We develop further the method for constructing hod mice from UIJ and EJ.

In particular, we assume

1. $\kappa_0$ is a measurable limit of Woodin cardinals, with $j: V \rightarrow M$ witnessing its measurability.
2. $g$ is col($\omega, \kappa_0$) - generic over $V$, and $\Gamma$ is a Wedge initial segment of $\text{HOD}^+$ such that $\Gamma = P(R_g^\ast) \cap L(\Gamma, R_g^\ast)$ and $L(\Gamma, R_g^\ast) \models \text{AD}_{R^+} + \text{DC}$.
3. Letting $H_0 = \text{HOD}^{L(\Gamma, R_g^\ast)} \cap V_{H_0(\alpha)}$, and $E_{H_0}$ be the join of the strategies for $H_0(\alpha)$, $\alpha < o(H_0)$, stretched by $\hat{j}: V_{E_{H_0}} \rightarrow M_{E_{H_0}}$, we have $0(\Sigma_{H_0}^1(\Gamma)) < \kappa_0^+$. 
In (3), when we speak of the "strategies for the HOD"**, we are implicitly assuming that HOD\(\mathcal{L}(\Gamma, \mathcal{R}_\mathcal{P})\) has been analyzed as a hod-limit. What we are doing here will then figure in the analysis of HOD\(\mathcal{L}(\Gamma, \mathcal{R}_\mathcal{P})\), for \(\Gamma \subseteq \mathcal{R}_\mathcal{P} \subseteq \text{Hod}^*\).

We shall describe \(\Pi (\mathcal{N}, \Sigma)\) as a hod pair, and \(\Sigma\) as a "full" initial segment of \(\mathcal{N}\), then we write \(\Sigma \mathcal{P}\) for the part of \(\Sigma\) that rolls us how to iterate \(\mathcal{P}\).

**Definition** Let \((\mathcal{N}, \Sigma)\) be a \(\Gamma\)-hod-pair. We say \((\mathcal{N}, \Sigma)\) is a \(\Gamma\)-LST-pair if for there is a \(\delta\) such that

1. \(\mathcal{N} = L^\mathcal{P} (\mathcal{N} \mathcal{P}) \mathcal{F}\), where \(\mathcal{L} = \bigoplus_{\alpha < \delta} \Sigma \mathcal{P}(\alpha)\), and

2. \(\mathcal{N} = \delta\) is Woodin, and
3. For \(\kappa < \delta\) the least \(<\delta\)-strong of \(\mathcal{N}\), \(\mathcal{N} = K\) is a limit of Woodins.
We say that $(\mathcal{N}, \Sigma)$ is below $\text{LST}^- \iff$ no initial segment $\mathcal{P}$ of $\mathcal{N}$ is such that $(\mathcal{P}, \Sigma_{\mathcal{P}})$ is a $\Gamma$-$\text{LST}^-$ pair.

Of course, more detail is needed in order to convert 0.1 into a full definition, we shall give that detail in sections 5 and 2. Our goal in this set of notes is to give a proof of

Theorem 0.2. Assume (1) - (3) above, and that for all $\Gamma \in \Gamma$, there are no $\Gamma_0$-$\text{LST}^-$ pairs; then $\Gamma \neq \text{Hom}^*$ and in fact there is a $\Gamma$-hod-pair $(\mathcal{N}, \Sigma)$ such that $\Sigma \in \text{Hom}^*$ and $(\mathcal{N}, \Sigma)$ is a pointclass generator for $\Gamma$.

The proof of 0.2 involves analyzing $\text{HOD}$ as a direct limit of hod mice up
to the point we reach $\Omega - LST$-pahs, for some $\Omega$. At the same time, it is part of that analysis, the part where one proves HP and (cf. II J.) Those notes will therefore also deal with the other main parts of the hod-analysis: condensation for stationary strategies, comparison and fine structure for hod-pairs, and mouse capturing. We shall do this below $\Omega - LST$, but we believe that many of the arguments extend much further.

As a corollary to the proof of 0.2, we shall obtain

Theorem 0.3. Suppose $\kappa$ is a limit of Woodins and $\langle \kappa - strong cardinals, \kappa is measurable, \rangle \rightarrow \forall \kappa$; then the derived model $D(V, \kappa)$ satisfies: "for some $\Gamma$, there is a $\Gamma - LST$-pair."
We believe that our novel model induction techniques will enable one to construct LST—pairs assuming only \( k \) is measurable.

Let us explain how our restriction to hod pairs below LST — simplifies the theory. 

Suppose that \( \mathbb{M}(\mathbb{Z}) \) is a \( \Gamma \)-hod—pair, and that \( \mathcal{M} \) has a longest cardinal \( \aleph_1 \) and regular in \( \mathcal{V} \).

Let \( 2^\omega \) be a normal tree on \( \omega_1 \) with \( \omega \) normal.

We can extend the idea to fragments of \( \mathcal{Z} \). One can extend the notion of fragments of normal trees, by \[ \{ \alpha \leq \omega_1 \mid \text{for all } \alpha \leq \omega_1 \text{ with } \text{finer scopes of normal trees}, \] or \[ \mathcal{Z}(\alpha) \not\subseteq \mathcal{Z}(\beta), \] for all \( \alpha < \omega_1 \).
We do not give any further detail, because our only purpose is to explain that the restriction to hod pairs below LST means that we never have to insert short type strategy information into our hod premise in any nontrivial way.

More precisely, for $(\mathcal{U}, \Sigma)$ and $\mathcal{S}$ as in the last paragraph, let $\mathcal{L} = \Sigma_{\mathcal{U} \mathcal{S}} = \bigoplus_{\alpha \in \mathcal{S}} \Sigma_\alpha(x)$ be the join of the lower level strategies. If $\mathcal{L} \upharpoonright (\mathcal{U} \mathcal{S})^x \not= \emptyset$ is not Woodin, then we don't need to insert $\Sigma_{\mathcal{U} \mathcal{S}}$ as we go on from $\mathcal{U} \mathcal{S}$, because we can figure it out using $\mathcal{Q}$-structures from $\mathcal{U} \mathcal{S}$.

If $\mathcal{L} \upharpoonright (\mathcal{U} \mathcal{S})^x \not= \emptyset$ is Woodin, but for $x$ the least strong to \( \mathcal{S} \), $x$ is not a limit of Woodins, then $\mathcal{L}$ is really equivalent to
\( \Sigma \subseteq \mathbb{N} \) where \( \mathbb{N} \) is the sup of the Woodin of \( M \) below \( \mathbb{N} \). In this case again, \( \Sigma \) can be determined from \( \mathbb{N} \) using \( \mathcal{Q} \)-structures. It is only in the remaining case, i.e., that \( (\mathbb{N}, \mathcal{E}) \) is a \( \Gamma \)-LST-pair, that \( \Sigma \) is truly new information, and must be inserted as we go on.

The problem here is that if \( \Sigma \) is not Woodin, we can't just start inserting the full \( \Sigma \) above \( \mathbb{N} \), because we might project across \( \mathbb{N} \) in \( L(\mathbb{N}) \), and then the structure might go bad. It's ok to project across \( \mathbb{N} \) in the \( L(\mathbb{N}) \) hierarchy; coming down works out. So first we must climb the \( L(\mathbb{N}) \) hierarchy, and only if that does not project strictly across \( \mathbb{N} \), then start inserting \( \Sigma \) for \( \mathbb{N} \). We'll have
to then prove that $L^2(\mathcal{A}/\mathcal{S})$ does not project across $S$, if $L^2(\mathcal{A}/\mathcal{S})$ does not.

However, inserting $\mathcal{S}/\mathcal{L}$ without inserting $\mathcal{Z}$ is delicate, because we must avoid non-short trees coming down to short ones. Sahsyan proposes a solution to this difficulty (see [8]), but it introduces many complexities we would like to avoid here. This is why we restrict attention here to hod pairs below $LST$.

In fact, we shall begin with a more severe restriction.

**Definition 0.4.** A $\Gamma$-hod-pair $(\mathcal{P}, \mathcal{S})$ is below $0^\mathcal{P}$ iff whenever $\mathcal{G}_k$ is a limit of woodins, and $\mathcal{G}_k$ is full, and $k < \mu^+$, and $\mu$ is the critical point of an extender on the $\mathcal{P}$-sequence, then $0(\mathcal{G}_k)^{\mathcal{G}_k} < \mu$. 
So one can have strong cardinals that are limits of $\Gamma$-full Woodin below $\mathcal{O}^\mathcal{P}_h$, but not much more. The construction of hod pairs that is our focus here is simpler below $\mathcal{O}^\mathcal{P}_h$, because below $\mathcal{O}^\mathcal{P}_h$ all levels of the construction will be sound, and so every level will be its own core.

In sections 1 and 2 we describe the first order structure of hod premice below $LST^-$. In sections 2-5, we prove theorem 0.2, with its hypothesis strengthened to: all $\Gamma$-hod-pairs are below $\mathcal{O}^\mathcal{P}_h$, and its large cardinal hypothesis strengthened as well. In part II of this paper, we shall prove the full theorem 0.2.
Historical note: We wrote these notes in the period May–October 2013. They went through many revisions, as the reader can easily see. We wish to thank Nam Trang for reading them, and pointing out a number of unclear passages and mistakes.

There are minor errors left of which we are aware, but it seems better to write up part II before attempting a polished version of the whole thing. This manuscript should be considered a preliminary draft.

Finally, the theory of hod mice below LST was first blocked out by Grigor Sargsyan and the author.
in late 2008 and early 2009.

This paper owes a great deal to
theses discussions, as well as innumerable
other ones between Sargsyan and the
author. It also relies heavily on
Sargsyan's papers 217 and 227.
§1. Hodge preimage - informal description

We shall describe a hierarchy for hodge preimage that can reach LST-pairs.

The hierarchy of \( \mathcal{L}_I \), which deals with preimage below measurable limits of Woodin, has the property that a hodge preimage \( M \) that has started insuring an iteration strategy for \( M/8 \) will never later project strictly across \( \mathcal{B} \).

Even below \( 0^\# \), one must give that up. We shall maintain enough "\( L^\# \)-bras", however, that such an \( M \) now projects all the way down to the least \( R \) that is strong to \( \mathcal{B} \).

Before giving the definition, let us describe the hierarchy of a hodge preimage \( M \) informally. \( M \) will be a \( T \)-structure, constructed from preimage...
extender sequence and iteration strategy information. The fine structural notions of I[6] will apply literally.

Def. 1 (a) $O(K)^M$ is the strict sup of all $\text{lh}(E)$ of extenders $E$ on the $M$-sequence with crit$(E)=k$.
(b) $\gamma$ is a cutpoint of $M$ if

$\forall k < \gamma \quad O(K)^M \leq \gamma$.

For hod mice in general, the cutpoints are more important than the Woodin cardinals when it comes to timing the insertion of strategy information. Indeed, it seems we should ignore the Woodin cardinals, some of $\text{Ext}$, and maintain the extender-bias of [27], but this might be better called "LP-bias" in the present situation. It means that if $\mathcal{N}$ is our current level, and its strategy $E$ has been
activated, then our hierarchy continues with \( L_p^\preceq (\mathcal{M}) \) before it does anything else.

Assume \( AD^+ \), and let \( \Gamma \) be a reasonably closed pointclass. For simplicity, let's assume \( \Gamma = P(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) \), and \( L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} \). We describe the hierarchy of a \( \Gamma \)-hod-pair \((\mathcal{M}, \Xi)\) in an informal way. Hod preserve will have certain of the first order properties of such \( \mathcal{M} \).

Certain levels of \( \mathcal{M} \) will have been designated by \( \mathcal{M} \) as "full". We might better call them "strategy-activation" levels. Once it passes a full level \( \mathcal{N} \), the \( \mathcal{M} \)-hierarchy starts inserting new \( \Xi \)-hierarchies into \( \mathcal{N} \) that is part of \( \Xi \). It never stops doing this, even if it later
projects across \( o(N) \), as it may do.

The hierarchy of \( M \) is partitioned into some intervals we call blocks. Let \( k \in M \); then \( k \) begins a block of \( M \) if there are arbitrarily large \( g < k \) such that \( M^g k \) is full, and \( g \) is a cutpoint of \( M \). In this case, \( M^g k \) is also full, by definition. Our \( L_\mu \)-bias will also insure that \( k \) is a limit cardinal in \( M \) if it begins a block. The \( k \)-block of \( M \) will now extend until we reach the sup limit of the first \( w \) full levels past \( o(\kappa)^M \), at which point the next block begins.

Within its \( k \)-block, the hierarchy of \( M \) looks as follows.
us write $M(\eta)$ for the $\eta$th full level of $M$, and $\Sigma^\eta$ for the strategy for $M(\eta)$ induced by $\Sigma$. Let $\Sigma^\eta$ be the join of all $\Sigma^\alpha$ for $\alpha \leq \eta$, and $\Sigma^\eta_{<\eta}$ be the join of all $\Sigma^\alpha$ for $\alpha < \eta$. Now fix $\eta$ such that $M(k) = M(\eta)$. (In the interesting cases $\eta = k$.) Then the $M$-hierarchy proceeds from $k$ by stacking $\Sigma^\eta$-mice with strategies in $\Gamma$ that project to $k$, and the result of this is its next full level. That is

$$M(\eta+1) = \text{lp}_{\Sigma^\eta}(M(\eta))^{\Gamma},$$

where the superscript $\Gamma$ indicates that we demand the collapsing mice we stack have $\Gamma$-iteration strategies in $\Gamma$.

(*) This is not quite accurate; $\Sigma^\eta_{<\eta}$ will have a little more information in the formal development. We call it the "complete strategy."
Remarks

(1) $\Sigma^\circ\eta$ will be a $\Sigma_0$-iteration strategy for $M(\eta)$; i.e. the embeddings will be all $0$-embs., so continuous at $\omega(M(\eta))$. Thus $\Sigma^\circ\eta = \Sigma^\eta$ in the present situation, where $\kappa$ is a limit of countable ordinals.

(2) We need $\Sigma^\eta \in \Gamma$ to make sense of $M(\eta+1)$. This is part of being a $\Gamma$-hood-pair. The full $\Sigma$ may not be in $\Gamma$, but the induced strategies $\Sigma^\eta$ must all be in $\Gamma$.

We shall then have

$0(M(\eta+1)) = (K^+)^M$

That is, the $M$-hierarchies never project across $0(M(\eta+1))$ as it goes on.
We now stack $\mathcal{L}_p$ one more time:

$$M(\gamma+2) = \mathcal{L}_p^\Sigma_{\gamma+1} (M(\gamma+1)).$$

Notice that if $K$ is regular in $M(\gamma+1) = M_{1(\kappa^+)^\kappa}$, then iterating $M_{1(\kappa^+)^\kappa}$ and $M_{1(\kappa^+)^\kappa}$ are basically equivalent (modulo questions about how to iterate the extra collapsing $\operatorname{w}_\varphi$, that $L_{\mathcal{L}_p}^\Sigma_{\gamma+1} (M(\gamma+1))$ can answer.)

So if $K$ is regular in $M(\gamma+2)$, it is equivalent to the double-stack $L_{\mathcal{L}_p^2} (M(\eta))$. But if $K$ is singular in $M(\gamma+1) = M_{1(\kappa^+)^\kappa}$, then $\Sigma_{\eta+1}$ will not be $\Sigma_\Delta^1(\Gamma, \mathbb{R})$ from $\Sigma_\eta$. Nevertheless, the new information will not collapse $(\kappa^+)^\kappa$ or add new subsets of $K$. The argument for this is in Sargsyan's thesis $\Sigma_\Delta^1$. 
If \( \kappa \) is singular in \( M(\eta+1) \), then in fact

\[
M(\eta+1+1) = \bigcup_{\xi<\kappa_1} (M(\eta+1))
\]

for all \( \xi < \omega \), and these are the next \( \omega \) full levels of \( M \), and they are cutpoints, so the \( k \)-block ends at \( \sup \{ \alpha(M(\eta+1)) \mid \text{view } \alpha = \kappa \} \), a limit cardinal of \( M \) beginning the next block.

If \( \kappa \) is regular in \( M(\eta+1) \), then \( \kappa = \text{cof}(M(\eta+2)) \) is eligible to be the index on the \( M \)-sequence of an extender with critical point \( \kappa \), the order 0 total measure of \( M \) on \( \kappa \). If it is not, i.e., \( E^M = \emptyset \), then the \( M \) hierarchy proceeds as in the \( \kappa \) singular case. That is,
\[ M(\eta+1) = L^\infty(M(\eta+1)) \text{ for all } i < \omega \] and this completes the \( k \)-block of \( M \).

Suppose then that \( E^m_\emptyset \neq \emptyset \)
whose \( \chi = 0(M(\eta+2)) \). Then we set

\[ M(\eta+3) = M \upharpoonright \chi = (M(\eta+2), E), \]
where \( E = E^m_\emptyset \). That is, we declare that
\( M \upharpoonright \chi \) is full (i.e., its strategy is now to be activated). We then set

\[ M(\eta+4) = L^\infty(\eta+3)(M(\eta+3)), \]
and go on. That is, \( \beta(\eta) \) is eligible to be the index of an order one measure on \( K \). If it is not, then the \( k \)-block ends with \( \omega \) iterations of \( L^\infty \). If it is,
\[ M(\eta+5) = (M(\eta+4), E), \]
where \( E = E^m_\emptyset \)
for \( \chi = 0(M(\eta+4)) \).
We now describe how the block of \( M \) at \( k \) proceeds, in general. The main new thing is that as we increase \( k \) upward, new functions with domain \( k \) may show up, giving us new ways to iterate a full level that we have passed long ago.

Suppose

\[
\alpha_0 = \sup \{ \beta + 1 \mid \beta < \alpha_0 \text{ and } \text{crit}(E^m_\beta) = k^3 \}
\]

is a "local \( o(k) \)", i.e., \( o(k) \) is a.

We describe what the \( M \)-hierarchy looks like up to the next \( \beta \) such that

\[
\text{crit}(E^m_\beta) = k^3.
\]

\[\text{Notation} \]

(i) \( M^{\alpha} \) is the \( \alpha \)-level of \( M \).

(ii) \( \mathcal{N}^{-} \) is \( \mathcal{N} \), but with its least extendible predicate set \( \emptyset \). So \( \mathcal{N} \) is \( e \)-passive iff \( \mathcal{N} = \mathcal{N}^{-} \).

(iii) \( M^{\alpha} = (M^{\alpha})^{-} \).
Case 1. \( \alpha_0 \) is a successor ordinal.

Let \( \alpha_0 = \beta + 1 \). We shall have an \( \gamma \) such that

\[ M(\gamma) = (M(\gamma), E_\beta) \]

\[ = M_\beta \text{ with its last extender removed,} \]

and

\[ M(\gamma+1) = M_\beta = (M(\gamma), E_\beta) \).

We have a "complete strategy" \( \Sigma_{\gamma+1} \) for \( M(\gamma+1) \).

The next full level of \( M \) will be

\[ M(\gamma+2) = \Lambda_{\gamma+1} (M(\gamma+1)). \]

Near the bottom of the \( k \)-block, \( \Phi(0,M(\gamma+2)) \) would be appropriate as the next index of an extender on the \( M \)-sequence with critical point \( k \), and if it were not such an index, the \( k \)-block of \( M \) would end, with \( o(k) \leq \alpha_0 \) in \( M(\gamma+2) \).

But higher up, using functions in \( M(\gamma+2) \)
may yield stronger extrapolates of $M(\eta + 1)$ than one gets using only functions in $M(\eta + 1)$. So passing to $M(\eta + 2)$ gave us new ways to iterate $M(\eta + 1)$. We don't want to add another extender with critical point $\kappa$ to $\mathcal{M}$ until we have added further information about how to iterate $M(\eta + 1)$.

So we here define $\tau_{\alpha + 1}$ for $\alpha \geq 2$ by

$$\tau_2 = O(M(\eta + 2)) = \beta,$$

$$\Lambda_\alpha = \text{complete strategy of } M(\eta + 2) \text{ from } \Sigma_\eta = \Sigma_{\eta + 2}.$$

If $\tau_\alpha$ and $\Lambda_\alpha$ are defined, then $\tau_{\alpha + 1}$ and $\Lambda_{\alpha + 1}$ are undefined iff

$$L^P \Lambda_\alpha (M[\eta + 2]) = |T_\alpha| = 1/\beta.$$
with

\[ M_{\Lambda_k} = L^p (\mathbb{M}_{\Lambda_k})^\gamma. \]

If \( \Lambda_k \) is defined for all \( \alpha < \lambda \), let

\[ \gamma = \sup \beta < \lambda \gamma_\beta, \]

\[ \Lambda_{\gamma} = \text{complete strategy of } M_{\Lambda_k}. \]

Remark: It will work out that \( \Lambda_{\gamma} = \bigoplus_{\alpha < \lambda} \Lambda_{\alpha} \).

Now let \( \alpha \) be largest such that \( \Lambda_k \) is defined. Equivalently, \( \alpha \) is least such that \( L^\infty (\mathbb{M}_{\Lambda_k})^\gamma = \Lambda_k = \beta^+ \).

Then we set

\[ M(\gamma + 3) = M_{\Lambda_{\gamma}}, \]

\[ \gamma_{\gamma + 3} = \Lambda_{\gamma}. \]

for all \( \gamma \) such that \( 3 \leq \gamma \leq \alpha \). (Note \( \alpha \geq 2 \).) We shall have that
$M(x, \xi)$ is e-passive, for all $\xi$ such that $a \leq \xi < x$; that is,

$M(x, \xi) = M(y, \xi)$ for $a \leq \xi < x$.

Our potential next index for an extremal with critical point $x$ is

$\gamma = \tau_x = (p^+)_{x \rightarrow 0} (M(x, \xi))$.

If $E_x^m \neq \emptyset$, then we shall have

$\text{crit}(E_x^m) = K$. (Otherwise our coherence condition will give $\gamma$ a limit of indices $\gamma$ such that $\text{crit}(E_x^m) = K$. )

Thus we have done what we set out to do, i.e., reach the first index $\geq a_0$ of an extremal on $M$ with $\text{crit} = K$. If $E_x^m = \emptyset$

then there are no such indices, i.e., $0(K)^m = a_0$. In this case, the $K$-block ends with a iteration of $E_x^m$.
as before: \( M(\eta + \alpha + 2) = M \uparrow \alpha \), and
\[
M(\eta + \alpha + i + 1) = \bigcup_{\tau \geq \eta + \alpha + i} (M(\eta + \alpha + i))^\tau
\]
for \( \alpha \leq i < \omega \), and these are all e-passive initial segments of \( M \), with \( o(M(\eta + \alpha + i)) \) a cardinal of \( M \) for \( 1 \leq i < \omega \).

Case 2: \( \xi_0 \) is a limit ordinal.

In this case, \( (M \uparrow \xi_0)^- \) is a limit of full levels, so we declare it full. Let \( M(\eta) = (M \uparrow \xi_0)^- \).
Subcase 29 \( E^m_{d_0} \neq \emptyset \)

Our indexing is such that we must have \( \text{crit}(E^m_{d_0}) > k \) in this case. (See immediately below.) Thus this case does not occur below \( O^p_k \). If we are beyond \( O^p_k \), we could have, for \( E = E^m_{d_0} \) and \( \text{crit}(E) > k \):

\[ M(n+1) = (M(n), E). \]

We then proceed exactly as in case 5. We never actually used \( \text{crit}(E) = k \) in the description of the \( M(n+1) \)'s.

Subcase 2b \( E^m_{d_0} = \emptyset \).

We proceed exactly as in cases 1 and 29, but we start from \( M(n) = M|d_0 = M|d_0 \) rather than the active \( M(n+1) \)'s of those cases.
Remark: Subcase 26 is where our below-LST hypothesis applies. We are assuming that $M(y + 1) = x_0$ is not Woodin.
This completes our informal description of the $k$-block of $M_I$, and thus of the first order structure of $M$. The main remaining points to clarify are:

1) How do we index expressons on the $M$-sequence? The answer here depends on whether $\text{crit}(E)$ begins a block of $M$:

(a) If $E$ is on the $M$-sequence, and $\text{crit}(E)$ does not begin a block of $M$, then $E$ must fit according to the rules of $\Sigma I$, as repaired in $\Sigma I$.

(b) If $k = \text{crit}(E)$ begins a block of $M$, then our indexing convention is a bit closer to Jensen's. Namely, let $\gamma$ be the least cutpoint of $\text{crit}(M, E)$ such that $(k^+)^M \leq \gamma < \text{id}(k)$. Since $k$ is a limit of cutpoints of $M$, $\gamma$ exists.
We then index $E$ at $(\eta^+)_{\mathfrak{U}_0}(\mathfrak{m}, E)$. 

It is possible that the $\mathfrak{m}$-index $(\mathcal{E}(E)^+)^{\mathfrak{U}_0}(\mathfrak{m}, E)$ is strictly below $(\eta^+)_{\mathfrak{U}_0}(\mathfrak{m}, E)$, though this does not happen near the bottom of the $\kappa$-block. Of course, the Jensen index $(i_\infty(\kappa)^+)^{\mathfrak{U}_0}(\mathfrak{m}, E)$ is strictly larger. We shall explain why $(\eta^+)_{\mathfrak{U}_0}(\mathfrak{m}, E)$ works best when we get to the construction of hod pairs.

Note that "begins a block" is locally definable over $\mathfrak{M}$, so there is no problem amalgamating the two conventions.
(2) What is the iteration game for which $\Sigma^0_\eta$ is a winning strategy? Roughly, the usual $G_{\omega_1}(\Lambda_\eta(\delta_{\Lambda_\eta}))(\lambda_\eta)\cdot G_\eta(\Lambda_\eta(\delta_{\Lambda_\eta}))(\lambda_\eta)\cdot G_\eta(\Lambda_\eta(\delta_{\Lambda_\eta}))(\lambda_\eta)\cdot$

(3) What is a "complete" strategy for $\eta(\lambda_\eta)$? Roughly: $\Sigma^0_\eta$, together with all complete strategies for all $\eta$ proper full initial segments of all $\Sigma^0_\eta$ invariant of $\eta(\lambda_\eta)$.

(4) When do we tell $M$ the value of $\Sigma^0_\eta(\bar{\bar{\xi}})$? Roughly: we tell $M$ the value of $\Sigma^0_\eta(\bar{\bar{\xi}})$ for $\eta, \bar{\bar{\xi}}$ least such that we have not yet done so, in the usual amenable fashion.

Before we go on to the formal definitions, here are a few more remarks.
Remarks

(4) The $k$-block of $M$ may not end. In this case, we call $(k, \alpha \circ (M))$ the top block of $M$. It is a first-order property of $ML$ that its $k$-block has ended at $\lambda$, uniformly in $k, \lambda, M$. (This is because we have in its language of $M$ a symbol indicating its full levels.)

(2) Let $E = E^m \neq \emptyset$, where $K < \gamma$ and $\gamma$ is in the $k$-block of $M$. Thus $K \leq \text{crit}(E)$.

Def 3. $E$ is $k$-relevant iff

\[ \text{crit}(E) = K, \quad \text{or} \quad M/\alpha(E) = 0(K) \geq \text{crit}(E). \] (Equivalently, $0(K) \geq \text{lh}(E)$. )
(a) If $M(\gamma + \eta) \in L_p(M(\gamma))$, then all extenders on the sequence of $M(\gamma + \eta)$ with index $0(M(\gamma + \eta)) > 0$ are $k$-irrelevant. Here we assume $0(M(\gamma + \eta)) > 0$ is in the $k$-block.

This is because the extenders stacked in $L_p^{\infty}(N)$ all have cores $0(N)$, so $0(N)$ is a cutpoint of $L_p^{\infty}(N)$.

In particular, there are no partial measures with critical point $k$ on the $M$-sequence, the way we are setting things up. They are all reached in $L^{2_k}(M\mid k)$, which basically because some $O(\epsilon(P,\mu))$ for $M\mid k \in P$ if $O(2^\epsilon)$.

(b) If $E(\eta) \in E$ is indexed in the block of $M$ and $o(E) \in o(M)$ then $E$ is $k$-relevant. In particular, if $\text{crit}(E) = k$, then $E$ is $k$-relevant.
(c) There can be partial extenders that are k-relevant indexed in the k-block of M. Indeed, if E is not in an Lp-internal as in (a), then E is k-relevant.

(d) If M is below O^p_k, then the k-relevant extenders are just close with crit = k. (Here k begins a block of M.) the active last extender of O^p_k is k-relevant. Here k begins the top block of O^p_k.

The k-relevant extenders are just the ones we might use if we are iterating M above k, trying to produce an extender E with crit(E) = k that can be used applied to M. Partial extenders with crit > k can be used to do this, but not the ones we start stack in Lp(\mathbb{V}(\mathcal{M})).

(3) One hierarchy might be described as "L^p_\Sigma^* - biased". Given we have reached a full level M of the M-hierarchy in \mathcal{M} as its k-block,
\( \mathcal{M} \) goes on with \( L^\kappa (\mathcal{M}) \) for as long as it can. When it can no longer insert \( \kappa \)-sequences into \( \mathcal{M} \), it either adds a next \( \kappa \)-relevant extender, or it ends the \( \kappa \)-block.

(4) The insertion of strategy information into \( \mathcal{M} \) is not tied to its Woodin cardinals.

So even below \( AD_\kappa + \Theta \) regular, there is no \( \text{hod}-\text{principle} \) in the sense of \( \Sigma_2 \mathcal{J} \). They are however inter-translatable with \( \text{hod}-\text{principle} \) in the sense of \( \Sigma_2 \mathcal{J} \), in that region.

For example, let \( \mathcal{P} \) be a \( \text{hod}-\text{principle} \) in the sense of \( \Sigma_2 \mathcal{J} \), with two Woodin cardinals \( \delta_0 \) and \( \delta_1 \). \( \mathcal{P} \) gets re-organized into a \( \text{hod}-\text{principle} \) \( \mathcal{M} \) as above in the following way:
(1) $M$ and $\mathcal{P}$ have the same universe, the same cardinals, the same total extenders or their successors.

(2) The blocks of $M$ are the intervals $(\kappa, \lambda)$ where $\kappa$ and $\lambda$ are limits of cardinals and of cutpoints in $\mathcal{P}$.

For example, the first block of $M$ is $(\aleph_\omega, \omega_1 + \omega)$. We have full level's $M(\eta)$ whose strategy is being revised here, but this never happen unless $M(\eta)$ has already reached all the $\mathbb{Q}$-structures necessary to identify its strategy $E^\eta$. So $\mathcal{P}$ can figure out locally what $M$ is doing.

The first phase nontrivial strategy information goes into $M$ in the $K_0$-block,
Where

\[ k_0 = \text{least } k \text{ that is } < \delta_0 \text{-strong in } \mathcal{P}. \]

We have \( M(\delta_0) = M/\delta_0 = " \mathcal{P}/\delta_0 \)

and \( \Sigma_0 \) is "trivial," i.e., not really

new information, \( M(\delta_0 + 1) = L^{\Sigma_0}(M/\delta_0) \)

is called \( \mathcal{P}(\delta_0) \) in \( \Sigma_Z \), and its

strategy \( \Sigma_0 \) is what \( \mathcal{P} \) starts inspecting

and where we inspect there, an inspection at the

same place. The \( k_0 \)-block of \( M \)

now ends at the \( \omega \)-cardinal of \( \mathcal{P} \)

above \( \delta_0 \).
§2. **Hod preimages, formal description**

Formally, hod preimage are acceptable $J$-structures $\langle J^A_\alpha, B \rangle$ in the sense of 1.8 and 1.20 of [57]. It will be convenient to decompose the two amenable predicates. The decomposition makes our hod preimage also models with parameter $0$ in the sense of [67].

Thus the language of hod preimage has $E$, unary predicate symbols $\hat{E}, \hat{B}, \hat{S}$, and constant symbols $\hat{e}$ and $\hat{f}$. A structure

$$M = (M, \in, \hat{E}^m, \hat{B}^m, \hat{S}^m, \hat{e}^m, \hat{f}^m)$$

is a hod preimage just in case $M$ is transitive, $\in$-closed, and the conditions (1) - (4) below are met.

Condition (1) determines the meanings of $\hat{S}$ and $\hat{e}$.
(1) $M$ is a model with parameters $\alpha$ in the sense of [BJ]. That is, $\hat{S}^m$ is the sequence of levels of $M$, and we write $l(m)$ for $\text{dom}(\hat{S}^m)$. We set

$$M/\alpha = \hat{S}^m(\alpha).$$

We demand the base model $\hat{S}^m(0)$ be $(\forall \omega, \epsilon, \phi, \psi, \varphi, 0, 0)$. We set

$$M/\ell(M) = M.$$ 

We have $\hat{m}^M = \ell(M) - 1$ if this exists, and $\hat{m}^M = 0$ otherwise.

Condition (2) tells us how we index extensions. We use $\hat{m}^M = \omega$ to mean "$M$ is full."

(2) Either $\hat{E}^m = \phi$, or $\hat{m}^M = \omega$ and $\hat{E}^m = E$ is an extension over $M$ satisfying the conditions of [BJ], that is
Condition (2) tells us how we index extenders. We use $f^M < \omega$ to mean that $M$ is full.

(2) Either $E^M = \emptyset$, or $f^M = 0 > B^M = \emptyset$, and $E^M = E$ is an extender over $M$ with $\mu(E) = \lambda = 0(M)$, and

(a) If $k$ does not begin a block of $M$, then the conditions of $\Sigma^1_4 J$ are met, i.e.,

(i) $M = k^+$ exists,

(ii) (coherence) $\text{Ult}_0(M, E) \downarrow \ell(M) = (M, E, \emptyset, \emptyset, f^M, 0, 0)$,

(iii) $\ell(M) = \mu(E) + \text{Ult}_0(M, E)$,

(iv) $E$ is not type $\mathcal{Z}$,

(v) $E$ satisfies the initial segment condition of $\Sigma^1_4 J$, as repaired in $\text{ET}_J$.

(b) If $k$ begins a block of $M$, then

(i) $M = k^+$ exists,

(ii) (coherence) $\text{Ult}_0(M, E) \downarrow \ell(M) = (M, E, \emptyset, \emptyset, f^M, 0, 0)$, and

(iii) $\ell(M) = \eta^+ \text{Ult}_0(M, E)$, where $\eta$ is the least cutpoint of $\text{Ult}_0(M, E)$ s.t. $\mathcal{C} \cup \Delta(k)$. 
Remark. Notice there is no initial segment condition in 2(b). As in Jensen indexing, we don’t need one because if $E$ and $M$ are as in 2(b), then for no $\mathcal{E} \prec \mathcal{H}(E)$ is $(M \mathcal{E}, E \mathcal{E})$ as in (2). That is, the literally-stated initial segment condition would be vacuous.

It will still be true that the "trivial completions" of various $E \mathcal{E}$ are on the $M$-sequence, and hence those $E \mathcal{E} \in M$. But we don’t need an axiom keeping track of which ones. For comparison-termination, it is enough that no $E \mathcal{E}$ is beaten on the $M$-sequence.

Note also that if we use $E$ that is a comparison, then $\text{Ult}(M, E) / \mathcal{H}(E)$ is a coprime, so on the $M$-side, we’ll never have another critical point $\leq \mathcal{H}(E)$.

Notation. If $E^M \neq \emptyset$, we say $M$ is $E$-active; and otherwise, we say $M$ is $E$-passive. The extraorder sequence of $M$ is given by $E^M \mapsto E \oplus M \mathcal{E}$, for $\alpha \leq \mathcal{L}(M)$.
If $M$ is $e$-active, then

$$M^e = (M_3, e, \phi, \phi, S^m, 0, 0)$$

is the associated $e$-passive structure. We write $M^e = M$ if $M$ is already $e$-passive. Let also

$$M11g = (M18)^e.$$ 

The levels of $M$ are just the structures $M18$ and $M11g$. The full levels are just those $M$ such that $\varphi^M \in V$. (Otherwise, $\varphi^M = \omega$.) We shall use $\varphi^M = k < \omega$ as a signal that the strategy in iteration strategy for the game $G^*_k(\omega, \omega, \omega)$ should be inspected.

Let $k < \omega$ and let $M$ be $k$-sound, with $\mathbb{N}$ its $\Sigma_k$-master-code structure. $G^*_k(\omega, \omega, \omega)$ is the following variant of the usual $(0, \omega, \omega)$-iteration game on $\mathbb{N}$:
(4) In round $\alpha$, I can play $G_0 (\alpha, w)$, where if $\omega \geq 0$, $\alpha$ is given by the previous rounds. So it's true if $\alpha$ is being played on $\alpha$ is normal.

(b) I am not allowed to exit round $\alpha$ at stage $\beta$ if $\omega \beta J_\alpha$ drops, since

Thus each $\alpha$ is a master-code structure, and we have a $k$-embedding from $\beta$ to the decoding of $\alpha$.

Remark: We do allow drops within the $\alpha$'s, but not on their main branches. Allowing drops off the main branches is needed when we use the strategies to form $\text{HOD}$-limits.

Disallowing drops on the main branches seems necessary when it comes to comparing strategies.
Def: Let $N$ be k-sound, and have a supposition $p_k(N) > k$. The top block has begins with k. The top-block-normal game $G^\text{top}_k(N, w, \mathcal{I})$ is the usual length $w$, normal situation game on the $\Sigma_k$ master cost of $N$ when $\mathcal{I}$ is restricted to playing extendors with critical point $> k$. A $(k, w_1)$-top-block-normal strategy is a winning strategy for $\mathcal{I}$ in $G^\text{top}_k(N, w, \mathcal{I})$.

In general, it might be (we're not sure) unnecessary to insert the top-block-normal strategy for $\mathcal{I}$ (and use it to come down) before inserting a full strategy for $G_k(N, w, \mathcal{I})$. But notice that below $G^\text{top}_k$, there are no extendors I can play in $G^\text{top}_k(N, w, \mathcal{I})$. 
so the top-block-normal strategy is trivial.
Indeed, $G^\top_k (M, \pi, w)$ is trivial for $k$ ways past $0^\top_k$. So we shall ignore the
top-block-normal strategy here.

The restrictions of $G^*_k (M, \pi, w, \omega)$ are such that if $\Sigma$ is a winning strategy for $\Pi$
and $M$ is a full level of $M$, then $\Sigma$
may not determine a strategy for $\Pi$ in
$G^*_k (M, \pi, w, \omega)$. This can happen if some
level of $M$ projects across $o(N)$. In this
case we will have

$$(k^+)^M \leq p_{\omega}(M, \pi, \omega) < o(N) < \gamma$$

where $k$ begins the top block of $M$. The strategy
for $M$ lets us stack normal trees on the
window $2p, o(N)^f$, whereas the strategy for
$Mf\gamma$ does not.

We remedy this by defining the notion of a “complete” iterational strategy.
Def. 2.0 Let $M$ be a full hod prime, and $k = f^M$. A complete strategy for $M$ is a pair $(\Omega, F)$ such that:

(a) $\Omega$ is a winning strategy for $M$ in $C^*_k(M, w, w)$, and

(b) whenever $\Omega$ is a stack by $\Omega$ with last model $P$, and $N$ is a full level proper of $P$ such that $N \neq P$, then $F(\Omega, N)$ is a complete strategy for $N$.

Remark: It is a tacit assumption about $(\Omega, F)$ in 2.0 that the obvious "mouse descent" relation generated by $(\Omega, F)$ is well-founded. This justifies defining "complete strategy" in terms of itself.

Notice that in (b), if $M$-to-$P$ does not drop in model or degree, then $(\Omega, F)$
also determines a complete strategy for \( P \). The difference between a complete strategy \( \Sigma \) for \( M \) and a strategy for the game \( G^*_{k}(M, w_1, w_1) \) is the variant of \( G_{k}(M, w_1, w_1) \) where I can exit a round after dropping, whereas occurs in (b) when \( M \rightarrow P \) drops. Then \( (L, F) \) does not tell us how to iterate \( P \) whereas \( \Sigma \) does tell us that. \( \text{Rank}_{L} \) this distinction between \( (L, F) \) and \( \Sigma \) is not relevant below \( \phi_{M} \).

Let \( \Sigma = (L, F) \) be a complete strategy to \( M \), and let \( F \) be a stack on \( M \) with last model \( P \). We write

\[
\Sigma_{L, \eta} = F(L, \eta)
\]

if \( \eta \neq P \) is a full level of \( P \). We also write \( \Sigma_{L, P} \) for the tail of \( L \), if \( M \rightarrow P \) does not drop. These notations follow that of \( \varepsilon \), of course. Let us write \( \Sigma_{\eta} \) for \( \Sigma_{L, \eta} \), when \( \eta \) is a full level of \( M \).
Let \( \Sigma \) be a complete iteration strategy for \( \mathcal{M} \), and let \( \mathcal{N} \) be a full level of \( \mathcal{M} \). Let \( k = \mathcal{N} \) and \( i = \mathcal{M} \). Suppose the \( \Sigma \mathcal{E} \) master code of \( \mathcal{N} \) has universe that is an initial segment of the \( \Sigma \mathcal{E} \) master code of \( \mathcal{M} \). (That is, between \( \mathcal{N} \) and \( \mathcal{M} \) we have projected

\[
< \rho_k(\mathcal{N}) \].
\]

Then plays of \( G^*_k(N, w, w_i) \) can be regarded as plays of \( G^*_c(M, w, w_i) \), by thinking of the trees as being on \( \mathcal{M} \) to start.

Let \( \Sigma = (\Omega, F) \). We say \( \Sigma \) is self-consistent iff whenever \( \mathcal{N} \) is as above, and \( \Sigma_{\eta} = (\Psi, G) \), then

(a) \( \Psi \leq \Omega \),

and in fact

(b) \( \Sigma_{\eta}, \rho \leq (\Sigma_{\eta})_{\Omega, \rho} \), whenever

\[
(\Sigma_{\eta})_{\Omega, \rho} \text{ is defined},
\]
Def. 2.0.1 (Strategy coherence) Let $\Sigma$ be a complete strategy for $M$. We say $\Sigma$ is coherent iff

(a) $\Sigma$ is self-consistent, and

(b) whenever $\Sigma \circ \mathcal{P}$ and $(\Sigma \circ \mathcal{P}) \tau, \eta$ are defined, and $\mathcal{N}$ is a full level at both $\mathcal{P}$ and $\mathcal{Q}$, then $\Sigma_{\tau, \eta} = (\Sigma \circ \mathcal{P})_{\tau, \eta}$.

Self-consistency is an obvious requirement. Part (b) of strategy coherence holds for the hierarchy of $\Sigma\mathcal{F}$ by pullback consistency. This is because there, if $\tau: \mathcal{P} \to \mathcal{R}$ and $\mathcal{N}$ is a full level of both $\mathcal{P}$ and $\mathcal{R}$, then $\text{crit } \tau \geq 0(\mathcal{N})$ because $\mathcal{N}$ is full and no extenders overlap full levels. Since $\Sigma_{\tau, \eta}$ pulls back by $\tau$ to $\Sigma_{\tau^\mathcal{P}, \mathcal{P}}$, we get (b).

But once we have extenders overlapping full
levels, part (b) of strategy coherence seems to go beyond pullback consistency.

Def 2.0.2 A good iteration strategy for a had premise $M$ is a complete, coherent strategy for $M$ that has pull condensation.

For pull condensation, see E2J. E2J shows that pull condensation implies pullback consistency. All iteration strategies we consider for had premise will be good, so we may sometimes forget to say "good".

Had premise are required to have an opinion as to their own good iteration strategy. For $\eta$, a full level of $M$, $\Sigma^M_{\eta}$ is the information about the "external $\Sigma^{\eta}$" that has been
coded into $B^m$ in the way we are about to describe.

(2) Let $M$ be a bad premouse, and $\alpha < \ell(M)$ such that $M^{\alpha + \gamma} = \text{KP}$, and there are bounded many $\beta < \alpha$ such that $M^{\beta} = \text{KP}$. Let $<n, \vec{a}_n>$ be least in $<^m$ such that $M$ is full, $\vec{a}_n$ is a play by $\Sigma^m_{\alpha + \gamma}$ with $\Pi$ to move, and $\Sigma^m_{\alpha + \gamma} (\vec{a}_n)$ is undefined. Suppose $y = h(U)$, where $U$ is the last component of $\vec{a}_n$, and $\alpha + y \leq \ell(M)$.

Then we require that

$$B^{\alpha + y} = 3 \alpha + \xi \mid \xi \in b^3$$

for some cofinal branch $b$ of $U$, and write

$$\Sigma^m_{\alpha + \gamma} (\vec{a}_n) = b \quad \text{for all } n \geq \alpha + y.$$
Finally, we have first order conditions that pin down which levels $M$ of $M$ can be declared full by $\mathcal{M}$, and what $\mathfrak{M}$ can be. These were discussed in the previous section, so we omit further details for now.

(3) Restrictions on fullness declarations, details later.

(4) If $M_{18+1}$ is not covered by (1)-(5), then $M_{18+1} = (\text{Red}(M_{18}), \mathcal{E}, \mathcal{P}, \mathcal{A}, I, \mathcal{L}, \mathcal{D})$, for $\mathcal{I}$ and $\mathcal{L}$ describing $(M_{18}/a \leq 8)$.

It is easy to see that we can look at these structures as in (1)-(4) as $\mathcal{F}$-structures in the sense of ETJ. Thus projecta, standard parameters, and cones/soundness make sense.
Def 2.1 A hod-principle is a structure $\mathcal{M}$ satisfying (1)-(4) above, and such that each proper initial segment of $\mathcal{M}$ is $\omega$-sound.

We let

$$\mathcal{M}(\alpha) = \alpha \uparrow \text{ full level of } \mathcal{M},$$

and $\omega^\mathcal{M}$ be the order-type of the full levels of $\mathcal{M}$. Set $\mu_\alpha = \text{ORD} \cap \mathcal{M}(\alpha)$, so that either $\mathcal{M}(\alpha) = \mathcal{M}/\mu_\alpha$ or $\mathcal{M}(\alpha) = (\mathcal{M}/\mu_\alpha)^-$. This notation is parallel to 

Sargsyan Z2.5.
The truly interesting object is the hod pair.

**Definition 2.2** \((M, \Sigma)\) is a hod pair just in case

1. \(M\) is a full hod pommaret, and \(k\)-sound, where \(k = f^* M\),

2. \(\Sigma\) is a good iteration strategy for \(M\), and

3. if \(\Sigma \Rightarrow_\delta \rho\) exists, and \(\eta\) is a full level of \(\rho\), then \((\Sigma \eta)^\rho \equiv \Sigma \Rightarrow_\delta \eta\).

Clause (3) says that the strategy information coded into the \(\delta\)-predicate of an iterate \(\rho\) is consistent with \(\Sigma\).

This definition is meant to be used in the AD\(^+\) context, where \(\omega_1\)-iterability
implies \( \omega_{\eta+1} \)-iterability. In that context, we can define fullness preservation.

**Def 2.3** Let \((\mathcal{M}, \Sigma)\) be a hbb pair, and \(\Gamma\) a pointclass. We say \(\Sigma\) is \(\Gamma\)-fullness preserving iff whenever \(\mathcal{P}\) is a \(\Sigma\)-iterate of \(\mathcal{M}\), and \(\mathcal{N}\) is a full level of \(\mathcal{P}\) that is not of the form \(Q^r\) for any active level \(Q\), and \(R\) is the next full level of \(\mathcal{P}\) after \(\mathcal{N}\), then whenever letting \(\mathcal{L}\) be the \((k, \nu, w, \lambda)\) strategy for \(\mathcal{N}\) induced by \(\Sigma\) (where \(k = 2^n\)), then either

(a) \(R = \text{first level of } L_{2^n}(\mathcal{N})\) with projectum \(\leq \rho_k(\mathcal{N})\), and \(\exists \mathcal{R} \leq \rho_k(\mathcal{N})\), or

(b) \(R = L_{2^n}(\mathcal{N})\), if no level as in (a) exists.
We are requiring as part of 2.3 that if $\Sigma$ is $\Gamma$-fullness preserving, then all "lower-level-strategies" $\Sigma_n$ induced on any full levels of iteration by $\Sigma$ are such that $\Sigma_n \in \Gamma$. However, $\Sigma$ itself may not be in $\Gamma$.

Def. 2.3.1 A $\Gamma$-hod-pair is a hod-pair $(M, \Sigma)$ such that $\Sigma$ is $\Gamma$-fullness preserving.
Some of the central arguments of [22] do not require that we are below $\mathsf{AD}_\kappa$ + $\Theta$ measurable. Let us record one example:

**Definition** Let $(\mathbf{P}, \mathbf{E})$ be a hod pair. We say that $\mathbf{E}$ is *positioned* iff whenever $\vec{\mathbf{F}}$ and $\vec{\mathbf{U}}$ are non-dropping stacks by $\mathbf{E}$ with common last model $\mathbf{Q}$, then $\mathbf{E}_{\mathbf{Q}, \vec{\mathbf{F}}} = \mathbf{E}_{\mathbf{Q}, \vec{\mathbf{U}}}$. 

(b) $\mathbf{E}$ has *branch condensation* iff whenever $\vec{\mathbf{F}} \uparrow \mathbf{b}$ and $\vec{\mathbf{U}}$ are by $\mathbf{E}$, and $\vec{\mathbf{F}}$ and $\vec{\mathbf{U}}$ exist, and we have

\[
\begin{array}{c}
\mathbf{P} \\
\downarrow^\mathbf{F} \\
\mathbf{M}_b \\
\downarrow^\mathbf{U} \\
\mathbf{M}_c \end{array}
\]

for some $\mathbf{F}$, then $\text{dom}(\mathbf{E}) = \mathbf{E}(\mathbf{U})$. 

\[\text{cylindrical} \quad \therefore \quad \mathbf{E} = \mathbb{E}(\vec{\mathbf{U}})\]
The iteration strategies we shall construct will be positional and have branch condensation. Sargsyan [12] shows such a strategy is commutating, in that if

\[
P \xrightarrow{i} Q \xrightarrow{f} R \xrightarrow{k}
\]

are such that all join and koi are by the strategy, then \( f = k \). This implies that

is a direct limit \( \text{Lim}(P, \Sigma) \) of all countable, non-dropping \( \Sigma \)-iterates of \( P \). See §2.6 of the July 2013 version of [17] for more on the basics of how iteration strategies can behave well.

If \( \Sigma \) is positional and \( \mathcal{N} \) is a \( \Sigma \)-iterate of \( P \) via a non-dropping tree (i.e., \( \mathcal{N} \in \text{I}(P, \Sigma) \), in the notation of [17]), then we write \( \Sigma_{\mathcal{N}} \) for the common value of all \( \Sigma \), and call \( \Sigma_{\mathcal{N}} \) the \( \mathcal{N} \)-tail of \( \Sigma \).
Comparison (essentially Sorgyon 27.) Assume AD+.

Lemma 2. Let \((P, \Sigma)\) and \((Q, \Lambda)\) be \(\Gamma\)-had pairs. Suppose \(\Sigma\) and \(\Lambda\) are \(\Gamma\)-finiteness preserving, have branch condensation, and are positional. Then there are \(\Gamma\)-names \((P^*, \Sigma^*)\) and \((Q^*, \Lambda^*)\) of \(\text{the}(P, \Sigma)\) and \((Q, \Lambda)\) by \(\Sigma\) and \(\Lambda\), respectively, such that

\[
\begin{align*}
\text{(a)} & \quad (P^*, \Sigma^*) = (Q^*(\alpha), \Lambda^*_\alpha), \text{ some } \alpha \leq Q^* \\
\text{(b)} & \quad (Q^*, \Lambda^*) = (P^*(\alpha), \Sigma^*_\alpha), \text{ some } \alpha \leq P^*
\end{align*}
\]

Proof (Sketch.) Assume for simplicity there is a good scaled pointclass \(\Gamma^* \supseteq \Gamma\). Let \(N^*\) be a coarse \(\Gamma^*\)-Woodin that captures a universal \(\Gamma\)-set, with \(P, Q \in N^*\). Let \(N^*_\beta\) be the \(\beta\)-th model of the maximal \(\Gamma\)-had-mouse-construction of \(N^*\). This construction has to reach \(\text{ord}^+(\alpha)\) of \((P, \Sigma)\)
and \((Q, \Lambda)\) before it breaks down.

The key to this is that there are necessary strategy disagreements, by branch condensation (see 2.2.27 of EZJ).

But then \((P, \Sigma)\) and \((Q, \Lambda)\) have been compared with a single construction, and hence with each other.

Remark The reason this is not a general comparison lemma is its branch condensation hypothesis. Once our hod mice have extended overlapping Woodin's branches, branch condensation will probably fail.

Nonetheless, we would guess that hod-pairs always iterate into the hod-mouse construction of a sufficiently Woodin background universe \(N^*\), and

There is a natural weakening of branch condensation (**photon condensation** for \(2\)-generated phalanxes) even.
That implies this. Below O, branch condensation will hold, and indeed the theory of E27 seems to go over without much change. Clearly, another level of detail is required beyond what we have provided above, but let us go on, and provide further detail as it becomes important.
Repeating def. 0.1, we record a notion of smallness weaker than "below $O_k"; 

**Def 2.3.3** A hod-promise $M$ is **below LST** iff whenever $N$ is a full level of $M$, and $\kappa$ has a top block beginning with $K$, and $K$ is a limit of Woodin's in $N$, then there is no $S \in N$ such that $K < S$ and $N \leq S$ is Woodin.

An $M$ that is below LST can have extenders overlapping local Woodins on its sequence, but those must occur inside $Lp$-intervals, e.g. below its $M^\lambda_j$.

The following lemma will be useful in comparing strategies below LST.
Lemma 2.3.4

Let \((M, \Sigma)\) and \((M, \Upsilon)\) be \(\Gamma\)-good pairs such that \(M \subseteq \text{UST}\).

Let \(U\) be a normal tree on \(M\) that is according to both \(\Sigma\) and \(\Upsilon\). Suppose \(b \neq c\),

where \(b = \Sigma(U)\) and \(c = \Upsilon(U)\). Suppose also

\[\Sigma_{U, \eta} = \Upsilon_{U, \eta}\]

for every full proper initial segment of \(M(U)\).

Then

1. Neither \(b\) nor \(c\) drops
2. \(\Sigma(U)\) is a Woodin and a cutpoint in both \(M^b\) and \(M^c\)
3. \(U = S^T\), where \(S\) and \(T\) are normal, \(\lambda : M \rightarrow R\) exists, and for some \(\eta < \delta < \xi \in R\)
   (a) \(b^*(\delta) = c^*(\delta) = \delta(S)\), where \(b^*\) and \(c^*\)
   are the branches of \(\Sigma\) induced by \(b\) and \(c\)
   (b) \(R = \{\text{there are no Woodins in the interval } (\eta, \delta)\}\)
   (c) all critical points in \(T\) are \(> \eta\).
Proof sketch. $s(U)$ must be a limit of full levels of $M(U)$, for otherwise $U = S \setminus T$ where $T$ is a normal tree on some $L_p^\infty(R)$, with all crits $> 0(R)$, and

$$\Lambda = \bigvee_{S,R} = \mu.$$ But just as

iteration strategies for sound mice projecting to $\mathcal{U}$ are unique, there can be only one strategy for $L_p^\infty(R)$.

Letting $K$ be the least cardinal of $M(U)$ that is strong so $s(U)$, we then get that $K$ is a limit of full levels of $M(U)$, and of cutpoints of $M(U)$. That is, $K$ begins the top block of $M(U)$.

Let

$$\Lambda = \text{join of all } S_{\alpha,\beta}, \text{ for } \alpha \in \Lambda \text{ a full proper initial segment of } M(U).$$
Then \( L^p(M(U))^5 \cong M_b \), as otherwise \( b \) drops, and \( Q(b, U) \) exists, and \( Q(b, U) \subset L^p(M(U))^5 \). If \( c \) drops, then we get \( Q(c, U) = Q(b, U) \), a contradiction. If \( c \) does not drop, then \( L^p(M(U))^5 \cong M_c \), so \( Q(b, U) \subset M_c \), again a contradiction.

Similarly, \( L^p(M(U))^5 \cong M_c \). But

\[
L^p(M(U))^5 \subset S(U) \text{ is Woodin},
\]
so applying 2.17 with \( N = L^p(M(U))^5 \), we see that \( K \) is not a limit of Woodins in \( M(U) \). Let \( \eta \) be the strict sup of the Woodins of \( M(U) \) that are \( < K \).

Since \( K \) is a cutpoint of \( M(U) \), we can write

\[
U = D^\omega F
\]
where \( S(A) \leq k \) and all others in \( J \) are \( \geq k \). Let \( R \) be the least model of \( S \). Let \( S \) be the least Woodin at \( R \). \( \eta \).

We leave the rest to the reader.

\[ \square \]

Remark: We don’t get that \( M^u \) and \( M^\mathbf{b} \) have the same subsets of \( S(\mathbf{u}) \). Letting \( \mathbf{u} = \mathbf{u}^{M^\mathbf{b}(\mathbf{u})(\mathbf{u})} \), it can be that \( \Sigma_{\mathbf{u}^{\mathbf{b}}, \mathbf{u}^{\mathbf{b}}(\mathbf{u})} \neq \Sigma_{\mathbf{u}^{\mathbf{b}}, \mathbf{u}^{\mathbf{b}}(\mathbf{u})} \). \( \Sigma \) and the two \( \mathcal{L} \)’s produce different subsets of \( S(\mathbf{u}) \). Those two \( \mathcal{L} \)’s will not kill the Woodinness of \( S(\mathbf{u}) \). However, this would happen if the case \( S(\mathbf{u}) = \text{least Woodin at} \ M^\mathbf{b} \).