

# HOD mice below $LST^-$ , I.

J.R. Steel

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§0.

We develop further the method for constructing hod mice from [1] and [3].

In particular, we assume

- (1)  $\kappa_0$  is a measurable limit of Woodin cardinals, with  $j: V \rightarrow M$  witnessing its measurability.
- (2)  $g$  is  $\text{col}(\omega, < \kappa_0)$ -generic over  $V$ , and  $\Gamma$  is a Wadge initial segment of  $\text{Hbun}^*$  such that  $\Gamma = P(\mathbb{R}_g^*) \cap L(\Gamma, \mathbb{R}_g^*)$  and  $L(\Gamma, \mathbb{R}_g^*) = \text{AD}_R + \text{DC}$ .
- (3) Letting  $H_0 = \text{HOD}^{L(\Gamma, \mathbb{R}_g^*)} \upharpoonright \Theta^{L(\Gamma, \mathbb{R}_g^*)}$  and  $\Sigma_{H_0}$  be the join of the strategies for  $H_0(\alpha)$ ,  $\alpha < o(H_0)$ , stretched by  $\hat{j}: V[g] \rightarrow M[h]$ , we have  $o(L_p^{\Sigma_{H_0}}(H_0))^{\hat{j}(\Gamma)} < \kappa_0^+$ .

(2)

In (3), when we speak of the "strategies for de Ho( $\alpha$ )", we are implicitly assuming that  $\text{HOD}^{L(\Gamma, \mathbb{R}_g^*)}$  has been analyzed as a hod-limit. What we are doing here will then figure in the analysis of  $\text{HOD}^{L(\Omega, \mathbb{R}_g^*)}$ , for  $\Gamma \subsetneq \Omega \subseteq \text{Hom}_g^*$ .

~~We shall describe~~ If  $(\mathcal{N}, \Sigma)$  is a hod pair, and  $P$  is a "full" initial segment of  $\mathcal{N}$ , then we write  $\Sigma_P$  for the part of  $\Sigma$  that tells us how to iterate  $P$ .

Definition 0.1 Let  $(\mathcal{N}, \Sigma)$  be a  $\Gamma$ -hod-pair.

We say  $(\mathcal{N}, \Sigma)$  is a  $\Gamma$ -LST-pair iff there is a  $\delta$  such that

$$(1) \mathcal{N} = L_p^\Omega(\mathcal{N} \upharpoonright \delta)^\Gamma, \text{ where } \Omega =$$

$$\bigoplus_{\alpha < \delta} \Sigma \upharpoonright \alpha, \text{ and}$$

(2)  $\mathcal{N} \upharpoonright \delta$  is Woodin, and

(3) For  $\kappa < \delta$  the least  $< \delta$ -strong of  $\mathcal{N}$ ,  $\mathcal{N} \upharpoonright \kappa$  is a limit of Woodins.

We say that  $(\mathcal{N}, \Sigma)$  is below LST (3)  
iff no initial segments  $P$  of  $\mathcal{N}$  is such  
that  $(P, \Sigma_P)$  is a  $\Gamma$ -LST-pair.

Of course, more detail is needed in  
order to convert 0.1 into a full definition.  
We shall give that detail in sections 1  
and 2. Our goal in this set of notes  
is to give a proof of

Theorem 0.2 Assume (1) - (3) above  
and that for all  $\Gamma_0 \subsetneq \Gamma$ , there are no  
 $\Gamma_0$ -LST-pairs; then  $\Gamma \neq \text{Hom}_g^*$ ,  
and in fact there is a  $\Gamma$ -hod-pair  
 $(\mathcal{N}, \Sigma)$  such that  $\Sigma \in \text{Hom}_g^*$  and  $(\mathcal{N}, \Sigma)$   
is a pointclass generator for  $\Gamma$ .

The proof of 0.2 involves analyzing  
HOD as a direct limit of hod mice up

to the point we reach  $\Omega$ -LST-pairs, 4  
for some  $\Omega$ . At the same time, it is  
part of that analysis, the part where  
one proves HPH. (cf. 217.) These notes  
will therefore also deal with the other  
main parts of the hod-analysis:  
condensation for iteration strategies, comparison  
and fine structure for hod-pairs, and  
mouse capturing. We shall do this below  
LST-, but we believe that many of the  
arguments extend much further.

As a corollary to the proof of 0.2,  
we shall obtain

Theorem 0.3 Suppose  $\kappa$  is a limit of Woodins  
and  $\kappa$ -strong cardinals,  $\kappa$  is measurable,  
and  $\neg \square_\kappa$ ; then the derived model  
 $D(V, \kappa)$  satisfies: "for some  $\Gamma$ , there  
is a  $\Gamma$ -LST-pair".

We believe that core model induction techniques will enable one to construct  $LST^-$  - pairs assuming only  $\kappa$  is measurable and  $\rightarrow \square_\kappa$ , but do not claim to have done this.

Let us explain how our restriction to hod pairs below  $LST^-$  simplifies the theory.

Suppose that  $(M, \Sigma)$  is a  $\Gamma$ -hod-pair, and that  $M$  has a largest cardinal  $\delta$ , with  $\delta$  being a cutpoint of  $M$  and regular in  $\delta M$ .

Let  $\mathcal{U}$  be a normal tree on  $M/\delta$  that is by  $\Sigma$ ; then we say  $\mathcal{U}$  is by  $\Sigma_{sh}^n$  iff for all  $\alpha < lh(\mathcal{U})$ , either  $\Sigma_{0, \alpha} \mathcal{U}$  drops, or  $i_{\alpha}^{\mathcal{U}}(\delta) > \delta(\mathcal{U})$ .  $\Sigma_{sh}^n$  is the normal short tree

fragment of  $\Sigma$ . One can extend the idea to finite stacks of normal trees, giving us the full short tree fragment of  $\Sigma$ , or  $\Sigma_{sh}$ .

~~doing this, one makes use of the fact that if  $b = \delta(\mathcal{U})$  where  $\mathcal{U}$  is normal and  $i_b^{\mathcal{U}}(\delta) = \delta(\mathcal{U})$ , then  $M_b^{\mathcal{U}}$  can be recovered from  $\mathcal{U}$  without knowing~~

In

(6)

~~What is it?~~ We do not give any further detail, because our only purpose is to explain that the restriction to hod pairs below LST- means that we never have to insert short-tree strategy information into our hod promise in any nontrivial way.

More precisely, for  $(\mathcal{N}, \Sigma)$  and  $\delta$  as in the last paragraph, let  $\Omega = \Sigma_{\mathcal{N}18} = \bigoplus_{\alpha < \delta} \Sigma_{\mathcal{N}(\alpha)}$  be the join of the lower level strategies. If  $L_p^\Omega(\mathcal{N}18)^\Gamma \models \delta$  is not Woodin, then we don't need to insert  $\Sigma_{sh}$  as we go on from  $\mathcal{N}18$ , because we can figure it out using  $\mathcal{Q}$ -structures from ~~the  $\mathcal{Q}$ -structures~~  $\Omega$ . If  $L_p^\Omega(\mathcal{N}18)^\Gamma \models \delta$  is Woodin, but for  $\kappa$  the least strong to  $\delta$ ,  $\kappa$  is not a limit of Woodins, then  $\Omega$  is really equivalent to

(7)

$\Sigma_{\mathcal{M}18}$ , where  $\gamma$  is the sup of the  
Woodin of  $\mathcal{M}$  below  $\kappa$ . In this case  
again,  $\Sigma_{sh}$  can be determined from  $\Omega$   
using  $\mathcal{Q}$ -structures. It is only in the  
remaining case, i.e. that  $(\mathcal{M}, \Sigma)$  is a  $\Gamma$ -LST-  
pair, that  $\Sigma_{sh}$  is truly new information, and  
must be inserted as we go on.

The problem here is that if  $L_p^{\Sigma_{sh}}(\mathcal{M}18) \neq \delta$   
 $\delta$  is not Woodin, we can't just start  
inserting the full  $\Sigma$  above  $\mathcal{M}18$ , because we  
might project across  $\delta$  in  $L_p^{\Sigma}(\mathcal{M}18)$ , and then  
fine structure might go bad. It's ok to  
project across  $\delta$  in the  $L_p^{\Sigma_{sh}}(\mathcal{M}18)$  hierarchy;  
coming down works out. So first we must  
climb the  $L_p^{\Sigma_{sh}}(\mathcal{M}18)$  hierarchy, and only  
if that does not project strictly across  $\delta$ ,  
then start inserting  $\Sigma$  for  $\mathcal{M}18$ . We'll have

to then prove that  $L_p^\Sigma(\mathcal{T}/\delta)$  does not project across  $\delta$ , if  $L_p^{\Sigma_s}(\mathcal{T}/\delta)$  does not.

However, inserting  $\Sigma_s$  without inserting  $\Sigma$  is delicate, because we must avoid non-short trees coming down to short ones. Saegsyon has a solution to this difficulty (see [8]), but it introduces many complexities we would like to avoid here.

This is why we restrict attention here to hod pairs below LST.

We are NOT claiming to have worked out the theory of hod mice beyond LST! There are many difficulties.

In fact, we shall begin with a more severe restriction.

Definition 0.4 A  $\Gamma$ -hod-pair  $(\mathcal{T}, \Sigma)$  is below  $O_n^p$  iff whenever  $\mathcal{T} \restriction K$  is a limit of Woodruff, and  $\mathcal{T} \restriction K$  is full, and  $K \in \mu$ , and  $\mu$  is the critical point of an extender on the  $\mathcal{T}$ -sequence, then  $O(K) \restriction \mathcal{T} \restriction \mu < \mu$ .



So one can have strong cardinals that are limits of  $\Gamma$ -full Woodins below  $O_h^{\text{IP}}$ , but not much more. The construction of hod pairs that is our focus here is simpler below  $O_h^{\text{IP}}$ , because below  $O_h^{\text{IP}}$  all levels of the construction will be sound, and so every level will be its own core.

In sections 1 and 2 we describe the first order structure of hod premice below  $LST^-$ . In sections 2-5, we prove theorem 0.2, with its hypothesis strengthened to: all  $T_0$ -hod-pairs are below  $O_h^{\text{IP}}$ , and its large cardinal hypothesis strengthened as well. In part II of this paper, we shall prove the full theorem 0.2.

9.1

Historical note We wrote these notes in the period May - October 2013. They went through many revisions, as the reader can easily see. We wish to thank Nam Trang for reading them, and pointing out a number of unclear passages and mistakes.

There are minor errors left of which we are aware, but it seems better to write up part II before attempting a polished version of the whole thing. This manuscript should be considered a preliminary draft.

Finally, the theory of hod mice below LST was first blocked out by Grigor Sargsyan and the author

in late 2008 and early 2009.

9.2

This paper owes a great deal to those discussions, as well as innumerable other ones between Sargsyan and the author. It also relies heavily on Sargsyan's papers [17] and [23].

§1. Hod premisses - informal description

We shall describe a hierarchy for hod premisses that can reach LST-pairs.

The hierarchy of [2], which deals with hod premisses below measurable limits of Woodin, has the property that a hod premouse  $M$  that has started inspecting an iteration strategy for  $M|_\delta$  will never later project strictly across  $\delta$ . Even below  $O_L^P$ , one must give that up. We shall maintain enough "Lp-bras", however, that such an  $M$  never-projects all the way down to the least  $\kappa$  that is strong to  $\delta$ .

Before giving the definition, let us describe the hierarchy of a hod premouse  $M$  informally.  $M$  will be a J-structure, constructed from predicates

extender sequence and iteration strategy information. The fine structural notions of [6] will apply literally.

(14)  
(96)

Def. 1 (a)  $O(K)^M$  is the strict sup of all  $\#$  indices  $lh(E)$  of extenders  $E$  on the  $M$ -sequence with  $crit(E) = K$ .

(b)  $\gamma$  is a cutpoint of  $M$  iff

$$\forall K < \gamma \quad O(K)^M \leq \gamma.$$

For hod mice in general, the cutpoints are more important than the Woodin cardinals when it comes to timing the insertion of strategy information. Indeed, it seems

~~We should ignore the Woodin cardinals.~~

We shall maintain <sup>some of</sup> the "extender-bias" of [2], but this might be better called "L<sub>p</sub>-bias" in the present situation. It means that if  $\mathcal{N}$  is our current level, and its strategy  $\Sigma$  has been

activated, then our hierarchy continues with  $L_p^\Sigma(\mathcal{N})$  before it does anything else.

(1)  
9c

Assume  $AD^+$ , and let  $\Gamma$  be a reasonably closed pointclass. For simplicity, let's assume  $\Gamma = P(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ , and  $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}}$ . We describe the hierarchy of a  $\Gamma$ -hod-pair  $(\mathcal{M}, \Sigma)$  in an informal way. Hod premice will have certain of the first order properties of such  $\mathcal{M}$ .

Certain levels  $\mathcal{N}$  of  $\mathcal{M}$  will have been designated by  $\mathcal{M}$  as "full". We might better call them "strategy-activation" levels. Once it passes a full level  $\mathcal{N}$ , the  $\mathcal{M}$ -hierarchy starts inserting iteration the iteration strategy for  $\mathcal{N}$  that is part of  $\Sigma$ . It never stops doing this, even if it later

projects across  $o(\mathcal{M})$ , as it may do.

(11)  
9/2

The hierarchy of  $\mathcal{M}$  is partitioned into ~~rows~~ intervals we call blocks.

Let  $\kappa \in \mathcal{M}$ ; then  $\kappa$  begins a block of  $\mathcal{M}$

iff there are arbitrarily large  $\gamma < \kappa$  such that  $\mathcal{M} \upharpoonright \gamma$  is full, and  $\gamma$  is

a cutpoint of  $\mathcal{M}$ . In this case,

$\mathcal{M} \upharpoonright \kappa$  is also full, by definition. Our

$\mathcal{L}_p$ -bias will also insure that  $\kappa$  is

a limit cardinal in  $\mathcal{M}$  if it begins

a block. The  $\kappa$ -block of  $\mathcal{M}$

will now extend until we reach the sup

limit of the first  $\omega$  full levels

past  $o(\kappa)^{\mathcal{M}}$ , at which point the

next block begins.

Within its  $\kappa$ -block, the hierarchy of  $\mathcal{M}$  looks as follows. ~~Then~~ Let

us write  $M(\eta)$  for the  $\eta^{\text{th}}$  full level of  $M$ , and  $\Sigma_\eta^0$  for the strategy for  $M(\eta)$  induced by  $\Sigma$ .

Let  $Z_\eta$  be the join<sup>(\*)</sup> of all  $\Sigma_\alpha^0$  for  $\alpha \leq \eta$ , and  $Z_{<\eta}$  be the join of all  $\Sigma_\alpha^0$  for  $\alpha < \eta$ . Now fix  $\eta$  such that  $M|_K = M(\eta)$ . (In the interesting cases,  $\eta = K$ .) Then the  $M$ -hierarchy proceeds from  $K$  by stacking  $\Sigma_\eta$ -mice with strategies in  $\Gamma$  that project to  $K$ , and the result of this is its next full level. That is

$$M(\eta+1) = \text{hp}^{Z_\eta} (M(\eta))^\Gamma,$$

where the superscript  $\Gamma$  indicates that we demand the collapsing mice we stack have  $\omega_1$ -iteration strategies in  $\Gamma$ .

(\*) This is not quite accurate;  $Z_\eta$  will have a little more information in the formal development. We call it the "closure strategy".



Remarks

(1)  $\Sigma_\eta^0$  will be a  $\Sigma_0$ -iteration strategy for  $\mathcal{M}(\eta)$ ; i.e. the embeddings will be all 0-embs., so continuous at  $o(\mathcal{M}(\eta))$ . Thus  $\Sigma_{\leftarrow \eta} = \Sigma_\eta$  in the present situation, where  $K$  is a limit of cutpoints of  $\mathcal{M}$ .

(2) We need  $\Sigma_\eta \in \Gamma$  to make sense of  $\mathcal{M}(\eta+1)$ . This is part of being a  $\Gamma$ -hod-pair. The full  $\Sigma$  may not be in  $\Gamma$ , but the induced strategies  $\Sigma_\eta$  must all be in  $\Gamma$ .

We shall then have

$$o(\mathcal{M}(\eta+1)) = (K^+)^{\mathcal{M}}$$

that is, the  $\mathcal{M}$ -hierarchy never projects across  $o(\mathcal{M}(\eta+1))$  as it goes on.

We now stack  $L_p$  one more time:

$$M(\eta+2) = L_p^{\Sigma_{\eta+1}}(M(\eta+1)).$$

Notice that if  $K$  is regular in  $M(\eta+1) = M(K^+)^m$ , then iterating  $M \circ K$  and  $M(K^+)^m$  are basically equivalent (modulo questions about how to iterate the extra collapsing mice, that  $L_p^{\Sigma_\eta}(M(\eta+1))^\Gamma$  can answer.)

So if  $K$  is regular,  $M(\eta+2)$  is equivalent to the double-stack

$$L_{p_2}^{\Sigma_\eta}(M(\eta))^\Gamma. \text{ But if } K \text{ is singular}$$

in  $M(\eta+1) = M(K^+)^m$ , then  $\Sigma_{\eta+1}^0$

will not be  $OD^{L(\Gamma, \mathbb{R})}$  from  $\Sigma_\eta$ .

Nevertheless, the new information will not

collapse  $(K^+)^m$  or add new subsets

of  $K$ . The argument for this is

in Sargsyan's thesis [2].

If  $\kappa$  is singular in  $\mathcal{M}(\eta+1)$ , then in fact

$$\mathcal{M}(\eta+i+1) = L_p^{\Sigma_{\eta+i}}(\mathcal{M}(\eta+i)) \uparrow$$

for all  $i < \omega$ , and these are the next  $\omega$  full levels of  $\mathcal{M}$ , and they are cutpoints, so the  $\kappa$ -block ends at  $\sup \{o(\mathcal{M}(\eta+i)) \mid i \in \omega\} = \kappa'$ , a limit cardinal of  $\mathcal{M}$  beginning the next block.

If  $\kappa$  is regular in  $\mathcal{M}(\eta+1)$ , then  $\gamma = o(\mathcal{M}(\eta+2))$  is eligible to be the index on the  $\mathcal{M}$ -sequence of an extender with critical point  $\kappa$ , the order  $\alpha$  total measure of  $\mathcal{M}$  on  $\kappa$ . If it is not, i.e.  $E_\gamma^\mathcal{M} = \emptyset$ , then the  $\mathcal{M}$  hierarchy proceeds as in the  $\kappa$  singular case. That is,

$M(\eta+i+1) = L_p^{\Sigma_{\eta+i+1}}(M(\eta+i))$  for all  $i < w$ , and this completes the  $K$ -block of  $M$ .

Suppose then that  $E_\gamma^m \neq \emptyset$ , where  $\gamma = o(M(\eta+2))$ . Then we set

$$M(\eta+3) = M|\gamma = (M(\eta+2), E),$$

where  $E = E_\gamma^m$ . That is, we declare that  $M|\gamma$  is full (i.e. its strategy is now to be activated). We then set

$$M(\eta+4) = L_p^{\Sigma_{\eta+3}}(M(\eta+3)),$$

and go on. That is,  $o(M(\eta+4))$  is eligible to be the index of an order one measure on  $K$ . If it is not, then the  $K$ -block ends with  $w$  iterations of  $L_p$ . If it is,

$M(\eta+5) = (M(\eta+4), E)$ , where  $E = E_\gamma^m$  for  $\gamma = o(M(\eta+4))$ .

We now describe how the block of  $\mathcal{M}$  at  $\kappa$  proceeds, in general. The main new thing is that as we ~~iterate~~ proceed upward, new functions with domain  $\kappa$  may show up, giving us new ways to iterate a full level that we have passed long ago.

Suppose

$$\alpha_0 = \sup \{ \beta + 1 \mid \beta < \alpha_0 \text{ and } \text{crit}(E_\beta^{\mathcal{M}}) = \kappa \}$$

is a "local  $o(\kappa)$ ", i.e.  $o(\kappa)^{\mathcal{M} \upharpoonright \alpha_0} = \alpha_0$ .

We describe what the  $\mathcal{M}$ -hierarchy looks like up to the next ~~index~~  $\beta$  such that  $\text{crit}(E_\beta^{\mathcal{M}}) = \kappa$ .

- Notation
- (i)  $\mathcal{M} \upharpoonright \alpha$  is the  $\alpha^{\text{th}}$  level of  $\mathcal{M}$ .
  - (ii)  ~~$\mathcal{M} \upharpoonright \alpha$~~   $\mathcal{N}^-$  is  $\mathcal{N}$ , but with its last extender predicate set  $= \emptyset$ . So  $\mathcal{N}$  is e-passive iff  $\mathcal{N} = \mathcal{N}^-$ .
  - (iii)  $\mathcal{M} \parallel \alpha = (\mathcal{M} \upharpoonright \alpha)^-$ .

Case 1.  $\alpha_0$  is a successor ordinal.

Let  $\alpha_0 = \beta + 1$ . We shall have an  $\eta$

such that

$$\begin{aligned}
 \mathcal{M}(\eta) &= (\mathcal{M} \upharpoonright \beta)^- \\
 &= \mathcal{M} \upharpoonright \beta \text{ with its last extender removed,}
 \end{aligned}$$

and

$$\mathcal{M}(\eta+1) = \mathcal{M} \upharpoonright \beta = (\mathcal{M}(\eta), \bar{E}_\beta^{\mathcal{M}}).$$

We have a "complete strategy"  $\Sigma_{\eta+1}$  for  $\mathcal{M}(\eta+1)$ .

The next full level of  $\mathcal{M}$  will be

$$\mathcal{M}(\eta+2) = L_p^{\Sigma_{\eta+1}}(\mathcal{M}(\eta+1))^\Gamma.$$

Near the bottom of the  $\kappa$ -block,  $\mathcal{M} \circ (\mathcal{M}(\eta+2))$  would be appropriate as the next index of an extender on the  $\mathcal{M}$ -sequence with critical point  $\kappa$ , and if it were not such an index, the  $\kappa$ -block of  $\mathcal{M}$  would end, ~~with~~  $\text{crit}(\kappa)^{\mathcal{M}} = \alpha_0$ , with  $\alpha_0$  as the next  $\mathcal{M}$ -critical.

But higher up, using functions in  $\mathcal{M}(\eta+2)$

may yield stronger ultrapowers of  $M(\eta+1)$   
 than one gets using only functions in  $M(\eta+1)$ .  
 So passing to  $M(\eta+2)$  gave us new ways  
 to iterate  $M(\eta+1)$ . We don't want to  
 add another extender with critical point  $\kappa$  to  
 $M$  until we have added ~~the~~ <sup>further information</sup> ~~the~~ about  
 how to iterate  $M(\eta+1)$ .

So we let's define  $\tau_\alpha$  for  $\alpha \geq 2$  by

$$\tau_\alpha = o(M(\eta+2)) = \beta,$$

$$\begin{aligned} \Lambda_\alpha &= \text{complete strategy of } M(\eta+2) \text{ from } \Sigma \\ &= \Sigma_{\eta+2}, \end{aligned}$$

If  $\tau_\alpha$  and  $\Lambda_\alpha$  are defined, then  
 $\tau_{\alpha+1}$  and  $\Lambda_{\alpha+1}$  are ~~not~~ defined iff

$$L_p^{\Lambda_\alpha}(M \parallel \tau_\alpha) \models |\tau_\alpha| = |\beta|.$$

In this case

$$\tau_{\alpha+1} = o(L_p^{\Lambda_\alpha}(M \parallel \tau_\alpha)),$$

and

$$\Lambda_{\alpha+1} = \text{complete strategy of } M \parallel \tau_{\alpha+1},$$

with

$$\mathcal{M} \parallel \tau_{\alpha+1} = L_p^{\Lambda_\alpha} (\mathcal{M} \parallel \tau_\alpha)^\Gamma.$$

If  $\tau_\alpha$  is defined for all  $\alpha < \lambda$ , we

let

$$\tau_\lambda = \sup_{\alpha < \lambda} \tau_\alpha,$$

$$\Lambda_\lambda = \text{complete strategy of } \mathcal{M} \parallel \tau_\lambda.$$

Remark It will work out that  $\Lambda_\lambda = \bigoplus_{\alpha < \lambda} \Lambda_\alpha$ .

Now let  $\alpha$  be largest such that  $\tau_\alpha$  is defined. Equivalently,  $\alpha$  is least such that  $L_p^{\Lambda_\alpha} (\mathcal{M} \parallel \tau_\alpha)^\Gamma \neq \tau_\alpha = \beta^+$ .

Then we set

$$\mathcal{M}(\eta+\xi) = \mathcal{M} \parallel \tau_\xi$$

$$\Sigma_{\eta+\xi} = \Lambda_\xi$$

for all  ~~$\alpha \geq 2$~~  ~~such that  $2 \leq \alpha \leq \alpha$~~

$\xi$  such that  $2 \leq \xi \leq \alpha$ . ~~(Note  $\alpha \geq 2$ )~~

(Note  $\alpha \geq 2$ .) We shall have that



$M \upharpoonright T_\xi$  is  $\epsilon$ -passive, for all  $\xi$  such that  $\alpha \leq \xi < \alpha$ ; that is,

$M \upharpoonright T_\xi = M \upharpoonright T_\alpha$  for  $\alpha \leq \xi < \alpha$ . Our potential next index for an extender with critical point  $\kappa$  is

$$\gamma = \tau_\alpha = (\beta^+)^{L_p^{\lambda_\alpha}(M \upharpoonright T_\alpha)}$$

If  $E_\gamma^m \neq \emptyset$ , then we shall have  $\text{crit}(E_\gamma^m) = \kappa$ . (Otherwise our coherence conditions will give  $\gamma$  a limit of indices  $\delta$  such that  $\text{crit}(E_\delta^m) = \kappa$ .) Thus we have done what we set out to do, i.e. reach the first index  $\geq \alpha_0$  of an extender on  $M$  with  $\text{crit} = \kappa$ . If  $E_\gamma^m = \emptyset$ , then there are no such indices, i.e.  $\circ(\kappa)^m = \alpha_0$ . In this case, the  $\kappa$ -block ends with  $\omega$  iterations of  $L_p$ ,

as before:  $M(\eta + \alpha + 2) = M \upharpoonright \delta$ , and

(19)

$$M(\eta + \alpha + i + 1) = L_{\rho}^{\Sigma_{\eta + \alpha + i}} (M(\eta + \alpha + i))^{\Gamma}$$

for  $1 \leq i < \omega$ , and these are all  $e$ -passive initial segments of  $M$ , with  $\text{co}(M(\eta + \alpha + i))$  a cardinal of  $M$  for  $1 \leq i < \omega$ .

Case 2  $\alpha_0$  is a limit ordinal.

In this case,  $(M \upharpoonright \alpha_0)^-$  is a limit of full levels, so we declare it full.

Let  $M(\eta) = (M \upharpoonright \alpha_0)^-$ .

Subcase 2a  $E_{d_0}^m \neq \emptyset$

20

Our indexing is such that we must have  $\text{crit}(E_{d_0}^m) > K$  in this case. (See immediately below.) Thus this case does not occur below  $O_h^{\text{IP}}$ . If we are beyond  $O_h^{\text{IP}}$ , we could have, for  $E = E_{d_0}^m$  and  $\text{crit}(E) > K$ :

$$\mathcal{M}(\gamma_{+1}) = (\mathcal{M}(\gamma), E).$$

We then proceed exactly as in case I. We never actually used  $\text{crit}(E) = K$  in the description of the  $\mathcal{M}(\gamma_{+i})$ 's.

Subcase 2b  $E_{d_0}^m = \emptyset$ .

We proceed exactly as in cases 1 and 2a, but we start from  $\mathcal{M}(\gamma) = \mathcal{M}|_{d_0} = \mathcal{M}|_{d_0^-}$ , rather than the active  $\mathcal{M}(\gamma_{+1})$ 's of those cases.

20a

Remark Subcase 2b is where our  
below - LST - hypothesis applies.

We are assuming that  $M(\gamma+1) \neq \alpha_0$  is  
not Woodin.

This completes our informal description of the  $\kappa$ -block of  $\mathcal{M}$ , ~~and thus of the~~ and thus of the first order structure of  $\mathcal{M}$ . The main remaining points to clarify are

(1) How do we index extenders on the  $\mathcal{M}$ -sequence? The answer here depends on whether  $\text{crit}(E)$  begins a block of  $\mathcal{M}$ :

(a) If  $E$  is on the  $\mathcal{M}$ -sequence, and  $\text{crit}(E)$  does not begin a block of  $\mathcal{M}$ , then  $E$  must fit according to the rules of [4], as repaired in [7].

(b) If  $\kappa = \text{crit}(E)$  begins a block of  $\mathcal{M}$ , then our indexing convention is a bit closer to Jensen's. Namely, let  $\eta$  be the least cutpoint of  $\text{Ult}_0(\mathcal{M}, E)$  such that  $(\kappa^+)^{\mathcal{M}} \leq \eta < i_E(\kappa)$ . Since  $\kappa$  is a limit of cutpoints of  $\mathcal{M}$ ,  $\eta$  exists.

We then index  $E$  at  $(\eta^+)^{U_{K_0}(\mathcal{M}, E)}$ . (22)

It is possible that ~~the~~ the ms-index  $(\nu(E)^+)^{U_{K_0}(\mathcal{M}, E)}$  is strictly below

$(\eta^+)^{U_{K_0}(\mathcal{M}, E)}$ , though this does not happen near the bottom of the  $K$ -block.

Of course, the Jensen index  $(i_E(K)^+)^{U_{K_0}(\mathcal{M}, E)}$  is strictly larger. We shall explain why  $(\eta^+)^{U_{K_0}(\mathcal{M}, E)}$  works best when we get to the construction of hod pairs.

Note that "begins a block" is locally definable over  $\mathcal{M}$ , so there is no problem amalgamating the two conventions.

(2) What is the iteration game for which  $\Sigma_\eta^0$  is a winning strategy?

Roughly, the usual  $G_0(M(\eta), w_1, w_2)$ .  
 $G_0(M(\eta), w_1, w_2)$ .

(3) What is a "complete" strategy for  $M(\eta)$ ?

Roughly:  $\Sigma_\eta^0$ , together with ~~other~~ complete strategies for all  $\rho$  a proper full initial segment of a  $\Sigma_\eta^0$ -iterate of  $M(\eta)$ .

(4) When do we tell  $M$  the value of  $\Sigma_\eta(\vec{I})$ ? Roughly: we tell  $M$

the value of  $\Sigma_\eta(\vec{I})$  for  $(\eta, \vec{I})$  least such that we have not yet done so, in the usual amenable fashion.

Before we go on to the formal definitions, here are a few more remarks.

Remarks

(1) The  $\kappa$ -block of  $\mathcal{M}$  may not end.

In this case, we call  $(\kappa, o(\mathcal{M}))$  the top block of  $\mathcal{M}$ . It is a first-order property of  $\mathcal{M}|_2$  that its  $\kappa$ -block has ended at 2, uniformly in  $\kappa, 2, \mathcal{M}$ . (This is because we have in its language of  $\mathcal{M}$  a symbol indicating the full levels.)

(2) Let  $E = E_\gamma^m \neq \emptyset$ , where  $\kappa < \gamma$  and  $\gamma$  is in the  $\kappa$ -block of  $\mathcal{M}$ . Thus  $\kappa \leq \text{crit}(E)$ .

Def 3.  $E$  is  $\kappa$ -relevant iff

$\text{crit}(E) = \kappa$ , or  $\mathcal{M}|_{lh(E)} \models o(\kappa) \geq \text{crit}(E)$ . (Equivalently,  $o(\kappa) \geq lh E$ .)



(a) If  $M(\gamma+1) \in L_p^{\Sigma_\eta}(M(\gamma))^\Gamma$ , then all extenders on the sequence of ~~the sequence~~  $M(\gamma+1)$  with index  $\geq o(M(\gamma+1))$  are  $\kappa$ -irrelevant.

Here we assume ~~the sequence~~  $o(M(\gamma+1))$  is in the  $\kappa$ -block.

This is because the extenders stacked in  $L_p^{\Sigma_\eta}(M)$  all have crits  $\geq o(M)$ , so  $o(M)$  is a cutpoint of  $L_p^{\Sigma_\eta}(M)$ .

In particular, there are no partial measures with critical point  $\kappa$  on the  $M$ -sequence, the way we are setting things up. They are all reached in  $L_p^{\Sigma_\eta}(M)^\Gamma$ , in a basically because some  $OD(P, \mu)$  for  $M \in P$  is  $OD(\Sigma_\kappa)^\Gamma$ .

(b) If  $EA(M) \in E$  is indexed in the  $\kappa$  block of  $M$  and  $EA \in E$  is total over  $M$ , then  $E$  is  $\kappa$ -relevant. In particular, if  $crit(E) = \kappa$ , then  $E$  is  $\kappa$ -relevant.

(c) There can be partial extenders that are  $\kappa$ -relevant indexed in the  $\kappa$ -block of  $\mathcal{M}$ . Indeed, if  $E$  is not in an  $L_p^\Sigma$ -interval as in (a), then  $E$  is  $\kappa$ -relevant.

(d) If  $\mathcal{M}$  is below  $O_h^{\text{IP}}$ , then the  $\kappa$ -relevant extenders are just those with  $\text{crit} \leq \kappa$ . (Here  $\kappa$  begins a block of  $\mathcal{M}$ .) The active last extender of  $O_h^{\text{IP}}$  is  $\kappa$ -relevant, where  $\kappa$  begins the top block of  $O_h^{\text{IP}}$ .

The  $\kappa$ -relevant extenders are just the ones we might use if we are iterating  $\mathcal{M}$  above  $\kappa$ , trying to produce an extender  $E$  with  $\text{crit}(E) = \kappa$  that can be used applied to  $\mathcal{M}$ . Partial extenders with  $\text{crit} > \kappa$  can be used to do this, but not the ones we stack in  $L_p^{\Sigma_0}(\mathcal{M}(\gamma))$ .

(3) Our hierarchy might be described as " $L_p^\Sigma$ -biased". Given we have reached a full level  $\mathcal{N}$  of the  $\mathcal{M}$ -hierarchy in  $\mathcal{M}$  its  $\kappa$ -block,

$M$  goes on with  $L_p^\Sigma(\mathcal{N})$  for as long as it can. When it can insert no further ~~bounded~~ subsets of  $\mathfrak{o}(\mathcal{N})$  this way, it either adds a next  $\kappa$ -relevant extender, or it ends the  $\kappa$ -block.

(4) The insertion of strategy information to  $M$  is not timed to its Woodin cardinals. So even below  $AD_R + \Theta$  regular, those hod- $\mathcal{P}$  are not hod- $\mathcal{P}$  in the sense of [2]. They are however inter-translatable with hod- $\mathcal{P}$  in the sense of [2], in that region.

For example, let  $\mathcal{P}$  be a hod- $\mathcal{P}$  in the sense of [2], with two Woodin cardinals  $\delta_0$  and  $\delta_1$ .  $\mathcal{P}$  gets re-organized into a hod- $\mathcal{P}$   $M$  as above in the following way:

(1)  $\mathcal{M}$  and  $\mathcal{P}$  have the same universe,  
the same cardinals, the same total  
extenders on their sequences.

(2) The blocks of  $\mathcal{M}$  are the intervals  
 $(\kappa, \lambda)$  where  $\kappa$  and  $\lambda$  are limits  
of cardinals and of cutpoints in  $\mathcal{P}$ .

For example, the first block of  $\mathcal{M}$  is  
 $(\tau_w, \kappa_{w+w})^{\mathcal{P}}$ . We have full levels  
 $\mathcal{M}(\eta)$  whose strategy is being inserted here,  
but this never happens unless  $\mathcal{M}(\eta)$  has  
already reached all its  $\mathbb{Q}$ -structures necessary  
to identify its strategy  $\Sigma_\eta$ . So  $\mathcal{P}$  can  
figure out locally what (i.e.  $\mathcal{P} \upharpoonright \tau_w^{\mathcal{P}}$  can  
figure out) what  $\mathcal{M}$  is doing.

The first place nontrivial strategy  
information goes into  $\mathcal{M}$  is in the  $\kappa_0$ -block,

where

27

$K_0 =$  least  $K$  that is  $< \delta_0$ -strong  
in  $\mathcal{P}$ .

We have  $\mathcal{M}(\delta_0) = \mathcal{M} \upharpoonright \delta_0 = \mathcal{P} \upharpoonright \delta_0$   
and  $\Sigma_{\delta_0}^0$  is "trivial", i.e. not really  
new information,  $\mathcal{M}(\delta_{0+1}) = L_{\mathcal{P}}^{\Sigma_{\delta_0}^0}(\mathcal{M} \upharpoonright \delta_0)$   
is called  $\mathcal{P}(\delta_0)$  in [2], and its  
strategy  $\Sigma_{\delta_{0+1}}^0$  is what  $\mathcal{P}$  starts inserting,  
and what we insert here, ~~as~~ starting at the  
same place. The  $K_0$ -block of  $\mathcal{M}$   
now ends at the  $\omega^{\text{th}}$  cardinal of  $\mathcal{P}$   
above  $\delta_0$ .

## §2. Mod premisses, Formal Description

Formally, mod premisses are acceptable  $\mathcal{J}$ -structures  $\langle \mathcal{J}_\alpha^A, \mathcal{B} \rangle$  in the sense of 1.8 and 1.20 of [5]. It will be convenient to decompose the two acceptable predicates. The decomposition makes our mod premisses also models with parameter 0, in the sense of [6].

Thus the language of mod premisses has  $\epsilon$ , unary predicate symbols  $\dot{E}, \dot{B}, \dot{S}$ , and constant symbols  $\dot{l}$  and  $\dot{f}$ . A structure

$$\mathcal{M} = (\mathcal{M}, \epsilon, \dot{E}^m, \dot{B}^m, \dot{S}^m, \dot{l}^m, \dot{f}^m)$$

is a mod premiss just in case  $\mathcal{M}$  is transitive,  $\epsilon$ -closed, and the conditions (1) - (4) <sup>and 2.1</sup> below are met.

Condition (1) determines the meanings of  $\dot{S}$  and  $\dot{l}$ .

(1)  $\mathcal{M}$  is a model with  $\aleph_0$  parameters in the sense of [6]. That is,  $\dot{J}^{\mathcal{M}}$  is the sequence of levels of  $\mathcal{M}$ , and we write  $l(\mathcal{M})$  for  $\text{dom}(\dot{J}^{\mathcal{M}})$ . We set

$$\mathcal{M} \upharpoonright \alpha = \dot{J}^{\mathcal{M}}(\alpha).$$

We ~~also~~ demand the base model  $\dot{J}^{\mathcal{M}}(0)$  be  $(\forall w, \epsilon, \phi, \phi, \phi, 0, 0)$ . We set

$$\mathcal{M} \upharpoonright l(\mathcal{M}) = \mathcal{M}.$$

We have  $\dot{J}^{\mathcal{M}} = l(\mathcal{M}) - 1$  if this exists, and  $\dot{J}^{\mathcal{M}} = 0$  otherwise.

~~Condition (2) tells us how we index extenders.~~

~~We use  $\dot{J}^{\mathcal{M}} \in \omega$  to mean " $\mathcal{M}$  is full".~~

~~(2) Either  $\dot{E}^{\mathcal{M}} = \emptyset$ , or  $\dot{J}^{\mathcal{M}} \upharpoonright \beta \uparrow \dot{B}^{\mathcal{M}} = \emptyset$~~

~~and  $\dot{E}^{\mathcal{M}} = \dot{E}$  is an extender over  $\mathcal{M}$  satisfying the conditions of [4]. That~~

~~is~~

Condition (2) tells us how we index extenders. We use  $\dot{f}^M < \omega$  to mean that  $M$  is full.

(2) Either  $\dot{E}^M = \emptyset$ , or  $\dot{f}^M = 0$ ,  $\dot{B}^M = \emptyset$ , and  $\dot{E}^M = E$  is a  $(\kappa, \lambda)$ -extender over  $M$  with  $\text{lh}(E) = \lambda = o(M)$ , and

(a) If  $\kappa$  does not begin a block of  $M$ , then the conditions of  $\Sigma 4J$  are met, i.e.

(i)  $M \models \kappa^+$  exists,

(ii) (coherence)  $\text{Ult}_0(M, E) \upharpoonright \mathcal{L}(M) = (M, \epsilon, \emptyset, \emptyset, \dot{J}^M, 0, 0)$ ,

(iii)  $\mathcal{L}(M) = \nu(E)^+ \text{Ult}_0(M, E)$ ,

(iv)  $E$  is not type  $\mathcal{E}$ ,

(v)  $E$  satisfies the initial segment condition of  $\Sigma 4J$ , as repaired in  $\Sigma 7J$ .

(b) If  $\kappa$  begins a block of  $M$ , then

(i)  $M \models \kappa^+$  exists,

(ii) (coherence)  $\text{Ult}_0(M, E) \upharpoonright \mathcal{L}(M) = (M, \epsilon, \emptyset, \emptyset, \dot{J}^M, 0, 0)$ , and

(iii)  $\mathcal{L}(M) = (\eta^+)^{\text{Ult}_0(M, E)}$ , where  $\eta$  is the least cutpoint of  $\text{Ult}_0(M, E)$  s.t.  $\kappa \leq \eta \leq i(\kappa)$ .



Remark Notice there is no initial segment condition in 2(b). As in Jensen indexing, we don't need one because if  $E$  and  $\mathcal{M}$  are as in 2(b), then for no  $\xi < lh(E)$  is  $(\mathcal{M} \upharpoonright \xi, E \upharpoonright \xi)$  as in (2). That is, the literally-stated initial segment condition would be vacuous.

It will still be true that the "trivial completions" of various  $E \upharpoonright \xi$  are on the  $\mathcal{M}$ -sequence, and hence those  $E \upharpoonright \xi \in \mathcal{M}$ . But we don't need an axiom keeping track of which ones. For Comparison-termination, it is enough that no  $E \upharpoonright \xi$  is below on the  $\mathcal{M}$ -sequence.

Note also that if we use  $E$  ~~then~~ is a comparison, then  $Ult(\mathcal{M}, E) \upharpoonright lh(E)$  is a cutpoint, so on the  $\mathcal{M}$ -side, we'll never have another critical point  $\leq lh(E)$ .

Notation If  $\dot{E}^{\mathcal{M}} \neq \emptyset$ , we say  $\mathcal{M}$  is e-active, and otherwise, we say  $\mathcal{M}$  is e-passive. The extender sequence of  $\mathcal{M}$  is given by  $E_{\alpha}^{\mathcal{M}} = \dot{E}^{\mathcal{M}} \upharpoonright \alpha$ , for  $\alpha \leq l(\mathcal{M})$ .

If  $\mathcal{M}$  is e-active,  
then

(31)

$$\mathcal{M}^- = (\mathcal{M}, e, \emptyset, \emptyset, \dot{S}^{\mathcal{M}}, 0, 0)$$

is the associated e-passive structure. We write  $\mathcal{M}^- = \mathcal{M}$  if  $\mathcal{M}$  is already e-passive.

Let also

$$\mathcal{M} \parallel \gamma = (\mathcal{M} \parallel \gamma)^-$$

The levels of  $\mathcal{M}$  are just the structures  $\mathcal{M} \parallel \gamma$  and  $\mathcal{M} \parallel \gamma$ . The full levels are just those  $\mathcal{N}$  such that  $f^{\dot{\mathcal{N}}} \in \omega$ . (Otherwise,  $f^{\dot{\mathcal{N}}} = \omega$ .) We shall use  $f^{\dot{\mathcal{N}}} = k < \omega$  as a signal that the ~~strategy~~ on iteration strategy for the game  $G_k^*(\mathcal{N}, \omega, \omega_1)$  should be inserted,

Let  $k < \omega$  and let  $\mathcal{N}$  be  $k$ -sound, with  $\mathcal{N}_0$  its  $\Sigma_k$ -master-code structure.  $G_k^*(\mathcal{N}, \omega, \omega_1)$  is the following variant of the usual  $(0, \omega_1, \omega_1)$ -iteration game on  $\mathcal{N}_0$ :

(4) In round  $\alpha$ , I and II play  $G_0(N_\alpha, w_1)$ , where if  $\alpha > 0$ ,  $N_\alpha$  is given by the previous rounds. So the tree  $T_\alpha$  being played on  $N_\alpha$  is normal.

(b) I is not allowed to exit round  $\alpha$  at stage  $\beta$  if  $\sum_{0, \beta} J_{T_\alpha}$  drops, ~~in~~ Thus each  $N_\alpha$  is a ~~master~~ <sup>$Z_k$</sup> -code structure, and we have a  $k$ -embedding from  $\mathcal{N}$  to the decoding of  $N_\alpha$ .

Remark We do allow drops within the  $T_\alpha$ 's, but not on their main branches. Allowing drops off the main branches is needed when we use the strategies to form hod-limits. Dis-allowing drops on the main branches seems necessary when it comes to comparing strategies.

Def Let  $\mathcal{N}$  be  $k$ -sound, and <sup>Suppose  $\rho_k(\mathcal{N}) > k$ .</sup> have a top block that begins with  $k$ . The

top-block-normal game  $G_k^{top}(\mathcal{N}, w_1)$

is the usual length  $w_1$  normal iteration game on the  $\Sigma_k$  master code of  $\mathcal{N}$ , where  $I$  is restricted to playing extenders with critical point  $> k$ . A  $(k, w_1)$ -top-block-normal

strategy is a winning strategy for  $II$  in  $G_k^{top}(\mathcal{N}, w_1)$ .

In general, it ~~seems~~ <sup>might be (we're not sure)</sup> necessary to insert the top-block-normal strategy for  $\mathcal{N}$  (and use it to come down) before inserting a full strategy for  $G_k(\mathcal{N}, w_1, w_1)$ .

But notice that below  $O_k^{\#}$ , there are no extenders  $I$  can play in  $G_k^{top}(\mathcal{N}, w_1)$ .

so the top-block-normal strategy is trivial. Indeed,  $G_R^{top}(\mathcal{N}, w_1)$  is trivial for a ways past  $O_h^P$ . So we shall ignore the top-block-normal strategy here.

The restrictions of  $G_R^*(\mathcal{M}, w_1, w_1)$  are such that if  $\Sigma$  is a winning strategy for  $\Pi$ , and  $\mathcal{N}$  is a full level of  $\mathcal{M}$ , then  $\Sigma$  may not determine a strategy for  $\Pi$  in  $G_{fn}^*(\mathcal{N}, w_1, w_1)$ . This can happen if some level of  $\mathcal{M}$  projects across  $O(\mathcal{N})$ . In this case we will have

$$(K^+)^M \in \rho_w(M1\gamma) < O(\mathcal{N}) < \gamma$$

where  $K$  begins the top block of  $\mathcal{N}$ . The strategy for  $\mathcal{N}$  lets us stack normal trees on the window  $[\rho, O(\mathcal{N})]$ , whereas the strategy for  $M1\gamma$  does not.

We remedy this by defining the notion of a "complete" iteration strategy.

Def, 2.0 Let  $M$  be a full hod premouse,  
and  $k = \dot{f}^M$ . A complete strategy for  $M$   
is a pair  ~~$(\Omega, F)$~~   $(\Omega, F)$  such that

(a)  $\Omega$  is a winning strategy for  $\Pi$  in  
 $G_k^*(M, w, w, )$ , and

(b) whenever  $\vec{Q}$  is a stack by  $\Omega$  with  
last model  $P$ , and  $\mathcal{N}$  is a full level  
~~proper~~ of  $P$  such that  $\mathcal{N} \neq P$ , then  
 $F(\vec{Q}, \mathcal{N})$  is a complete strategy for  $\mathcal{N}$ .

Remark It is a tacit assumption about  $(\Omega, F)$   
in 2.0 that the obvious ~~relation~~ "mouse descent"  
relation generated by  $(\Omega, F)$  is wellfounded.  
This justifies defining "complete strategy" in  
terms of itself.

Notice that in (b), if  $M$ -to- $P$   
does not drop in model or degree, then  $(\Omega, F)$

also determines a complete strategy for  $\mathcal{P}$ . The difference between a complete strategy  $(\Omega, F)$  for  $\mathcal{M}$  and a strategy  $\Sigma$  for the game  $G_k^+(M, w_1, w_1)$ , ~~an~~ the variant of  $G_k^*(M, w_1, w_1)$  where I can exit a round after dropping, ~~is that~~ occurs in (b) when  $M$ -to- $\mathcal{P}$  drops. Then  $(\Omega, F)$  does not tell us how to reward  $\mathcal{P}$ , whereas  $\Sigma$  does tell us that. Remark This distinction between  $(\Omega, F)$  and  $\Sigma$  is not relevant below  $\mathcal{O}_h^P$ .

Let  $\Sigma = (\Omega, F)$  be a complete strategy for  $\mathcal{M}$ , and let  $\vec{\mathcal{I}}$  be a stack on  $\mathcal{M}$  with last model  $\mathcal{P}$ . We write

$$\Sigma_{\vec{\mathcal{I}}, \mathcal{N}} = F(\vec{\mathcal{I}}, \mathcal{N})$$

if  $\mathcal{N} \neq \mathcal{P}$  is a full level of  $\mathcal{P}$ . We also write  $\Sigma_{\vec{\mathcal{I}}, \mathcal{P}}$  for the tail of  $\Omega$ , if  $M$ -to- $\mathcal{P}$  does not drop. These notations follow that of  $[\Sigma \vec{\mathcal{I}}]$ , of course. Let us write  $\Sigma_{\mathcal{N}}$  for

$\Sigma_{\emptyset, \mathcal{M}}$ , when  $\mathcal{N}$  is a full level of  $\mathcal{M}$ .

Let  $\Sigma$  be a complete iteration strategy for  $\mathcal{M}$ , and let  $\mathcal{N}$  be a full level of  $\mathcal{M}$ .

Let  $k = f^{\dot{\mathcal{N}}}$  and  $i = f^{\dot{\mathcal{M}}}$ . Suppose the  $\Sigma_k$  master code of  $\mathcal{N}$  has universe that is an initial segment of the  $\Sigma_i$  master code of  $\mathcal{M}$ .

(That is, between  $\mathcal{N}$  and  $\mathcal{M}$  we never projected

$\prec \rho_k(\mathcal{N})$ .) Then plays of  $G_k^*(\mathcal{N}, w_1, w_2)$  can be regarded as plays of  $G_i^*(\mathcal{M}, w_1, w_2)$ , by thinking of the trees as being on  $\mathcal{M}$  to start.

Let  $\Sigma = (\Omega, F)$ . We say  $\Sigma$  is self-consistent iff whenever  $\mathcal{N}$  is as

above, and  $\Sigma_{\mathcal{N}} = (\Psi, G)$ , then

(a)  $\Psi \subseteq \Omega$ ,

and in fact

(b)  $\Sigma_{\vec{\mathcal{A}}, \rho} \cong (\Sigma_{\mathcal{N}})_{\vec{\mathcal{A}}, \rho}$ , whenever

$(\Sigma_{\mathcal{N}})_{\vec{\mathcal{A}}, \rho}$  is defined.



(34d)

Def. 2.0.1 (Strategy coherence) Let  $\Sigma$  be a complete strategy for  $\mathcal{M}$ . We say  $\Sigma$  is coherent iff

- (a)  $\Sigma$  is self-consistent, and
- (b) whenever  $\Sigma_{\vec{I}, P}$  and  $(\Sigma_{\vec{I}, P})_{\vec{u}, Q}$  are defined, and  $\mathcal{N}$  is a full level of both  $P$  and  $Q$ , then  $\Sigma_{\vec{I}, \mathcal{N}} = (\Sigma_{\vec{I}, P})_{\vec{u}, \mathcal{N}}$ .

Self-consistency is an obvious requirement. Part (b) of strategy coherence holds for the hierarchy of  $\Sigma \mathbb{Z}$  by pullback consistency. This is because there, if  $i_{\vec{u}}: P \rightarrow R$  and  $\mathcal{N}$  is a full level of both  $P$  and  $R$ , then  $\text{crit } i_{\vec{u}} \geq o(\mathcal{N})$  because  $\mathcal{N}$  is full and no extenders overlap full levels. Since  $\Sigma_{\vec{I}, \vec{u}, R}$  pulls back by  $i_{\vec{u}}$  to  $\Sigma_{\vec{I}, P}$ , we get (b).

But once we have extenders overlapping full

levels, part (b) of strategy coherence seems to go beyond pullback consistency.

(34e)

Def. 2.0.2 A good iteration strategy for a full hod premouse  $\mathcal{M}$  is a complete, coherent strategy for  $\mathcal{M}$  that has hull condensation.

For hull condensation, see [2]. [2] shows that hull condensation implies pullback consistency.

All iteration strategies we consider for hod premice will be good, so we may sometimes forget to say "good".

Hod premice are required to have an opinion as to their own good iteration strategy. For  $\pi$  a full level of  $\mathcal{M}$ ,  $\Sigma_{\pi}^{\mathcal{M}}$  is the information about the "external  $\Sigma_{\pi}$ " that has been

coded into  $\dot{B}^m$  in the way we are about to describe.

(2) Let  $M$  be a hod premouse, and  $\alpha < \ell(M)$  such that  $M \upharpoonright \alpha \models KP$ , ~~and there is a length~~ and there are boundedly many  $\beta < \alpha$  such that  $M \upharpoonright \beta \models KP$ . Let  $\langle n, \vec{a} \rangle$  be least in  $\dot{C}^m$  such that  $n$  is full,  $\vec{a}$  is a play by  $\Sigma_n^{M \upharpoonright \alpha}$  with  $\#$  to move, and  $\Sigma_n^{M \upharpoonright \alpha}(\vec{a})$  is undefined. Suppose  $\gamma = \text{lh}(U)$ , where  $U$  is the last component of  $\vec{a}$ , and  $\alpha + \gamma \leq \ell(M)$ .

Then we require that

$$\dot{B}^{M \upharpoonright \alpha + \gamma} = \{ \alpha + \xi \mid \xi \in b \}$$

for some cofinal branch  $b$  of  $U$ ,

and write

$$\Sigma_n^{M \upharpoonright \alpha + \gamma}(\vec{a}) = b, \quad \text{for all } \alpha + \gamma \leq \ell(M).$$

Finally, we have first order conditions that pin down which levels  $\mathcal{M}$  of  $\mathcal{M}$  can be declared full by  $\mathcal{M}$ , and what  $f^{\mathcal{M}}$  can be. These were discussed in the previous section, so we omit further detail for now.

(3) Restrictions on fullness declarations, details later.

(4) If  $\mathcal{M}_{\gamma+1}$  is not covered by (1)-(3), then  $\mathcal{M}_{\gamma+1} = (\text{Rad}(\mathcal{M}_{\gamma}), \epsilon, \emptyset, \emptyset, \mathcal{I}, \mathcal{L}, 0)$ , for  $\mathcal{I}$  and  $\mathcal{L}$  describing  $\langle \mathcal{M}_{\alpha} \mid \alpha \leq \gamma \rangle$ .

It is easy to see that we can think of the structures as in (1)-(4) as  $\mathcal{T}$ -structures, in the sense of [5]. Thus projecta, standard parameters, and cones/soundness make sense.

Def 2.1 A hod-promouse is a structure  $\mathcal{M}$  satisfying (1)-(4) above, and such that each proper initial segment of  $\mathcal{M}$  is  $\omega$ -sound.

We let

$$\mathcal{M}(\alpha) = \alpha^{\text{th}} \text{ full level of } \mathcal{M},$$

and  $\lambda^{\mathcal{M}}$  be the order-type of the full levels of  $\mathcal{M}$ . Set  $\mu_\alpha = \text{ORD} \cap \mathcal{M}(\alpha)$ ,

so that either  $\mathcal{M}(\alpha) = \mathcal{M} / \mu_\alpha$  or

$$\mathcal{M}(\alpha) = (\mathcal{M} / \mu_\alpha)^- .$$

This notation is parallel

to Sargsyan [2].

The truly interesting object is the  
hod pair.

38

Definition 2.2  $(M, \Sigma)$  is a hod pair

just in case

- (1)  $M$  is a full hod premouse, and  $k$ -sound, where  $k = f^{*M}$ ,
- (2)  $\Sigma$  is a good iteration strategy for  $M$ ,  
and
- (3) if  $\Sigma_{\vec{J}, P}$  exists, and  $N$  is a full  
level of  $P$ , then  $(\Sigma_N)^P \subseteq \Sigma_{\vec{J}, N}$ .

Clause (3) says that the strategy information  
coded into the  $\dot{B}$ -predicate of an iterate  $P$   
is consistent with  $\Sigma$ .

This definition is meant to be used in  
the  $AD^+$  context, where  $\omega_1$ -iterability

implies  $\omega, \tau$  - iterability. In that (39) (40) context, we can define fullness preservation.

Def 2.3 Let  $(M, \Sigma)$  be a hod pair, and  $\Gamma$  a pointclass. We say  $\Sigma$  is  $\Gamma$ -fullness preserving iff whenever  $P$  is a  $\Sigma$ -iterate of  $M$ , and  $R$  is a full level of  $P$  that is not of the form  $Q^-$  for any  $\bar{e}$ -active level  $Q$ , and  $R$  is the next full level of  $P$  after  $\eta$ , then ~~with~~ letting  $\Omega$  be the  $(k, w, w_1)$  strategy for  $\mathcal{M}$  induced by  $\Sigma$  (where  $k = f^{\eta}$ ), then either

(a)  $R = \text{first level of } L_p^{\Omega}(\mathcal{M})^{\Gamma}$  with  $\text{projctum} < \rho_k(\mathcal{M})$ , and  $f^{\circ R} \equiv$  at least  $\delta$  str.  $\rho_0(\mathcal{M}) < \rho_k(\mathcal{M})$ , or

(b)  $R = L_p^{\Omega}(\mathcal{M})^{\Gamma}$ , if no level as in (a) exists.

We are requiring as part of 2.3 that if  $\Sigma$  is  $\Gamma$ -fullness preserving, then all "lower-level-strategies"  $\Sigma_n$  induced on ~~some~~ full levels of iterates by  $\Sigma$  are such that  $\Sigma_n \in \Gamma$ . However,  $\Sigma$  itself may not be in  $\Gamma$ .

Def. 2.3.1 A  $\Gamma$ -hod pair is a hod-pair  $(M, \Sigma)$  such that  $\Sigma$  is  $\Gamma$ -fullness preserving.

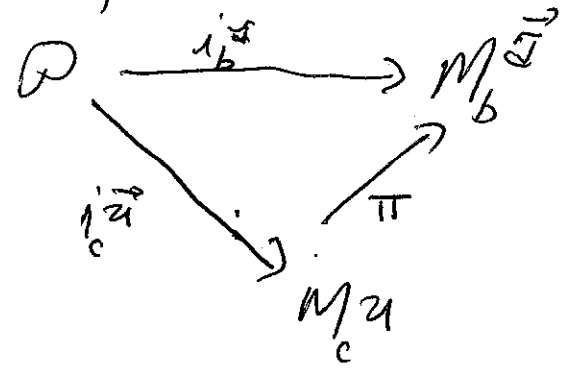


Some of the central arguments of [2] do not <sup>fully</sup> require that we are below  $AD_{\mathbb{R}} + \theta$  measurable. Let us record one ~~extra~~ thing.

~~2.3.1~~ 2.3.2

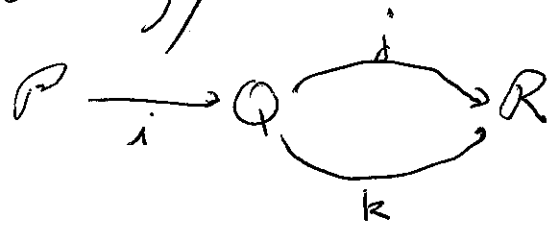
Definition (a) Let  $(P, \Sigma)$  be a hod pair. We say that  $\Sigma$  is positional iff whenever  $\vec{\mathcal{I}}$  and  $\vec{\mathcal{U}}$  are non-dropping stacks by  $\Sigma$  with common last model  $Q$ , then  $\Sigma_{Q, \vec{\mathcal{I}}} = \Sigma_{Q, \vec{\mathcal{U}}}$ .

(b)  $\Sigma$  has branch condensation iff whenever  $\vec{\mathcal{I}} \upharpoonright b$  and  $\vec{\mathcal{U}}$  are by  $\Sigma$ , and  $i_b^{\vec{\mathcal{I}}}$  and  $i_c^{\vec{\mathcal{U}}}$  exist, and we have



for some  $\pi$ , then ~~some~~  $c = \Sigma(\vec{\mathcal{U}})$ .

The iteration-strategies we shall construct will be positional and have branch condensation. Saegsyon [2] shows such a strategy is commuting, in that if



are such that  $jk \circ i$  and  $k \circ i$  are by the strategy, when  $j = k$ . This implies there is a direct limit  $M_\infty(P, \Sigma)$  of all countable, non-dropping  $\Sigma$ -iterations of  $P$ . See § 2.6 of the July 2013 version of [1] for more on the basics of how iteration strategies can behave well.

If  $\Sigma$  is positional and  $\mathcal{N}$  is a  $\Sigma$ -iterate of  $P$  via a non-dropping tree (i.e.  $\mathcal{N} \in I(P, \Sigma)$ , in the notation of [1]), then we write  $\Sigma_{\mathcal{N}}$  for the common value of all  $\Sigma_{\vec{I}, \mathcal{N}}$ , and call  $\Sigma_{\mathcal{N}}$  the  $\mathcal{N}$ -tail of  $\Sigma$ .

Comparison (essentially Sargsyan EZJ.) Assume  $AD^+$ .

Lemma  $\uparrow$  Let  $(P, \Sigma)$  and  $(Q, \Lambda)$

~~(P, \Sigma)~~ ~~(Q, \Lambda)~~

43

be  $\Gamma$ -hod pairs. Suppose  $\Sigma$  and  $\Lambda$  are  ~~$\Gamma$ -fullness preserving~~ have branch condensation, and are positional. Then there are iterates  $(P^*, \Sigma^*)$  and  $(Q^*, \Lambda^*)$  of the  $(P, \Sigma)$  and  $(Q, \Lambda)$  by  $\Sigma$  and  $\Lambda$  respectively such that

(a)  $(P^*, \Sigma^*) = (Q^*(\alpha), \Lambda^*_\alpha)$ , some  $\alpha \in L^{Q^*}$ ,

or  
(b)  $(Q^*, \Lambda^*) = (P^*(\alpha), \Sigma^*_\alpha)$ , some  $\alpha \in L^{P^*}$ .

Proof (Sketch.) Assume for simplicity there is

a good scaled pointclass  $\Gamma^* \not\subseteq \Gamma$ . Let

$N^*$  be a co-se  $\Gamma^*$ -mouse that captures and contains  $\Gamma$ .

a universal  $\Gamma$  set, with  $P, Q \in N^*$ . Let

$\mathcal{M}_\xi$  be the  $\xi$ -th model of the maximal

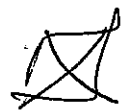
$\Gamma$ -hod-mouse-construction of  $N^*$ . This

construction has to reach iterates of  $(P, \Sigma)$

and  $(Q, \Lambda)$  before it breaks down,  
The key to that is that there are never  
any strategy disagreements, by branch  
condensation (see 2.2.27 of [EZ]).

(18)  
44

But then  $(P, \Sigma)$  and  $(Q, \Lambda)$  have been  
compared with a single construction, and  
hence with each other.



Remark The reason this is not a general  
comparison lemma is its branch condensation  
hypothesis. Once our hod mice have extensive  
overlapping Woodin, branch condensation will probably  
fail.

Nonetheless, we would guess that hod-pairs  
always iterate into ~~the~~ to hod-mouse construction  
of a sufficiently Woodin background universe  $N^*$ .  
There is a natural weakening of branch condensation  
(“phalanx condensation” for  $\Sigma$ -generated phalanxes) which

That implies this.

Below  $O_h^P$ , branch condensation  
will hold, and indeed the theory of  
[2] seems to go over without much change.

Clearly, another level of detail is required  
beyond what we have provided above, but  
let us go on, and provide further detail  
as it becomes important.

191  
45

Repeating def. 0.1,  
we need a notion of smallness  
weaker than "below  $O_k^p$ ":

45a

Def <sup>2.3.3</sup> A hod-premouse  $M$  is below LST<sup>-</sup>  
iff whenever  $\mathcal{N}$  is a full level of  $M$ ,  
and  $\mathcal{N}$  has a top block beginning with  $\kappa$ , and  
 $\kappa$  is a limit of Woodin's in  $\mathcal{N}$ ,  
then there is no  $\delta \in \mathcal{N}$  such that  $\kappa < \delta$   
and  $\mathcal{N} \upharpoonright \delta$  is Woodin.

An  $M$  that is below LST<sup>-</sup> can have  
extender overlapping ~~the~~ local Woodin's  
on its sequence, but these must occur  
inside  $L_p$ -intervals, e.g. below its  $w_1$ .

The following lemma will be useful  
in comparing strategies below LST<sup>-</sup>.

Lemma 2.3.4

Let  $(M, \Sigma)$  and  $(M, \Psi)$  be  $\Gamma$ -hod-pairs such that  $M$  is below LST.

Let  $\mathcal{U}$  be a normal tree on  $M$  that is according to both  $\Sigma$  and  $\Psi$ . Suppose

$$b \neq c,$$

where  $b = \Sigma(\mathcal{U})$  and  $c = \Psi(\mathcal{U})$ . Suppose also

$$\Sigma_{\mathcal{U}, \pi} = \Psi_{\mathcal{U}, \pi}$$

for every full proper initial segment of  $\mathcal{U}$ .

Then

- (1) neither  $b$  nor  $c$  drops
- (2)  $\delta(\mathcal{U})$  is a Woodin and a cutpoint in both  $M_b^\pi$  and  $M_c^\pi$
- (3)  $\mathcal{U} = \mathcal{A} \hat{\cap} \mathcal{I}$ , where  $\mathcal{A}$  and  $\mathcal{I}$  are normal,  $\iota^\mathcal{A}: M \rightarrow \mathcal{R}$  exists, and for some  $\eta < \delta \in \mathcal{R}$

(a)  $\iota_{b^*}^\delta(\delta) = \iota_{c^*}^\delta(\delta) = \delta(\mathcal{U})$ , where  $b^*$  and  $c^*$  are the branches of  $\mathcal{I}$  induced by  $b$  and  $c$

(b)  $\mathcal{R} \models$  there are no Woodins in the interval  $(\eta, \delta)$ ,

(c) all critical points in  $\mathcal{I}$  are  $> \eta$ .

Proof sketch

$\delta(\mathcal{U})$  must be

a limit of full levels of  $M(\mathcal{U})$ ,

for otherwise  $\mathcal{U} = \mathcal{A} \wedge \mathcal{I}$  where

$\mathcal{I}$  is a normal tree on some  $L_p^{-\lambda}(\mathbb{R})^\Gamma$

with all crits  $> 0(\mathbb{R})$ , and

$$\underline{\Lambda} = \sum_{\mathcal{S}, \mathcal{R}} = \Psi_{\mathcal{S}, \mathcal{R}}. \text{ But just as}$$

iteration strategies for sound mice projecting to  $\mathcal{W}$  are unique, there can be only one strategy for  $L_p^{-\lambda}(\mathbb{R})^\Gamma$ .

Letting  $\kappa$  be the least cardinal of  $M(\mathcal{U})$  that is strong to  $\delta(\mathcal{U})$ , we then get that  $\kappa$  is a limit of full levels of  $M(\mathcal{U})$ , and of cutpoints of  $M(\mathcal{U})$ . That is,  $\kappa$  begins the top block of  $M(\mathcal{U})$ .

Let

$$\underline{\Lambda} = \text{join of all } \sum_{\mathcal{U}, \pi}, \text{ for } \mathcal{N} \text{ a full proper-initial segment of } M(\mathcal{U}).$$



(45d)

Then  $L_p^{\wedge}(M(\mathcal{U}))^{\Gamma} \not\subseteq M_b$ , as otherwise  $b$  drops, and  $Q(b, \mathcal{U})$  exists, and  $Q(b, \mathcal{U}) \in L_p^{\wedge}(M(\mathcal{U}))^{\Gamma}$ . If  $c$  drops, then we get  $Q(c, \mathcal{U}) = Q(b, \mathcal{U})$ , a contradiction. If  $c$  does not drop, then  $L_p^{\wedge}(M(\mathcal{U}))^{\Gamma} \subseteq M_c$ , so  $Q(b, \mathcal{U}) \in M_c$ , again a contradiction.

Similarly,  $L_p^{\wedge}(M(\mathcal{U}))^{\Gamma} \not\subseteq M_c$ . But

then

$L_p^{\wedge}(M(\mathcal{U}))^{\Gamma} \in \delta(\mathcal{U})$  is Woodin.

So applying 2.7 with  $\mathcal{M} = L_p^{\wedge}(M(\mathcal{U}))^{\Gamma}$ , we see that  $\kappa$  is not a limit of Woodins in  $M(\mathcal{U})$ . Let  $\eta$  be the strict sup of the Woodins of  $M(\mathcal{U})$  that are  $< \kappa$ .

Since  $\kappa$  is a cutpoint of  $M(\mathcal{U})$ , we can write

$$\mathcal{U} = \mathcal{A} \hat{\wedge} \mathcal{I}$$

where  $\delta(S) \leq \kappa$  and all ordinals in  $\mathcal{I}$  are  $\geq \kappa$ . Let  $R$  be the last model of  $\mathcal{I}$ . Let  $\delta$  be the least Woodin at  $R > \eta$ .

We leave the rest to the reader.



Remark We don't get that  $M_b^u$  and  $M_c^u$  have the same subsets of  $\delta(\mathcal{U})$ . Letting  $\eta = L_p^{\lambda}(\mathcal{M}(\mathcal{U}))^{\Gamma}$ , it can be that  $\Sigma_{u,b,\eta} \neq \Psi_{u,c,\eta}$ , and the two  $L_p$ 's produce different subsets of  $\delta(\mathcal{U})$ . Those two  $L_p$ 's will not kill the Woodinness of  $\delta(\mathcal{U})$ , however. This could happen. Cf. the case  $\delta(\mathcal{U}) = \text{least Woodin at } M_b^u$ .