iff letting $\beta = \max((\kappa^{\mathcal{M}})^{+\mathcal{M}}, o(\kappa)^{\mathcal{M}})$, we have $\beta < o(\mathcal{M})$, and $L_p^{\Psi}(\mathcal{M})^{\hat{\delta}(T)} \models o(\mathcal{M}) = \beta^+$.

If $(\mathcal{N}_{\xi+2}, \Psi_{\xi+2})$ is not θ^0-extender-ready, then we obtain $(\mathcal{N}_{\xi+3}, \Psi_{\xi+3})$

<u>footnote</u>: It's possible that $\Psi_{\xi+2}$ would be our pointclass generator. We assume otherwise.

from $(\mathcal{N}_{\xi+2}, \Psi_{\xi+2})$ in the very same way that we obtained $(\mathcal{N}_{\xi+2}, \Psi_{\xi+2})$ from $(\mathcal{N}_{\xi+1}, \Psi_{\xi+1})$. And so on. That is, we define $(\mathcal{N}_\alpha, \Psi_\alpha)$ for $\alpha \geq \xi+1$ by induction on α. For α a limit, $\mathcal{N}_\alpha = \text{"}\bigcup_{\beta<\alpha}\text{"}\, \mathcal{N}_\beta$, and $\Psi_\alpha = \bigoplus_{\beta<\alpha} \Psi_\beta$. If $(\mathcal{N}_\alpha, \Psi_\alpha)$ is θ^0-extender ready, then we stop the induction. Otherwise, $\mathcal{N}_{\alpha+1} = Lp^{\Psi_\alpha}(\mathcal{N}_\alpha)^{\hat{\lambda}(\Gamma)}$, and $\Psi_{\alpha+1}$ is the j-realization strategy we have indicated above. Lemmas 3.2 and 3.3 insure that the fine structure works out.

Eventually, we reach a θ^0-extender-ready pair $(\mathcal{N}_\alpha, \Psi_\alpha)$.

§4 The existence of the next extender

Let $(\mathcal{N}_\alpha, \mathring{\Psi}_\alpha)$ be Θ^0-extender-ready. Let us also assume that $\Vdash_{Col(\omega,<\kappa_1)}$ "$\mathring{\Psi}_\alpha$ is a j-realization strategy", in the following sense.

<u>Definition 4.1</u> Let $(\mathcal{N}, \Sigma)$ be a $\hat{j}(\tau)$-hod-pair, with $j \upharpoonright \mathcal{N} \in M$. We say Σ is a <u>j-realization strategy</u> iff whenever $i : \mathcal{N} \xrightarrow{\vec{\mathcal{T}}} Q$ is a non-dropping Σ-iteration, then there is a $\tau : Q \to j(\mathcal{N})$ such that $j = \tau \circ i$, and $\Sigma_{\vec{\mathcal{T}}, Q} = j(\Sigma)^\tau$.

All the strategies we have constructed have been j-realization strategies. We now complete the cycle in sections 2 and 3 by showing

Lemma 4.2 Let $(\mathcal{M}, \Sigma)$ be a $\hat{j}(\Gamma)$-hod-pair ~~such that~~ from our construction such that Σ is a j-realization strategy, and $(\mathcal{M}, \Sigma)$ is θ^0-extender-ready; then there is an E such that E is j-certified over $(\mathcal{M}, \Sigma)$.

Proof Let $\beta = \max((\kappa^{\mathcal{M}})^{+\mathcal{M}}, o(\kappa^{\mathcal{M}})^{\mathcal{M}})$, and $\gamma = o(\mathcal{M})$. Let $\pi^{\Sigma} : \mathcal{M} \to H_1$ be the map of the direct limit system. For $c \in [\gamma]^{<\omega}$ and $A \in H_0^+$, we put

$$(c, A) \in E \text{ iff } \pi^{\Sigma}(c) \in j(A).$$

This is of course the only possibility for E. What we need to see is that $(\mathcal{M}, E)$ is a hod premouse. Most of the proof was given in [3], in the case that E is the order zero measure.

First, amenability:

<u>Claim ‡</u> For every $\eta < o(\mathcal{M})$ and $\xi < o(H_0^+)$,

$E \cap ([\eta]^{<\omega} \times \mathcal{M}|\xi) \in \mathcal{M}$.

<u>Proof</u>. Let $\langle A_\alpha \mid \alpha < \theta^0 \rangle$ enumerate $\mathcal{M}|\xi$, and $A(\alpha, u)$ iff $u \in A_\alpha$. Let

$$B = j(A) \cap \left(\pi^\Sigma(\eta) \times \pi^\Sigma(\eta)\right).$$

Since $B \in H_1$, B is $OD^{\hat{j}(\Gamma)}$. But then for $c \in [\eta]^{<\omega}$ and $A_\alpha \subseteq [\theta^0]^{|c|}$, with $\alpha < \theta^0$:

$$(c, A_\alpha) \in E \text{ iff } \pi^\Sigma(c) \in j(A_\alpha),$$

$$\text{iff } \pi^\Sigma(c) \in B_{\pi^\Sigma(\alpha)}.$$

Thus $E \cap ([\eta]^{<\omega} \times \mathcal{M}|\xi)$ is $(OD_\Sigma)^{\hat{j}(\Gamma)}$, and hence in $\mathcal{M}$ by Σ-mouse capturing and the fact that $(\mathcal{M}, \Sigma)$ is θ^0-extender-ready.

Claim 2 Each E_c is $\mathcal{N}$-θ^0-complete.

Proof Let $\beta < \theta^0$, and $\langle A_\alpha \mid \alpha < \beta \rangle \in \mathcal{N}$, with each $A_\alpha \in E_c$. We need to see $\bigcap_{\alpha<\beta} A_\alpha \in E_c$, that is, $\pi^\Sigma(c) \in j(\bigcap_{\alpha<\beta} A_\alpha) = \bigcap_{\alpha<j(\beta)} j(\vec{A})_\alpha$. We use j-condensation, lemma 2.4. Let $\alpha < j(\beta)$, and $\alpha = \pi^{\Phi}_{Q,H_1}(\bar{\alpha})$, where Q is a Σ-iterate of $\mathcal{N}$, and Φ is the Q-tail of Σ. We claim $\pi^\Sigma(c) \in j(A)_\alpha = j(A)_{\pi^\Phi(\bar{\alpha})}$. For

$$L(Hom^*_h, j(\mathcal{N})) \models \forall \nu < \beta \left(\pi^\Sigma_-(c) \in j(A)_{\pi^\Sigma(\nu)} \right).$$

The right hand side is a statement about $\mathcal{N}$, c, β, and the strategy $\Sigma_* = j(\Sigma)^j$.

Since Σ is a j-realization strategy, we have $j = \tau \circ i$ and $\Phi = j(\Sigma)^\tau$, for some realization r. By 2.4 then,

$$L(\mathrm{Hom}^*_h, j(M)) \models \forall \gamma < i(\beta)\ \left(\pi^\Phi(i(c)) \in j(A)_{\pi^\Phi(\gamma)}\right).$$

But $\pi^\Phi(i(c)) = \pi^\Sigma(c)$, and $\bar{\alpha} < i(\beta)$, so this is what we need.

$\boxtimes$

<u>Claim 3</u> E is N-normal.

<u>Proof</u> Let $f : [\theta^0]^{|c|} \to \theta^0$, $f \in H_0^+$, and $c \in [\gamma]^{<\omega}$, and $f(u) < \max(u)$ for E_c a.e. u. So $j(f)(\pi^\Sigma(c)) < \max \pi^\Sigma(c)$. We must find $\xi < \max(c)$ such that $j(f)(\pi^\Sigma(c)) = \pi^\Sigma(\xi)$. But we can find an iteration $i : N \to Q$ by Σ as above, a j-realization $r : Q \to j(N)$, and for $j(\Sigma)^r$ = the Q-tail of $\Sigma = \Phi$, $j(f)(\pi^\Sigma(c)) \in \operatorname{ran} \pi^\Phi \restriction i(\max(c))$. That is,

$j(f)(\pi^{\Phi}(i(c))) \in \operatorname{ran} \pi^{\Phi} \restriction \max(i(c))$

Pulling back via j-condensation, we get $j(f)(\pi^{\Sigma}(c)) \in \operatorname{ran} \pi^{\Sigma} \restriction \max(c)$, as desired.

$\boxtimes$

<u>Claim 4</u> $\mathrm{Ult}_0(\mathcal{N}, E)|\gamma = \mathcal{N}$.

<u>Proof</u> Let $\tau : \mathrm{Ult}_0(\mathcal{N}, E) \to j(\mathcal{N})$ be given by

$$\tau([a, f]) = j(f)(\pi^{\Sigma}(a)).$$

Since $a = [a, id]$, $\tau \restriction \gamma = \pi^{\Sigma} \restriction \gamma$.

Thus $\mathrm{Ult}_0(\mathcal{N}, E)|\gamma$ is isomorphic to $\pi^{\Sigma}``\mathcal{N}$, and hence to $\mathcal{N}$.

$\boxtimes$

Finally, we show that γ is the proper place to index E.

<u>Claim 5.</u> β is a cutpoint of $Ult(N,E)$, and $\gamma = \beta^{+Ult(N,E)}$.

<u>Proof</u> Toward contradiction, let F be the first extender on the sequence of $Ult(N,E)$, with $crit(F) = \theta^0$ and $lh F \geq \beta$. By claim 4, we have $lh F \geq \gamma$. Let $\tau : Ult_0(N,E) \longrightarrow j(N)$ as in claim 4.

Let $j_1 = j(j) : M \longrightarrow N$. Consider the diagram

$$\begin{tikzcd}
 & & j_1(j(N)) \\
N \arrow[r, "j"] \arrow[dr, "i"'] & j(N) \arrow[r, "u"'] \arrow[ur, "j_1"] & Ult(j(N), \tau(F)) \arrow[u, "\sigma"'] \\
 & Ult(N,E) \arrow[u, "\tau"'] \arrow[r, "t"'] \arrow[d, equal] & Ult(M,F) \arrow[u, "k"'] \\
 & M &
\end{tikzcd}$$

Since $\tau(F)$ is on the sequence of H_1 (and hence $j(\mathcal{N})$), it is j_1-certified over $(j(\mathcal{N})||\mathrm{lh}\,\tau(F), \Lambda)$, where $\Lambda = j(\Sigma)_{j(\mathcal{N})||\mathrm{lh}F}$. This is what gives us the j_1-realization map σ. The maps t, u, and k come from copying.

<u>Subclaim</u> $\mathrm{lh}(F) = \gamma$.

<u>Proof</u> Since $\mathrm{lh}(F)$ was least, there are no $\mathrm{crit} = \theta^0$ extenders with lengths $\in [\beta, \mathrm{lh}F)$ on the sequence of $\mathcal{M}$ ~~...~~ $= \mathrm{Ult}(\mathcal{N}, E)$. Since $\mathcal{M}$ is a hod premouse, β is a cutpoint of $\mathrm{Ult}(\mathcal{M}, F)$, and $\mathrm{lh}(F) = \beta^{+\mathrm{Ult}(\mathcal{M},F)}$. So if $\mathrm{lh}(F) > \gamma$, there a least $\mathcal{P} \triangleleft \mathcal{M}||\mathrm{lh}(F)$ such that $\mathcal{P} \models |\gamma| = \beta$. We have $\mathcal{N} \trianglelefteq \mathcal{P}$. Note also that $\mathcal{P}$ is a level of $Lp^{\Sigma}(\mathcal{N})$. This is because $j(\Sigma)^{\tau}_{\mathcal{N}} = \Sigma$ (because

$\tau \restriction \mathcal{N} = \pi^{\Sigma} \restriction \mathcal{N}$, and Σ is pullback consistent), and the existence of τ guarantees the required iterability. But $(\mathcal{N}, \Sigma)$ was Θ^0-extender-ready, so there is no level of $Lp^{\Sigma}(\mathcal{N})$ collapsing $o(\mathcal{N})$.

⊠

We now show that F is certified by j over $(\mathcal{N}, \Sigma)$, which then gives $F = E$, contrary to $F \in Ult(\mathcal{N}, E)$, a contradiction.

Let $H_2 = j_1(H_1)$ be the HOD of $L(j_1''(j(\Gamma)))$. Note that $(\mathcal{N}, \hat{j}_1(\Sigma))$ is a point in the system converging to H_2, and since $\tau(\mathcal{N}) = \pi^{\Sigma}(\mathcal{N})$ and $\Sigma \subseteq \hat{j}_1(\Sigma)$, $j(\Sigma)_{\tau(\mathcal{N})}$ is a tail of $\hat{j}_1(\Sigma)$. Thus

$$\pi^{\hat{j}_1(\Sigma)}_{\mathcal{N}, H_2} = \pi^{\hat{j}(\Sigma)}_{\tau(\mathcal{N}), H_2} \circ \pi^{\Sigma}_{\mathcal{N}, H_1}.$$

By the way we defined our realizing maps

$$\pi^{\Sigma}_{\eta, H_1} = \tau \restriction \eta$$

and

$$\pi^{j(\Sigma)}_{\tau(\eta), H_2} = \sigma \restriction \overline{\eta}.$$

Returning now to the diagram above, for $c \in [\Sigma o(\eta)]^{<\omega}$ and $A \in H_0^+$, we have

$$
\begin{aligned}
A \in F_c \quad &\text{iff} \quad c \in t(A) \\
&\text{iff} \quad c \in t(i(A))
\end{aligned}
$$

(since $i(A) \cap H_0 = A \cap H_0$)

$$
\begin{aligned}
&\text{iff} \quad \tau(c) \in u(\tau(i(A))) \\
&\text{iff} \quad \tau(c) \in u(j(A)) \\
&\text{iff} \quad \sigma(\tau(c)) \in \sigma(u(j(A))) \\
&\text{iff} \quad \pi^{\hat{j}_1(\Sigma)}_{\eta, H_2}(c) \in j_1(j(A)) \\
&\text{iff} \quad \pi^{\Sigma}_{\eta, H_1}(c) \in j(A).
\end{aligned}
$$

This shows that F is certified by j over (η, Σ). That completes the proof of Claim 5. $\boxtimes$

This completes the proof of lemma 4.2. ☒

§5 Limit points of the §2–§4 cycle.

We have shown how the $\aleph_0$ construction proceeds from the moment one extender with critical point θ^0 is added, until the moment the next such extender is added. (That is, we have done so at stages $< \kappa_0^+$, or more generally, at stages η such that $j \upharpoonright \eta \in M$.)

Now suppose we are at a limit λ of such stages. ~~We put~~ If we are below O_λ^P, we put

$$\eta_\lambda = \text{"}\bigcup_{\alpha<\lambda}\text{"}\, \eta_\alpha$$

$$\dot{\varphi}_\lambda = \text{name for } \bigoplus_{\alpha<\lambda} \dot{\varphi}_\alpha .$$

Our indexing is such that $o(\mathcal{N}_\lambda)$ cannot be an index for an extender with critical point θ^0. So we simply proceed from $(\mathcal{N}_\lambda, \dot{\psi}_\lambda)$ to the next θ^0-extender-ready stage, just as we did in §3.

If we are beyond O^P_λ, there may have been coring down on the way to $\mathcal{M}_\lambda$, so we take $\mathcal{N}_\lambda$ to be the usual lim inf, and $\dot{\psi}_\lambda$ to be the complete Σ_0-strategy we get from the $\dot{\psi}_\alpha$ for $\alpha < \lambda$. We may be able to put an extender with critical point $> \theta^0$ on at $o(\mathcal{N}_\lambda)$, subject to some background condition. If so, we do it, and core down. If not, we proceed as in §3, but again, $Lp^{\dot{\psi}_\lambda}(\mathcal{N}_\lambda)$-closure may cause more coring down. We go into this further in ~~§7 (later)~~ part II.

References

[1] G. Sargsyan, Covering with Universally Baire operators, Feb. 2013.

[2] G. Sargsyan, A tale of hybrid mice, Ph.D. thesis, Berkeley 2009.

[3] J. Steel, Remarks on a paper by Sargsyan, April 2013

[4] W. Mitchell, J. Steel, Fine structure and iteration trees, Springer LNL.

[5] R. Schindler, M. Zeman, Fine structure, Handbook of Set Theory.

[6] T. Wilson, Contributions to descriptive inner model theory, Ph.D. thesis, Berkeley 2012.

[7] J. Steel, ~~Ou~~ An Outline of inner model theory, Handbook of Set Theory.

[8] G. Sargsyan, The hierarchy of LSA hod mice, informally circulated.