§2. Construction of $\mathcal{H}_0$: adding an extender with length below κ_0^+

We have $j : V \to M$, $\mathrm{crit}(j) = \kappa_0$,

as in section 0. Let g be V-generic over $\mathrm{Col}(\omega, <\kappa_0)$. ~~is a model of $AD_{\mathbb{R}} + AD$~~ Let $\Gamma \subseteq \mathrm{Hom}^*_g$

be such that

$$L(\Gamma, \mathbb{R}^*_g) \models AD_{\mathbb{R}} + DC + \text{" } HOD \upharpoonright \Theta$$

is the direct limit of all hod-pairs (P, Σ) s.t. Σ is fullness-pres. and has branch condensation, under the comparison maps".

Put

$$\Theta^0 = \Theta^{L(\Gamma, \mathbb{R})},$$

$$H_0 = HOD^{L(\Gamma, \mathbb{R})} | \Theta^0 .$$

Let $\hat{j} : V[g] \to M[h]$ extend j, and

$$H_0^+ = Lp^{\Sigma_{H_0}}(H_0)^{\hat{j}(\Gamma)},$$

where $\Sigma_{H_0} = \bigoplus_{\alpha < \theta^0} \Sigma_{H_0(\alpha)}$, and

$\Sigma_{H_0(\alpha)}$ is the common tail of all $\hat{j}(\Sigma_P)$, where $H_0(\alpha)$ is a Σ_P-iterate of P. If $\Gamma = Hom_g^*$, then $\hat{j}(\Gamma) = Hom_h^*$. Otherwise, any complete Γ set A has a Hom_g^* code, and $\hat{j}$ moves this code to the code of the complete $\hat{j}(\Gamma)$ set $\hat{j}(A)$.

Working in $M[h]$, we shall build a $\hat{j}(\Gamma)$-hod-pair $(\mathcal{H}, \Sigma_{\mathcal{H}})$ that extends (H_0^+, Σ_{H_0}). $\mathcal{H}$ will have a top block, and ~~that~~ θ^0 will begin that block. We shall show that if $\Gamma \neq Hom_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ either reaches $\Theta_h^{\mathbb{P}}$, or is a pointclass generator for $\hat{j}(\Gamma)$ in $M[h]$. We shall show that if $\Gamma = Hom_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ reaches $\Theta_h^{\mathbb{P}}$.

In fact, $\mathcal{H}$ and $\Sigma_{\mathcal{H}}$ are constructed in M, and $M[h]$ is just used to record properties of them. So we'll have $\mathcal{H} \in M$, and an iteration strategy Ψ for $\mathcal{H}$ with $\Psi \in M$, and defined on all $\vec{\mathcal{T}} \in M$ of size $< j(\kappa)$. Ψ will "determine itself on generic extensions" in such a way that for all l on $col(\omega, <j(\kappa))$ there is $\dot{\Psi}^l \supseteq \Psi$ defined on $HC^{M[l]}$. What we are calling $\Sigma_{\mathcal{H}}$ is then $\dot{\Psi}^h$. (So $M \models \Vdash^{col(\omega, <j(\kappa))} \check{\Psi} \subseteq \dot{\Psi}$.) $\dot{\Psi}$ is a "symmetric name", in that $\dot{\Psi}^l = \dot{\Psi}^m$ whenever $HC^{M[l]} = HC^{M[m]}$.

So in M, we are constructing by induction on ξ pairs $(\mathcal{N}_\xi, \dot{\Psi}_\xi)$ such that the following induction hypotheses hold. We call them $(\dagger)$.

Induction hypotheses $(\dagger)_\xi$: In M, the following hold, for $(\mathcal{N}, \dot\varphi) = (\mathcal{N}_\xi, \dot\varphi_\xi)$:

(a) $\mathcal{N}$ is a hod premouse extending H_0^+,

(b) $\dot\varphi$ is a $\mathrm{col}(\omega, <\kappa_1)$-name such that $\dot\varphi^\ell = \dot\varphi^m$ whenever $HC^{M[\ell]} = HC^{M[m]}$,

(c) $\Vdash^{\mathrm{col}(\omega,<\kappa_1)} (\check{\mathcal{N}}, \dot\varphi)$ is a $j(\dot\Gamma)$-hod-pair such that $\dot\varphi$ is $j(\dot\Gamma)$-fullness-preserving and has branch condensation and is positional.

It is easiest to describe the construction of $\mathcal{N}_{\xi+1}$ and $\dot\varphi_{\xi+1}$ if we have $j \restriction \mathcal{N}_\xi \in M$. This is of course true if $o(\mathcal{N}_\xi) < \kappa_0^+$, and may be true beyond that if j witnesses more than measurability of κ_0.

Remarks

(1) If $\neg \Box_{\kappa_0}$, ~~then~~ and we reach ξ such that $o(\mathcal{N}_\xi) = \kappa_0^+$, then

$\mathcal{N}_\xi \models ZFC +$ "θ_0 is a strong limit of Woodins".

So although we haven't reached $O_h^{\mathbb{P}}$, we're close.

(2) If j witnesses κ_0 is huge, then we can go up to $\mathcal{N}_{\kappa_1}$, and we will definitely reach $O_h^{\mathbb{P}}$ before that.

(3) In clause $(\dagger)(c)$, $\dot{\Gamma}$ is a symmetric name for Γ. We assume $\dot{\Gamma} \in V$ for simplicity; in general it will be in some size $< \kappa_0$ extension of V.

We have one further induction hypothesis: for $(\mathcal{N}, \dot{\varphi}) = (\mathcal{N}_\xi, \dot{\varphi}_\xi)$,

$(*)_\xi$ if $j \restriction \mathcal{N} \in M$, then

$$M \models \Vdash^{Col(\omega, <\kappa_1)} \dot{\varphi} = j(\dot{\varphi})^j.$$

(Here on the right side, "j" should be replaced by "$\check{j \restriction \mathcal{N}}$", to be precise.) (In the superscript only.)

Some explanation is in order. Let $j_1 = j(j)$, with $j_1 : M \to N$, and $\kappa_2 = j_1(\kappa_1)$. We have that $j(\dot{\varphi})$ is a $\mathrm{col}(\omega, <\kappa_2)$ name in N for a strategy for $j(\mathcal{N})$. But letting h be $\mathrm{col}(\omega, <\kappa_1)$-generic $/M$, we can make sense of $j(\dot{\varphi})_h$ by the symmetry of $j(\dot{\varphi})$: it is the common value of all $j(\dot{\varphi})_\ell \cap V_{\kappa_2}^{N[\ell]}$ for $h \subseteq \ell$, ℓ on $\mathrm{col}(\omega, <\kappa_2)$. So $j(\dot{\varphi})_h \in M[h]$.

Thus its $j \restriction \mathcal{N}$ - pullback $j(\dot{\varphi})_h^j$ makes sense in $M[h]$. It is defined on all $HC^{M[h]}$. (Note $HC^{M[h]} \cup \{j \restriction \mathcal{N}\} \subseteq N[h]$.)

$(*)_\xi$ then says that for any such h,

$$\dot{\varphi}_h = \left(j(\dot{\varphi})_h\right)^j .$$

This almost follows from ~~branch~~ hull condensation. Namely, let $\mathring{\varphi}_g$ be the common value of all $\mathring{\varphi}_h \cap V_{\kappa_1}^{M[g]}$, for $g \leq h$, h on $Col(\omega, <\kappa_1)$ and g on $Col(\omega, <\kappa_0)$. $\mathring{\varphi}_g \in V[g]$, and has branch condensation. Moreover, with $\hat{j} : V[g] \to M[h]$, we have $\hat{j}(\mathring{\varphi}_g) = j(\mathring{\varphi})_h$. But letting $\mathcal{T} \in V[g]$ be by $\mathring{\varphi}_g$, we have $\hat{j}(\mathcal{T})$ is by $j(\mathring{\varphi})_h$, so $\hat{j}\,\mathcal{T} \subseteq \hat{j}\,\mathcal{T}$ is by $j(\mathring{\varphi})_h$. [The fact that $\hat{j}\,``\mathcal{T} \notin M[h]$ can be overcome with an absoluteness argument.] Thus $\mathcal{T}$ is by $(j(\mathring{\varphi})_h)^{\hat{j}}$.

However, the argument of the last paragraph falls short of proving $(*)_\xi$ from $(+)_\xi$, because it only works for $\mathcal{T}$ in $V[g]$, so that $\hat{j}(\mathcal{T})$ makes sense. So it only gives $\dot{\psi}_g \subseteq (j(\dot{\psi})_h)^{\xi}$, not the full $(*)_\xi$.

$\boxed{\text{Insert p. 48f.}}$

Set

$$\mathcal{N}_0 = H_0^+$$

$\dot{\psi}_0$ = canonical $\mathrm{col}(\omega, <\kappa_1)$-name in M for $\bigoplus_{\alpha<\theta^0} \Phi_\alpha$, where

Φ_α = $H_0(\alpha)$-tail of $\hat{j}(\Lambda)$, for any and all (P, Λ) s.t. $H_0(\alpha) = M_\infty(P, \Lambda)$ in $V[g]$.

We have $o(\mathcal{N}_0) < \kappa_0^+$, so $j \restriction \mathcal{N}_0 \in M$. The reader can easily check $(\dagger)_0$ (see [1] and [3]). The main thing is that $\mathcal{N}_0 \models \theta^0$ is regular. For $(*)_0$, this says in our earlier notation that Σ_{H_0}, as defined on $HC^{M[h]}$, is the

We need two further induction hypotheses: (48f)

$(\dagger)_\xi$ (d): If E is on the $\mathcal{N}$-sequence and $\mathrm{crit}(E) = \theta^0$, then in M

$$\Vdash^{\mathrm{Col}(\omega,<\kappa_1)} \check{E} \text{ is certified by } j \restriction H_0^+ \text{ over } (\mathcal{N} \| \mathrm{lh}\, E, \mathring{\Psi}_{\mathcal{N}\|\mathrm{lh}(E)}).$$

(See definition 2.5 below for "certifies".)

$(*)_\xi$ (b): (Absolute condensation to pullbacks)

Let $\pi : R \to V_\gamma^M$ with γ large, $\pi \in M$, R transitive, $V_{\kappa_0+1} \cup \{\mathcal{N}\} \subseteq R$, and $|R| < \kappa_1$. Suppose $\pi(\overline{j(\mathring{\Psi})}) = j(\mathring{\Psi})$. Let h be M-generic over $\mathrm{col}(\omega, <\kappa_1)$, and h_0 be R-generic over $\mathrm{col}(\omega, <\bar{\kappa}_1)$, with $h_0 \in M[h]$; then

$$\overline{j(\mathring{\Psi})}_{h_0} \subseteq \left(j(\mathring{\Psi})_h\right)^\pi$$

j-pullback of Σ_{H_1}, as defined on $V_{K_2}^{N[h]}$ (where $j_1 : M \to N$).

That follows from the fact that $j \upharpoonright H_0$ is the iteration map $\pi : H_0 \to H_1$ by Σ_{H_0}, and Σ_{H_1} is the H_1-tail of Σ_{H_0} wherever they are defined, and Σ_{H_0} is pullback consistent, wherever it is defined.

Because we are just shooting for $O_h^{\mathbb{P}}$, we shall never add Θ^0-relevant extenders with critical point $> \Theta^0$. As a consequence, all levels of our construction will be fully sound, and we'll never have to core down. Thus we'll set

$$\mathcal{N}_\lambda = \text{“}\bigcup_{\alpha<\lambda}\text{”}\, \mathcal{N}_\alpha$$

and

$$\dot{\Psi}_\lambda = \text{canonical name for} \bigoplus_{\alpha<\lambda} \dot{\psi}_\alpha$$

for λ limit.

Now suppose we are given $(\mathcal{N}_\xi, \dot{\Psi}_\xi) =$ $(\mathcal{N}, \dot{\Psi})$ satisfying $(\dagger)_\xi$ and $(*)_\xi$.

We assume that $j \restriction \mathcal{N} \in M$, so that $(*)_\xi$ is not vacuous, and deal with the more general case in ~~the next~~ a subsequent section. We ~~will~~ shall obtain $(\mathcal{N}_{\xi+1}, \dot{\Psi}_{\xi+1})$ in one of two ways:

(i) close under $(\text{Lp-strategy})^\Gamma$, as on p. 16 ff., or

(ii) add an extender with critical point θ°.

The main tool in our arguments will be "j-condensation", i.e. lemma 11.15 of [1], generalized slightly as lemma 2 in [3]. The form we need is

<u>Lemma 2.4</u> Let $(\mathcal{N}, \mathring{\Psi})$ satisfy $(\dagger)_\xi$ and $(*)_\xi$, and suppose $j \restriction \mathcal{N} \in M$. Let h be $Col(\omega, {<}\kappa_1)$ generic, and suppose we have

$$\begin{tikzcd}
\mathcal{N} \arrow[rrr, "j"] \arrow[dr, "i"'] & & & j(\mathcal{N}) \\
& P \arrow[r, "k"'] \arrow[urr, "\sigma"] & Q \arrow[ur, "\tau"'] &
\end{tikzcd}$$

commuting, with $P, Q, \sigma, \tau \in M[h]$ and countable there. Let $\Sigma_P = \left(j(\mathring{\Psi})_h\right)^\sigma$ and $\Sigma_Q = \left(j(\mathring{\Psi})_h\right)^\tau$ be the pullback strategies.

<u>Then</u>

(i) $\Sigma_P, \Sigma_Q \in \hat{j}(\Gamma)$

(ii) $\Theta^0 = \Theta^{L(\Gamma, \mathcal{N})}$ is regular in $L(\Gamma, \mathcal{N})$, and

(iii) for all wffs φ, all $s \in P$, all $X \in \mathcal{N}$:

$$L(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[P, \Sigma_P, s, j(X)]$$

iff

$$L(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[Q, \Sigma_Q, k(s), j(X)].$$

<u>Proof sketch</u> We shall prove conclusion (i). The rest is proved just as in [1] and [3], so we do not give any details here. For (i), we let $j_1 = j(j)$ and $j_1 : M \to N$, and let g, h, ℓ be on ~~the~~ $col(\omega, <\kappa_0)$, $col(\omega, <\kappa_1)$, $col(\omega, <\kappa_2)$ respectively, with $g \subseteq h \subseteq \ell$. We have, for Q, T as in the hypotheses

$$\Sigma_Q = (\text{common value of all } j(\dot{\Psi})_m \cap HC^{M[h]}, \text{ for } m \geq h, m \text{ on } col(\omega, <\kappa_2))^{T}$$

by definition. So

$$\hat{j}_1(\Sigma_Q) = \left(\begin{array}{l}\text{common value of all } j_1(j(\dot{\Psi}))_m \cap HC^{N[\ell]}, \\ \text{for } m \geq \ell \text{ on } col(\omega, <j_1(\kappa_2))\end{array}\right)^{\hat{j}_1(T)}$$

$$= \left(\begin{array}{l}\text{common value of all } j_1(j(\dot{\Psi}))_m \cap HC^{N[\ell]}, \\ \text{for } m \geq \ell \text{ on } col(\omega, <j_1(\kappa_2))\end{array}\right)^{j_1 \circ T},$$

(since $j_1(T) = j_1 \circ T$)

$$= \left(\left(\text{common value of all } j_1(j(\dot{\psi}))_m \cap H(^{N[\ell J]}) \right)^{j_1} \right)^{\gamma}$$

$$= j(\dot{\psi})^{\gamma}_{\ell} \ .$$

The last step is $(*)_\xi$, moved from M ~~[illegible]~~ where it holds of $\dot{\psi}$ ~~[illegible]~~ and $j \restriction \eta$, to N ~~[illegible]~~, where it holds of $j(\dot{\psi})_*$ and $j(j \restriction \eta) = j_1 \restriction j(\eta)$.

But then

$$N[\ell J] \models j(\dot{\psi})_\ell' \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$N[\ell J] \models j(\dot{\psi})^{\gamma}_{\ell} \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$N[\ell J] \models \hat{j}_1(\Sigma_Q) \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$M[h J] \models \Sigma_Q \in \hat{j}(\Gamma),$$

as desired.

$\boxtimes$

We add extenders when they fit on the sequence, i.e. yield hod premice, and are "certified by j" in the following sense.

<u>Def 2.5</u> Let $(\mathcal{P}, \Lambda)$ be a $\hat{j}(\Gamma)$ hod pair in $M[h]$, and suppose $\mathcal{P}$ has a top block beginning at κ. Suppose

$$k : \mathcal{P} | (\kappa^+)^{\mathcal{P}} \longrightarrow H_1^+$$

is fully elementary. We say that <u>E is k-certified over $(\mathcal{P}, \Lambda)$</u> iff

(a) $(\mathcal{P}, E)$ is a hod premouse, and

(b) for all $a \in [lh(E)]^{<\omega}$ and $X \in \mathcal{P}|(\kappa^+)^{\mathcal{P}}$,

$$X \in E_a \text{ iff } \pi^{\Lambda}_{\mathcal{P},\infty}(a) \in k(X),$$

where $\pi^{\Lambda}_{\mathcal{P},\infty} : \mathcal{P} \longrightarrow H_1$ is the map given by $(\mathcal{P}, \Lambda)$ being in the hod-limit-system of $L(\hat{j}(\Gamma), \mathbb{R}^*_h)$.

We consider first the case in which $\mathcal{N}_{\xi+1}$ is obtained by adding an extender with critical point Θ^0 to $\mathcal{N}_\xi$.

Let h be $col(\omega, <\kappa_1)$-generic.

We have $(\mathcal{N}_\xi, \dot{\Psi}_\xi) = (\mathcal{N}, \dot{\Psi})$ our current pair satisfying $(T)_\xi$ and $(*)_\xi$.

Set

$$\Sigma = \dot{\Psi}_h .$$

So in $M[h]$, where we will be working most of the time, $(\mathcal{N}, \Sigma)$ is a $\hat{j}(\Gamma)$ hod pair.

<u>Case 1</u> There is an E that is j-certified over $(\mathcal{N}, \Sigma)$.

In this case, there is a unique such E, (so $E \in M$). We set

$$\mathcal{N}_{\xi+1} = (\mathcal{N}, E) .$$

Our goal now is to construct an iteration strategy Ω for $(\mathcal{N}, E)$ such that in $M[h]$, $((\mathcal{N}, E), \Omega)$ is a $\hat{j}(\Gamma)$

hod pair, and Ω is fullness preserving and has branch condensation. The construction will give a symmetric term $\dot{\Omega}$ such that $\dot{\Omega}^h = \Omega$, and we'll show (*) holds for $((\mathcal{N}, E), \dot{\Omega})$.

<u>Notation</u> If $\mathcal{P}$ is a hod premouse having a top block, then $\kappa^{\mathcal{P}}$ is the ordinal that begins the top block of $\mathcal{P}$.

What we need about $(\mathcal{N}, E)$ to construct an iteration strategy is just that its top block is not too complicated.

<u>Definition 2.6</u> Let $\mathcal{P}$ be a hod premouse having a top block. We say that the <u>top block of $\mathcal{P}$ is below $0^{\mathcal{P}}$</u> iff whenever E is a $\kappa^{\mathcal{P}}$-relevant extender in the top block of $\mathcal{P}$, then $\mathrm{crit}(E) = \kappa^{\mathcal{P}}$.

Note that if $\mathcal{U}$ is a normal tree on a hod premouse $\mathcal{P}$, and $[0, \alpha]_{\mathcal{U}}$ does not drop, and $E_\alpha^{\mathcal{U}}$ is taken from the top block of $\mathcal{M}_\alpha^{\mathcal{U}}$ & with $\mathrm{crit}(E_\alpha^{\mathcal{U}}) = \kappa^{\mathcal{M}_\alpha^{\mathcal{U}}}$, then $E_\alpha^{\mathcal{U}}$ is applied to $\mathcal{M}_\alpha^{\mathcal{U}}$ (i.e. $\mathcal{M}_{\alpha+1}^{\mathcal{U}} = \mathrm{Ult}(\mathcal{M}_\alpha^{\mathcal{U}}, E_\alpha^{\mathcal{U}})$), and the rest of $\mathcal{U}$ is based on $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$. This is because $\mathrm{lh}(E_\alpha^{\mathcal{U}})$ is a cutpoint of $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$.

Thus if $\mathcal{P}$ has a top block that is below $o^{\mathcal{P}}$, ~~and~~ any non-dropping iteration of $\mathcal{P}$ can be given by a sequence $\langle (\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ where $\mathcal{P}_0 = \mathcal{P}$, $\vec{\mathcal{T}}_\alpha$ is a stack of normal trees with base model $\mathcal{P}_\alpha$ and last model $\mathcal{P}_{\alpha+1}$, $\mathcal{P}_\lambda = \lim_{\alpha<\lambda} \mathcal{P}_\alpha$ for λ limit, and for each α, either

(i) $\vec{\mathcal{T}}_\alpha$ uses no extenders in the top block of $\mathcal{P}_\alpha$ or its images, or

(ii) $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, G)$, for some G in the top block of $\mathcal{P}_\alpha$, with $\mathrm{crit}(G) = \kappa^{\mathcal{P}_\alpha}$.

<u>Definition</u> 2.6b1 An iteration $\langle (\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ satisfying (i) and (ii) is said to be in <u>normal form</u>.

~~We now describe our complete iteration strategy for $(\mathcal{N}, E)$. Set~~

~~Ω $Q_0 = \mathcal{N}$, $E_0 = E$,~~

~~$\mathcal{P}_0 = (Q_0, E_0)$;~~

~~and $\Omega_0 = \Sigma$.~~

Before describing our iteration strategy for $(\mathcal{N}, E)$, we make some definitions and prove a few simple things.

<u>Definition 2.7</u> (a) Let $\sigma : R \to H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$-hod pair in $M[h]$. We say Λ is <u>σ-consistent</u> iff letting $\pi : R \to H_1$ be the iteration map by Λ of the $\hat{j}(\Gamma)$-hod-limit system, $\pi \restriction K^R = \sigma \restriction K^R$.

(b) Let $\sigma : R \to j(\mathcal{N})$ be fully elem., and (R, Λ) a $\hat{j}(\Gamma)$-hod-pair in $M[h]$. We say Λ is <u>locally σ-consistent</u> iff $\Lambda_{R|(K^R)^{+R}}$ is σ-consistent.

<u>Remark</u> If $\pi : R \to H_1$ is the map of R into $M_\infty(R, \Lambda)^{\hat{j}(\Gamma)}$, the hod limit, then $\pi \restriction K^R$ is the map of $R|K^R$ into $M_\infty(R|K^R, \Lambda_{R|K^R})$ (We are assuming K^R regular in R.) So really, it's $\Lambda_{R|K^R}$ that determines local σ-consistency. We call it plain

σ-consistency in (a) because then all iterations of $\mathcal{R}$ (that don't drop) are by $\Lambda_{\mathcal{R}|k^{\mathcal{R}}}$.

<u>Remark</u> If $\sigma : \mathcal{R} \to H_1^+$, and $(\mathcal{R}, \Lambda)$ is a $\hat{j}(\Gamma)$ hod pair such that Λ is σ-consistent, then $\Lambda = \Sigma_{H_1}^{\sigma}$. This is because, letting $j_1 = \hat{j}(j)$, Σ_{H_1} is pullback-consistent in $\hat{j}_1(M[h])$, and $\Sigma_{H_1(\pi(k^{\mathcal{R}}))}$ is a tail of $\hat{j}_1(\Lambda)$ there. Thus $\Sigma^{\sigma}_{H_1(\pi(k^{\mathcal{R}}))} \cap HC^{M[h]} = \hat{j}_1(\Lambda) \cap HC^{M[h]}$

$= \Lambda$.

Thus σ-consistency determines Λ in this case.

<u>Remark</u> There are probably examples of $\sigma : \mathcal{R} \to \hat{j}(\eta)$ and $(\mathcal{R}, \Lambda)$ locally σ-consistent, but $\Lambda \neq \Sigma^{\sigma}_{j(\eta)}$. We do not know one at the moment, but surely local σ-consistency is not enough to determine Λ.

Here are some simple facts about σ-consistency.

Definition 2.7 Let $\sigma : R \to H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$-hod pair in $M\Sigma hJ$. We say that Λ is <u>σ-consistent</u> iff letting $\pi : R \to H_1$ be the iteration map, we have $\pi \restriction \kappa^R = \sigma \restriction \kappa^R$.

Notice that if Λ is σ-consistent, then $\Lambda = \Sigma^{\sigma}_{H_1}$. This is because $\Sigma_{H_1(\pi(\kappa^R))}$ is a tail of $\hat{j}_1(\Lambda)$, where $j_1 = j(j)$, and $\hat{j}_1(\Lambda)$ is pullback-consistent, and $\Lambda \subseteq \hat{j}_1(\Lambda)$.

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Proposition 2.8 Let $\sigma : R \to H_1^+$ be elem., and (R, Λ) a $\hat{j}(\Gamma)$ hod pair in $M\Sigma hJ$. The following are equivalent:

(1) Λ is σ-consistent,

(2) whenever $i : R \to S$ is by Λ, then the extender of i is given by

$$E_i = E_\sigma \restriction \pi^{\Lambda}_{S,\infty}{}'' \kappa^S$$

where $\pi^{\Lambda}_{S,\infty} : S \to H_1$ is the iteration map by Λ,

(3) $E_\pi \restriction \pi(\kappa^R) = E_\sigma \restriction \pi(\kappa^R)$, for $\pi = \pi^{\Lambda}_{R,\infty}$.

Remark In clause (2) of 2.6, the equation

$E_i = E_\sigma \restriction \pi^{\Lambda}_{\mathcal{S},\infty}$ " $K^{\mathcal{R}}$ means: for all $a \in [K^{\mathcal{S}}]^{<\omega}$ and all $X \in \mathcal{R}$, $(a, X) \in E_i$ iff $(\pi^{\Lambda}_{\mathcal{S},\infty}(a), X) \in E_\sigma$.

Equivalently

$$a \in i(X) \text{ iff } \pi^{\Lambda}_{\mathcal{S},\infty}(a) \in \sigma(X),$$

for $a \in [K^{\mathcal{S}}]^{<\omega}$ and $X \in \mathcal{R}$. Notice that i is continuous at $K^{\mathcal{R}}$, so it suffices here to consider $X \in \mathcal{R} | K^{\mathcal{R}}$.

Proof of 2.8

(1) $\Rightarrow$ (2). Let $i : \mathcal{R} \to \mathcal{S}$ by Λ, $a \in [K^{\mathcal{S}}]^{<\omega}$, and $X \in \mathcal{R} | K^{\mathcal{R}}$. Then

$$a \in i(X) \text{ iff } \pi^{\Lambda}_{\mathcal{S},\infty}(a) \in \pi^{\Lambda}_{\mathcal{S},\infty}(i(X)),$$

$$\text{iff } \pi^{\Lambda}_{\mathcal{S},\infty}(a) \in \sigma(X),$$

because $\sigma \restriction (\mathcal{R} | K^{\mathcal{R}}) = \pi^{\Lambda}_{\mathcal{R},\infty} \restriction (\mathcal{R} | K^{\mathcal{R}})$.

(2) $\Rightarrow$ (3) Let $X \in \mathcal{R} | K^{\mathcal{R}}$ and $b \in [\pi(K^{\mathcal{R}})]^{<\omega}$, where $\pi = \pi^{\Lambda}_{\mathcal{R},\infty}$. ~~Then~~ Let $b = \pi^{\Lambda}_{\mathcal{S},\infty}(a)$, where $i : \mathcal{R} \to \mathcal{S}$ is by Λ. Then

$$(b,X)\in E_\sigma \text{ iff } \pi_{\mathcal{S},\infty}(a)\in\sigma(X)$$
$$\text{iff } a\in i(X) \quad (\text{by } (2))$$
$$\text{iff } \pi_{\mathcal{S},\infty}(a)\in\pi_{\mathcal{S},\infty}(i(X))$$
$$\text{iff } b\in\pi_{R,\infty}(X)$$
$$\text{iff } (b,X)\in E_\pi,$$

as desired.

(3) $\to$ (1) is clear.

$\boxtimes$

Let (R,Λ) be a $j(\Gamma)$-hod pair in $M[h]$, and $\sigma: R\to j(\mathcal{N})$ be such that Λ is locally σ-consistent. Let $i: R\to\mathcal{S}$ be an iteration map by $\Lambda_{R|k^R}$; that is, an iteration not using any extenders in the top block or its images. There is then a natural factor map τ from $\mathcal{S} = Ult(R, E_\sigma \upharpoonright \pi_{\mathcal{S},\infty}{}''k^{\mathcal{S}})$ to $j(\mathcal{N}) = Ult(R, E_\sigma)$, given by

$$\tau(i(f)(a)) = \sigma(f)(\pi_{\mathcal{S},\infty}(a)).$$

These maps commute with the iteration maps by $\Lambda_{R|K^R}$, yielding

$$\begin{tikzcd} R \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & j(N) \\ & S \arrow[ur, "\tau"'] & \end{tikzcd}$$

Remark We didn't need anything about the full Λ to get τ.

Proposition 2.9 Let $\sigma : R \to j(N)$ and (R, Λ) be locally σ-consistent. Let $i : R \to S$ be an iteration map by $\Lambda_{R|K^R}$, and $\tau : S \to j(N)$ the factor map; then the S-tail of Λ is locally τ-consistent.

Proof Let $\eta < K^S$. Since i is continuous at K^R, we have $\eta = i(f)(a)$ where $f \in R|K^R$ and $a \in [K^S]^{<\omega}$. Then

$$\begin{aligned} \pi_{S,\infty}(\eta) &= \pi_{S,\infty}(i(f)(a)) \\ &= \pi_{S,\infty}(i(f))(\pi_{S,\infty}(a)) \\ &= \sigma(f)(\pi_{S,\infty}(a)) \\ &= \tau(\eta), \end{aligned}$$

as desired. $\boxtimes$

Now we want to see what happens when we touch the top block.

Lemma 2.10 Let $(R, \bar{\Psi})$ be a $\hat{j}(\Gamma)$-hod-pair in $M[h]$, and $\sigma : R \to j(\mathcal{N})$ be such that $\bar{\Psi}$ is locally σ-consistent. Suppose we have i such that

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\ & R|(\kappa^R)^{+R} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes. Suppose also G is on the R-sequence, $\mathrm{crit}(G) = \kappa^R$, and G is σ-certified over $(R\|\mathrm{lh}(G), \bar{\Psi}_{R\|\mathrm{lh}(G)})$. Let $\mathcal{S} = \mathrm{Ult}(R, G)$, and $\tau : \mathcal{S} \to j(\mathcal{N})$ be given by $\tau(i_G(f)(a)) = \sigma(f)(\pi^{\Psi_{R\|\mathrm{lh}(G)}}_{R\|\mathrm{lh}G,\infty}(a))$; then

(1) τ is well-defined,

(2) $\begin{tikzcd} R \arrow[rr, "\sigma"] \arrow[dr, "i_G"'] & & j(\mathcal{N}) \\ & \mathcal{S} \arrow[ur, "\tau"'] & \end{tikzcd}$ commutes,

(3) the $\mathcal{S}$-tail of $\bar{\Psi}$ is locally τ-consistent.

Remark The same proof yields the same conclusions if we assume only that G is an amenable predicate such that $(\mathcal{R}, G)$ is a hod premouse, and G is σ-certified over $(\mathcal{R}, \Psi)$.

Proof of 2.10 Parts (1) and (2) follow at once from the fact that G is σ-certified over $(\mathcal{R} \| h(G), \Psi_{\mathcal{R} \| h(G)})$.

For (3), we apply j-condensation, as stated in Lemma 2 of [3]. (This is why we assumed i exists.) Let $\eta = i_G(f)(a) < \kappa^{\mathcal{S}}$. Let $Q = \mathcal{R} | (\kappa^a)^{+\mathcal{R}}$ and $W = \mathcal{S} | (\kappa^{\mathcal{S}})^{+\mathcal{S}}$.

We may assume $a \in [h(G)]^{<\omega}$, and we can write

~~(with $a \in [lh\,G]^{<\omega}$) we can write~~

$$\eta = i_G \circ i(g)(b, a), \text{ where } b \in [K^\Phi]^{<\omega}.$$

Then

$$L(\mathrm{Hom}^*_h, H_1^+) \models \pi^{\Sigma_{H_0}}_{H_0^+,\infty} {}'' g \subseteq j(g),$$

a statement about Σ_{H_0}, g, H_0^+, and $j(g)$.

So by j-condensation

$$L(\mathrm{Hom}^*_h, H_1^+) \models \pi^{\Phi_W}_{W,\infty} {}'' \, i_G \circ i(g) \subseteq j(g)$$

Let us write $\pi^\Psi = \pi^{\Psi_Q}_{Q,\infty}$, $\pi^\Phi = \pi^{\Phi_W}_{W,\infty}$.

Then

$$\begin{aligned}
\pi^\Phi(\eta) &= \pi^\Phi(i_G \circ i(g)(b, a)) \\
&= j(g)(\pi^\Phi(b), \pi^\phi(a)) \\
&= j(g)(\pi^\Psi(b), \pi^\Psi(a))
\end{aligned}$$

(by strategy coherence for Ψ, and $lh(G)$ being a cutpoint of W --- this is why we indexed that way!)

$$\begin{aligned}
&= \sigma(i(g))(\sigma(b), \pi^\Psi(a)) \\
&= \sigma(i(g)(b))(\pi^\Psi(a)) = \sigma(f)(\pi^\Psi(a)) \\
&= \tau(\eta),
\end{aligned}$$

as desired.

$\boxtimes$

The next lemma deals with how certification in the top block is preserved.

<u>Lemma 2.11</u> Suppose in $M[\Sigma]$ we have R, i, σ with

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i"'] & & j(M) \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commuting, R countable, suppose $j \restriction M \in M$. Let $\Psi = \hat{j}(\Sigma)^\sigma$, and suppose that Ψ is locally σ-consistent. Let G be an extender from the top block of R, with $\mathrm{crit}(G) = \kappa^R$; then G is σ-certified over $(R \| lh(G), \Psi_{R\| lh(G)})$.

<u>Proof</u> For $R = M$ and $\Psi = \Sigma = \hat{j}(\Sigma)^j$, this is true. We express it as a collection of statements involving parameters $j(A)$ for $A \in H_0^+$, and then apply j-condensation in the form of lemma 2.4.

Note that since j is continuous at $(\kappa^M)^{+M} = o(H_0^+)$, i is continuous at $(\kappa^R)^{+R}$.

Let $\theta^0 < \xi < o(H_0^+)$, and let

$A^\xi = \langle A^\xi_\alpha \mid \alpha < \theta^0 \rangle \in H_0^+$ be an enumeration of $H_0^+ | \xi$. Then

$$L(\mathrm{Hom}^*_h, j(\mathcal{M})) \models \text{for all extenders } G \text{ on the sequence of } \mathcal{M} \text{ with } \mathrm{crit}(G) = \theta^0, \text{ for all } a \in [lh G]^{<\omega} \text{ and all } \alpha < \theta^0$$

$$A^\xi_\alpha \in G_a \iff \pi^{\Sigma_{\mathcal{M}\|lhG}}_{\mathcal{M}\|lhG,\infty}(a) \in j(A^\xi)_{\pi^{\Sigma_{H_0}}_{H_0,\infty}(\alpha)},$$

which is a statement φ about the parameters $\mathcal{M}, \Sigma, A^\xi$, and $j(A^\xi)$. By 2.4, $\varphi(\mathcal{R}, \Psi, i(A^\xi), j(A^\xi))$ holds in $L(\mathrm{Hom}^*_h, j(\mathcal{M}))$, i.e.

$$L(\mathrm{Hom}^*_h, j(\mathcal{M})) \models \text{for all extenders } G \text{ on the sequence of } \mathcal{R} \text{ with } \mathrm{crit}(G) = \kappa^{\mathcal{R}}, \text{ for all } a \in [lh(G)]^{<\omega} \text{ and all } \alpha < \kappa^{\mathcal{R}}$$

$$i(A^\xi)_\alpha \in G_a \iff \pi^{\Psi_{\mathcal{R}\|lhG}}_{\mathcal{R}\|lhG,\infty}(a) \in j(A^\xi)_{\pi^{\Psi_{\mathcal{Q}}}_{\mathcal{Q},\infty}(\alpha)}$$

However, notice $\sigma(i(A^\xi)_\alpha) = j(A^\xi)_{\sigma(\alpha)} = j(A^\xi)_{\pi^{\Psi_{\mathcal{Q}}}_{\mathcal{Q},\infty}(\alpha)}$, since $\Psi_{\mathcal{Q}}$ is locally σ-consistent.

It follows that all G on the R-sequence such that $\mathrm{crit}(G) = \kappa^R$ are σ-certified over $(R \| lh(G), \Psi_{R\|lh(G)})$, so far as sets to be measured in $i(H_0^+|\xi)$ go. But i is continuous at $o(H_0^+)$, and ξ was arbitrary.

$\boxtimes$

Virtually the same proof gives

<u>Lemma 2.12</u> Suppose in $M[h]$ we have R, i, σ with

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "j"] \arrow[dr, "i"'] & & j(\mathcal{N}) \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

and $j \restriction \mathcal{N} \in M$.

commuting and R countable. Suppose $j(\Sigma)^\sigma$ is locally σ-consistent. Let E be j-certified over $(\mathcal{N}, \Sigma)$, and $F = \bigcup\{i(G) \mid G \subseteq E \wedge G \in \mathcal{N}\}$; then F is σ-certified over $(R, j(\Sigma)^\sigma)$.

<u>Proof</u> We apply the proof of 2.11 to the fragments $i(G)$ of F, fragment-by-fragment. We leave the details to the reader.

$\boxtimes$

We would like to apply 2.11 and 2.12 with $i : \mathcal{N} \to R$ an iteration map by Σ, or more generally, by the iteration strategy Ω for $(\mathcal{N}, E)$ we are trying to construct. The problems are, in the case of Σ-iterations

(a) how do we know $\sigma : R \to j(\mathcal{N})$ with $j = \sigma \circ i$ exists?

(b) ~~how do we know~~ why is $j(\Sigma)^{\sigma}$ the R-tail of Σ?

There are parallel problems in the case i is by Ω, ~~and~~ our strategy for $(\mathcal{N}, E)$.

More generally, let

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "j"] \arrow[dr, "k"'] & & j(\mathcal{N}) \\ & P \arrow[ur, "l"'] \arrow[rr, "i"'] & & R \end{tikzcd}$$

be given in $M[h]$, where $j = l \circ k$, and i is an iteration by $j(\Sigma)^{l} =_{df} \Psi$, and Ψ is locally l-consistent. We'd like to find τ such that

$$\begin{tikzcd} & j(\mathcal{M}) & \\ P \arrow[ur, "\ell"] \arrow[rr, "i"'] & & R \arrow[ul, "\tau"'] \end{tikzcd}$$

commutes, $j(\Sigma)^\tau$ is the R-tail of Ψ, and $j(\Sigma)^\tau$ is locally τ-consistent. To do this, we put our iteration in normal form (cf. Def. 2.6.1). So we have $\langle (P_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ with $P = P_0$, and last model R. We write $R = P_\eta$. (So R is the last model of $\vec{\mathcal{T}}_{\eta-1}$ if $\eta - 1$ exists, and $R = \lim_{\alpha<\eta} P_\alpha$ otherwise.) Let $\ell = \tau_0$.

We define embeddings $\tau_\alpha : P_\alpha \longrightarrow j(\mathcal{M})$ by induction so that

(1) $\tau_\gamma = \tau_\alpha \circ i_{\gamma\alpha}$ for $\gamma < \alpha$, $i_{\gamma\alpha} : P_\gamma \to P_\alpha$,

(2) $j(\Sigma)^{\tau_\alpha}$ is the P_α-tail of Ψ,

(3) the P_α-tail of Ψ is locally τ_α-consistent.

Suppose first that we have τ_α such that

(1) – (3) hold, and we want $\tau_{\alpha+1}$.

<u>Case 1</u> $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, G)$, for $\mathrm{crit}(G) = \kappa^{\mathcal{P}_\alpha}$ on the $\mathcal{P}_\alpha$-sequence.

Then G is τ_α-certified over $(\mathcal{P}_\alpha \| lh G, j(\Sigma)^{\tau_\alpha}_{\mathcal{P}_\alpha \| lh(G)})$ by 2.11. So letting

$$\tau_{\alpha+1}(i_G(f)(a)) = \tau_\alpha(f)(\pi^{\Phi}_{\mathcal{P}_\alpha \| lh(G), \infty}(a))$$

for $\Phi = j(\Sigma)^{\tau_\alpha}$, we have by 2.10 that ~~$\tau_{\alpha+1}$~~ $\tau_\alpha = \tau_{\alpha+1} \circ i_{\alpha,\alpha+1}$, and $j(\Sigma)^{\tau_{\alpha+1}}$ is locally $\tau_{\alpha+1}$-consistent. So it is enough to show that $j(\Sigma)^{\tau_{\alpha+1}}$ is the $\mathcal{P}_{\alpha+1}$-tail of Ψ. **Remark** See p. 73a

But this follows from theorem 3.76 in the 3/25/00 version of Sargsyan's thesis [2]. ("Branch condensation pulls back".) To jog the reader's memory, here is the briefest sketch: we get a hod mouse W

<u>Remark</u> Let $\Phi = j(\mathcal{E})^{\tau_\alpha}$, and Λ be the $P_{\alpha+1}$-tail of Φ. Let $W = P_{\alpha+1} \mid (K^{P_{\alpha+1}})^{+P_{\alpha+1}}$. Lemma 2.10 then says that Λ_Q is $\tau_{\alpha+1}$-consistent. But then $\Lambda_Q = j(\mathcal{E})_Q^{\tau_{\alpha+1}}$, so $j(\mathcal{E})^{\tau_{\alpha+1}}$ is locally $\tau_{\alpha+1}$-consistent.

with ω Woodins extending $\mathcal{N}$, and having a UB representation of Σ that is moved properly. $i_{0,\alpha+1} \circ l$ ~~act~~ extends to act on W, yielding

$$\begin{tikzcd}
W \arrow[rr, "j"] \arrow[dr, "l"'] & & j(W) \\
& W_0 \arrow[ur, "\varphi_0"] \arrow[dr] & \\
& & i_{0,\alpha+1} \circ l(W) \arrow[uu, "\varphi"']
\end{tikzcd}$$

with $\varphi \restriction \mathcal{P}_{\alpha+1} = \tau_{\alpha+1}$, $\varphi_0 \restriction \mathcal{P}_0 = \tau_0$.

The W-representation of Σ gets moved to the W_0-representation of its $\mathcal{P}_0$-tail, and ~~Sargsyan show~~ also to $j(\Sigma)^{\tau_0}$, as Sargsyan shows. It is then further moved to its $\mathcal{P}_{\alpha+1}$-tail, and to $j(\Sigma)^{\tau_{\alpha+1}}$.

Case 2 $\vec{\mathcal{T}}_\alpha$ involves no extenders in the top block of $\mathcal{P}_\alpha$, or its images.

In this case we get $\tau_{\alpha+1} : \mathcal{P}_{\alpha+1} \to j(\mathcal{N})$ as in (1) - (3) by using proposition 2.9 where we used 2.10 in case 1. We leave the details to the reader.

Finally, suppose $\lambda \leq \eta$ is a limit ordinal. We define $\tau_\lambda(i_{\alpha\lambda}(x)) = \tau_\alpha(x)$, for $\alpha < \lambda$. This gives $\tau_\lambda : P_\lambda \to j(\mathcal{M})$, and it is clear that (1) holds. For (2), we use Sargsyan's 3.76 of [2] again.

For (3), let $\nu < \kappa^{P_\lambda}$, and $\pi : P_\lambda \to H_1$ the iteration map by $j(\Sigma)^{\tau_\lambda} = P_\lambda$-tail of Ψ.

Let $\nu = i_{\alpha\lambda}(\bar{\nu})$, and ~~$\varphi = j(\Sigma)^{\tau_\alpha} = P_\alpha$-tail~~ $\varphi : P_\alpha \to H_1$ be the iteration map by $j(\Sigma)^{\tau_\alpha} = P_\alpha$-tail of Ψ. Then

$$\tau_\lambda(\nu) = \tau_\alpha(\bar{\nu}) = \varphi(\bar{\nu})$$
$$= \pi(\nu),$$

as desired.

We have shown

<u>Lemma 2.13</u> Assume $j \upharpoonright \mathcal{N} \in M$. Let

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "j"] \arrow[dr, "k"'] & & j(\mathcal{N}) \\ & \mathcal{P} \arrow[ur, "l"] \arrow[rr, "i"'] & & \mathcal{R} \end{tikzcd}$$

be given in $M[h]$, where $j = l \circ k$ and i is an iteration map by $j(\Sigma)^{l} = \Psi$. Suppose Ψ is locally l-consistent. Let Φ be the $\mathcal{R}$-tail of Ψ. Then there is a unique embedding τ such that

$$\begin{tikzcd} & j(\mathcal{N}) & \\ \mathcal{P} \arrow[ur, "l"] \arrow[rr, "i"'] & & \mathcal{R} \arrow[ul, "\tau"'] \end{tikzcd}$$

commutes, $\Phi = j(\Sigma)^{\tau}$, and Φ is locally τ-consistent.

<u>Proof</u> All that's left is uniqueness of τ. But since the top block of $\mathcal{P}$ is below $O^{\mathcal{P}}$, any $x \in \mathcal{R}$ has the form $i(f)(a)$, where $a \in [\kappa^{\mathcal{R}}]^{<\omega}$. But then $\tau(x) = \tau(i(f)(a)) = l(f)(\pi^{\Phi}_{\mathcal{R},\infty}(a))$, so τ is determined by l and the $\mathcal{R}$-tail of Ψ, hence by l and $\mathcal{R}$. ☒

We are now ready to define our iteration strategy Ω for $(\mathcal{N}, E)$, where E is j-certified over $(\mathcal{N}, \Sigma)$.

We are assuming ~~witnesses hugeness~~ $j \restriction \mathcal{N} \in M$. Let

$Q_0 = \mathcal{N}$, $E_0 = E$, and

$$P_0 = (Q_0, E_0).$$

Let $\Lambda_0 = \Sigma$, and $\tau_0 = j \restriction \mathcal{N}$. Let $\langle (P_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ be an iteration of P_0 in normal form, played according to the strategy Ω that we are defining. We maintain by induction that there are $\tau_\alpha : Q_\alpha \to j(\mathcal{N})$ so that

$$\begin{tikzcd} Q_\beta \arrow[rr, "\tau_\beta"] \arrow[dr, "i_{\beta\alpha}"'] & & j(\mathcal{N}) \\ & Q_\alpha \arrow[ur, "\tau_\alpha"'] & \end{tikzcd}$$

commutes when $\beta < \alpha$, where $i_{\beta\alpha} : P_\beta \to P_\alpha$ is the iteration map, and strategies Λ_α for Q_α

such that

(1) Ω_α is good, and has branch condensation,

(2) $\Omega_\alpha = \hat{j}(\Sigma)^{\tau_\alpha}$, and

(3) Ω_α is locally τ_α-consistent,

(4) if $\vec{\mathcal{I}}_\alpha$ is ∃ on Q_α, then it is by Ω_α, and $\Omega_{\alpha+1} = (\Omega_\alpha)_{\vec{\mathcal{I}}_\alpha, Q_{\alpha+1}}$, and

(5) if $P_{\alpha+1} = \mathrm{Ult}(P_\alpha, E_\alpha)$, then $\Omega_\alpha = (\Omega_{\alpha+1})_{Q_\alpha}$.

Note that Q_α is a cutpoint full initial segment of $Q_{\alpha+1}$ when (5) applies.

Suppose we have $\langle \tau_\beta \mid \beta \leq \alpha \rangle$.

Case 1 $\vec{\mathcal{I}}_\alpha$ is on Q_α.

In this case, lemma 2.13, with $P = Q_\alpha$ and $R = Q_{\alpha+1}$, gives $\tau_{\alpha+1}$ and (1)–(5).

<u>Case 2</u> $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, E_\alpha)$.

We have that E_α is τ_α-certified over $(Q_\alpha, \mathcal{R}_\alpha)$, by 2.12 and induction hypotheses (2) and (3). By ~~the~~ the remark after 2.10, if we set

$$\tau_{\alpha+1}\left(i_{E_\alpha}(f)(a)\right) = \tau_\alpha(f)\left(\pi^{\mathcal{R}_\alpha}_{Q_\alpha,\infty}(a)\right),$$

then $\tau_{\alpha+1}$ is well-defined and $\tau_\alpha = \tau_{\alpha+1} \circ i_{\alpha,\alpha+1}$.

Moreover, setting $\mathcal{R}_{\alpha+1} = j(\Sigma)^{\tau_{\alpha+1}}$, we have as in the proof of 2.10 that $\mathcal{R}_{\alpha+1}$ is good, has branch condensation, and is locally $\tau_{\alpha+1}$ consistent. (This all uses j-condensation, lemma 2.4.)

Since Q_α is a cutpoint of $Q_{\alpha+1}$

$$\pi^{(\mathcal{R}_{\alpha+1})_{Q_\alpha}}_{Q_\alpha,\infty} \restriction o(Q_\alpha) = \pi^{\mathcal{R}_{\alpha+1}}_{Q_{\alpha+1},\infty} \restriction o(Q_\alpha)$$

$$= \tau_{\alpha+1} \restriction o(Q_\alpha) \quad \mathcal{R}_{\alpha+1} \text{ locally consis.}$$

(since $\Omega_{\alpha+1}$ is locally $\tau_{\alpha+1}$-consistent)

$$= \pi^{\Omega_\alpha}_{Q_\alpha,\infty} \restriction o(Q_\alpha)$$

(by the definition of $\tau_{\alpha+1}$). Pullback condensation for Ω_α and $\Omega_{\alpha+1}$ then give us (5).

Finally, if λ is a limit, we define τ_λ by $\tau_\lambda(i_{\alpha\lambda}(x)) = \tau_\alpha(x)$, and ~~let~~ set $\Omega_\lambda = j(\Sigma)^{\tau_\lambda}$. We leave the rest to the reader.

This then tells us how to define Ω for one more step. We only need worry about the case that $\eta = \alpha+1$, and $\vec{T}_\alpha$ is on Q_α. (Otherwise, there is no choice of branch to be made.) But in this case, we let Ω choose $\Omega_\alpha(\vec{T}_\alpha)$.

This completes our definition of Ω for (N, E), assuming j ~~witnesses hugeness~~ that $j \restriction \eta \in M$.

Lemma 2.15 Assume that $j \restriction \mathcal{N} \in M$, ~~j witnesses that κ_0 is huge~~, and let Ω be the iteration strategy for $(\mathcal{N}, E)$ defined above; then Ω is good.

Proof That Ω is self-consistent and coherent is just clauses (4) and (5) of our induction hypotheses. That Ω is $\hat{j}(\Gamma)$-fullness preserving is also implicit in the induction hypotheses. (We get it from j-condensation.)

$\boxtimes$

Lemma 2.16 Assume that $j \restriction \mathcal{N} \in M$, ~~j witnesses that κ_0 is huge~~, and let Ω be the iteration strategy for $(\mathcal{N}, E)$ defined above; then Ω has branch condensation.

Proof. Let $\mathcal{P} = (M, E)$, and suppose branch condensation fails. We can then find a stack $\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle$ on $\mathcal{P}$ such that $\vec{\mathcal{T}}$ is by Ω, and there is an iteration $i \colon \mathcal{P} \to \mathcal{Q}$ by Ω and cofinal branches b and c of $\mathcal{U}$ and $\pi \colon M_b^{\mathcal{U}} \to \mathcal{Q}$ such that

(1) $i = \pi \circ i_b^{\mathcal{U}} \circ i^{\vec{\mathcal{T}}}$,

(2) $c = \Omega(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle)$, and

(3) $b \neq c$.

Letting $\mathcal{R}$ be the base model of $\mathcal{U}$, the picture is

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "i"] \arrow[dr, "i^{\vec{\mathcal{T}}}"'] & & \mathcal{Q} \\
& \mathcal{R} \arrow[r, "i_b"] \arrow[dr, "i_c"'] & M_b^{\mathcal{U}} \arrow[u, "\pi"'] \\
& & M_c^{\mathcal{U}}
\end{tikzcd}$$

b is $i_b \circ i^{\vec{\mathcal{T}}}$-realized, while c is by Ω.

b does not drop because that is part of the hypothesis in branch condensation. We shall show that c does not drop.

<u>Claim I.</u> $M_b^{\mathcal{U}} \models \delta(\mathcal{U})$ is Woodin.

<u>Proof</u> If η is a cardinal of a hod premouse $\mathcal{M}$, then $\mathcal{M}|\eta$ is a full level of $\mathcal{M}$. Thus $\mathcal{M}(\mathcal{U})$ is a limit of full levels of $M_b^{\mathcal{U}}$ and $M_c^{\mathcal{U}}$.

By minimizing our counterexample to branch condensation, we can arrange (letting $i = i^{\vec{w}}$)

(*) Let $\mathcal{A}$ be a full proper initial segment of $\mathcal{M}(\mathcal{U})$; then

$$\left(\Omega_{\vec{w},\pi(\mathcal{A})}\right)^{\pi} = \left(\Omega_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{\mathcal{A}}.$$

<u>Proof</u> Note that $\left(\Omega_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{M(\mathcal{U})} \in \hat{j}(\Gamma)$ by our construction. (All tails of Ω, projected to proper initial segments of their base models, are in $\hat{j}(\Gamma)$.) Suppose we have chosen our

counterexample $\vec{T}, \mathcal{U}, b, c, \pi, \vec{W}$ so that $\left(\Omega_{\vec{T}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{M(\mathcal{U})}$ has minimal possible Wadge rank. We claim that (*) holds.

For let $\mathcal{S}$ be a counterexample to (*), and let $\alpha \in b$ be such that $\mathcal{S}$ is a proper initial segment of $M_\alpha^{\mathcal{U}} \mid \mathrm{crit}(i_{\alpha b}^{\mathcal{U}})$. Then we have that for $\sigma = \pi \circ i_{\alpha b}$, $\sigma \restriction \mathcal{S} \cup \{\mathcal{S}\} = \pi$, and so $\left(\Omega_{\vec{W},Q}\right)_{\mathcal{S}}^{\sigma} = \left(\Omega_{\vec{W},\pi(\mathcal{S})}\right)^{\pi}$, so

$\left(\Omega_{\vec{W},Q}\right)_{\mathcal{S}}^{\sigma} \neq \Omega_{\vec{T}^\frown\langle\mathcal{U}|\alpha+1\rangle,\mathcal{S}}$, since the latter is just $\Omega_{\vec{T}^\frown\langle\mathcal{U}\rangle^\frown c,\mathcal{S}}$ by strategy coherence. This means we get a counterexample of the form

$$\begin{tikzcd}
 & & Q \arrow[rr, "\sigma\vec{T}^*"] & & \cdot \arrow[r, "\sigma\mathcal{U}^*"] & b^*\ Q^* \\
P \arrow[urr, "i^{\vec{W}}"] \arrow[dr, "i^{\vec{T}}"'] & & & & & \\
 & R \arrow[r] & M_\alpha^{\mathcal{U}} \arrow[uu, "\sigma"] \arrow[rr, "i^{\vec{T}^*}"'] & & \cdot \arrow[r] \arrow[dr] & b^* \arrow[uu, "\pi^*"'] \\
 & & & & & c^* \\
 & & & & \mathcal{U}^* &
\end{tikzcd}$$

with $\vec{T}^{*\frown}\mathcal{U}^*$ based on $\mathcal{S}$ being according to both $(\Omega_{\vec{w},Q})^{\sigma}_{\Delta}$ and $\Omega_{\vec{d}^\frown u|\alpha+1,\Delta}$, but the former choosing b^*, the latter choosing c^*, and $b^* \neq c^*$. But now

$\Omega_{\vec{d}^\frown\langle u|\alpha+1\rangle^\frown\vec{d}^{*\frown}\langle u^*\rangle^\frown c^*, M(u^*)}$ is a tail of $\Omega_{\vec{d}^\frown\langle u\rangle^\frown c,\Delta}$, so is projective in $\Omega_{\vec{d}^\frown\langle u\rangle^\frown c,\Delta}$, so has Wadge rank strictly less than that of $\Omega_{\vec{d}^\frown\langle u\rangle^\frown c, M(u)}$. This contradiction yields (*).

$\boxtimes$

~~Let Φ be the join of~~

<u>Remark</u> Our notation from §1 is such that if ν is a limit of full levels of $\mathcal{M}$, where $(\mathcal{M}, \Sigma)$ is a hod pair, then Σ_ν is just the join of the $\Sigma_{\nu'}$ for ν' a proper full initial segment of ν. So another way to write (*) is: $(\Omega^{\pi}_{\vec{w},Q})_{M(u)} = \Omega_{\vec{d}^\frown\langle u\rangle^\frown c, M(u)}$

Now let $\Phi = \left(\Omega^{\pi}_{\vec{w},q}\right)_{M(u)} = \Omega_{\vec{\Gamma}\langle u\rangle \frown c,\, M(u)}$.

We claim

> **Remark** The remainder of this proof, up to Claim 2 on p. 89, is written out in more generality as 2.3.4, p. 45a, etc.

(**) $L_p^{\Phi}(M(u))^{\hat{j}(\Gamma)}$ is the next full level of M^u_b after $M(u)$.

This follows from j-condensation, i.e. 2.4. For recall that $\mathcal{J}^-$ is $\mathcal{J}$ with its last extender predicate removed. We then have

$$\begin{tikzcd}
N = P^- \arrow[rr, bend left, "j"] \arrow[r, "i"] \arrow[dr, "k"'] & Q^- \arrow[r, "r"] & j(N) \\
& \left(M^u_b\right)^- \arrow[u, "\pi"'] &
\end{tikzcd}$$

where $k = i_b \circ i^{\vec{\Gamma}}$. Here r is the "realization map" that is part of the definition of Ω. We have $\Omega_{\vec{w},Q^-} = j(\Sigma)^r$, so $\Phi = j(\Sigma)^{r\circ\pi}_{M(u)}$.

This gives us, via 2.4, that $\left(M^u_b\right)^-$ is $L_p^{\Phi}(M(u))^{\hat{j}(\Gamma)}$ – full, as desired.

We then get

(***) $Lp^{\Phi}(\mathcal{M}(\mathcal{U}))^{\hat{j}(\Gamma)}$ is the next full level of $M_c^{\mathcal{U}}$ after $\mathcal{M}(\mathcal{U})$.

If not, since Ω is $\hat{j}(\Gamma)$-fullness preserving, we must have that c dropped, and $M_c^{\mathcal{U}}$ is the first level of $Lp^{\Phi}(\mathcal{M}(\mathcal{U}))^{\hat{j}(\Gamma)}$ that is not in $M_c^{\mathcal{U}}$, and this level projects strictly accross $\delta(\mathcal{U})$. But that means $\delta(\mathcal{U})$ is not a cardinal in $M_b^{\mathcal{U}}$, contradiction.

By (**) and (***), $Lp^{\Phi}(\mathcal{M}(\mathcal{U}))^{\hat{j}(\Gamma)} \models \delta(\mathcal{U})$ is Woodin. Let κ be the least cardinal strong to $\delta(\mathcal{U})$ in $\mathcal{M}(\mathcal{U})$. Let $\mu = o(Lp^{\Phi}(\mathcal{M}(\mathcal{U}))^{\hat{j}(\Gamma)})$.

Let $\Psi_b = (\Omega^{\pi}_{\vec{w},Q})_{M_b|\mu}$ and $\Psi_c = \Omega_{\mathcal{T}^\frown\langle\mathcal{U}\rangle^\frown c,\, M_c|\mu}$.

It is possible that $\Psi_b \neq \Psi_c$. Nevertheless, because we are below Θ_n^P, ~~...~~ in both

M_b and M_c the κ-blocks end with ω ~~iteration~~ more full levels above ~~[illegible]~~ $M_c | \mu = M_b | \mu$, and moreover $\delta(\mathcal{U})$ remains Woodin in M_b and M_c. (But $\mu < \delta(\mathcal{U})^{+M_b}$ and $\mu < \delta(\mathcal{U})^{+M_c}$.) [~~For example~~ κ is not a limit of Woodins in $M(\mathcal{U})$, as otherwise we are past O^P_h. By 22J then, $Lp^{\Psi_b}(M_b \upharpoonright \mu)^{M_b} \models \delta(\mathcal{U})$ is Woodin. But then the κ-block ends in M_b with ω more Lp's, unless ~~[illegible]~~ $o(Lp^{\Psi_b}(M_b | \mu))^{M_b}$ is an index on M_b of an extender with critical point κ. Since κ is not a limit of Woodins, $o(Lp^{\Psi_b}(M_b|\mu))^{M_b}$ is not such an index. Similarly on the c-side.]

This finishes our proof of claim 2. We leave it to the reader to check that we actually proved: