

§2. Construction of \mathcal{H}_0 : adding an extender with length below K_0^+

(46) ~~(47)~~

We have $j: V \rightarrow M$, $\text{crit}(j) = K_0$,

as in section 0. Let g be V -generic over

$\text{Col}(w, < K_0)$. ~~Let $\mathcal{D}(V, K_0) = \text{HOD}_g^+(R_g^+)$~~

~~is a model of $\text{AD}_R^+ + \text{AD}_G^+$~~ Let $\Gamma \subseteq \text{Hom}_g^+$

be such that

$$L(\Gamma, R_g^+) = \text{AD}_R + \text{DC} + \text{"HOD} \cap \theta$$

is the direct limit of all
hop-pairs (P, Σ) s.t.

Σ is fullness-pres. and
has branch condensation,

under the comparison maps".

Put

$$\theta^0 = \theta^{L(\Gamma, R)},$$

$$H_0 = \text{HOD}^{L(\Gamma, R)} \upharpoonright \theta^0.$$

Let $\hat{j}: V \mathcal{E}gJ \rightarrow M \mathcal{E}hJ$ extend j , and

$$H_0^+ = L_P^{\Sigma_{H_0}}(H_0)^{\hat{j}(\Gamma)},$$

where $\Sigma_{H_0} = \bigoplus_{\alpha \leq \theta^0} \Sigma_{H_0(\alpha)}$, and

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$\Sigma_{H_0(\alpha)}$ is the common tail of all $\hat{j}(\Sigma_P)$, where $H_0(\alpha)$ is a Σ_P -iterate of P . If $\Gamma = \text{Hom}_g^*$, then $\hat{j}(\Gamma) = \text{Hom}_h^*$. Otherwise, any complete Γ set A has a Hom_g^* code, and \hat{j} moves this code to the code of the complete $\hat{j}(\Gamma)$ set $\hat{j}(A)$.

Working in MLHJ, we shall build a $\hat{j}(\Gamma)$ -hod-pair $(\mathcal{H}, \Sigma_{\mathcal{H}})$ that extends (H_0^+, Σ_{H_0}) . \mathcal{H} will have a top block, and θ^0 will begin that block. We shall show that if $\Gamma \neq \text{Hom}_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ either reaches \mathcal{O}_h^P , or is a pointclass generator for $\hat{j}(\Gamma)$ in MLHJ. We shall show that if $\Gamma = \text{Hom}_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ reaches \mathcal{O}_h^P .

In fact, \mathcal{H} and $\Sigma_{\mathcal{H}}$ are constructed

in M , and $MZHJ$ is just used to record properties of them. So we'll have $H \in M$, and an iteration strategy Ψ for H with $\Psi \in M$, and defined on all $\vec{I} \in M$ of size $< j(\kappa)$. Ψ will "determine itself on generic extensions" in such a way that for all \mathcal{A} on $\text{coll}(\omega, < j(\kappa))$ there is $\dot{\Psi}^{\mathcal{A}} \cong \Psi$ defined on HC^{MZHJ} .

What we are calling Σ_H is then $\dot{\Psi}^h$. (So $M \models \text{H}^{\text{coll}(\omega, < \kappa, \mathcal{A})} \dot{\Psi} \subseteq \dot{\Psi}$.) Ψ is a "symmetric name", in that $\dot{\Psi}^{\mathcal{A}} = \dot{\Psi}^m$ whenever $HC^{MZHJ} = HC^{M\Sigma mJ}$.

So in M , we are constructing by induction on ξ pairs $(\mathcal{N}_\xi, \dot{\Psi}_\xi)$ such that the following induction hypotheses are hold. We call them (†).

Induction Hypotheses $(H)_{\xi}$: In M , the following hold, for $(\pi, \dot{\psi}) = (\pi_{\xi}, \dot{\psi}_{\xi})$:

- (a) π is a hod premouse extending H_0^+ ,
- (b) $\dot{\psi}$ is a $\text{col}(\omega, \leq \kappa_1)$ -name such that $\dot{\psi}^{\mathcal{L}} = \dot{\psi}^{\mathcal{M}}$ whenever $\text{HC}^{\mathcal{M} \text{REG}} = \text{HC}^{\mathcal{M} \text{SUMT}}$,
- (c) $\underset{\text{H}}{\text{col}(\omega, \leq \kappa_1)} (\overset{\vee}{\pi}, \dot{\psi})$ is a $j(\overset{\circ}{\Gamma})$ -hod-pair such that $\dot{\psi}$ is $j(\overset{\circ}{\Gamma})$ -fullness-preserving and has branch condensation and is positional.

It is easiest to describe the construction of $\pi_{\xi+1}$ and $\dot{\psi}_{\xi+1}$ if we have $j \upharpoonright \pi_{\xi} \in M$. This is of course true if $o(\pi_{\xi}) < \kappa_0^+$, and may be true beyond that if j witnesses more than measurability of κ_0 .

Remarks

- (1) If $\rightarrow \square_{\kappa_0}$, then and we reach ξ such that $o(\pi_{\xi}) = \kappa_0^+$, then

$\mathcal{N}_\xi \models ZFC + "$ θ_0 is a strong limit of Woodin's"

So although we haven't reached $O_h^{\mathbb{P}}$, we're close.

(2) If j witnesses K_0 is huge, then we can go up to \mathcal{N}_{K_1} , and we will definitely reach $O_h^{\mathbb{P}}$ before that.

(3) In clause (†)(c), $\dot{\Gamma}$ is a symmetric name for Γ . We assume $\dot{\Gamma} \in V$ for simplicity; in general it will be in some size $< K_0$ extension of V .

We have one further induction

hypothesis: for $(\mathcal{N}, \dot{\Psi}) = (\mathcal{N}_\xi, \dot{\Psi}_\xi)$,

(*) $_\xi$ if $j \upharpoonright \mathcal{N} \in M$, then

$$M \models \text{IH}^{\text{col}(\dot{\omega}, < K_1)} \quad \dot{\Psi} = j(\dot{\Psi})^{\dot{j}}$$

(Here on the right side, "j" should be replaced by " $j \upharpoonright \mathcal{N}$ ", to be precise.) (In the superscript only.)

Some explanation is in order. Let $d_1 = j(j)$, with $j_1: M \rightarrow N$, and $K_2 = j_1(K_1)$. We have that $j(\dot{\Psi})$ is a $\text{col}(\omega, \langle K_2 \rangle)$ name in N for a strategy for $j(\mathcal{M})$. But letting h be $\text{col}(\omega, \langle K_1 \rangle)$ -generic / M , we can make sense of $j(\dot{\Psi})_h$ by the symmetry of $j(\dot{\Psi})$: it is the common value of all $j(\dot{\Psi})_d \cap \text{col} V_{K_2}^{M \cap H}$ for $h \in d$, d on $\text{col}(\omega, \langle K_2 \rangle)$. So $j(\dot{\Psi})_h \in M \cap H$.

Thus its $j \upharpoonright \mathcal{M}$ - pullback $j(\dot{\Psi})_h^d$ makes sense in $M \cap H$. It is defined on all $H \in M \cap H$. (Note $H \in M \cap H \cup \{j \upharpoonright \mathcal{M}\} \in N \cap H$.)

(*)_h then says that for any such h , $\dot{\Psi}_h = (j(\dot{\Psi})_h)^d$.

This almost follows from ~~branch~~ ^{hull} condensation. Namely, let $\dot{\Psi}_g$ be the common value of all $\dot{\Psi}_h \cap V_{K_1}^{MEGT}$, for $g \leq h$, h on $Coll(\omega, \leq K_1)$ and g on $Coll(\omega, \leq K_0)$. $\dot{\Psi}_g \in V_{\Sigma g T}$, and has branch condensation.

Moreover, with $\hat{j}: V_{\Sigma g T} \rightarrow M[h]$, we have $\hat{j}(\dot{\Psi}_g) = j(\dot{\Psi})_h$. ~~and~~ But letting $\mathcal{I} \in V_{\Sigma g T}$ be by $\dot{\Psi}_g$, we have $\hat{j}(\mathcal{I})$ is by $j(\dot{\Psi})_h$, so $\hat{j} \mathcal{I} \cong \hat{j}'' \mathcal{I}$ is by $j(\dot{\Psi})_h$. [The fact that $\hat{j}'' \mathcal{I} \notin M[h]$ can be overcome with an absoluteness argument.]

Thus \mathcal{I} is by $(j(\dot{\Psi})_h)^{\dot{i}}$.

However, the argument of the last paragraph falls short of proving $(*)_{\xi}$ from $(+)_{\xi}$,

because it only works for \mathcal{I} in $V\mathcal{E}g\mathcal{I}$,
so that $j^{\uparrow}(\mathcal{I})$ makes sense. So it only
gives $\dot{\Psi}_g \subseteq (j^{\uparrow}(\dot{\Psi}_h))^{\downarrow}$, not the full $(*)$.

Insert p. 48f.

Set

$$\mathcal{N}_0 = H_0^+$$

$\dot{\Psi}_0 =$ canonical coll $(\omega, \langle K_1 \rangle)$ - name
in M for $\bigoplus_{\alpha \in \theta^0} \Phi_{\alpha}$, where

$\Phi_{\alpha} = H_0(\alpha)$ - tail of $j^{\uparrow}(\Delta)$,
for any and all (P, Δ) s.t.
 $H_0(\alpha) = M_{\infty}(P, \Delta)$ in $V\mathcal{E}g\mathcal{I}$.

We have $o(\mathcal{N}_0) \prec K_0^+$, so $j^{\uparrow} \mathcal{N}_0 \in M$.

The reader can easily check $(\dagger)_0$ (see [13] and [23]). The main thing is that $\mathcal{N}_0 \neq \theta^0$ is regular.

For $(*)_0$, this says in our earlier notation
that Σ_{H_0} , as defined on $HC^{M\mathcal{E}g\mathcal{I}}$, is the

We need two further induction hypotheses:

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(†)_ε (d): If E is on the \mathcal{N} -sequence and $\text{cmr}(E) = \theta^0$, then in \mathcal{M}

$\text{col}(\omega, < \kappa_1)$ \check{E} is certified by $j^* H_0^+$ over $(\mathcal{M} \parallel \mathcal{M} E, \dot{\Psi}_{\mathcal{M} \parallel \mathcal{M}(E)})$.

(See Definition 2.5 below for "certifies".)

(*)_ε (b): (Absolute condensation to pullbacks)

Let $\pi: R \rightarrow V_\gamma^{\mathcal{M}}$ with γ large, $\pi \in \mathcal{M}$,

R transitive, $\forall_{\kappa_0+1} \{ \mathcal{N} \} \in R$, and $|R| < \kappa_1$.

Suppose $\pi(\overline{j(\dot{\Psi})}) = j(\dot{\Psi})$. Let h be \mathcal{M} -generic

over $\text{col}(\omega, < \kappa_1)$, and h_0 be R -generic over

$\text{col}(\omega, < \kappa_1)$, with $h_0 \in \mathcal{M}[h]$; then

$$\overline{j(\dot{\Psi})}_{h_0} \subseteq (j(\dot{\Psi})_h)^\pi$$

j -pullback of Σ_{H_1} , as defined
on $V_{K_2}^{N \times H_1}$ (where $j_1: M \rightarrow N$).

That follows from the fact that $j_1^* H_0$ is
the iteration map $\pi: H_0 \rightarrow H_1$ by Σ_{H_0} ,
and Σ_{H_1} is the H_1 -tail of Σ_{H_0} whenever
they are defined, and Σ_{H_0} is pullback
consistent, whenever it is defined.

Because we are just shooting for
 $O_h^{\mathbb{P}}$, we shall never add Θ^0 -relevant
extenders with critical point $> \Theta^0$. As a
consequence, all levels of our construction
will be fully sound, and we'll never have
to core down. Thus we'll set

$$\mathcal{N}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{N}_\alpha$$

and

$$\dot{\Psi}_\lambda = \text{canonical name for } \bigoplus_{\alpha < \lambda} \dot{\Psi}_\alpha$$

for λ limit.

Now suppose we are given $(\mathcal{N}_\xi, \dot{\Psi}_\xi) =$
~~the~~ $(\mathcal{N}, \dot{\Psi})$ satisfying $(\dagger)_\xi$ and $(*)_\xi$.

We assume that $\exists \mathcal{N} \in \mathcal{M}$, so that $(*)_\xi$
is not vacuous, and deal with the more
general case in ~~the next~~ ^{a subsequent} section. We

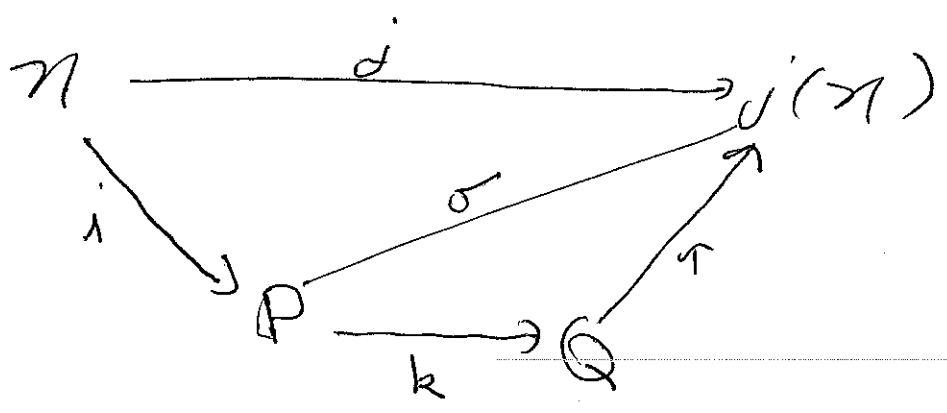
Moreover shall obtain $(\mathcal{N}_{\xi+1}, \dot{\Psi}_{\xi+1})$

in one of two ways:

- (i) close under $(L_p\text{-strategy})^\Gamma$, as on p. 16 ff., or
- (ii) add an extender with critical point θ° .

The main tool in our arguments will
be "j-condensation", i.e. lemma 11.15
of $\Sigma 1J$, generalized slightly as lemma
2 in $\Sigma 3J$. The form we need is

Lemma 2.4 Let $(\mathcal{N}, \dot{\Psi})$ satisfy $(\dagger)_{\xi}$ and $(*)_{\xi}$, and suppose $j^{\uparrow} \mathcal{N} \in \mathcal{M}$. Let h be $\text{Col}(\omega, \tau K_1)$ generic, and suppose we have



commuting, with $P, Q, \sigma, \tau \in \mathcal{M}[h]$ and countable there. Let $\Sigma_P = (j(\dot{\Psi})_h)^{\sigma}$ and $\Sigma_Q = (j(\dot{\Psi})_h)^{\tau}$ be the pullback strategies.

Then

- (i) $\Sigma_P, \Sigma_Q \in \hat{j}(\Gamma)$
- (ii) $\theta^{\circ} = \theta^{\downarrow}(\Gamma, \mathcal{N})$ is regular in $\mathcal{L}(\Gamma, \mathcal{N})$, and
- (iii) for all wfts φ , all $s \in P$, all $X \in \mathcal{N}$:

$$\mathcal{L}(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[P, \Sigma_P, s, j(X)]$$

iff

$$\mathcal{L}(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[Q, \Sigma_Q, k(s), j(X)].$$

Proof sketch We shall prove conclusion (i).

The rest is proved just as in [17] and [31], so we do not give any details here. For (i), we let $j_1 = j(j)$ and $j_1: M \rightarrow N$, and let g, h, ℓ be on ~~the~~ $\text{col}(\omega, \leq \kappa_0)$, $\text{col}(\omega, \leq \kappa_1)$, $\text{col}(\omega, \leq \kappa_2)$ respectively, with $g \subseteq h \subseteq \ell$. We have, for Q, τ as in \mathcal{A}_0 hypotheses

$$\Sigma_Q = \left(\text{common value of all } j(\Psi)_m \cap HC^{M \times 2 \times J}, \text{ for } m \geq h, m \text{ on } \text{col}(\omega, \leq \kappa_2) \right)^\tau$$

by definition. So

$$\hat{j}_1(\Sigma_Q) = \left(\text{common value of all } \hat{j}_1(j(\Psi))_m \cap HC^{N \times 2 \times J}, \text{ for } m \geq \ell \text{ on } \text{col}(\omega, \leq j_1(\kappa_2)) \right) \hat{j}_1(\tau)$$

$$= \left(\text{common value of all } \hat{j}_1(j(\Psi))_m \cap HC^{N \times 2 \times J}, \text{ for } m \geq \ell \text{ on } \text{col}(\omega, \leq j_1(\kappa_2)) \right) \hat{j}_1 \circ \tau$$

(since $\hat{j}_1(\tau) = \hat{j}_1 \circ \tau$)

$$= \left(\text{common value of all } f_i(j(\dot{\Psi}))_m \in HC^{N \times J} \right)_{d_i}^T$$

$$= j(\dot{\Psi})_Q^T$$

The last step is $(*)_f$, moved from M ~~where~~ where it holds of $\dot{\Psi}$ ~~and~~ and $j^T \pi$, to N ~~where~~, where it holds of $j(\dot{\Psi})_x$ and $j(j^T \pi) = f_i^T j(\pi)$.

But then

$$N \times J \models j(\dot{\Psi})_Q \in \hat{f}_1(\hat{j}(\Gamma)),$$

so

$$N \times J \models j(\dot{\Psi})_Q^T \in \hat{f}_1(\hat{j}(\Gamma)),$$

so

$$N \times J \models \hat{f}_1(\Sigma_Q) \in \hat{f}_1(\hat{j}(\Gamma)),$$

so

$$M \times J \models \Sigma_Q \in \hat{j}(\Gamma),$$

as desired,



We add extenders when they fit on the sequence, i.e. yield hod premice, and are "certified by \mathcal{J} " in the following sense.

Def 2.5 Let (\mathcal{P}, Λ) be a $\hat{\mathcal{J}}(\Gamma)$ hod pair in $M\mathcal{E}H\mathcal{J}$, and suppose \mathcal{P} has a top block beginning at κ . Suppose

$$k: \mathcal{P}/(\kappa^+)^{\mathcal{P}} \longrightarrow H_{\mathcal{J}}^+$$

is fully elementary. We say that

E is k -certified over (\mathcal{P}, Λ) iff

(a) (\mathcal{P}, E) is a hod premouse, and

(b) for all $a \in [\text{lh}(E)]^{\lt \omega}$ and $X \in \mathcal{P}/(\kappa^+)^{\mathcal{P}}$,

$$X \in E_a \text{ iff } \pi_{\mathcal{P}, \infty}^{\Lambda}(a) \in k(X),$$

where $\pi_{\mathcal{P}, \infty}^{\Lambda}: \mathcal{P} \rightarrow H_{\mathcal{J}}$ is the map given by (\mathcal{P}, Λ) being in the hod-limit-system of $L(\hat{\mathcal{J}}(\Gamma), \mathcal{R}_{\mathcal{J}}^*)$.

We consider first the case in which $\mathcal{N}_{\xi+1}$ is obtained by adding an extender with critical point θ^0 to \mathcal{N}_{ξ} .

Let h be $\text{col}(\omega, \langle \kappa_1 \rangle)$ -generic.

We have $(\mathcal{N}_\xi, \dot{\Psi}_\xi) = (\mathcal{N}, \dot{\Psi})$ our current pair satisfying $(T)_\xi$ and $(*)_\xi$.

Set

$$\Sigma = \dot{\Psi}_h$$

So in $\mathcal{M}[h]$, where we will be working most of the time, (\mathcal{N}, Σ) is a $\hat{J}(\Gamma)$ hod pair.

Case 1 There is an E that is \hat{J} -certified over (\mathcal{N}, Σ) .

In this case, there is a unique such E , (so $E \in \mathcal{M}$). We set

$$\mathcal{N}_{\xi+1} = (\mathcal{N}, E)$$

Our goal now is to construct an iteration strategy \mathcal{Q} for (\mathcal{N}, E) such that in $\mathcal{M}[h]$, $((\mathcal{N}, E), \mathcal{Q})$ is a $\hat{J}(\Gamma)$

hod pair, and \mathcal{Q} is fullness preserving
 and has branch condensation. The construction
 will give a symmetric term $\dot{\mathcal{Q}}$ such
 that $\dot{\mathcal{Q}}^h = \mathcal{Q}$, and we'll show
 (*) holds for $((\mathcal{N}, E), \dot{\mathcal{Q}})$.

Notation If \mathcal{P} is a hod premouse having
 a top block, then $\kappa^{\mathcal{P}}$ is the ordinal
 that begins the top block of \mathcal{P} .

What we need about (\mathcal{P}, E) to construct an iteration strategy is just that its top block is not too complicated.


Definition 2.6 Let \mathcal{P} be a hod premouse having a top block. We say that the top block of \mathcal{P} is below $O^{\mathcal{P}}$ iff whenever E is a $K^{\mathcal{P}}$ -relevant extender in the top block of \mathcal{P} , then $\text{crit}(E) = K^{\mathcal{P}}$.

Note that if \mathcal{U} is a normal tree on a hod premouse \mathcal{P} , and $[0, \alpha]_{\mathcal{U}}$ does not drop, and $E_{\alpha}^{\mathcal{U}}$ is taken from the top block of $\mathcal{M}_{\alpha}^{\mathcal{U}}$ with $\text{crit}(E_{\alpha}^{\mathcal{U}}) = K^{\mathcal{M}_{\alpha}^{\mathcal{U}}}$, then $E_{\alpha}^{\mathcal{U}}$ is applied to $\mathcal{M}_{\alpha}^{\mathcal{U}}$ (i.e. $\mathcal{M}_{\alpha+1}^{\mathcal{U}} = \text{Ult}(\mathcal{M}_{\alpha}^{\mathcal{U}}, E_{\alpha}^{\mathcal{U}})$), and the rest of \mathcal{U} is based on $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$. This is because $\text{lh}(E_{\alpha}^{\mathcal{U}})$ is a cutpoint of $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$.

Thus if P has a top block that is below O^P , and any non-dropping iteration of P can be given by a sequence $\langle (P_\alpha, \vec{T}_\alpha) \mid \alpha < \eta \rangle$ where $P_0 = P$, \vec{T}_α is a stack of normal trees with base model P_α and last model $P_{\alpha+1}$, $P_\lambda = \lim_{\alpha < \lambda} P_\alpha$ for λ limit, and for each α , either

- (i) \vec{T}_α uses no extenders in the top block of P_α or its images, or
- (ii) $P_{\alpha+1} = \text{Ult}(P_\alpha, G)$, for some G in the top block of P_α , with $\text{crit}(G) = \kappa^{P_\alpha}$.

Definition ^{Zelbol} An iteration $\langle (P_\alpha, \vec{T}_\alpha) \mid \alpha < \eta \rangle$ satisfying (i) and (ii) is said to be in normal form.

~~We now describe our complete iteration strategy  for (M, E) . Set $Q_0 = M$, $E_0 = E$, $P_0 = (Q_0, E_0)$, and $\Omega_0 = \Sigma$.~~

Before describing our iteration strategy for (π, E) , we make some definitions and prove a few simple things.

Definition 2.7 (a) Let $\sigma: R \rightarrow H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$ -hod pair in MLHJ. We say Λ is σ -consistent iff letting $\pi: R \rightarrow H_1$ be the iteration map by Λ of the $\hat{j}(\Gamma)$ -hod-limit system, $\pi \upharpoonright K^R = \sigma \upharpoonright K^R$.

(b) Let $\sigma: R \rightarrow j(\pi)$ be fully elem., and (R, Λ) a $\hat{j}(\Gamma)$ -hod-pair in MLHJ. We say Λ is locally σ -consistent iff $\Lambda \upharpoonright_{R \upharpoonright (K^R) \upharpoonright R}$ is σ -consistent.

Remark If $\pi: R \rightarrow H_1$ is the map of R into $M_{\infty}(R, \Lambda) \upharpoonright^{\hat{j}(\Gamma)}$, the hod limit, then $\pi \upharpoonright K^R$ is the map of $R \upharpoonright K^R$ into $M_{\infty}(R \upharpoonright K^R, \Lambda \upharpoonright_{R \upharpoonright K^R})$

~~It~~ (We are assuming K^R regular in R .) So really, it's $\Lambda \upharpoonright_{R \upharpoonright K^R}$ that determines local σ -consistency. We call it plain

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σ -consistency in (a) because then all iterations of R (that don't drop) are by $\Lambda R \Lambda R$.

Remark If $\sigma: R \rightarrow H_1^+$, and (R, Λ) is a $j(\Gamma)$ hod pair such that Λ is σ -consistent, then $\Lambda = \Sigma_{H_1}^\sigma$. This is because, letting $j_1 = j(j)$, $\Sigma_{H_1}^\sigma$ is pullback-consistent in $\hat{j}_1(M \Sigma H J)$, and $\Sigma_{H_1}^\sigma(\pi(KR))$ is a tail of $\hat{j}_1(\Lambda)$ there.

Thus $\Sigma_{H_1}^\sigma(\pi(KR)) \cap_{HC} M \Sigma H J = \hat{j}_1(\Lambda) \cap_{HC} M \Sigma H J = \Lambda$.

Thus σ -consistency determines Λ in this case.

Remark There are probably examples of $\sigma: R \rightarrow j(\Gamma)$ and (R, Λ) locally σ -consistent, but $\Lambda \neq \Sigma_{j(\Gamma)}^\sigma$. We do not know one at the moment, but surely local σ -consistency is not enough to determine Λ .

Here are some simple facts about σ -consistency.

Definition 2.7 Let $\sigma: R \rightarrow H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$ -hod pair in $\mathcal{M}\mathcal{E}\mathcal{H}\mathcal{J}$. We say that Λ is σ -consistent iff letting $\pi: R \rightarrow H_1$ be the iteration map, we have $\pi \upharpoonright K^R = \sigma \upharpoonright K^R$.

Notice that if Λ is σ -consistent, then $\Lambda \cup \sigma = \sum_{H_1} \sigma$. This is because $\sum_{H_1} (\pi(K^R))$ is a tail of $\hat{j}_1(\Lambda)$, where $j_1 = j(j)$, and $\hat{j}_1(\Lambda)$ is pullback-consistent, and $\Lambda \subseteq \hat{j}_1(\Lambda)$.

Proposition 2.8 Let $\sigma: R \rightarrow H_1^+$ be elem., and (R, Λ) a $\hat{j}(\Gamma)$ hod pair in $\mathcal{M}\mathcal{E}\mathcal{H}\mathcal{J}$. The following are equivalent:

- (1) Λ is σ -consistent,
- (2) whenever $i: R \rightarrow \mathcal{A}$ is by Λ , then the extender of i is given by

$$E_i = E_\sigma \upharpoonright \pi_{\mathcal{A}, \infty}^{\Lambda} \upharpoonright K^{\mathcal{A}}$$

where $\pi_{\mathcal{A}, \infty}^{\Lambda}: \mathcal{A} \rightarrow H_1$ is the iteration map by Λ ,

- (3) $E_\pi \upharpoonright \pi(K^R) = E_\sigma \upharpoonright \pi(K^R)$, for $\pi = \pi_{R, \infty}^{\Lambda}$.

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Remark In clause (2) of 2.6, the equation

$E_i = E_\sigma \uparrow \pi_{\mathcal{S}, \infty}^\wedge \uparrow K^R$ means: for all $a \in [K^{\mathcal{A}} J]^{\leq \omega}$ and all $X \in R$, $(a, X) \in E_i$ iff $(\pi_{\mathcal{S}, \infty}^\wedge(a), \overset{X}{\sigma(X)}) \in E_\sigma$.

Equivalently

$$a \in i(X) \text{ iff } \pi_{\mathcal{S}, \infty}^\wedge(a) \in \sigma(X),$$

for $a \in [K^{\mathcal{A}} J]^{\leq \omega}$ and $X \in R$. Notice that i is continuous at K^R , so it suffices here to consider $X \in R \upharpoonright K^R$.

Proof of 2.6

(1) \Rightarrow (2). Let $i: R \rightarrow \mathcal{S}$ by \wedge , $a \in [K^{\mathcal{A}} J]^{\leq \omega}$, and $X \in R \upharpoonright K^R$. Then

$$a \in i(X) \text{ iff } \pi_{\mathcal{S}, \infty}^\wedge(a) \in \pi_{\mathcal{S}, \infty}^\wedge(i(X)),$$

$$\text{iff } \pi_{\mathcal{S}, \infty}^\wedge(a) \in \sigma(X),$$

$$\text{because } \sigma \uparrow (R \upharpoonright K^R) = \pi_{R, \infty}^\wedge \uparrow (R \upharpoonright K^R).$$

(2) \Rightarrow (3) Let $X \in R \upharpoonright K^R$ and $b \in [\pi(K^R) J]^{\leq \omega}$,

where $\pi = \pi_{R, \infty}^\wedge$. Then let $b = \pi_{\mathcal{S}, \infty}^\wedge(a)$, where

$i: R \rightarrow \mathcal{S}$ is by \wedge . Then

$$\begin{aligned}
(b, X) \in E_\sigma & \text{ iff } \pi_{\mathcal{L}, \infty}(a) \in \sigma(X) \\
& \text{ iff } a \in i(X) \quad (\text{by (2)}) \\
& \text{ iff } \pi_{\mathcal{L}, \infty}(a) \in \pi_{\mathcal{L}, \infty}(i(X)) \\
& \text{ iff } b \in \pi_{\mathcal{R}, \infty}(X) \\
& \text{ iff } (b, X) \in E_\pi,
\end{aligned}$$

as desired.

(3) \rightarrow (1) is clear.



Let (R, Δ) be a $j(\bar{r})$ -hod pair in $\mathcal{M}[hJ]$, and $\sigma: R \rightarrow j(\mathcal{M})$ be such that Δ is locally σ -consistent. Let $i: R \rightarrow \mathcal{L}$ be an iteration map by $\Delta \upharpoonright_{R \times R}$; that is, an iteration not using any extenders in the top block or its images. There is then a natural factor map \uparrow from $\mathcal{L} = \text{Ult}(R, E_\sigma \upharpoonright \pi_{\mathcal{L}, \infty} \text{ " } \kappa^{\mathcal{L}})$ to $j(\mathcal{M}) = \text{Ult}(R, E_\sigma)$, given by

$$r(i(f)(a)) = \sigma(f)(\pi_{\mathcal{L}, \infty}(a)).$$

These maps commute with the iteration maps by $\Lambda_{R|K^R}$, yielding

$$\begin{array}{ccc} R & \xrightarrow{\sigma} & j(\pi) \\ & \searrow i & \nearrow \tau \\ & S & \end{array}$$

Remark We didn't need anything about the full Λ to get τ .

Proposition 2.9 Let $\sigma: R \rightarrow j(\pi)$ and (R, Λ) be locally σ -consistent. Let $i: R \rightarrow S$ be an iteration map by $\Lambda_{R|K^R}$, and $\tau: S \rightarrow j(\pi)$ the factor map; then the S -tail of Λ is locally τ -consistent.

Proof Let $\eta \in K^S$. Since i is continuous at K^R , we have $\eta = i(f)(a)$ where $f \in R|K^R$ and $a \in \Sigma_{K^S} J^{<\omega}$. Then

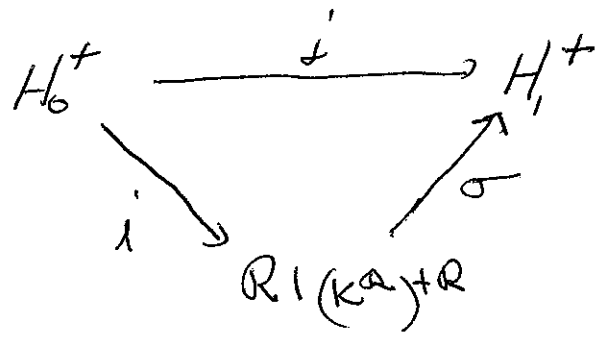
$$\begin{aligned} \pi_{S,\infty}(\eta) &= \pi_{S,\infty}(i(f)(a)) \\ &= \pi_{S,\infty}(i(f))(\pi_{S,\infty}(a)) \\ &= \sigma(f)(\pi_{S,\infty}(a)) \\ &= \tau(\eta), \end{aligned}$$

as desired. ◻

Now we want to see what happens when we touch the top block.

Lemma 2.10 Let $(R, \underline{\Psi})$ be a $\hat{f}(\Gamma)$ -hod-pair in MEHJ, and $\sigma: R \rightarrow j(\pi)$ be such that $\underline{\Psi}$ is locally σ -consistent.

Suppose we have i such that



commutes. Suppose also G is on the R -sequence, $\text{crit}(G) = KR$, and G is σ -certified

over $(R // \text{th}(G), \underline{\Psi}_{R // \text{th}(G)})$. Let $\Delta = \cup_{\mathcal{I}} (R, G)$, and $\tau: \Delta \rightarrow j(\pi)$ be given by $\tau(i_G^{\mathcal{I}}(t)(a)) = \sigma(f)(\pi_{R // \text{th}(G, \mathcal{I})}^{\underline{\Psi}_{R // \text{th}(G)}}(a))$; then

- (1) τ is well-defined,
- (2) $R \begin{array}{ccc} \xrightarrow{\sigma} & j(\pi) \\ \searrow i_G & \nearrow \tau \\ & \Delta \end{array}$ commutes,

(3) the Δ -tail of $\underline{\Psi}$ is locally τ -consistent.

Remark The same proof yields the same conclusions if we assume only that G is an amenable predicate such that (R, G) is a hod premouse, and G is σ -certified over $(R, \overline{\Psi})$. ~~THE FACT~~

Proof of 2.10 Parts (1) and (2) follow at once from the fact that G is σ -certified over $(R \parallel h(G), \overline{\Psi}_{R \parallel h(G)})$.

For (3), we apply j -condensation, as stated in Lemma 2 of [3]. (This is why we assumed it exists.) Let

$$\eta = j_G(f)(a) \in K^\delta. \text{ Let } Q = R \upharpoonright (K^a)^{+\mathbb{R}} \text{ and } W = \mathcal{S} \upharpoonright (K^\delta)^{+\mathbb{S}}.$$

We may assume $a \in [h(G)]^{<\omega}$, and we can write

(with $a \in \mathbb{Z}h(G)^{<\omega}$) we can write η (67) ~~(68)~~

$$\eta = i_G \circ i(g)(b, a), \text{ where } b \in \mathbb{Z}K^{\otimes J}^{<\omega}.$$

Then

$$L(\text{Hom}_h^*, H_1^+) \simeq \pi_{H_0^+, \infty}^{\Sigma_{H_0}} " g \subseteq j(g),$$

a statement about Σ_{H_0}, g, H_0^+ , and $j(g)$.

So by j -condensation

$$L(\text{Hom}_h^*, H_1^+) \simeq \pi_{W, \infty}^{\bar{\Phi}_W} " i_G \circ i(g) \subseteq j(g)$$

$$\text{Let us write } \pi^\Psi = \pi_{Q, \infty}^{\Psi_Q}, \quad \pi^{\bar{\Phi}} = \pi_{W, \infty}^{\bar{\Phi}_W}.$$

Then

$$\begin{aligned} \pi^{\bar{\Phi}}(\eta) &= \pi^{\bar{\Phi}}(i_G \circ i(g)(b, a)) \\ &= j(g)(\pi^{\bar{\Phi}}(b), \pi^{\bar{\Phi}}(a)) \\ &= j(g)(\pi^{\Psi}(b), \pi^{\Psi}(a)) \end{aligned}$$

(by strategy coherence for Ψ , and $h(G)$ being a cutpoint of W --- this is why we indexed that way!)

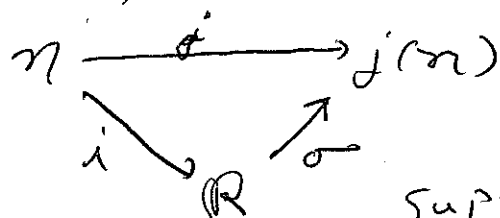
$$\begin{aligned} &= \sigma(i(g))(\sigma(b), \pi^\Psi(a)) \\ &= \sigma(i(g)(b))(\pi^\Psi(a)) = \sigma(f)(\pi^\Psi(a)) \\ &= \uparrow(\eta), \end{aligned}$$

as desired.



The next lemma deals with how certification in the top block is preserved.

Lemma 2.11 Suppose in \mathcal{MKT} we have \mathcal{R}, i, σ with



commuting, \mathcal{R} countable, ~~and i continuous~~.
 Let $\Psi = \hat{j}(\Sigma)^\sigma$, and suppose that Ψ is locally σ -consistent. Let G be an extension from the top block of \mathcal{R} , with $\text{crit}(G) = \mathcal{K}^{\mathcal{R}}$, then G is σ -certified over $(\mathcal{R} \parallel \mathcal{H}(G), \Psi \parallel \mathcal{H}(G))$.

Proof For $\mathcal{R} = \mathcal{N}$ and $\Psi = \Sigma = \hat{j}(\Sigma)^\hat{j}$, this is true. We express it as a collection of statements involving parameters $j(A)$ for $A \in H_0^+$, and then apply j -condensation in the form of lemma 2.4.

Note that since j is continuous at $(\mathcal{K}^{\mathcal{N}})^+ \mathcal{N} = o(H_0^+)$, i is continuous at $(\mathcal{K}^{\mathcal{R}})^+ \mathcal{R}$.

Let $\theta^0 < \xi < \omega(H_0^+)$, and let

$A^\xi = \langle A_\alpha^\xi \mid \alpha < \theta^0 \rangle \in H_0^+$ be an enumeration of $H_0^+ \mid \xi$. Then

$L(\text{Hom}_h^* j(\mathcal{N})) \models$ for all extenders G on the sequence of \mathcal{N} with $\text{crit}(G) = \theta^0$, for all $a \in [lh(G)]^{<\omega}$ and all $\alpha < \theta^0$

$$A_\alpha^\xi \in G_a \iff \prod_{\mathcal{N} \parallel lh(G), \infty}^{\Sigma_{\mathcal{N} \parallel lh(G)}} (a) \in j(A^\xi) \prod_{H_0, \infty}^{\Sigma_{H_0}} (\alpha),$$

which is a statement φ about the parameters $\mathcal{N}, \Sigma, A^\xi$, and $j(A^\xi)$. By 2.4,

$\varphi(\mathcal{R}, \mathcal{I}, i(A^\xi), j(A^\xi))$ holds in

$L(\text{Hom}_h^* j(\mathcal{N}))$, i.e.

$L(\text{Hom}_h^* j(\mathcal{N})) \models$ for all extenders G on the sequence of \mathcal{R} with $\text{crit}(G) = \kappa^{\mathcal{R}}$, for all $a \in [lh(G)]^{<\omega}$ and all $\alpha < \kappa^{\mathcal{R}}$

$$i(A^\xi)_\alpha \in G_a \iff \prod_{\mathcal{R} \parallel lh(G), \infty}^{\mathcal{I} \parallel \mathcal{R} \parallel lh(G)} (a) \in j(A^\xi) \prod_{\mathcal{R}, \infty}^{\mathcal{I} \parallel \mathcal{R}} (\alpha)$$

However, notice ~~$\sigma(A^\xi)$~~ $\sigma(i(A^\xi)_\alpha) = j(A^\xi)_{\sigma(\alpha)} =$

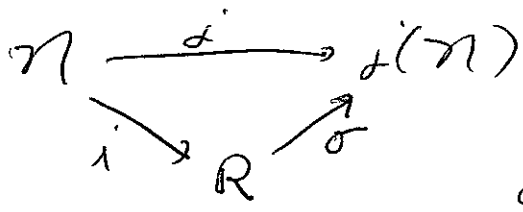
$j(A^\xi) \prod_{\mathcal{Q}, \infty}^{\mathcal{I} \parallel \mathcal{Q}} (\alpha)$, since $\mathcal{I} \parallel \mathcal{Q}$ is σ -consistent locally.

It follows that all G on the \mathcal{R} -sequence such that $\text{crit}(G) = K^{\mathcal{R}}$ are σ -certified over $(\mathcal{R} \parallel \text{th}(G), \mathbb{F}_{\mathcal{R} \parallel \text{th}(G)})$, so far as sets to be measured in $i(H_0^+ | \xi)$ go. But i is continuous at $0(H_0^+)$, and ξ was arbitrary.



Virtually the same proof gives

Lemma 2.12 Suppose in M&hJ we have \mathcal{R}, i, σ with



and $j \upharpoonright \mathcal{N} \in M$.

commuting and \mathcal{R} countable. Suppose $j(\mathcal{E})^\sigma$ is locally σ -consistent. Let E be j -certified over $(\mathcal{N}, \mathcal{E})$, and $F = \bigcup \{i(G) \mid G \in E \wedge G \in \mathcal{N}\}$; then F is σ -certified over $(\mathcal{R}, j(\mathcal{E})^\sigma)$.

Proof We apply the proof of 2.11 to the fragments $i(G)$ of F , fragment-by-fragment. We leave the details to the reader.

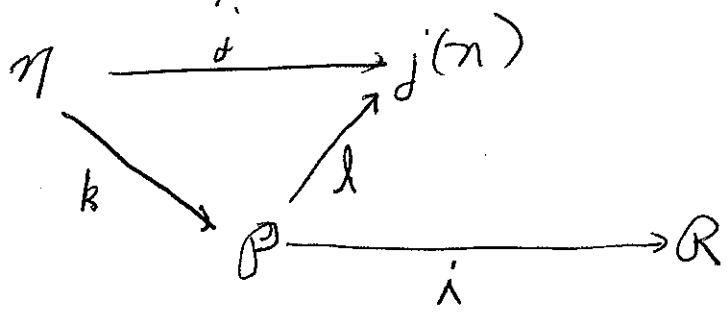


We would like to apply 2.11 and 2.12 with $i: \mathcal{N} \rightarrow \mathcal{R}$ an iteration map by Σ , or more generally, by the iteration strategy σ for (\mathcal{N}, E) we are trying to construct. The problems are, in the case of Σ -iterations

- (a) how do we know $\sigma: \mathcal{R} \rightarrow j(\mathcal{N})$ with $j = \sigma \circ i$ exists?
- (b) how do we know why is $j(\Sigma)^\sigma$ the \mathcal{R} -tail of Σ ?

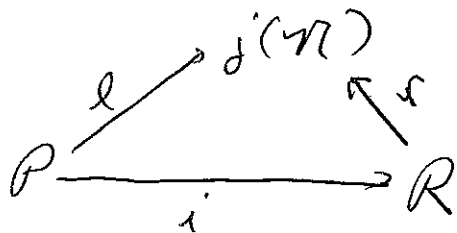
There are parallel problems in the case i is by Ω , and our strategy for (\mathcal{N}, E) .

More generally, let



be given in $M[E, j]$, where $j = l \circ k$, and i is an iteration by $j(\Sigma)^\sigma = \Psi$, and Ψ is locally l -consistent. We'd like to

find τ such that



commutes, $j(\Sigma)^\tau$ is the R -tail of \mathcal{I} , and $j(\Sigma)^\tau$ is locally τ -consistent. To do this, we put our iteration in normal form

(cf. Def. 2.6.1). So we have

$\langle (P_\alpha, \vec{T}_\alpha) \mid \alpha < \eta \rangle$ with $P = P_0$, and last model R . We write $R = P_\gamma$. (So R is the last model of $\vec{T}_{\gamma-1}$ if $\gamma-1$ exists, and $R = \lim_{\alpha < \eta} P_\alpha$ otherwise.) Let $\mathcal{I} = \mathcal{T}_0$.

We define embeddings $T_\alpha: P_\alpha \rightarrow j(\Sigma)$ by induction so that

- (1) $T_\gamma = j_{T_\alpha} \circ T_\alpha$ for $\gamma < \alpha$, $j_{T_\alpha}: P_\gamma \rightarrow P_\alpha$,
- (2) $j(\Sigma)^{T_\alpha}$ is the P_α -tail of \mathcal{I} ,
- (3) the P_α -tail of \mathcal{I} is locally T_α -consistent.

Suppose first that we have T_α such that

(1) - (3) hold, and we want

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$\tau_{\alpha+1}$.

Case 1 $P_{\alpha+1} = \text{Ult}(P_\alpha, G)$, for $\text{crit}(G) = \kappa^{P_\alpha}$
on the P_α -sequence.

Then G is τ_α -certified over
 $(P_\alpha // H, j(\Sigma)_{P_\alpha // H}^{\tau_\alpha})$ by 2.11. ~~and~~ So letting

$$\tau_{\alpha+1}(i_G(f)(a)) = \tau_\alpha(f) \left(\pi_{P_\alpha // H, \infty}^\Phi(a) \right)$$

for $\Phi = j(\Sigma)_{P_\alpha // H}^{\tau_\alpha}$, we have by 2.10 that

~~$\tau_{\alpha+1} \circ \tau_\alpha = \tau_{\alpha+1} \circ j_{\alpha, \alpha+1}$~~ and $j(\Sigma)_{P_\alpha // H}^{\tau_{\alpha+1}}$

is locally $\tau_{\alpha+1}$ -consistent. So it is
enough to show that $j(\Sigma)_{P_\alpha // H}^{\tau_{\alpha+1}}$ is the

$P_{\alpha+1}$ -tail of Ψ . Remark See p. 73a

But this follows from theorem 3.76
in the 3/25/00 version of Sargsyan's
thesis [2]. ("Branch condensation pulls back".)

To jog the reader's memory, here is the
barest sketch: we get a hod mouse M

Remark Let $\Phi = j(\Sigma)^{\tau_\alpha}$, and Λ be the $P_{\alpha+1}$ -tail of Φ . Let

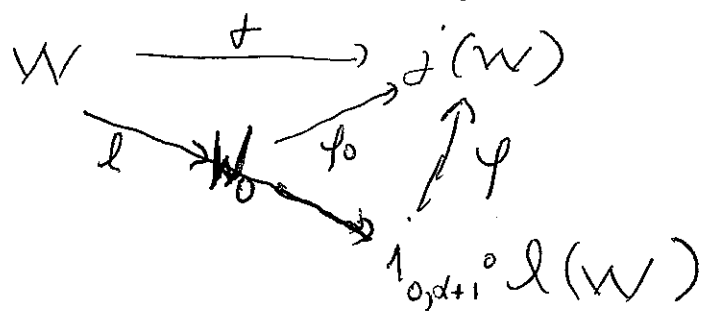
$$W = P_{\alpha+1} | (K^{P_{\alpha+1}})^{+P_{\alpha+1}} \text{ . Lemma 2.10}$$

then says that Λ_Q is $\tau_{\alpha+1}$ -consistent.

But then $\Lambda_Q = j(\Sigma)_Q^{\tau_{\alpha+1}}$, so $j(\Sigma)^{\tau_{\alpha+1}}$

is locally $\tau_{\alpha+1}$ -consistent.

with w Woodruff extending \mathcal{N} , and having a UB representation of Σ that is moved properly. $i_{\alpha+1} \circ l$ ~~and~~ extends to ac on W , yielding



with $\varphi \upharpoonright P_{\alpha+1} = \tau_{\alpha+1}$, $f_0 \upharpoonright P_0 = \tau_0$.

The W -representation of Σ gets moved to the W_0 -representation of its P_0 -tail, and ~~Sageev~~ show also to $j(\Sigma)^{\tau_0}$, as Sageev shows. It is then further moved to its $P_{\alpha+1}$ -tail, and to $j(\Sigma)^{\tau_{\alpha+1}}$.

Case 2 $\overline{\tau_\alpha}$ involves no extenders in the top block of P_α , or its images.

In this case we get $\tau_{\alpha+1}: P_{\alpha+1} \rightarrow j(\mathcal{N})$ as in (1) - (3) by using proposition 2.9 where we used 2.10 in case 1. We leave

the details to the reader.

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Finally, suppose $\lambda \leq \eta$ is a limit ordinal. We define $\tau_\lambda(i_{\alpha\lambda}(x)) = \tau_\alpha(x)$, for $\alpha < \lambda$. This gives $\tau_\lambda: P_\lambda \rightarrow j(\mathcal{M})$, and it is clear that (1) holds. For (2), we use Sargsyan's 3.76 of [2] again.

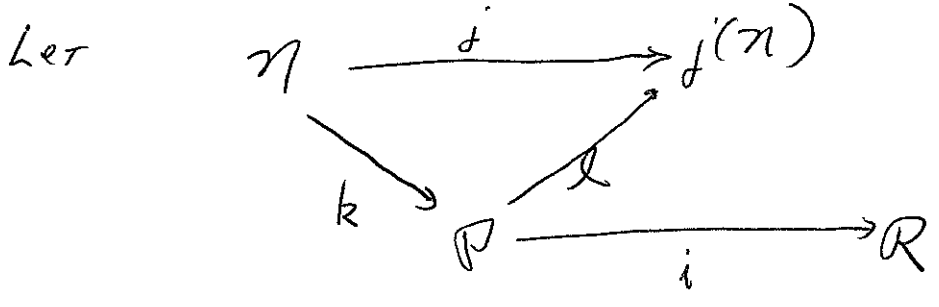
For (3), let $\nu < \kappa^{P_\lambda}$, and $\pi: P_\lambda \rightarrow H_1$ the iteration map by $j(\Sigma)^\nu = P_\lambda$ -tail of \mathcal{I} . Let $\bar{\nu} = i_{\alpha\lambda}(\bar{\nu})$, and ~~$\varphi = j(\Sigma)^{\bar{\nu}} = P_\alpha$ -tail~~ $\varphi: P_\alpha \rightarrow H_1$ be the iteration map by $j(\Sigma)^{\bar{\nu}} = P_\alpha$ -tail of \mathcal{I} . Then

$$\begin{aligned}\tau_\lambda(\bar{\nu}) &= \tau_\alpha(\bar{\nu}) = \varphi(\bar{\nu}) \\ &= \pi(\bar{\nu}),\end{aligned}$$

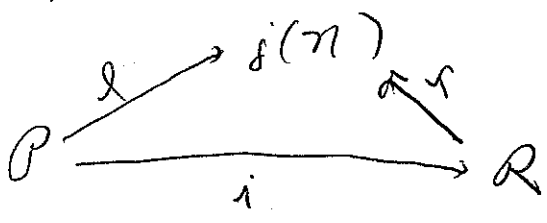
as desired.

We have shown

Lemma 2.13 Assume ~~...~~ $j \uparrow \pi \in M_j$.



be given in $M \uparrow h J$, where $j^- = \varphi \circ k$ and i is an iteration map by $j(\Sigma)^l = \Psi^-$. Suppose Ψ is locally \mathcal{I} -consistent. Let Φ be the \mathbb{R} -tail of Ψ . Then there is a unique embedding τ such that



commutes, $\Phi = j(\Sigma)^\tau$, and Φ is locally τ -consistent.

Proof All that's left is uniqueness of τ . But since the top block of \mathbb{P} is below $\mathbb{O}^\mathbb{P}$, any $x \in \mathbb{R}$ has the form $i(f)(a)$, where $a \in [K \uparrow J]^{<\omega}$. But then $\tau(x) = \tau(i(f)(a)) = l(f)(\pi_{\mathbb{R}, \infty}^\Phi(a))$, so τ is determined by l and the \mathbb{R} -tail of Ψ , hence by l and \mathbb{R} . \square

We are now ready to define our iteration strategy Ω for (\mathcal{N}, E) , where E is j -certified over (\mathcal{N}, E) .

We are assuming ~~that $j \upharpoonright \mathcal{M}$ is a Σ_1 -elementary submodel~~ $j \upharpoonright \mathcal{M} \in \mathcal{M}$. Let

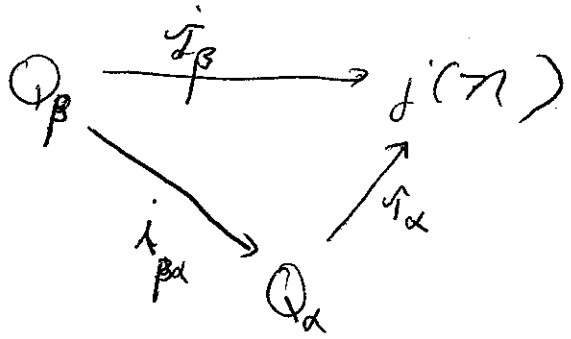
$Q_0 = \mathcal{N}$, $E_0 = E$, and

$P_0 = (Q_0, E_0)$.

Let $\mathcal{L}_0 = \Sigma$, and $\gamma_0 = j \upharpoonright \mathcal{N}$. Let

$\langle (P_\alpha, \vec{\mathcal{L}}_\alpha) \mid \alpha < \eta \rangle$ be an iteration of P_0

in normal form, played according to the strategy Ω that we are defining. We maintain by induction that there are $\gamma_\alpha : Q_\alpha \rightarrow j(\mathcal{N})$ so that



where $i_{\beta\alpha}$ is the iteration map, and strategies Ω_α for Q_α

such that

(78)

- (1) Ω_α is good, and has branch condensation,
- (2) $\Omega_\alpha = \vec{f}(\Sigma)^{\tau_\alpha}$, and
- (3) Ω_α is locally τ_α -consistent.
- (4) if \vec{T}_α is on Q_α , then it is by Ω_α ,
and $\Omega_{\alpha+1} = (\Omega_\alpha)_{\vec{T}_\alpha, Q_{\alpha+1}}$, and
- (5) if $P_{\alpha+1} = \cup_{\tau} (P_\alpha, E_\alpha)$, then ~~Ω_α~~
 $\Omega_\alpha = (\Omega_{\alpha+1})_{Q_\alpha}$.

Note that Q_α is a cutpoint ^{full} initial segment
of $Q_{\alpha+1}$ ~~in case (5)~~ when (5) applies.

Suppose we have $\langle \tau_\beta \mid \beta \leq \alpha \rangle$.

Case 1 \vec{T}_α is on Q_α .

In this case, lemma 2.13, with $P = Q_\alpha$
and $Q = Q_{\alpha+1}$, gives $T_{\alpha+1}$ and (1)-(5).

Case 2 $P_{\alpha+1} = \cup/\pi(P_\alpha, E_\alpha)$.

We have that E_α is γ_α -certified over $(Q_\alpha, \mathbb{R}_\alpha)$, by 2.12 and induction hypotheses (2) and (3). By ~~the~~ the remark after 2.10, if we set

$$\tau_{\alpha+1} (j_{E_\alpha}^{-1}(f)(a)) = \tau_\alpha(f)(\pi_{Q_\alpha, \infty}^{\mathbb{R}_\alpha}(a)),$$

then $\tau_{\alpha+1}$ is well-defined and $\gamma_\alpha = \gamma_{\alpha+1} \circ j_{\alpha, \alpha+1}^{-1}$.

Moreover, setting $\mathbb{R}_{\alpha+1} = j(\Sigma)^{\gamma_{\alpha+1}}$, we have as in the proof of 2.10 that $\mathbb{R}_{\alpha+1}$ is good, has branch condensation, and is locally $\tau_{\alpha+1}$ consistent. (This all uses j -condensation, lemma 2.4.)

Since Q_α is a cutpoint of $Q_{\alpha+1}$

$$\begin{aligned} \pi_{Q_\alpha, \infty}^{(Q_{\alpha+1})_{Q_\alpha}} \uparrow \circ (Q_\alpha) &= \pi_{Q_{\alpha+1}, \infty}^{\mathbb{R}_{\alpha+1}} \uparrow \circ (Q_\alpha) \\ &= \tau_{\alpha+1} \uparrow \circ (Q_\alpha) \quad \text{locally consistent} \end{aligned}$$

(since $\Omega_{\alpha+1}$ is locally $\tau_{\alpha+1}$ -consistent)

$$= \prod_{Q_{\alpha, \infty}} \Omega_{\alpha} \uparrow 0(Q_{\alpha})$$

(by the definition of $\tau_{\alpha+1}$), Pullback condensation for Ω_{α} and $\Omega_{\alpha+1}$ then give us (5).

Finally, if λ is a limit, we define τ_{λ} by $\tau_{\lambda}(\iota_{\alpha\lambda}(x)) = \tau_{\alpha}(x)$, and set $\Omega_{\lambda} = j(\Sigma)^{\tau_{\lambda}}$. We leave the rest to the reader.

This then tells us how to define Ω for one more step. We only need worry about the case that $\eta = \alpha+1$, and \vec{T}_{α} is on Q_{α} . (Otherwise, there is no choice of branch to be made.) But in this case, we let Ω choose $\Omega_{\alpha}(\vec{T}_{\alpha})$.

This completes our definition of Ω for (\mathcal{N}, E) , assuming ~~j witnesses hugeness~~ that $j \upharpoonright \mathcal{N} \in \mathcal{M}$.

Lemma 2.15 Assume ~~j witnesses that κ is huge~~ that $j \upharpoonright \mathcal{M} \in \mathcal{M}$, and let Ω be the iteration strategy for (\mathcal{N}, E) defined above; then Ω is good.

Proof That Ω is self-consistent and coherent is just clauses (4) and (5) of our induction hypothesis. That Ω is $\tilde{j}(\Gamma)$ -fullness preserving is also implicit in the induction hypothesis. (We get it from \tilde{j} -condensation.)

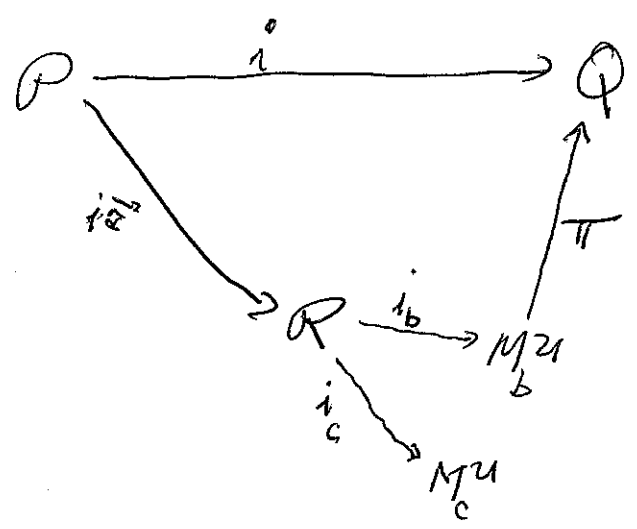


Lemma 2.16 Assume ~~j witnesses that κ is huge~~ that $j \upharpoonright \mathcal{M} \in \mathcal{M}$, and let Ω be the iteration strategy for (\mathcal{N}, E) defined above; then Ω has branch condensation.

Proof. Let $\mathcal{P} = (\mathcal{N}, E)$, and suppose branch condensation fails. We can then find a stack $\vec{\mathcal{T}} \sim \langle \mathcal{U} \rangle$ on \mathcal{P} such that $\vec{\mathcal{T}}$ is by Ω , and there is an iteration $i: \mathcal{P} \rightarrow \mathcal{Q}$ by Ω and cotinal branches b and c of \mathcal{U} and $\pi: \mathcal{M}_b^{\mathcal{U}} \rightarrow \mathcal{Q}$ such that

- (1) $i = \pi \circ i_b^{\mathcal{U}} \circ i^{\vec{\mathcal{T}}}$,
- (2) $c = \Omega(\vec{\mathcal{T}} \wedge \langle \mathcal{U} \rangle)$, and
- (3) $b \neq c$.

Letting \mathcal{R} be the base model of \mathcal{U} , the picture is



b is $i_b \circ i^{\vec{\mathcal{T}}}$ -realized, while c is by Ω .

b does not drop because that is part of the hypothesis in branch condensation. We shall show that c does not drop.

Claim 1. $M_b^u \models \delta(u)$ is Woodin.

Proof If η is a cardinal of a hod premouse \mathcal{M} , then $\mathcal{M} \upharpoonright \eta$ is a full level of \mathcal{M} . Thus $\mathcal{M}(u)$ is a limit of full levels of M_b^u and M_c^u .

By minimizing our counterexample to branch condensation, we can arrange (letting $i = i^{\vec{w}}$)

(*) Let \mathcal{A} be a full proper initial segment of $\mathcal{M}(u)$; then

$$\left(\Omega_{\vec{w}, \pi(\mathcal{A})} \right)^\pi = \left(\Omega_{\vec{w} \upharpoonright \mathcal{A}, \pi \upharpoonright \mathcal{A}} \right)_{\mathcal{A}}$$

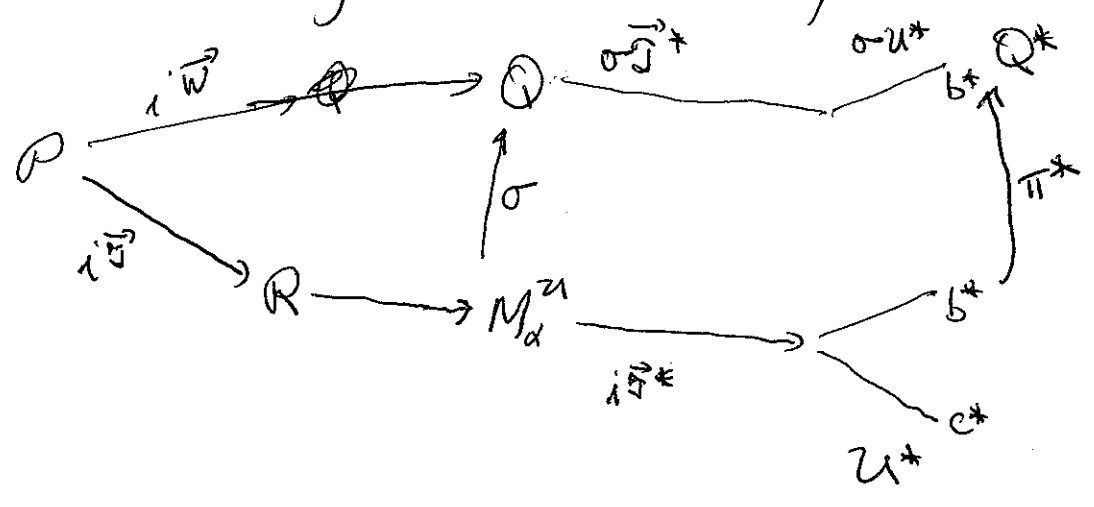
Proof Note that $\left(\Omega_{\vec{w} \upharpoonright \mathcal{A}, \pi \upharpoonright \mathcal{A}} \right)_{\mathcal{M}(u)} \in \mathcal{J}(\mathcal{M})$ by our construction. (All tails of Ω , projected to proper initial segments of their base models, are in $\mathcal{J}(\mathcal{M})$.) Suppose we have chosen our

counterexample $\vec{I}, u, b, c, \pi, \vec{v}$ so that $(\Omega_{\vec{I} \setminus \langle u \rangle}^c)_{M(u)}$ has minimal possible wedge rank. We claim that (*) holds.

For let \mathcal{A} be a counterexample to (*), and let $\alpha \in b$ be such that \mathcal{A} is a proper initial segment of $M_{\alpha}^u / \text{con}(\iota_{\alpha}^u)$. Then we have that for $\sigma = \pi \circ \iota_{\alpha}^u$, $\sigma \upharpoonright \mathcal{A} \cup \mathcal{B} = \pi$, and so $(\Omega_{\vec{w}, Q})_{\mathcal{A}}^{\sigma} = (\Omega_{\vec{w}, \pi(\mathcal{A})})^{\pi}$, so

$(\Omega_{\vec{w}, Q})_{\mathcal{A}}^{\sigma} \neq \Omega_{\vec{I} \setminus \langle u \rangle}^c, \mathcal{A}$, since the latter is just $\Omega_{\vec{I} \setminus \langle u \rangle}^c, \mathcal{A}$ by strategy coherence.

This means we get a counterexample of the form



with $\vec{I}^* \wedge u^*$ based on \mathcal{A} being according to

both $(\Omega_{\vec{w}, Q})_{\Delta}^{\sigma}$ and $\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, \Delta}$

but the former choosing b^* , the latter choosing c^* , and $b^* \neq c^*$. But now

$\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, \Delta}^{\sigma}$ is a tail of $\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, \Delta}$

of $\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, \Delta}$, so is projective in

$\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, \Delta}$, so has Wadge rank strictly less than that of $\Omega_{\vec{d} \wedge \langle u \rangle \wedge c, M(u)}$.

This contradiction yields (*).



~~Let Σ be the join of $\Sigma_{\vec{v}}$~~

Remark Our notation from §1 is such that if \vec{v} is a limit of full levels of M , where (M, Σ) is a hod pair, then $\Sigma_{\vec{v}}$ is just the join of the $\Sigma_{\vec{v}'}$ for \vec{v}' a proper full initial segment of \vec{v} .

So another way to write (*) is: $(\Omega_{\vec{w}, Q})_{M(u)}^{\pi} = \Omega_{\vec{d} \wedge \langle u \rangle \wedge c, M(u)}$

Now let $\Phi = \left(\Omega_{\vec{w}, \vec{q}}^\pi \right)_{M(u)} = \Omega_{\vec{w}, \vec{q}, M(u)}^\pi$. (86)

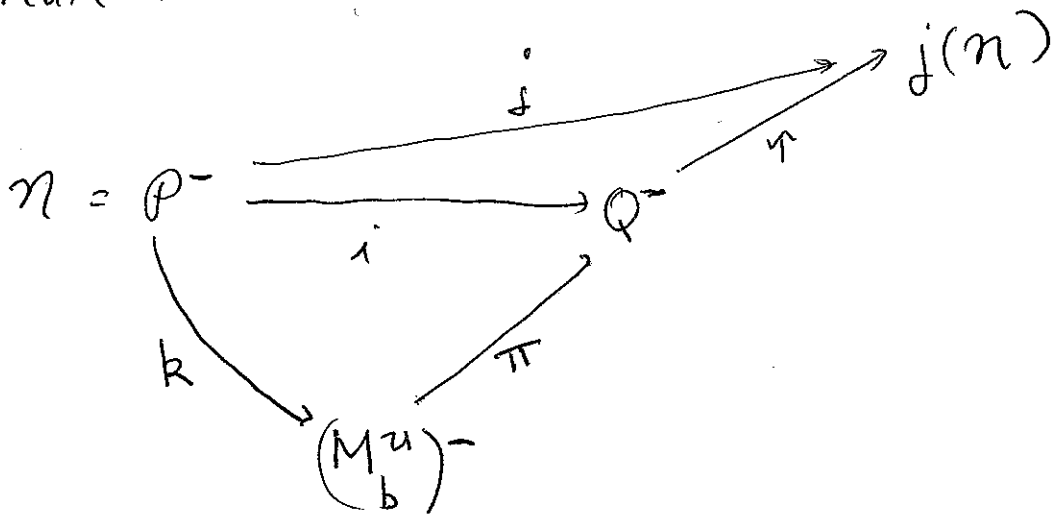
We claim

Remark The remainder of this proof, up to claim 2 on p. 89, is written out in more generality as 2.3.4, p. 45a, etc.

(**) $L_p^\Phi(M(u))^{j(\Gamma)}$ is the next full level of M_b^u after $M(u)$.

This follows from j -condensation, i.e. 2.4.

For recall that \mathcal{J}^- is \mathcal{J} with its last extender predicate removed. We then have



where $k = i_b \circ i^{\vec{J}}$. Here \uparrow is the "realization map" that is part of the definition of Ω .

We have $\Omega_{\vec{w}, \vec{q}}^\pi = j(\Sigma)^\uparrow$, so $\Phi = j(\Sigma)_{M(u)}^{\uparrow \circ \pi}$.

This gives us, via 2.4, that $(M_b^u)^-$ is $L_p^\Phi(M(u))^{j(\Gamma)}$ -full, as desired.

We then get

(***) $L_p^\Phi(M(\mathcal{U}))^{\hat{j}(\Gamma)}$ is the next full level of $M_c^\mathcal{U}$ after $M(\mathcal{U})$.

If not, since Ω is $\hat{j}(\Gamma)$ -fullness preserving, we must have that c dropped, and $M_c^\mathcal{U}$ is the first level of $L_p^\Phi(M(\mathcal{U}))^{\hat{j}(\Gamma)}$ that is not in $M_c^\mathcal{U}$, and this level projects strictly across $\delta(\mathcal{U})$. But that means $\delta(\mathcal{U})$ is not a cardinal in $M_b^\mathcal{U}$, contradiction.

By (**) and (***), $L_p^\Phi(M(\mathcal{U}))^{\hat{j}(\Gamma)} \models \delta(\mathcal{U})$ is Woodin. Let κ be the least cardinal strong to $\delta(\mathcal{U})$ in $M(\mathcal{U})$. Let $\mu = o(L_p^\Phi(M(\mathcal{U}))^{\hat{j}(\Gamma)})$.

Let $\Psi_b = (\Omega_{\vec{w}, Q}^\pi)_{M_b \upharpoonright \mu}$ and $\Psi_c = \mathcal{L}_{\vec{j}(\mathcal{U}) \upharpoonright c, M_c \upharpoonright \mu}$.

It is possible that $\Psi_b \neq \Psi_c$. Nevertheless, because we are below \mathcal{O}_h^P , Ψ_b in both

M_b and M_c the K -blocks end with w ~~iteration~~ more full levels above ~~$M(\mathcal{U})$~~
 $M_c | \mu = M_b | \mu$, and moreover $\delta(\mathcal{U})$ remains
 Woodin in M_b and M_c . (But $\mu \in \delta(\mathcal{U})^{M_b}$
 and $\mu \in \delta(\mathcal{U})^{M_c}$.) [For ~~example~~ K is not
 a limit of Woodins in $M(\mathcal{U})$, as otherwise
 we are past O_h^P . By Σ then,
 $L_p^{\bar{\Psi}_b}(M_b | \mu)^{M_b} \in \delta(\mathcal{U})$ is Woodin. But then
 the K -block ends in M_b with w more L_p 's,
 unless $\text{crit}(L_p^{\bar{\Psi}_b}(M_b | \mu)^{M_b})$ is an index
 on M_b of an extender with critical point K .
 Since K is not a limit of Woodins,
 $\text{crit}(L_p^{\bar{\Psi}_b}(M_b | \mu)^{M_b})$ is not such an index.
 Similarly on the c -side. \square

This finishes our proof of claim I.
 We leave it to the reader to check that
 we actually proved: