§2. Construction of $H_0$ : adding an extender with length below $K_0^+$

We have $j : V \rightarrow M$, $\text{crit}(j) = K_0$, as in section 0. Let $\mathcal{g}$ be $V$-generic over $\text{Col}(\omega, < K_0)$.

Let $\Gamma \subseteq \text{Hom}_\mathcal{g}^+$ be such that

$$ L(\Gamma, R^+g) = \text{AD}_{R^g} + DC + " \text{HOD} \theta \text{ is the direct limit of all } \text{hod}-\text{pairs } (P, \Sigma) \text{, s.t. }$$

$$ \Sigma \text{ is fullness-props and has branch condensation, under the comparison maps".}$$

Put

$$ \theta^0 = \theta(\Gamma, R^g),$$

$$ H_0 = \text{HOD}^{L(\Gamma, R^g)}_{\theta^0}.$$

Let $\hat{j} : V\mathcal{g}J \rightarrow M[\mathcal{h}J]$ extend $j$, and

$$ H_0^+ = L_{\mathcal{F}}(H_0)^{\hat{j}(\Gamma)}.$$
where $\mathcal{E}_H = \bigoplus_{x \in \mathcal{E}_0} \mathcal{E}_{H_0}(x)$, and

$\mathcal{E}_{H_0}(x)$ is the common tail of all $\mathcal{E}_{H}(x)$, where $H_0(x)$ is a $\mathcal{E}_p$-iterate of $P$. If $\Gamma = \text{Hom}^*$, then $\mathcal{E}(\Gamma) = \text{Hom}^*$. Otherwise, any complete $\Gamma$ set $A$ has a $\text{Hom}^*$ code, and $\mathcal{E}$ moves this code to the code of the complete $\mathcal{E}(\Gamma)$ set $\mathcal{E}(A)$.

Working in $\mathcal{MCHJ}$, we shall build a $\mathcal{E}(\Gamma)$-hood-pair $(\mathcal{H}, \mathcal{E}_H)$ that extends $(\mathcal{H}_0, \mathcal{E}_{H_0})$. $\mathcal{H}$ will have a top block, and when $\mathcal{E}$ will begin that block, we shall show that if $\Gamma \neq \text{Hom}^*$, then $(\mathcal{H}, \mathcal{E}_H)$ either reaches $\mathcal{O}_H^p$, or is a pointclass generator for $\mathcal{E}(\Gamma)$ in $\mathcal{MCHJ}$. We shall show that if $\Gamma = \text{Hom}^*$, then $(\mathcal{H}, \mathcal{E}_H)$ reaches $\mathcal{O}_H^p$.

In fact, $\mathcal{H}$ and $\mathcal{E}_H$ are constructed...
in $M$, and $M \kappa I$ is just used to record properties of them. So we'll have $H \in M$, and an iteration strategy $\Psi$ for $H$ with $\Psi \in M$, and defined on all $\xi \in M$ of size $< \kappa(k)$. $\Psi$ will "determine itself on generic extensions" in such a way that for all $\lambda \in \text{coll}(\omega, < \kappa(k))$ there is $\Psi \upharpoonright \lambda \in \Psi$ defined on $\text{HCM}_{\lambda} \kappa I$.

What we are calling $\Sigma H$ is then $\frac{\bar{\Psi}}{\bar{H}}$.

(So $M \models \Sigma H : \psi \iff \bar{\psi} \in \bar{\Psi}$.) $\bar{\psi}$ is a "symmetric name", in that $\Psi \upharpoonright \lambda = \psi^M$ whenever $\text{HCM}_{\lambda} \kappa I = \text{HCM}_{\lambda \upharpoonright \kappa I}$.

So in $M$, we are constructing by induction on $\xi$ pairs $(\eta, \psi)$ such that the following induction hypotheses are hold. We call them ($\dagger$).
Induction hypotheses (48a): In \( M \), the following hold, for \( (\mathcal{N}, \mathcal{V}) = (\mathcal{N}_\xi, \mathcal{V}_\xi) \):

(a) \( \mathcal{N} \) is a hod prime model extending \( H_0^+ \).

(b) \( \mathcal{V} \) is a \( col(\omega, \leq \kappa_1) \)-name such that \( \mathcal{V}_\xi = \mathcal{V}_\eta^m \) whenever \( HC \cap M_\xi = HC \cap M_\eta \).

(c) \( col(\omega, \leq \kappa_1) (\mathcal{N}, \mathcal{V}) \) is a \( j(\Gamma) \)-hod-pair such that \( \mathcal{V} \) is \( j(\Gamma) \)-fullness preserving and has branch condensation and is positional.

It is easiest to describe the construction of \( \mathcal{N}_{\xi+1} \) and \( \mathcal{V}_{\xi+1} \) if we have \( j \downarrow M \in M \).

This is of course true if \( 0(\mathcal{M}_\xi) < \kappa_0^+ \), and may be true beyond that if \( j \) witnesses more than measurability of \( \kappa_0 \).

Remarks

(1) If \( j(\Gamma) \), then and we reach \( \xi \) such that \( 0(\mathcal{M}_\xi) = \kappa_0^+ \), then
$\mathcal{N} = \text{ZFC } + \text{ " } \Theta_0 \text{ is a strong limit of Woodins" .} $

So although we haven't reached $\Theta_0^P$, we're close.

(2) If $i$ witnesses $\Theta_0$ is huge, then we can go up to $\mathcal{N}_{ki}$, and we will definitely reach $\Theta_0^P$ before that.

(3) In clause (t)(c), $\bar{i}$ is a symmetric name for $i$. We assume $\bar{i} \in V$ for simplicity; in general it will be in some size $< \Theta_0$ extension of $V$.

We have one further induction hypothesis: for $(\mathcal{N}, \check{\varphi}) = (\mathcal{M}, \check{\varphi})$,

(4) if $\check{j} \upharpoonright \mathcal{N} \in M$, then

$M = \text{Ult}(\omega, \check{\text{col}}(\omega, <\kappa_1))$ \quad $\check{\varphi} = j(\check{\varphi})^\ast$.

(Here on the right side, "$j$" should be replaced by "$\check{j}\upharpoonright \mathcal{N}$", to be precise. ) (In the superscript only.)
Some explanation is in order. Let $d_1 = j(j)$, with $j : M \rightarrow N$, and $\kappa_2 = j(j_1)$. We have that $j(j)$ is a col$(\omega, \kappa_2)$ name in $N$ for a strategy for $j(M)$. But letting $k$ be col$(\omega, \kappa_1)$ - generic over $M$, we can make sense of $j(j)$. By the symmetry of $j(j)$, it is the common value of all $j(j), j, \kappa_2 \in \text{MEHJ}$ for $h \leq \kappa_1$ on col$(\omega, \kappa_2)$. So $j(j)$ is defined in $\text{MEHJ}$. Thus its pullback $j(j) \downarrow$ makes sense in $\text{MEHJ}$. It is defined on all $HC_{\text{MEHJ}}$. (Note $HC_{\text{MEHJ}} \cup \forall j \forall M \subseteq \text{MEHJ}$.)

(4) Then says that for any such $h$,

$$\psi_h = (j(j)) \downarrow.$$
This almost follows from branch condensation. Namely, let \( \tilde{\psi} \) be the common value of all \( \psi_k \in \psi_{\mathcal{M}_k} \), for \( g \in h \) on \( \text{Col}(\omega, < \kappa) \) and \( g \) on \( \text{Col}(\omega_2, < \kappa_0) \). \( \tilde{\psi} \in \psi_{\mathcal{M}_k} \), and has branch condensation.

Moreover, with \( j: \psi_{\mathcal{M}_k} \to \text{Mek} \), we have \( j(\tilde{\psi}) = j(\psi)^{\text{Mek}} \). But letting \( \tilde{J} \in \psi_{\mathcal{M}_k} \) be by \( \tilde{\psi} \), we have \( j(\tilde{J}) = j(\psi)^{\text{Mek}} \), so \( j(\tilde{J}) = j(\tilde{\psi}) \). [The fact that \( j(\tilde{J}) \in \text{Mek} \) can be overcome with an absoluteness arguments.]

Thus \( J \) is by \( (j(\tilde{\psi}))^{\text{Mek}} \).

However, the argument of the last paragraph falls short of proving \( (*) \) from \( (4) \).
because it only works for $T$ in $\mathcal{V}_{2gT}$, so that $j^*(T)$ makes sense. So it only gives $\psi^* \in (j^*(T))^*$, not the full $\psi^*$.

So

$$\mathcal{N}_0 = H_0^\perp$$

$$\psi_0 = \text{canonical coll}(w_k, \kappa_1) - \text{name}$$

in $M$ for $\bigoplus_{\alpha < \theta^0} \Phi_\alpha$, where

$$\Phi_\alpha = H_0(\alpha) - \text{tail of } j^*(\Lambda)$$

for any and all $(P, \Lambda)$ s.t.

$$H_0(\alpha) = \mathcal{M}_0(P, \Lambda)$$

in $\mathcal{V}_{2gT}$.

We have $\chi(\mathcal{N}_0) < K_0^+$, so $j^\perp \mathcal{N}_0 \in M$. The reader can easily check $(\#)_0$ (see [IJ] and [EFJ]). The main thing is that $\mathcal{N}_0 \in \Theta_0$ is regular.

For $(\#)_0$, this says in our earlier notation that $\Sigma_{H_0}$, as defined on $HC_{A^2gT}$, is the
We need two further induction hypotheses:

(1') (d): If $E$ is on the $\mathcal{N}$-sequence and $\text{crit}(E) = 0^\alpha$, then in $\mathcal{N}$

$\text{col}(\omega, < k_1)$ \quad E is certified by $j \upharpoonright \text{Ho}^+$ over \n
$(\mathcal{N} \upharpoonright \text{Ho}(E), \dot{\Psi}^\omega \upharpoonright \text{Ho}(E))$.

(See definition 2.5 below for "certifies".)

(2') (b): (Absolute condensation to pullbacks)

Let $\pi: R \rightarrow V^N_\beta$ with $\beta$ large, $\pi \in \mathcal{N}$, $R$ transitive, $V_{\kappa_0 + 1}$ is $\mathcal{N}$, and $|R| < k_1$.

Suppose $\pi(j(\dot{\Psi})) = j(\dot{\Psi})$. Let $h$ be $\mathcal{M}$-generic over $\text{col}(\omega, < K_1)$, and let $h_0$ be $R$-generic over $\text{col}(\omega, < k_1)$, with $h_0 \in \mathcal{N} \upharpoonright \text{Ho}^+$, then

$\dot{j}(\dot{\Psi})_{h_0} \subseteq (j(\dot{\Psi})_h)^\pi$. 
\( j \)-pullback of \( \Sigma H \), as defined on \( V_{K_2} \) (where \( j : M \rightarrow N \)).

This follows from the fact that \( j \upharpoonright H_0 \) is the iteration map \( \pi : H_0 \rightarrow H_1 \) by \( \Sigma H_0 \) and \( \Sigma H \) is the \( H_1 \)-tail of \( \Sigma H_0 \) whenever they are defined, and \( \Sigma H_0 \) is pullback consistent whenever it is defined.

Because we are just shooting for \( D_k \), we shall never add \( \Theta^0 \)-relevant extenders with critical point \( > \Theta^0 \). As a consequence, all levels of our construction will be fully sound, and we'll never have to core down. Thus we'll set

\[ N_\alpha = \bigcup \{ N_\kappa \}_{\kappa < \lambda} \]

and

\[ \psi_\alpha = \text{canonical name for } \bigoplus \psi_\kappa \]

for \( \lambda \) limit.
Now suppose we are given \((N_p, \phi)\) = 

\((N, \phi)\) satisfying \((\gamma)\) and \((**)_p\).

We assume that \(j^* N \in M\), so that \((**)_p\) is not vacuous, and deal with its more

general case in a subsequent section. We

shall obtain \((N_{p+1}, \phi_{p+1})\)
in one of two ways:

(i) close under \((4p\text{-strategy})^{F}\), as on

p. 16 ff., or

(ii) add an extender with critical point \(\Theta\).

The main tool in our arguments will be "\(j\)-condensation", i.e. lemma 11.15

of \(\Sigma T\), generalized slightly as lemma

2 in \(\Sigma T\). The form we need is
Lemma 2.4. Let \((\mathcal{M}, \mathcal{N})\) satisfy (4)\(_{\bar{\times}}\) and (\((\ast)\)\(_{\bar{\times}}\)), and suppose \(j \upharpoonright \mathcal{M} \in \mathcal{M}\). Let \(h\) be \(Col(<\omega, \kappa_1)\) generic, and suppose we have

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow \\
\mathcal{N} \\
\downarrow \\
P \\
\downarrow \\
Q \\
\end{array}
\]

commuting, with \(P, Q, \sigma, \tau \in M2k I\) and countable there. Let \(\Sigma_h = (j(\mathcal{N}))^\sigma\) and \(\Sigma_h = (j(\mathcal{N}))^\tau\) be the pullback strategies.

Then

(i) \(\Sigma_h, \Sigma_h \in j(\mathcal{N})\)

(ii) \(\Theta^\omega = \Theta^k(\mathcal{N})\) is regular in \(L(\mathcal{N}, \mathcal{N})\) and

(iii) for all \(wfs \varphi\), all \(s \in P\), all \(x \in \mathcal{N}\):

\[L(j(\mathcal{N}), j(\mathcal{N})) = \varphi[P, \Sigma_h, s, j(x)]\]

iff

\[L(j(\mathcal{N}), j(\mathcal{N})) = \varphi[Q, \Sigma_h, k(s), j(x)].\]
Proof sketch. We shall prove conclusion (i).

The rest is proved just as in 217 and 237, so we do not give any details here. For (i),
we let \( j_1 = j \circ j \) and \( j_1 : M \to N \),
and let \( g, h, f \) be on \( \text{col}(w, < k_0), \text{col}(w, < k_1), \text{col}(w, < k_2) \) respectively,
with \( g \leq h \leq f \). We have, for \( Q, T \)
as in 20 hypotheses

\[
\Xi_Q = \left( \text{common value of all } j(\bar{\psi})_m \in \text{HC}^{\text{M2xJ}} \right)^T
\]

for \( m \geq h, m \in \text{col}(w, < k_2) \),

by definition. So

\[
j_1(Z_Q) = \left( \text{common value of all } j_1(j(\bar{\psi}))_m \in \text{HC}^{\text{N2xJ}} \right)^T
\]

for \( m \geq f \) on \( \text{col}(w, < j_1(k_2)) \),

\[
= \left( \text{common value of all } j_1(j(\bar{\psi}))_m \in \text{HC}^{\text{N2xJ}} \right)^T
\]

for \( m \geq f \) on \( \text{col}(w, < j_1(k_2)) \),

(since \( j_1(\bar{\psi}) = j \circ \bar{\psi} \)).
= \left( \text{common value of all } d_i \{ j(\psi) \}_{m \in \mathbb{N}} \cap \mathbb{N} \right)_{d_i}

= j(\psi)_{d_i}

The last step is (\star), moved from \mathbb{N} to \mathbb{N}^\ast, where it holds of \psi and \psi^\ast \cap \mathbb{N}, where it holds of \psi(\psi) * and \psi(\psi) * = \psi^\ast \cap \psi(\psi) *.

But then

\text{NEXT} = j(\psi)_{d_i} \in j(\psi(\psi) *)

so

\text{NEXT} = j(\psi)_{d_i} \in j(\psi(\psi))

so

\text{NEXT} = j(\psi(\psi)) \in j(\psi(\psi))

so

\text{NEXT} = \psi(\psi) \in j(\psi(\psi))

so

\text{NEXT} = \psi(\psi) \in j(\psi(\psi))

as desired.
We add extenders when they fit on the sequence, i.e., yield hod premises, and are "certified by $f$" in the following sense.

**Def 2.5** Let $(P, A)$ be a $f(\tau)$ hod pair in $\mathcal{M}_{\Sigma^1 J}$, and suppose $P$ has a top block beginning at $K$. Suppose
$$k: P_1(k^+) \to H^+$$
is fully elementary. We say that $E$ is $k$-certified over $(P, A)$ if and only if

(a) $(P, E)$ is a hod promouse, and

(b) for all $a \in \text{Id}(E) J \times \omega$ and $X \in P_1(k^+)$,
$$X \in E_a \iff \Theta^{\Lambda}_{P_0}(a) \in k(X),$$

where $\Theta^{\Lambda}_{P_0} : P \to H^+$ is the map given by $(P, A)$ being in the hod-limit-system of $L(f(\tau), R^*_x)$.

We consider first the case in which $\mathcal{N}_{\xi+1}$ is obtained by adding an extender with critical point $\Theta^0$ to $\mathcal{N}_\xi$. 

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Let \( h \) be \( \text{coll}(\omega, \aleph_1) \)-generic. We have \((\mathcal{N}_\xi, \psi_\xi) = (\mathcal{N}, \psi)\) our current pair satisfying \((T)_\xi\) and \((*)_\xi\).

Set
\[
\Sigma = \psi_k
\]

So in \( \mathcal{M} \mathcal{I} \mathcal{H} \mathcal{J} \), where we will be working most of the time, \((\mathcal{N}, \Sigma)\) is a \( \mathfrak{S} (T) \) hod pair.

**Case 1**: There is an \( E \) that is \( j \)-certified over \((\mathcal{N}, \Sigma)\).

In this case, there is a unique such \( E \), \((\text{so } E \in \mathcal{N})\). We set
\[
\mathcal{N}_{\xi+1} = (\mathcal{N}, E)
\]

Our goal now is to construct an iteration strategy \( \mathcal{L} \) for \((\mathcal{N}, E)\) such that in \( \mathcal{M} \mathcal{I} \mathcal{H} \mathcal{J} \), \(((\mathcal{N}, E), \mathcal{L})\) is a \( \mathfrak{S} (T) \) hod pair.
hod pair, and \( \mathcal{L} \) is fullness preserving and has branch condensation. The construction will give a symmetric term \( \mathcal{L}' \) such that \( \mathcal{L}'^* = \mathcal{L} \), and we'll show \((*)\) holds for \( (\mathcal{N}, E), \mathcal{L}' \).

**Notation** If \( \mathcal{P} \) is a hod premouse having a top block, then \( \mathcal{K}^{\mathcal{P}} \) is the ordinal that begins the top block of \( \mathcal{P} \).
What we need about \((N, E)\) to construct an iteration strategy is just that its top block is not too complicated.

**Definition 2.6** Let \(P\) be a hod premouse having a top block. We say that the top block of \(P\) is below \(O^P\) iff whenever \(E\) is a \(K^P\)-relevant extender in the top block of \(P\), then \(\text{crit}(E) = K^P\).

Note that if \(U\) is a normal tree on a hod premouse \(P\), and \(\text{cof} \cap U\) does not drop, and \(E_u\) is taken from the top block of \(M^u\) with \(\text{crit}(E_u) = K^M_u\), then \(E_u\) is applied to \(M^u\) (i.e. \(M^u_{\text{Adv}} = \text{Ult}(M^u, E_u)\)), and the rest of \(U\) is based on \(M^u_{\text{Adv}}\). This is because \(\text{lh}(E_u)\) is a cutpoint of \(M^u\).
Thus if \( P \) has a top block that is below \( O^p \), and any non-dropping iteration of \( P \) can be given by a sequence \( \langle (P_\alpha, \mathcal{F}_\alpha) \rangle_{\alpha < \eta} \) where, \( P_0 = P \), \( \mathcal{F}_\alpha \) is a stack of normal trees with base model \( P_\alpha \) and last model \( P_{\alpha+1} \), \( P_\lambda = \text{lim}_{\alpha < \lambda} P_\alpha \) for \( \lambda \) limit, and for each \( \alpha \), either

(i) \( \mathcal{F}_\alpha \) uses no extenders in the top block of \( P_\alpha \) or its images, or

(ii) \( P_{\alpha+1} = \text{Ult} (P_\alpha, G) \), for some \( G \) in the top block of \( P_\alpha \), with \( \text{crit}(G) = \kappa^{P_\alpha} \).

Definition: An iteration \( \langle (P_\alpha, \mathcal{F}_\alpha) \rangle_{\alpha < \eta} \) satisfying (i) and (ii) is said to be in normal form.

We now describe our complete iteration strategy for \( \langle \mathcal{M}, E \rangle \). Set

\[ \mathcal{M}_0 = \mathcal{M} \quad E_0 = E \]

and

\[ \sigma_0 = E. \]
Before describing our iteration strategy for $(\mathcal{H}, E)$, we make some definitions and prove a few simple things.

Definition 2.7 (a) Let $\sigma : R \to H_1$ be fully elementary, and suppose $(R, \Lambda)$ is a $\Lambda^*(\Gamma)$-hod pair in $\text{Mk}_{KJ}$. We say $\Lambda$ is $\sigma$-consistent iff letting $\Pi : R \to H_1$ be the iteration map by $\Lambda$ of the $\Lambda^*(\Gamma)$-hod-limit system $\Pi \uparrow K^R = \sigma \uparrow K^R$.

(b) Let $\sigma : R \to \Lambda^*(\Gamma)$ be fully elementary and $(R, \Lambda)$ a $\Lambda^*(\Gamma)$-hod-pair in $\text{Mk}_{KJ}$. We say $\Lambda$ is locally $\sigma$-consistent iff $\Lambda_{R \uparrow (K^R)_{t \in \Gamma}}$ is $\sigma$-consistent.

Remark. If $\Pi : R \to H_1$ is the map of $R$ into $\text{Mk}_{K^R}(R, \Lambda)^{\Lambda^*(\Gamma)}$, its hod limit, then $\Pi \uparrow K^R$ is the map of $R \uparrow K^R$ into $\text{Mk}_{K^R}(R \uparrow K^R, \Lambda)^{\Lambda^*(\Gamma)}$ (We are assuming $K^R$ regular in $\mathcal{R}_P$.) So really, it's $\Lambda_{R \uparrow K^R}$ that determines local $\sigma$-consistency. We call it plain
\( \sigma \)-consistency in (a) because then all iterations of \( R \) (that don't drop) are by \( \Lambda \).

Remark. If \( \sigma : R \to \mathcal{H}^+ \), and \((R, \Lambda)\) is a \( j^*(\tau) \) good pair such that \( \Lambda \) is \( \sigma \)-consistent, then \( \Lambda = \Sigma_\mathcal{H}^+ \). This is because, letting \( j = j^*(\tau) \), we have pullback-consistent in \( \mathcal{H}^+ \)(\( M2\mathcal{H}J \)), and \( \Sigma_{\mathcal{H}^+}(\pi(K\mathcal{R})) \) is a tail of \( j^*(\Lambda) \) there.

Thus \( \Sigma_{\mathcal{H}^+}(\pi(K\mathcal{R})) \Lambda \cap M2\mathcal{H}J = j^*(\Lambda) \cap M2\mathcal{H}J = \Lambda \).

Thus \( \sigma \)-consistency determines \( \Lambda \) in this case.

Remark. There are probably examples of \( \sigma : R \to \mathcal{j}^*(\tau) \) and \((R, \Lambda)\) locally \( \sigma \)-consistent, but \( \Lambda \neq \Sigma_{\mathcal{j}^*(\tau)} \). We do not know one at the moment, but surely local \( \sigma \)-consistency is not enough to determine \( \Lambda \).

Here are some simple facts about \( \sigma \)-consistency.
Remark. In clause (2) of 2.6, the equation
\[ E_i = E_0 \cup \pi_{d,\infty}^\Lambda \]  
"K^R" means: for all \( a \in [K^dJ]^{\infty} \) and all \( x \in R \), \((a, x) \in E_i \) iff \( \pi_{d,\infty}^\Lambda (a), x(x(a)) \in E_0 \).

Equivalently,
\[ a \in i(x) \text{ iff } \pi_{d,\infty}^\Lambda (a) \in \sigma(x), \]
for \( a \in [K^dJ]^{\infty} \) and \( x \in R \). Notice that \( i \) is continuous at \( K^R \), so it suffices here to consider \( x \in R \setminus K^R \).

Proof of 2.6

(1) \( \Rightarrow \) (2). Let \( i : R \to A \) by \( \Lambda \), \( a \in [K^dJ]^{\infty} \), and \( x \in R \setminus K^R \). Then
\[ a \in i(x) \text{ iff } \pi_{d,\infty}^\Lambda (a) \in \pi_{d,\infty}^\Lambda (i(x)), \]
iff \( \pi_{d,\infty}^\Lambda (a) \in \sigma(x) \),
because \( \sigma i(R \setminus K^R) = \pi_{R,\infty}^\Lambda i(R \setminus K^R) \).

(2) \( \Rightarrow \) (3) Let \( x \in R \setminus K^R \) and \( b \in [\pi_{R,\infty}^\Lambda (K^R)]^{\infty} \), where \( \pi = \pi_{R,\infty}^\Lambda \). Then let \( b = \pi_{d,\infty}^\Lambda (a) \), where \( i : R \to A \) is by \( \Lambda \). Then
(b, X) \in E_0 \text{ iff } \Pi_{j, \omega}(a) \in \sigma(X) \\
\text{iff } a \in i(X) \text{ (by (z)) } \\
\text{iff } \Pi_{j, \omega}(a) \in \Pi_{j, \omega}(i(X)) \\
\text{iff } b \in \Pi_{j, \omega}(X) \\
\text{iff } (b, X) \in E_{\Pi},

as desired.

(3) \rightarrow (1) \text{ is clear.}

\[ \bigstar \]

Let \((R, A)\) be a \(j(\mathcal{F})\)-local pair in \(\mathcal{L}(E_0, \pi)\) and \(\sigma: R \rightarrow j(\mathcal{M})\) be such that \(A\) is locally \(\sigma\)-consistent. Let \(i: R \rightarrow \mathcal{A}\) be an iteration map by \(\Delta_{\text{iter}}\); that is, an iteration not using any extenders in the top block or its images. There is then a natural factor map \(\tau\) from \(\mathcal{A} = \mathcal{U}_f(R, E_0, \Pi_{j, \omega}(\sigma, \Delta))\) to \(j(\mathcal{M}) = \mathcal{U}_f(R, E_0)\), given by

\[ \tau(i(f)(a)) = \sigma(f)(\Pi_{j, \omega}(a)). \]
These maps commute with the iteration maps by \( \Lambda R \), yielding

\[
\begin{array}{ccc}
\sigma & \rightarrow & \mathcal{I}(\mathcal{N}) \\
\downarrow & & \downarrow & \\
\Lambda & & \sigma & \rightarrow & \mathcal{I}(\mathcal{N})
\end{array}
\]

Remark: we didn't need anything about the full \( \Lambda \) to get \( \mathcal{I} \).

**Proposition 2.9** Let \( \sigma : \mathcal{R} \rightarrow \mathcal{I}(\mathcal{N}) \) and \( (\mathcal{R}, \Delta) \) be locally \( \sigma \)-consistent. Let \( i : \mathcal{R} \rightarrow \mathcal{I} \) be an iteration map by \( \Lambda R \), and \( \mathcal{I} : \mathcal{A} \rightarrow \mathcal{I}(\mathcal{N}) \) the factor map; then the \( \mathcal{A} \)-tail of \( \Lambda \) is locally \( \mathcal{I} \)-consistent.

**Proof** Let \( \eta \prec K \mathcal{A} \). Since \( i \) is continuous at \( K \mathcal{R} \), we have \( \eta = i(f)(a) \) where \( f \in \Lambda R \) and \( a \in \varepsilon K \mathcal{A} \). Then

\[
\Pi_{\mathcal{A},0}(\eta) = \Pi_{\mathcal{A},0}(i(f)(a)) = \Pi_{\mathcal{A},0}(i(f)) (\Pi_{\mathcal{A},0}(a)) = \sigma(f) (\Pi_{\mathcal{A},0}(a)) = \eta(\eta),
\]

as desired.
Now we want to see what happens when we touch the top block.

**Lemma 2.10** Let \((R, \overline{\psi})\) be a \((T, \beta)\)-body pair in \(M2\) and \(\sigma : R \rightarrow j(\mathcal{N})\) be such that \(\overline{\psi}\) is locally \(\sigma\)-consistent.

Suppose we have \(i\) such that

\[
\begin{array}{c}
H^+ \xrightarrow{i} H^+ \\
\downarrow \quad \downarrow \sigma \\
R \leftarrow (\mathcal{K}^\gamma \mathcal{H}) \\
\end{array}
\]

commutes. Suppose also \(G\) is on the \(R\)-sequence, \(\text{crit}(G) = \mathcal{K}^\gamma\), \(\text{and} \ G\) is \(\sigma\)-certified over \((R \ll \mathcal{H}(G), \overline{\psi})\). Let \(\mathcal{S} = \mathcal{O}_T(R, G)\), and \(\pi : \mathcal{S} \rightarrow j(\mathcal{N})\) be given by \(\pi(G_i(f)(a)) = \sigma(f)(\prod_{\mathcal{H}(G_i)} (a))\); then

1. \(\pi\) is well-defined,
2. \((R \xrightarrow{\sigma} \mathcal{S}) \xrightarrow{j(\mathcal{N})}\) commutes,
3. the \(\mathcal{S}\)-tail of \(\overline{\psi}\) is locally \(T\)-consistent.
Remark: The same proof yields the same conclusion if we assume only that $G$ is an amenable pseudocare such that $(R, G)$ is a bord premouse and $G$ is $\sigma$-certified on $(R, P)$. 

Proof of 2.10: Parts (1) and (2) follow at once from the fact that $G$ is $\sigma$-certified on $(R \text{I1}(a), \overline{P})$. 

For (3), we apply $j$-condensation, as stated in Lemma 2 of [FSJ]. (This is why we assumed $i$ exists.) Let 

$$J = j(f)(a) < K^a.$$ 

Let $Q = R \text{I1}(k^a)^+$ and $W = S \setminus (K^a)^+$. 

We may assume $a \in \text{I1}(G)^<\omega$ and we can write
\( \eta = i_\circ i(g) (b, a) \), where \( b \in \text{Obj}(\mathcal{C}) \).

Then

\[
\mathcal{L}(\text{Hom}_\mathcal{C}^\ast, \mathcal{C}_\mathcal{C}^+ \mapsto \prod_{\mathcal{C}_\mathcal{C}^+} \Gamma_{\mathcal{C}_\mathcal{C}^+}, \quad g \leq j(g),
\]

a statement about \( \mathcal{L}(\text{Hom}_\mathcal{C}^\ast, \mathcal{C}_\mathcal{C}^+) \mapsto \prod_{\mathcal{C}_\mathcal{C}^+} \Gamma_{\mathcal{C}_\mathcal{C}^+}(g) \leq j(g) \).

So by \( j \)-condensation

\[
\mathcal{L}(\text{Hom}_\mathcal{C}^\ast, \mathcal{C}_\mathcal{C}^+) \mapsto \prod_{\mathcal{C}_\mathcal{C}^+} \Gamma_{\mathcal{C}_\mathcal{C}^+}(g) \leq j(g)
\]

Let us write \( \prod \psi = \prod \psi_{i, \infty} \), \( \prod \Phi = \prod \Phi_{i, \infty} \).

Then

\[
\prod \Phi(\eta) = \prod \Phi(i_\circ i(g)(b, a))
\]

\[
= j(g)(\prod \Phi(b), \prod \Phi(a))
\]

\[
= j(g)(\prod \psi(b), \prod \psi(a))
\]

(by strategy coherence for \( \psi \), and \( \mathcal{L}(g) \) being a coproduct of \( \mathcal{C} \) — this is why we indexed that way!)

\[
= \sigma(i(g))(\sigma(b), \prod \psi(a))
\]

\[
= \sigma(i(g)(b))(\prod \psi(a)) = \sigma(f)(\prod \psi(a))
\]

\[
= \eta(\eta),
\]
The next lemma deals with how certification in the top block is preserved.

**Lemma 2.11** Suppose \( R_i \leq \lambda \) we have \( R, i, j \) with

\[
\eta \xrightarrow{\delta} j'(\eta) \xleftarrow{\delta} i' \rightarrow R
\]

Suppose \( j' \Vdash \mathcal{M} \in M_0 \)

commuting, \( R \) countable.

Let \( \mathcal{P} = j'(\Sigma) \delta \), and suppose that \( \mathcal{P} \) is locally \( \delta \)-consistent. Let \( G \) be an extender from the top block of \( R \), with \( \text{crit}(G) = K_R \).

Then \( G \) is \( \delta \)-certified over \( (R \upharpoonright \text{crit}(G), \mathcal{P} \upharpoonright R \upharpoonright \text{crit}(G)) \).

**Proof** For \( R = \eta \) and \( \mathcal{P} = \Sigma = j(\Sigma) \delta \), this is true. We express it as a collection of statements involving parameters \( j(A) \) for \( A \in H_0^+ \), and then apply \( j \)-condensation in the form of lemma 2.4.

Note that since \( j \) is continuous as \( (K^\ast)^{\#} = 0(H_0^+) \), \( i \) is continuous as \( (K^\ast)^{\#} = 0(H_0^+) \),
Let $\theta^0 < \varepsilon < o(H^+_0)$, and let

$$A^\varepsilon = \langle A^\varepsilon_\alpha \mid \alpha < \Theta^0 \rangle \in H^+_0$$

be an enumeration of $H^+_0/\mathcal{F}$. Then

$$L(Hom^*_{\mathcal{F}}(\mathcal{M})) = \text{for all extensions } G \text{ of the sequence of } \mathcal{M} \text{ with } \text{crit}(G) = \Theta^0, \text{ for all } a \in \prod_{\mathcal{F}} \mathcal{F} \text{ and all } \alpha < \Theta^0$$

$$A^\varepsilon_\alpha \in G_a \iff \prod_{\mathcal{F}} \mathcal{F}(a) \in \mathcal{J}(A^\varepsilon) \prod_{\mathcal{F}} \mathcal{W}_{\mathcal{F}}(\alpha),$$

which is a statement about the parameters $\mathcal{M}, \Sigma, A^\varepsilon$, and $\mathcal{J}(A^\varepsilon)$. By 2.4,

$$\mathcal{P}(R, \mathcal{T}, i(A^\varepsilon), j(A^\varepsilon)) \text{ holds in } L(Hom^*_{\mathcal{F}}(\mathcal{M})), \text{ i.e.,}$$

$$L(Hom^*_{\mathcal{F}}(\mathcal{M})) = \text{for all extensions } G \text{ of the sequence of } \mathcal{R} \text{ with } \text{crit}(G) = K_R, \text{ for all } a \in \prod_{\mathcal{F}} \mathcal{F}(G) \text{ and all } \alpha < K_R$$

$$i(A^\varepsilon_\alpha) \in G_a \iff \prod_{\mathcal{F}} \mathcal{F}(a) \in \mathcal{J}(A^\varepsilon) \prod_{\mathcal{F}} \mathcal{W}_{\mathcal{F}}(\alpha).$$

However, notice that

$$\sigma(i(A^\varepsilon_\alpha)) = j(A^\varepsilon_\sigma(\alpha)) = j(A^\varepsilon) \prod_{\mathcal{F}} \mathcal{W}_{\mathcal{F}}(\alpha),$$

since $\mathcal{T}$ is locally $\theta^0$-consistent.
It follows that all $G$ on the $R$-sequence such that $\text{crit}(G) = k^R$ are $\sigma$-certified over $(R \upharpoonright \text{lh}(G), G \upharpoonright \text{lh}(G))$, so far as sets to be measured in $i(H_0^+)$ go. But $i$ is continuous at $o(H_0^+)$, and $\xi$ was arbitrary.

Similarly the same proof gives

**Lemma 2.12** Suppose in $M[H]$ we have $R, i, \sigma$ with

\[
\begin{array}{c}
\mathcal{N} \xrightarrow{\sigma} i(\mathcal{N}) \\
\downarrow & \searrow \sigma \\
i & \Rightarrow R & \Rightarrow \xi \\
\end{array}
\]

and $\xi \in M$. Commuting and $R$ countable. Suppose $i(\xi)^{\sigma}$ is locally $\sigma$-consistent. Let $E$ be $i$-certified over $(\mathcal{N}, \xi)$, and $F = \bigcup \forall \xi' \in (\mathcal{N}, \xi) \forall G \subseteq E \cap G \in M$, then $F$ is $\sigma$-certified over $(R, i(\xi)^{\sigma})$.

**Proof** We apply the proof of 2.11 to the fragments $i(G)$ of $F$, fragment-by-fragment. We leave the details to the reader.
We would like to apply 2.11 and 2.12 with \( i : \mathbb{N} \to R \) or iteration map \( \Sigma \), or more generally, by the iteration strategy for \((\mathbb{N}, E)\) we are trying to construct. The problems are, in the case of \( E \)-iterations:

(a) how do we know \( \sigma : R \to j'(\mathbb{N}) \) with \( j = \sigma \circ i \) exists?

(b) how do we know why is \( j'(\Sigma)^E \) the \( R \)-tail of \( \Sigma \)?

There are parallel problems in the case \( i \) is by \( \mathbb{N} \) and our strategy for \((\mathbb{N}, E)\).

More generally, let

\[
\begin{array}{c}
\mathbb{N} \\
\downarrow i \\
\downarrow k
\end{array}
\quad \xrightarrow{\quad \lambda \quad \downarrow j \quad \downarrow \beta \quad \downarrow \lambda \quad \downarrow R
\]

be given in \( \mathbb{MEJ} \), where \( j = \lambda \circ k \) and \( i \) is an iteration by \( j'(\Sigma)^E \) of \( \Sigma \), and \( \Sigma \) is locally \( \lambda \)-consistent. We'd like to
find \( r \) such that
\[
\begin{array}{ccc}
\downarrow & \downarrow \text{fix} \\
P & \stackrel{r}{\rightarrow} & \mathcal{R} \\
\end{array}
\]
commutes, \( \text{fix} \) is the \( \mathcal{R} \)-tail of \( \mathcal{I} \), and \( \text{fix} \) is locally \( \mathcal{R} \)-consistent. To do this, we put our iteration in normal form (cf. Def. 2.6.1). So we have
\[
\langle (P_\alpha, \tau_\alpha) \rangle \ni \langle \eta \rangle 
\]
with \( P = P_0 \), and last model \( \mathcal{R} \). We write \( \mathcal{R} = \mathcal{R}_\eta \). (So \( \mathcal{R} \) is the last model of \( \mathcal{I}_{\eta-1} \) if \( \mathcal{I}_{\eta-1} \) exists, and \( \mathcal{R} = \lim_{\alpha < \eta} P_\alpha \) otherwise.)

Let \( J = T_0 \).

We define embeddings \( T_\alpha : P_\alpha \to \text{fix} \) by induction so that

1. \( T_\alpha = \alpha \tau_\alpha \) if \( \alpha \in \mathcal{I} \),
2. \( \text{fix} \) is the \( P_\alpha \)-tail of \( \mathcal{I} \),
3. the \( P_\alpha \)-tail of \( \mathcal{I} \) is locally \( \tau_\alpha \)-consistent.

Suppose first that we have \( T_\alpha \) such that
(1) - (3) hold, and we want $\mathcal{C}^{\mathcal{A}}_{t+1}$.

Case 1: $\mathcal{C}^{\mathcal{A}}_{t+1} = \mathcal{C}^{\mathcal{A}}(\mathcal{P}_t^G)$, for $\mathcal{C}^{\mathcal{A}}(G) = \mathcal{C}^{\mathcal{A}}$ on the $\mathcal{P}_t^G$ sequence.

Then $G$ is $\mathcal{C}^{\mathcal{A}}$-certified over $(\mathcal{P}_t^G, \mathcal{C}^{\mathcal{A}}_{t+1})$ by 2.11. So letting $\mathcal{C}^{\mathcal{A}}_{t+1}(\cdot^{\cdot}_{G}(a)) = \mathcal{C}^{\mathcal{A}}(f)(\frac{\tilde{t}}{\mathcal{P}_t^G}(\tilde{a}))$

for $\tilde{F} = \mathcal{C}^{\mathcal{A}}(\tilde{E})$, we have by 2.10 that $\mathcal{C}^{\mathcal{A}}_{t+1}(\cdot^{\cdot}_{G}(a)) = \mathcal{C}^{\mathcal{A}}_{t+1}(\cdot^{\cdot}_{G}(a))$ and $\mathcal{C}^{\mathcal{A}}_{t+1}$ is locally $\mathcal{C}^{\mathcal{A}}_{t+1}$-consistent. So it is enough to show that $\mathcal{C}^{\mathcal{A}}_{t+1}$ is $\mathcal{C}^{\mathcal{A}}_{t+1}$-tail of $\mathcal{C}^{\mathcal{A}}_{t+1}$. \[\text{Remark: See p. 73a}\]

But this follows from theorem 3.76 in the 3/25/60 version of Sargsyan's thesis [27]. ("Branch condensation pulls back.")

To jog the reader's memory, here is the barest sketch: we got a bad mouse W
Remark Let $\mathcal{F} = f(\mathcal{E})^T_{\alpha}$, and let $\mathcal{A}$ be its $P_{\alpha+1}$-tail of $\mathcal{F}$. Let $W = P_{\alpha+1} / (K_{\alpha+1})^T_{\alpha+1}$. Lemma 2.10

It then says that $\Lambda_{\mathcal{E}} \mathcal{G}$ is $T_{\alpha+1}$-consistent.

But then $\Lambda_{\mathcal{E}} = f(\mathcal{E})^T_{\alpha+1}$, so $f(\mathcal{E})^T_{\alpha+1}$ is locally $T_{\alpha+1}$-consistent.
with \( \omega \) Woodin extending \( \Pi \), and having a UB representation of \( \Sigma \) that is moved properly. Then \( \omega \) extends to \( \alpha \) or \( \omega \), yielding:

\[
\begin{array}{c}
W \\
\xleftarrow{f_0} j(W) \\
\xleftarrow{\eta} \eta_0 \\
\xleftarrow{\eta_{\alpha+1}} \eta(W)
\end{array}
\]

with \( \eta' \upharpoonright \eta_{\alpha+1} = \eta_{\alpha+1} \), \( \eta_0 \upharpoonright \eta_0 = \eta_0 \). The \( W \)-representation of \( \Sigma \) gets moved to the \( W_0 \)-representation of its tail, and Sageev shows also to \( j(\Sigma) \eta_0 \), as Sageev shows. It is then further moved to its \( \eta_{\alpha+1} \)-tail, and to \( j(\Sigma) \eta_{\alpha+1} \).

Case 2: \( \Pi \) involves no extenders in the top block of \( \Pi \), or its images.

In this case we get \( \eta_{\alpha+1} : \eta_{\alpha+1} \rightarrow j(\Pi) \) as in (1) - (3) by using proposition 2.9 where we used 2.10 in case 1. We leave
the details to the reader.

Finally, suppose \( \lambda \leq \eta \) is a limit ordinal. We define \( \tau_{\lambda}(\alpha, \xi) = \tau_{\alpha}(\lambda) \), for \( \alpha < \lambda \). This gives \( \tau_\lambda : \mathcal{P}_\lambda \rightarrow j^\prime(\mathcal{M}) \), and it is clear that (1) holds. For (2), we use Sargsyan's 3.76 of EZT again.

For (3), let \( \nu < \kappa^\mathcal{P}_\lambda \), and \( \pi : \mathcal{P}_\lambda \rightarrow H \). The iteration map by \( \tau_{\nu}(\xi) = \mathcal{P}_\nu \text{- tail of } \Psi \).

Let \( \nu = \lambda \uparrow (\xi) \), and \( \Phi = \tau_{\nu}(\xi, \tau_{\lambda}) = \mathcal{P}_\lambda \text{- tail of } \Psi \). Then

\[
\tau_{\lambda}(\Phi) = \tau_{\lambda}(\mathcal{P}_\lambda \uparrow \xi) = \Phi(\xi) = \pi(\lambda),
\]

as desired.

We have shown
Lemma 20.13

Assume \( \mathcal{M} \in \mathcal{M} \).

Let \( \eta \xrightarrow{t} j(\eta) \)

\[
\begin{array}{ccc}
\eta & \xrightarrow{t} & j(\eta) \\
\downarrow & & \downarrow \\
\Phi & \xrightarrow{i} & \mathcal{M} \\
\end{array}
\]

be given in \( \mathcal{M} \mathcal{H} \mathcal{J} \), where \( \xi = \Phi \circ k \) and \( i \) is an iteration map by \( j(E)^l = \mathcal{P} \). Suppose \( \mathcal{P} \) is locally \( \tau \)-consistent. Let \( \Phi \) be the \( \mathcal{P} \)-tail of \( \mathcal{P} \). Then there is a unique embedding \( \mathcal{P} \) such that

\[
\begin{array}{ccc}
\eta & \xrightarrow{t} & j(\eta) \\
\downarrow & & \downarrow \\
\Phi & \xrightarrow{i} & \mathcal{M} \\
\end{array}
\]

commutes, \( \Phi = j(E)^\tau \), and \( \Phi \) is locally \( \tau \)-consistent.

Proof: All that's left is uniqueness of \( \mathcal{P} \).

But since the top block of \( \Phi \) is below \( \Phi^0 \), any \( x \in \mathcal{P} \) has the form \( i(f)(a) \), where \( a \in \sum_{K \in J} \mathcal{P} \). But then \( \mathcal{T}(x) = \tau(i(f)(a)) = l(f)(\prod_{\mathcal{P}} (a)) \), so \( \mathcal{T} \) is determined by \( l \) and the \( \mathcal{P} \)-tail of \( \mathcal{P} \), hence by \( l \) and \( \mathcal{P} \).
We are now ready to define our iteration strategy \( \mathcal{I} \) for \((\mathcal{M}, E)\), where \( E \) is \( j \)-certified over \((\mathcal{M}, E)\). We are assuming \( \mathcal{M} \) is consistent. Let \( Q_0 = \mathcal{N} \), \( E_0 = E \), and

\[
\rho_0 = (Q_0, E_0).
\]

Let \( \delta_0 = \Sigma \), and \( \gamma_0 = j' \cap \mathcal{N} \). Let \( \langle (Q_\alpha, \overline{\alpha}) \rangle_{\alpha < \eta} \) be an iteration of \( \rho_0 \) in normal form, played according to the strategy \( \mathcal{I} \) that we are defining. We maintain by induction that there are \( \eta_\alpha : Q_\alpha \to j'(\mathcal{N}) \) so that

\[
\begin{align*}
\overline{\beta} &\rightarrow j'(\mathcal{N}) \\
\downarrow \beta &\quad \downarrow \gamma_\alpha \\
\overline{\beta} &\rightarrow \beta
\end{align*}
\]

where \( \beta \) commutes, where \( \overline{\beta} : Q_{\beta} \to Q_\alpha \) is the iteration map, and strategies \( \rho_\alpha \) for \( Q_\alpha \).
such that

(1) \( \mathcal{Q}_\alpha \) is good, and has branch condensation,
(2) \( \mathcal{Q}_\alpha = \mathcal{Q}(\Sigma) \mathcal{Q}_\alpha \) and
(3) \( \mathcal{Q}_\alpha \) is locally \( \mathcal{Q}_\alpha \)-consistent.
(4) if \( \mathcal{Q}_\alpha \) is \( \mathcal{Q}_\alpha \)-join \( \mathcal{Q}_\alpha \), then it is by \( \mathcal{L}_\alpha \), and
\( \mathcal{L}_\alpha+1 = (\mathcal{Q}_\alpha)^{\mathcal{Q}_\alpha+1} \), and

(5) if \( \mathcal{Q}_\alpha+1 = \mathcal{L}/(\mathcal{Q}_\alpha, E_\alpha) \), then
\( \mathcal{Q}_\alpha = (\mathcal{Q}_\alpha+1)^{\mathcal{Q}_\alpha} \).

Note that \( \mathcal{Q}_\alpha \) is a cutpoints initial segment
of \( \mathcal{Q}_\alpha+1 \), whenever \( (5) \) applies.

Suppose we have \( \langle \beta \mid \beta \leq \alpha \rangle \).

Case 1: \( \mathcal{Q}_\alpha \) is on \( \mathcal{Q}_\alpha \).

In this case, Lemma 2.13, with \( \mathcal{P} = \mathcal{Q}_\alpha \)
and \( \mathcal{R} = \mathcal{Q}_\alpha+1 \), gives \( \mathcal{L}_\alpha+1 \) and \( (1)-(5) \).
Case 2 \( P_{d+1} = U \tau (P_d, E_\alpha) \).

We have that \( E_\alpha \) is \( \tau_2 \)-certified over \((Q_\alpha, E_\alpha)\), by 2.12 and induction hypotheses (2) and (3). By the remark after 2.10, if we set

\[
T_{d+1} \left( \pi_{E_\alpha}^*(f)(a) \right) = Q(f)(\pi_{Q_\alpha, \infty}^*(a)),
\]

then \( T_{d+1} \) is well-defined and \( \tau_2 = \tau_2^{\circ} \circ T_{d+1} \).

Moreover, setting \( P_{d+1} = \pi_{Q_\alpha}^*(E_\alpha) \), we have as in the proof of 2.10 that \( P_{d+1} \) is good, has branch condensation, and is locally \( T_{d+1} \)-consistent. (This all uses \( \tau \)-condensation, lemma 2.4.)

Since \( Q_\alpha \) is a cutpoint of \( Q_{d+1} \)

\[
\pi_{Q_\alpha, \infty}^*(Q_\alpha) \cap 0(Q_\alpha) = \pi_{Q_{d+1}}^*(Q_{d+1} \cap 0(Q_\alpha))
\]

\[= \gamma_{d+1} \cap 0(Q_\alpha) \]

is locally \( \tau \).
(since \( S_{\alpha+1} \) is locally \( \eta_{\alpha+1} \)-consistent)

\[ = \prod_{\eta_{\alpha+1}} \top_{\eta_{\alpha+1}} (Q_{\eta_{\alpha+1}}) \]

(by the definition of \( T_{\alpha+1} \)). Pullback conservativeness
for \( Q_{\eta_{\alpha+1}} \) and \( S_{\alpha+1} \) then give us (5).

Finally, if \( \lambda \) is a limit \( \eta \)

define \( \tilde{\eta} \) by \( \tilde{\eta} (\lambda_\alpha (x)) = T_{\alpha} (x) \), and

the \( \tilde{\mathfrak{L}_\lambda} = \{ \tilde{\eta} (\tilde{\mathfrak{E}}) \} \). We leave the
rest to the reader.

This then tells us how to define \( \tilde{\mathfrak{L}} \)
for one more step. We only need worry
about the case that \( \eta = \alpha+1 \), and

\( S_{\alpha} \) is on \( \Omega_{\alpha} \). (Otherwise, there is no choice
of branch to be made.) But in this
case, we let \( \mathfrak{L} \) choose \( \Omega_{\alpha} (\tilde{\mathfrak{L}}_{\alpha}) \).

This completes our definition of \( \tilde{\mathfrak{L}} \) for \( (\lambda, \mathfrak{E}) \),
assuming \( j \mathfrak{L} \) extensions large enough that \( j \mathfrak{L} N \in M \).
Lemma 2.15 Assume that $\mathcal{M} \in M$, $\mathcal{N} \in M$, and let $\Sigma$ be the iteration strategy for $(\mathcal{N}, E)$ defined above.

Then $\Sigma$ is good.

Proof That $\Sigma$ is self-consistent and coherent is just clauses (4) and (5) of our induction hypotheses. That $\Sigma$ is $\xi(\Gamma)$-fullness preserving is also implicit in the induction hypotheses. (We get it from $\alpha$-condensation.)

\[\square\]

Lemma 2.16 Assume that $\mathcal{M} \in M$, $\mathcal{N} \in M$, and let $\Sigma$ be the iteration strategy for $(\mathcal{N}, E)$ defined above. Then $\Sigma$ has branch condensation.
Proof. Let $\mathcal{P} = (\mathcal{A}, E)$, and suppose branch condensation fails. We can find a stack $\tilde{\mathcal{F}} < \mathcal{U} >$ on $\mathcal{P}$ such that $
exists$ is by $\mathcal{P}$, and there is an iteration $i: \mathcal{P} \rightarrow \mathcal{Q}$ by $\mathcal{P}$ and cofinal branches $b$ and $c$ of $\mathcal{U}$ and $\pi: M_b^\mathcal{U} \rightarrow \mathcal{Q}$ such that:

1. $i = \pi \circ i_b \circ \hat{i}^\mathcal{U}$,
2. $c = \mathcal{P}(\tilde{\mathcal{F}}^\mathcal{U} < \mathcal{U} >)$, and
3. $b \neq c$.

Letting $\mathcal{R}$ be the base model of $\mathcal{U}$, the picture is

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{i} & \mathcal{Q} \\
| & | & | \\
\downarrow{i_b} & \downarrow{\pi} & \downarrow{\pi} \\
\mathcal{R} & \xrightarrow{i_b} & M_b^\mathcal{U} \\
| & | & | \\
\downarrow{i_c} & & \downarrow{\pi} \\
M_c^\mathcal{U} & & \\
\end{array}
\]

$b$ is $b$-realized, while $c$ is by $\mathcal{P}$. 
b does not drop because this is part of the hypothesis in branch condensation. We shall show that c does not drop.

**Claim.** $M^u_\beta = S(U)$ is Woodin.

**Proof** If $\eta$ is a cardinal of a hod promouse $M$, then $M/\eta$ is a full level of $M$. Thus $M(U)$ is a limit of full levels of $M^u_\beta$ and $M^u_\zeta$.

By minimizing our counterexample to branch condensation, we can arrange (letting $i = \bar{\omega}$)

\[(*): \text{let } A \text{ be a full proper initial segment of } M(U); \text{ then} \]
\[
\left(\prod_{\omega < \zeta} M^u_\zeta \right)_{M(U)} = \left(\prod_{\omega < \Sigma^u_\zeta} \right)_{\Delta}.
\]

**Proof** Note that $(\prod_{\Sigma^u_\zeta} M(U)) C (\mathcal{F}(\Gamma))$ by our construction. (All tails of $\mathcal{L}$, projected to proper initial segments of their base models, are in $\mathcal{F}(\Gamma)$.) Suppose we have chosen our-
counterexample \( \overline{\tau} \), \( \bar{U}, B, C, \overline{\nu}, \overline{W} \) so that \( (L_{\overline{\tau}}^{\bar{U} \times C}, \overline{\nu}) M(\nu) \) has minimal possible \nabla \text{duality rank}. We claim that (*) holds.

For let \( S \) be a counterexample to (*), and let \( x \neq b \) be such that \( S \) is a proper initial segment of \( M^2_{\bar{U}} \). Then we have that for \( \sigma = \overline{\tau} \circ \overline{\nu} \), \( \sigma \circ S = \overline{\nu} \), and so \( (L_{\overline{\tau}}^{\bar{U}, \sigma}) \sigma \neq (L_{\overline{\tau}}^{\bar{U}, \overline{\nu}}) \overline{\nu} \), so

\[
(L_{\overline{\tau}}^{\bar{U}, \sigma}) \sigma \neq (L_{\overline{\tau}}^{\bar{U}, \overline{\nu}}) \overline{\nu},
\]

since the latter is just \( L_{\overline{\tau}}^{\bar{U} \times C}, \sigma \) by strategy collapse.

This means we get a counterexample of the form

\[
\begin{tikzpicture}
  \node (Q) {Q};
  \node (R) [below left of=Q] {R};
  \node (P) [above left of=R] {P};
  \node (U) [below right of=Q] {U};
  \node (W) [above right of=Q] {W};
  \node (V) [below right of=W] {V};
  \node (S) [above right of=W] {S};
  \node (T) [above right of=S] {T};
  \node (U') [below right of=W] {U'};
  \node (V') [below right of=T] {V'};
  \node (W') [below right of=S] {W'};

  \draw[->] (P) -- (R);
  \draw[->] (R) -- (Q);
  \draw[->] (Q) -- (U);
  \draw[->] (Q) -- (W);
  \draw[->] (W) -- (V);
  \draw[->] (S) -- (T);
  \draw[->] (T) -- (U');
  \draw[->] (U') -- (V');
  \draw[->] (V') -- (W');
  \draw[->] (W') -- (S);
  \draw[->] (U') -- (S);
  \draw[->] (V') -- (S);
  \draw[->] (W') -- (S);
\end{tikzpicture}
\]

with \( \overline{\tau} \neq \overline{\nu} \neq \overline{W} \) based on \( \sigma \) being according to
both \( (\omega^\omega, \mathbb{Q})^\omega \) and \( (\omega^\omega, \mathbb{Q})^{\omega^\omega} \) but the former choosing \( b^* \), the latter choosing \( c^* \), and \( b^* \neq c^* \). But now

\[ \omega^\omega \cup \omega^\omega \rightarrow^\omega \langle u^* \rangle \rightarrow c^*, M(u^*) \]

is a tail of \( \omega^\omega \cup \omega^\omega \rightarrow^\omega c^*, M \), so is projective in \( \omega^\omega \cup \omega^\omega \rightarrow^\omega c, M \), so has Wade rank strictly less than that of \( \omega^\omega \rightarrow^\omega c, M(u) \).

This contradiction yields (\#).

Remark. Our notation from §1 is such that if \( \mathbb{V} \) is a limit of full levels of \( M \), whose \((M, \mathbb{V})\) is a bad pair, then \( \mathbb{V} \) is just the join of the \( \mathbb{V}_\alpha \) for \( \mathbb{V}' \), a proper full initial segment of \( \mathbb{V} \). So another way to write (\#) is:

\[ (\omega^\omega, \mathbb{Q})_{M(u)} = \omega^\omega \rightarrow^\omega c, M(u) \]
Now for \( \Phi = (\Omega_{W, q})_{M(\mathbb{U})} = \Omega_{G \times \mathbb{U}, c} \, M(\mathbb{U}) \).

We claim

\[ \Leftrightarrow \quad \text{Remark: up to claim 2 on p. 89, is written out in more generality as 2.3.4, p. 617, in this next full case of} \]

\[ \Leftrightarrow \quad \text{the next full case of} \]

\[ \Leftrightarrow \quad \text{the next full case of} \]

\[ \Leftrightarrow \quad \text{the next full case of} \]

This follows from \( j^- \)-condensation, i.e. 2.4.

For recall that \( \Delta^- \) is \( \Delta \) with its last extender predicate removed. We then have

\[ \begin{array}{ccc}
\mathcal{N} = P^- & \overset{\pi}{\rightarrow} & Q^- \\
\downarrow i & & \downarrow \eta \\
(M^U)^- & \overset{k}{\rightarrow} & (M^U)^- \\
\end{array} \]

where \( k = \iota_{\mathbb{U}} \circ \iota_{\mathbb{U}}^{-1} \). Here \( \eta \) is the "realization map" that is part of the definition of \( \Pi \).

We have \( \Omega_{W, q} = j(Z)^T \), so \( \Phi = j(Z)^T \, \Pi \, M(\mathbb{U}) \).

This gives us, via 2.4, that \( (M^U)^- \) is \( \Leftrightarrow \quad \text{full as desired.} \)
We then get

\[ (***) \quad L^2 \Phi (M(U))^d \hat{\mathcal{S}}(U) \text{ is the next full level of } M_c \text{ after } M(U). \]

If not, since \( \mathcal{S} \) is \( \hat{\mathcal{S}}(U) \)-fullness preserving, we must have that \( c \) dropped, and \( M_c \) is the first level of \( L^2 \Phi (M(U))^d \hat{\mathcal{S}}(U) \) that is not in \( M_c \), and this level projects strictly across \( \mathcal{S}(U) \). But that means \( \mathcal{S}(U) \) is not a cardinal in \( M_b \), contradiction.

By (**) and (***) \( L^2 \Phi (M(U))^d \hat{\mathcal{S}}(U) \) is Woodin. Let \( \kappa \) be the least cardinal strong to \( \mathcal{S}(U) \) in \( M(U) \). Let \( \mu = L^2 \Phi (M(U))^d \hat{\mathcal{S}}(U) \).

Let \( \Psi_b = (\mathcal{P}(\kappa \cup \mathcal{S}(U)))_{M_b}^\mu \) and \( \overline{\Psi} = L^2 \Phi (M(U))^d \hat{\mathcal{S}}(U) \cap \mathcal{S}(U). \)

It is possible that \( \overline{\Psi}_b \neq \overline{\Psi}_c \). Nonetheless, because we are below \( O^0_a \), that in both
$M_b$ and $M_c$ to $k$-blocks and with $w$ noticed more full levels about $M(b)$.

$M_b/\mu = M_b/\mu$, and moreover $s(\nu)$ remains Woodin in $M_b$ and $M_c$. (But $\mu < s(\kappa)^{M_b}$ and $\mu < s(\kappa)^{M_c}$.)

For example $k$ is not a limit of Woodins in $M(\nu)$, as otherwise we are past $O_b$. By $\varphi_T$ then,

$L_\varphi \bar{\psi}_b(\overline{M_b/\mu})^{M_b} = \check{s}(\nu)$ is Woodin. But then the $k$-block ends in $M_b$ with $w$ more $L_\varphi$'s unless $\varphi_T \bar{\psi}_b (\overline{L_\varphi \bar{\psi}_b (M_b/\mu)})^{M_b}$ is an indox on $M_b$ or an extendor with critical point $k$.

Since $k$ is not a limit of Woodins, $\varphi_T \bar{\psi}_b (\overline{L_\varphi \bar{\psi}_b (M_b/\mu)})^{M_b}$ is not such an indox.

Similarly on the $c$-side.

This finishes our proof of Claim $I$. We leave it to the readers to check that we actually proved.
Claim 2: $s(U)$ is a Woodin cutpoint in both $M^U_b$ and $M^U_c$, moreover neither $A_b$ and $C$ drops, and $(\mathcal{W}_{w, q})_{M^U(w)} = \mathcal{W}_{\mathcal{S}_{\text{Wood}}^U, w, q}$.

Let $\eta < s(U)$ be the strict sup of the Woodins of $M(U)$. We may and do assume that all critical points in $U$ are $> \eta$. Thus we have $s \in R_s$, $s(U) = \mathcal{S}_{b} (s) = \mathcal{S}_{c} (s)$.

By tracing back to where $s$ came from in $U$, we get that

$$3: \exists (f)(a) / f \in H_0 \land a \in \eta \cup \mathcal{W}$$

is cofinal in $s$.

Now we compare the $\mathcal{F}(U)$ hod pairs $(M_b^-, A_b)$ and $(M_c^-, A_c)$, where
$\Lambda_b$ and $\Lambda_c$ are the $\pi$-pullback and $\eta$-rail strategies respectively. We can do this because we've added the "minus", and because $\Lambda_b = \Phi(\xi)^{\text{ova}}$ and $\Lambda_c = \Phi(\xi)^{\sigma}$ for $\sigma$ and $\eta$ that make 2.4 apply, so that they are good and have branch condensation. In fact, it's enough just to compare $(M_b^{-1} \mu, \overline{\Phi}_b)$ with $(M_c^{-1} \mu, \overline{\Phi}_c)$, where $M_b^{-1} \mu = M_c^{-1} \mu = (p(M(U)))^{\psi(\gamma)}$, and $\overline{\Phi}_b$ and $\overline{\Phi}_c$ are the strategies induced by $\Lambda_b$ and $\Lambda_c$. Letting the comparison trees act on $M_b$ and $M_c$, we get the following diagram. Notice that the comparison trees are normal (you compare with a backgrounded construction, and no strategy disagreements show up), moreover they can be written as $T \wedge U_b$ and $T \wedge U_c$, where $T$ is normal on $M(U) \mid \eta$ and by both strategies, and all crits in $\Phi_b \overline{\Phi}$ are $\geq \psi(\eta)$. 
Here \( k \) and \( l \) are the embeddings of \( I \). They agree on \( M_b \cup \mu \). Let

\[
\lambda = \lambda_{b_1}^\nu (k(M_b \cup \mu)) = \lambda_{c_1}^\nu (l(M_c \cup \mu))
\]

be the common lined up part of \( \lambda \) on \( V \), and

\[
\psi = \text{common \( L \)-tail of \( \Psi_b \) and \( \Psi_c \)}
\]

The map \( \lambda \) of our diagram is given
by 2.13, which also tells us that the \( \psi \)-tail of \( \Omega \) is \( \check{\psi}(\Xi)^{\Psi} \). So

\[
\Psi = \left( \check{\psi}(\Xi)^{\Psi} \right)_{\Xi}.
\]

The map \( \sigma \) is part of the definition of \( \Omega \), and we therefore have

\[
\Psi = \left( \check{\psi}(\Xi)^{\sigma} \right)_{\Xi}.
\]

Let \( \Lambda_{\Psi} = \check{\psi}(\Xi)^{\psi} \) and \( \Lambda_{\sigma} = \check{\psi}(\Xi)^{\sigma} \). These are \( \check{\psi}(\Xi) \) fullness preserving, positional, and have branch condensation, by 2.4. \( \xi \) is a cutpoint inside both \( \Delta \) and \( \Xi \). So the maps \( \Pi_{\Delta, \psi} : \Delta \rightarrow \mathcal{H}_{1} \) and

\[
\Pi_{\Xi, \sigma} : \Xi \rightarrow \mathcal{H}_{1}
\]

agree on \( \mathcal{L} \) with \( \Pi_{\mathcal{L}, \psi} : \mathcal{L} \rightarrow \mathcal{H}_{1} \). We write

\[
\Pi_{\mathcal{L}, \psi} = \Pi_{\mathcal{L}, \sigma} = \Pi_{\Delta, \psi} \downarrow \mathcal{L} = \Pi_{\Xi, \sigma} \uparrow \mathcal{L}.
\]
Claim 3: Let $g \in G$ and $g = \iota_{\overrightarrow{c}}(f)(a)$, where $a \in \eta_{T,F}$ and $f \in H_0$, with $f : [\Theta \cup \mathcal{l}] \to \Theta$. Then

$$\iota_{b_1} \circ \iota_{b_1}(f) = \iota_{c_1} \circ \iota_{c_1}(f).$$

Proof: $\iota_b$ does not move $a_2$, and $\iota_{b_1}$ does not move $k(a)$, so

$$\iota_{b_1} \circ \iota_{b_1}(f) = \iota_{b_1} \circ \iota_{b_1} \circ \iota_{\overrightarrow{c}}(f)(k(a)).$$

Similarly

$$\iota_{c_1} \circ \iota_{c_1}(f) = \iota_{c_1} \circ \iota_{c_1} \circ \iota_{\overrightarrow{c}}(f)(l(a)).$$

But $k(a) = l(a)$. Let $a^+ = k(a) = l(a)$.

Now $\Pi_{H_0, \Theta}^\Sigma (f) \leq j(f)$, so by $j$-condensation (Lemma 2 of 237 is enough here),

$$\Pi_{A_0}^{\Sigma_1} \iota_{\iota_{b_1} \circ \iota_{b_1} \circ \iota_{\overrightarrow{c}}(f)} \leq j(f)$$

and

$$\Pi_{A_0}^{\Sigma_1} \iota_{\iota_{c_1} \circ \iota_{c_1} \circ \iota_{\overrightarrow{c}}(f)} \leq j(f).$$
But notice that $i_{b,1} \circ \text{col}_b(y) \in L$ and $i_{c,1} \circ \text{col}_c(y) \in L$. This is where $\pi_{\mathcal{A},\infty}$ and $\pi_{\mathcal{B},\infty}$ agree with $\pi_{\mathcal{A},\infty}$, so we get

$$\pi_{\mathcal{A},\infty}(i_{b,1} \circ \text{col}_b(y)) = j(f)(\pi_{\mathcal{A},\infty}(a^*)) = \pi_{\mathcal{A},\infty}(i_{c,1} \circ \text{col}_c(y)).$$

This implies $i_{b,1} \circ \text{col}_b(y) = i_{c,1} \circ \text{col}_c(y)$ as desired.

But then ran $i_{b,1}$ ran $i_{c,1}$ is cofinal in $\mathfrak{S}(\mathfrak{A},1)$, so $b_1 = c_1$. This gives

Claim 4: Let $y = \mathfrak{L}_0(f)(a) \in \mathfrak{S}$, where $a \in \mathfrak{S}_{\mathcal{A},\infty}$ and $f \in H_{\mathcal{A},\infty}^0$. Then $\text{col}_b(y) = \text{col}_c(y)$.
But $k \not\in S(u) = \ell \not\in S(u)$, so we have $i_b(y) = i_c(y)$ for all $y$ as in claim 4.
But such $y$ are cofinal in $\delta$, so $b = c$, a contradiction.

Lemma 2.16

A very similar proof yields

Lemma 2.17 Assume that $j \in N = M$, and let $\mathcal{L}$ be the iteration strategy for $(\mathcal{M}, E)$ defined above; then $\mathcal{L}$ is positional.

Proof Let $(Q,F)$ be a $\mathcal{L}$-iterant of $(\mathcal{M}, E)$ via two different stacks $\overrightarrow{T}$ and $\overrightarrow{U}$. Let $i = i_{\overrightarrow{T}}$ and $j = i_{\overrightarrow{U}}$. Suppose that $\overrightarrow{L}_{\overrightarrow{T}}(Q,F) \neq \overrightarrow{L}_{\overrightarrow{U}}(Q,F)$. Clearly there is no ambiguity about how to iterate via the top extension predicate, so by perhaps iterating further, we may assume

$\overrightarrow{L}_{\overrightarrow{T}}(Q) \neq \overrightarrow{L}_{\overrightarrow{U}}(Q).$
We may also assume \( \mathcal{F} \) and \( \mathcal{G} \) were in normal form, so that we have realization maps
\[
\tau : Q \to \mathcal{F}(\mathcal{V})
\]
and
\[
\sigma : Q \to \mathcal{G}(\mathcal{V})
\]
such that
\[
\tau_0 \kappa = \sigma_0 \kappa = \mathcal{G}(\mathcal{V}) \wedge \mathcal{V},
\]
and
\[
\tau_{\mathcal{F}} \mathcal{F} = \mathcal{F}(\mathcal{V})^{-1}
\]
and
\[
\tau_{\mathcal{G}} \mathcal{G} = \mathcal{G}(\mathcal{V})^{-1}.
\]
Moreover, \( \tau_{\mathcal{F}} \mathcal{F} \) is \( \tau \)-consistent, and \( \tau_{\mathcal{G}} \mathcal{G} \) is \( \sigma \)-consistent.  Would like to see \( \sigma = \tau \), a contradiction.  Note that
\[
\mathcal{Q} = \bigcup \{ \mathcal{F}(\mathcal{V}) \mid \mathcal{F} \in \mathcal{N} \} \quad \text{and} \quad \mathcal{Q} = \bigcup \{ \mathcal{G}(\mathcal{V}) \mid \mathcal{G} \in \mathcal{N} \}
\]
be \( \mathcal{Q} \).  So \( \tau \mathcal{Q} = \bigcup \{ \mathcal{F}(\mathcal{V}) \mid \mathcal{F} \in \mathcal{N} \} \) and \( \sigma \mathcal{Q} = \bigcup \{ \mathcal{G}(\mathcal{V}) \mid \mathcal{G} \in \mathcal{N} \} \).  It is enough to show \( \tau \mathcal{Q} = \sigma \mathcal{Q} \), so it is enough to show \( \tau \mathcal{Q} \cap \mathcal{Q} \mathcal{Q} = \tau \mathcal{Q} \).  But those are the iteration
maps by \( \Omega_{\overline{\Omega}, \Omega \Gamma \kappa} \) and \( \tilde{\Omega}_{\overline{\Omega}, \Omega \Gamma \kappa} \) respectively. So it is enough to show that \( \Omega_{\overline{\Omega}, \Omega \Gamma \kappa} = \tilde{\Omega}_{\overline{\Omega}, \Omega \Gamma \kappa} \).

Suppose not. Then we can get a stack \( \overline{\Omega} \) on \( \Omega \) by both strategies with base model \( R \) and \( \tilde{\overline{\Omega}} = \mathbf{d} : \Omega \to R \), and a normal test \( U \) on \( R \) such that for some control \( b \neq c \),

\[
\Phi \text{ by } \Lambda \chi
\]

\[
\overline{\Omega}^ U \{ b \} \text{ is by } \Omega_{\overline{\Omega}, \Omega \Gamma \kappa},
\]

and

\[
\overline{\Omega}^ U \{ c \} \text{ is by } \tilde{\Omega}_{\overline{\Omega}, \Omega \Gamma \kappa}.
\]

Let \( \overline{\Phi}^ U_b \) be the \( \Lambda U - \tau_a \) tail of \( \Omega_{\overline{\Omega}, \Omega \Gamma \kappa} \) (which is propositional), and \( \overline{\Phi}^ U_c \) the \( \Lambda U - \tau_a \) tail of \( \tilde{\Omega}_{\overline{\Omega}, \Omega \Gamma \kappa} \). By minimizing, as in the proof of 2.16, we may assume

\[
(\overline{\Phi}^ U_b)_{M(U)} = (\overline{\Phi}^ U_c)_{M(U)},
\]
Now using 2.3.4 or p. 459 and following (re-written in the proof of 2.16) we may assume

(i) neither b nor c drops, and

$S(U)$ is a wooded cutpoint in both $M_b$ and $M_c$,

(ii) there are $\eta < \delta \in R$ such that

$$i_b(\delta) = i_c(\delta) = 5(U)$$

and there are no wooded sets of $R$

in $(\eta, \delta)$, and all cuts in $U$ are $> \eta$

(iii) $\delta = \sup \left\{ \log(f)(a) \mid a \in \eta J_w \wedge f \in \mathcal{N} \right\}$

$$= \sup \left\{ \log(f)(a) \mid a \in \eta J_w \wedge f \in \mathcal{N} \right\}.$$

Further, we have $\mathcal{N}$-realizations
\[ \psi : M_b \to j(T) \] and \[ \rho : M_c \to j(T) \]

such that \[ \psi_b = j(\delta) \psi \] and \[ \psi_c = j(\delta) \rho \].

The picture is

Here, as in 2.16, \( U_1 \) is a single normal tree used to compare \( \langle m, (\psi_b)_m \rangle \).
with $(\mathcal{N}, (\psi_c)_n)$, where $\mathcal{N} = L^p(\mathcal{M}(\Omega)^\Gamma)$, for $\Lambda = (\psi_0)|_{\mathcal{M}(\Omega)} = (\psi_c)|_{\mathcal{M}(\Omega)}$.

Remark The reader may notice that the maps $k$ and $\ell$ of the proof of 2.1k (see diagram p. 91) have become the identity $\iota$ that can be arranged by comparing $(\mathcal{N}, (\psi_0)_n)$ and $(\mathcal{N}, (\psi_c)_n)$ with the $\Lambda \mathcal{R}_{\mathcal{N}}$-hoï-mouse construction of a sufficiently rich $\mathcal{N}^\sharp$. It's not necessary to do so, it just simplifies the diagram.

Claim 3. Let $\mathcal{N} = \delta$ and $\delta = \delta_k(f)(\alpha)$, where $f \in \mathcal{H}_0^\sharp$ and $\alpha \in 2\eta^\sharp$. Then

\[ \iota_{\mathcal{N}} \circ \iota_{\mathcal{N}}(\delta) \in \overline{\text{ran} \{ \iota_1, \iota_2 \}} \]

Proof. Suppose Claim 3 of 2.1b.
Proof. We have to arrange things a little differently than we did in the branch continuation proof, because the single embedding \( i^* \) of that proof has been replaced by two embeddings, \( k \) and \( i \).

Let

\[
   a^* = i_b \circ i_b (a)
\]

\[
   = i_{c_1} \circ i_{c_1} (a),
\]

and

\[
   a^{**} = \pi (a^*) = \gamma (a^*).
\]

We have

\[
   \pi \left( i_{b_1} \circ i_b \left( \pi (f(a)) \right) \right) = \pi \left( i_{b_1} \circ i_b \circ i_k (f)(a^*) \right)
\]

\[
   = j(f)(a^{**}).
\]
Also
\[ \psi(i_c \circ i_c \circ \text{lo}(f)(a)) \]
\[ = \psi(i_c \circ i_c \circ \text{lo}(f)(a^*)) \]
\[ = \psi(f)(a^{**}). \]

We are using here that \( \tau \) and \( \psi \) agree on the common lined up part of \( S \) and \( V \), where they are the iteration maps, to see that \( \psi(a^*) = \tau(a^*) \). Moreover, \( \psi^{-1}(f)(a^{**}) = \tau^{-1}(f)(a^{**}) \) is in the part of \( H \) where \( \psi^{-1} \) and \( \tau^{-1} \) agree. So applying \( \tau^{-1} \) and \( \psi^{-1} \)
\[ i_b \circ i_b \circ \text{lo}(f)(a) = i_b \circ i_c \circ \text{lo}(f)(a) \]
Thus \( i_b \circ i_b \circ f \) is ran \( (i_c \circ i_c) \).

Remark We are grateful to Nam Trang for pointing out that the branch condensation argument was not quite enough.
As before, Claim 2 gives $b_1 = c_1$, and then $b = c$. That contradiction completes the proof of 2.17.

Let $\mathfrak{I}_2$ be the natural name in $M$ for $\mathfrak{I} = \mathfrak{I}_h$. Let $(\mathfrak{N}_{k+1}, \mathfrak{I}_{k+1}) = ((\mathfrak{N}, E), \mathfrak{I}_2)$. We have now verified all of (1) (a)-(d). We turn to (2) (a+1).

**Lemma 2.18** Assume $j : N \subseteq M$. Then $M = \bigcup \{ \text{coll}(\omega, <k_1) \mid j(\mathfrak{I}_2) = \mathfrak{I}_2 \}$.

**Proof.** Let $h$ be $\text{coll}(\omega, <k_1)$ be arbitrary. Let $j_1 = j(j)$, with $j_1 : M \rightarrow N$. Let $k_2 = j_1(k_1)$, and let $\mathfrak{I}$ be $\text{coll}(\omega, <k_2)$ generic with $h \subseteq \mathfrak{I}$. We want to see
It's worth abstracting the strategy — uniqueness result behind the proofs of 2.16 and 2.17.

Lemma 2.17.1 (Uniqueness of pullbacks)

Assume (\(\star\)) and (\(\dagger\)), for \((M, \psi) = (M_1, \psi_1)\). Let \(k\) be \(M\)-generic over \(\text{col}(\omega, <k_1)\), and suppose in \(M[k]\) we have

\[
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{i} & P \\
\downarrow{k} & \nearrow{\sigma} & \downarrow{\eta} \\
\mathcal{Q} & \xrightarrow{\delta} & \mathcal{M}
\end{array}
\]

commuting, with \(P\) and \(Q\) countable. Suppose

\[P\models \mu = Q\models \nu = \mathbb{W},\]

where \(\mathbb{W}\) is a full superset in both \(P\) and \(Q\).
Suppose that whenever $a$ is a cutpoint of $V$ such that $e = \sup \exists_t(t)(a) / \forall e H^+$ and $\exists_t(t)(a) = \exists_t(t)(b)$, then

$$
(j(\psi))_Y = (j(\psi))_Y
$$

We leave the proof of 2.17.1 to the reader; it's right there in the proofs of 2.16 and 2.17.

Remark: Because of its demand that $V$ be a cutpoint, either we have $P = Q = V$, or $V$ is below the top blocks of $P$ and $Q$.

In the case $V$ is below the top blocks, only (4) and (1) matter. We have

$H^+_P \rightarrow H^+_1$ with $P \mu = Q \mu = V$ and we conclude that the two pullbacks of $\Sigma_h$ to $V$ are the same.
Let $I$ be an iteration of $P_0 = (\mathbb{N}, E)$ by $\mathbb{Q}$, that is, in normal form. Say $I = \langle (P_\alpha, T_\alpha) / \alpha < \mathbb{P} \rangle$. We show by induction on $\mathbb{P}$ that the $i$-lift $j^I$ is by $j(i(\mathbb{Q}))$.

(Note here that $j^I \mathbb{N} \in \mathbb{N}$, because $j^I \mathbb{N} = j_1 \circ j_0 (j^I \mathbb{N})$, so that $j^I \mathbb{N} \in \mathbb{NL}_I$.)

The induction is obvious if $\mathbb{P}$ is a limit, so let $\mathbb{P} = \alpha + 1$. Let $P_\alpha = (\mathbb{Q}_\alpha, E_\alpha)$. Let $P_\alpha = (S_\alpha, F_\alpha)$ be the $\alpha$th model of $j^I$. We then have the diagram
Here $i$ and $k$ are the inclusion maps of $I$ and $jI$, $\sigma$ is the copy map, and $\nu$ is the $j_i$-realization map. That we get out of our definition of $j_I(\mathcal{I})$ in $\text{N}ILJ$, from $j_I(j_i\mathcal{M}) = j_I(j(I))$. ($\nu$ is uniquely determined by $j_I$.)

We may assume $\mathcal{I} \subset \mathcal{Q}_a$, as otherwise $\mathcal{R}_a(\mathcal{I}) = \mathcal{U}_x(\mathcal{R}_a, \mathcal{I})$, $\mathcal{R}_a(\mathcal{I}) = \mathcal{U}_x(\mathcal{R}_a, \mathcal{I})$, and $j_I$ is by $j(I)$. Let

\[ \Phi = (\mathcal{R}_a - \text{tail of } j_I(\mathcal{I}))_{\mathcal{Q}_a} \]

and

\[ \Lambda = (\mathcal{R}_a - \text{tail of } j(I))_{\mathcal{S}_a} \].

We must see that $\Phi \subseteq \Lambda$. Recall that $\Phi$ is our name for the strategy of $\mathcal{I}$. We have that $\Lambda = j(I(\Phi))$. $\Phi$ is a pullback of $j(\Phi)_k$. Namely,
Let $\varphi : Q\rightarrow j(n)$, $\forall \varphi \in \text{MTHJ}$, be the $j$-realization such that $\Theta = (j(\varphi))_n$ and $\Theta$ is $\varphi$-consistent. Using $\hat{j} : \text{MTHJ} \rightarrow \text{NLLJ}$, we get the diagram:

From the point of view of $\text{NLLJ}$, $\Theta$ is the $\hat{j}_*(\varphi)$-pullback of $\hat{j}_1(j(\varphi))_n$ (intersected with $\text{HCMTHJ}$). This is because, by $(\ast)_\varepsilon (a)$ moved over $j(\varphi)_n = (j_1(j(\varphi)))_n \varepsilon (a)$. On the
other hand, $\Lambda$ is the pullback of $\hat{f}_i(j(\psi))\mathcal{L}$. We can now use the method of 2.16 and 2.17 to show these two pullback strategies are consistent with each other. We omit further detail for now.

Remark One can apply 2.17.1 in $\text{NII}^J$, with $P = Q = Q_k$.

Remark In the situation above, we won't have $\sigma_{0T} = \hat{f}_i(j(\psi))$ unless $i$ happened to be an iteration of $\Lambda$ in $\text{NII}$. Otherwise $\sigma_{0T}$ moves things up further than $\hat{f}_i(j(\psi))$, so $\Lambda$ is not $\sigma_{0T}$ consistent.
Lemma 2.19 (*) holds.

**Proof** Let \( \pi : R \to V_\alpha^m \) where \( R \) is transitive and \( V_{\alpha+1} \supseteq \mathcal{U} \subsetneq R \), \( 1 \mathcal{U} \subsetneq \mathcal{K}_1 \), and \( \pi \in M \).

Let \( h \) be \( \text{col}(w, < \mathcal{K}_1) \)-generic over \( M \), and \( h_0 \in M \mathcal{K} \) be \( \text{col}(w, < \mathcal{K}_1) \)-generic over \( R \). Here \( \pi(\mathcal{K}_1) = \mathcal{K}_1 \). Let \( \pi(\overline{j(\mathcal{G}_1)}) = \overline{j(\mathcal{G}_2)} \). We must show that \( \overline{j(\mathcal{G}_2)}_{h_0} \subseteq \overline{j(\mathcal{G}_2)}_h \).

If not, then we have an iteration \( i : (\overline{j(\mathcal{M})}, \overline{j(E)}) \to (P, F) \) that is by both \( \overline{j(\mathcal{G}_2)}_{h_0} \) and \( \overline{j(\mathcal{G}_2)}_h \), and such that the projected strategies \( \overline{(j(\mathcal{G}_2))}_{h_0} \) and \( \overline{(j(\mathcal{G}_2))}_h \) disagree on some normal tree \( \mathcal{U} \) on \( P \), with \( \mathcal{U} \in R \mathcal{K}_0 \mathcal{J} \).
Let $\Lambda = \left( \hat{j}(\hat{\mathcal{H}}) \right)_{\rho}$ and $\overline{\Phi} = \left( \hat{j}(\hat{\mathcal{H}}) \right)_{\Pi}$.

Let $j_1 : \mathcal{H} \to \mathcal{N} \mathcal{L} \mathcal{L} \mathcal{J}$, with $I$ on col $(\omega, \kappa_2)$. Note that $\omega \not\equiv \Pi \in \mathcal{N} \mathcal{L} \mathcal{L} \mathcal{J}$. So in $\mathcal{N} \mathcal{L} \mathcal{L} \mathcal{J}$, we have the diagram.

Here $\gamma$ is the realization we get from the definition of $\mathcal{L}$, or is the copy map.
and \( r \) is the realization we get from the definition of \( j'(\mathcal{X})_2 \).

Let 
\[
\Sigma = \psi^0_h,
\]
so that in \( \text{Nil} \)
\[
\Lambda = \bar{\hat{j}}_1(j(\mathcal{X}))^\lor_0 y
\]
and
\[
\overline{\Omega} = \hat{j}_1(j(\mathcal{X}))^\lor_0 
\]
The first inclusion takes a bit of proof, as we must see that 
\[
\hat{j}_1(j(\psi))_{ho} \supseteq \hat{j}_1(j(\mathcal{X}))^\lor_0
\]
But this just follows from \((*)_1(b)\) holding in \( N \) of \( j'(\psi) \).

Now note that \( R \cap N \cap N \cap N \cap N \subseteq \cap \), and that we can extend \( ho \) to a col \( (\psi)_{\leq K_2} \) -generic over \( R \cap N \cap N \cap N \).
But now $2 \cdot 17 \cdot 1$ applied in NLS gives that the two pullbacks of $\hat{\Psi}(\hat{j}(\mathcal{E})))$
by $\Psi$ and $\Psi^#$ respectively, are equal. So they agree on $M$, contradiction.

\[\Box\]

We have now verified all of $(\Psi)_9$, and $(\Psi)_9^\#$. The motivation for $(\Psi)_9^\#, (b)$
is brought out by the following.

\underline{Lemma 2.20} Let $h$ be $\text{col}(\omega, < k_1)$
generic over $M$; then $\hat{\Psi}_h^\# \in \text{Hom}_h^\#$.

\underline{Proof} Let $h_0$ be $\text{col}(\omega, \text{dist}(\mathcal{E}))$ - generic
with $h_0 \leq h$. We show that $\hat{\Psi}_h^\#$ has a $< k_1$ - UB code in
$M[h_0]$. For this, we need...
Let $\varphi(V_0, \ldots, V_4)$ be the formula:

"it is forced in col(f) that $V_1$ is a stack of iteration trees on $V_2$. That is according to the $V_3$-pullback of $(V_4)_f$",

thus for $\gamma$ reasonably closed and any size $< k_1$, generic in our $M_{\text{ho}}$, and any $\mathcal{F} \in M_{\text{ho}}$, $\mathcal{I}$,

$\mathcal{F}$ is by $\mathcal{R}_{\text{ho}}$ iff $\nu^{M_{\text{ho}}, \mathcal{I}}(\gamma) = \varphi(k_1, \mathcal{F}, \gamma, j, j^m, j(\mathcal{R}))$.

It is enough to see that for any such $\gamma$, and any

$\sigma : S \to \nu^{M_{\text{ho}}, \mathcal{I}}$

with $\sigma, S \in M_{\text{ho}}, \mathcal{I}$ and countable above, $S$ transitive, enough in ran $(\sigma)$, we have for all $\mathcal{F} \in S = R_{\text{ho}}, \mathcal{I}$

$\mathcal{F}$ is by $\mathcal{R}_{\text{ho}}$ iff $R_{\text{ho}}, \mathcal{I}(\gamma) = \varphi(k_1, \mathcal{F}, \gamma, j, j^m, j(\mathcal{R}))$.\"
Here \( \sigma(\langle k_1, j^n, j(\bar{\omega}) \rangle) = \langle k_1, j^n, j(\bar{\omega}) \rangle \).

We may assume \( \Re \lambda M \) and \( \sigma \wedge \mathcal{R} = \pi \in M \).

Suppose \( R\lambda\omega, nJ = \varphi [k_1, l^\infty, k^n, j^n, j(l^\infty)] \).

Let \( M \lambda \lambda J \) be \( R\lambda\omega, nJ \) - generic over \( \text{col}(\omega) \prec k_1 \). Note that

\[
\bar{j}(\bar{\omega})_{\lambda} \subseteq \langle j(\bar{\omega})_{\lambda} \rangle_{\bar{\lambda}}
\]

by \((*)_{\lambda t+1}\). Thus \( \bar{\omega} \) is by \( j\bar{l}(\bar{\omega})_{\lambda} = \bar{j}(\bar{\omega})_{\bar{\lambda}} \) as \( \pi \circ j^n = j^n \). But then \( \bar{\omega} \) is by \( \pi \bar{\lambda} \) by \((*)_{\lambda t+1}(a)\).

This gives the \( \leftarrow \) direction of the desired equivalence, and the \( \rightarrow \) direction has a similar proof.
Let \((M, \mathcal{D})\) be a \(\hat{\mathcal{D}}(\mathcal{F})\)-hod-pair in \(\mathcal{M}_{\mathcal{D}}\), with \(\mathcal{D}\) being \(\hat{\mathcal{D}}(\mathcal{F})\)-fullness preserving, and having branch condensation.

Let

\[ \tau : M \to \mathcal{H} \]

where \(M_\infty (M, \mathcal{D}) \leq \mathcal{H}\), and \(\mathcal{H}_+ \in \mathcal{H}\).

We say \(\mathcal{D}\) is \(\tau\)-consistent if

there is an embedding \(k\) such that

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & \mathcal{H} \\
\downarrow \Pi \mathcal{D} & & \downarrow k \\
M_\infty (M, \mathcal{D}) & \xrightarrow{k} & \mathcal{H}_+ \end{array}
\]

commutes, with \(\Pi \mathcal{D}\) the natural iteration map, and

\[ k \uparrow \Pi \mathcal{D}(K^M) = \text{identity} \].
(So we are assuming here that \( M \) has a top block.)

Notice that \( k \) is determined by \( \Phi \) and \( \eta \); if \( M \) is below \( \Omega \), that is because elements of \( M_0(M, \Phi) \) are then of the form \( \Pi_\Phi(t)(a) \) for \( a \in \Pi_\Phi(K^n)^{<\omega} \), and \( k(\Pi_\Phi(t)(a)) = T(t)(a) \) for such \( a \).

Now let \( i : M \rightarrow \mathbb{Q} \) be an iteration map by \( \Phi \). Let \( t : \mathbb{Q} \rightarrow M_0(M, \bar{\Phi}) \) be the iteration map by the \( \mathbb{Q} \)-tail of \( \Phi \).

Let \( \sigma = k \circ i \). So we have

\[
\begin{array}{ccc}
M & \xrightarrow{i} & \mathbb{Q} \\
\downarrow & & \downarrow t \\
\mathbb{Q} & \xrightarrow{\sigma} & M_0(M, \bar{\Phi}) \\
\downarrow k & & \downarrow \end{array}
\]

We then have, for \( \Lambda = \mathbb{Q} \)-tail of \( \Phi \),
(1) $\Lambda$ is locally $\sigma$-consistent, and

(2) If $E$ is on the $Q$-sequence, with $\text{crit}(E) = K^w$, then $E$ is $\sigma$-certained over $(Q^{-1}, h(E), \Lambda_{Q^{-1}, h(E)})$.

Part (1) comes from $kP \lambda(k^w) = \text{identity}$.

Part (2) comes from considering

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\sigma} & \mathbb{Z}^1 \\
\downarrow{i} & \sigma & \uparrow{k} \\
\mathcal{Q} & \xrightarrow{l} & \mathcal{M}_0(m, \bar{m}) \\
\downarrow{i_E} & \Upsilon(Q, E) & \uparrow{t} \\
\end{array}
\]

where $t$ is the iteration map. We have $t \Upsilon h(E)$ is the iteration map by $\Lambda_{Q^{-1}, h(E)}$ by strategy coherence and its fact that $\Upsilon h(E)$ is a cursor of $\Upsilon(Q, E)$. Moreover, for $x \leq 2k^Q J^{10}$ and $a \in [h(E)J^{<w}]$,
\[ a \in i_X(X) \text{ iff } t(a) \in i_X(X) \]
\[ \text{iff } t(a) \in \sigma(X) \]

because \( k(t(a)) = t(a) \).

What we have done in §2 is construct a \( j \)-consistent strategy \( \mathcal{L} \) for \((\mathcal{M}, E)\). This we did by maintaining (1) and (2) as we went along, with respect to inductively determined \( j \)-realization maps \( \sigma \).

The argument also showed that all the \( (\Psi_\gamma) \) for \( \gamma \leq \bar{\gamma} \) were also \( j \)-consistent.

This also follows at once from the \( j \)-consistency of \( \mathcal{L} \).