

Proof of theorem 10 in anomalous case 2

Again, we have that $\mu_0 \in p_1(M)$,
that $r = p_1(M) - (\mu_0 + 1)$ is solid, and
that $\alpha \leq \mu_0$ is least such that $\mathcal{H}_1^M(\alpha \cup r) \notin M$.

We have $\mathcal{H} = \mathcal{H}_1^M(\alpha \cup r)$, and α is a
cardinal of M . In anomalous case 2, α
is a "pseudo-index" of some \bar{G} . That
is, there is a least $\xi > \alpha$ such that
 $p_1(M|\xi) < \alpha$, and this $M|\xi$ is type \mathcal{E}_1 ,
with last extender G and stretching
extender $F = \dot{E}_i^{M|\xi}$; moreover

$$M|\xi \models \alpha = K_F^{++}$$

Let

$$\bar{G} = \bigcup_{\eta < \dot{E}_i^{M|\xi}} i_F^{-1} \text{ " } G \upharpoonright \eta$$

be the extender pseudo-indexed at α .

So $\nu(\bar{G}) = K_F^{+M} = K_F^{+H}$, and

$$\nu(\bar{G})^{+ \text{Ult}(M, \bar{G})} = \kappa.$$

Our goal is to ^{reach a contradiction} show that ~~is the~~ least element of $P_1(M)$, and that $P_1(M)$ is of type ~~\mathbb{Z}~~ $\mathbb{Z} \cup \mathbb{N}$. The proof is similar

to the non-anomalous case, but instead of comparing (M, H, d) with M , we compare (M, H, d) with

$$\mathbb{I} = (M, \text{Ult}(M, \bar{G}), K_F^+).$$

We let \mathcal{T} be the tree on

$$\bar{\mathbb{I}} = (M, H, d)$$

that is produced, and \mathcal{U} the tree on $\bar{\mathbb{I}}$. The rules for forming ~~\mathbb{I}~~ \mathcal{T} are similar to those in anomalous case \mathbb{I} .

Again, let $P_\eta = M_\eta^{\bar{\mathbb{I}}}$, with $P_0 = M$

and $P_1 = \mathcal{H}$. Let $E_\eta = E_\eta^{\mathcal{I}}$. Let $\lambda_\eta^{\mathcal{I}} = \lambda_{E_\eta^{\mathcal{I}}}$ for $\eta \geq 1$, and $\lambda_0^{\mathcal{I}} = \alpha$. We have

$$T\text{-pred}(\eta+1) = \text{least } \beta \text{ s.t. } \text{dom}(E_\eta) \subseteq P_\beta \upharpoonright \lambda_\beta.$$

E_η then gets applied to the longest initial segment of P_β possible, but with the following exception. Suppose that $\text{crit}(E_\eta) = \kappa_F$, and E_η is long. As in the proof of Claim 6, ~~parallel to~~ we can show that E_η has exactly one long generator. We shall set

~~$P_{\eta+1} = \text{Ult}_0(M \upharpoonright \alpha, \bar{E}, E_\eta)$~~
 in this case,

$$P_{\eta+1} = (\text{Ult}_0(M \upharpoonright \alpha, E_\eta), K)$$

where K is the extender of length $\|E_\eta$

determined by $i_{E_\eta} \circ i_G^{-1}$, in this case.

(65)

Claim 1b Let $\text{crit}(E_\eta) = K_F$, and suppose that E_η is long; then

- (i) E_η has exactly one long generator,
- (ii) if K is its extender of length $lh E_\eta$ over M generated by $i_{E_\eta} \circ i_G^{-1}$, then

(a) $(\text{Ult}_0(M \upharpoonright \alpha, E_\eta), K)$ is a type \mathcal{E}_1 premouse, with stretching extender $E_\eta \upharpoonright \beta$ ($\beta \parallel lh E_\eta$), and

(b) $(\text{Ult}_0(M \upharpoonright \alpha, E_\eta), K)$ adds \overline{G} , (See 217, definition 9.)

The proof of claim 1b is similar to the proof of claim 6, ~~precisely the same~~ so we omit it for now.

We shall show that if $P_{\eta+1} = (\text{Ult}_0(M \upharpoonright \alpha, E_\eta), K)$ as above, then

$$E_{\eta+1} = K,$$

and then of course

$$P_{\eta+2} = \text{Ult}_0(M, K).$$

Moreover, since $\lambda_K = \lambda_{E_\gamma}$, there are no models in \mathcal{I} above $P_{\gamma+1}$ — it is a dead node. To show this, we have to look at \mathcal{U} .

We set $Q_\gamma = M_\gamma^u$, with $Q_0 = \mathcal{M}$ and $Q_1 = \text{Ult}(M, \bar{G})$. We let $\lambda_\gamma^u = \lambda_{E_\gamma}$, where $F_\gamma = E_\gamma^u$, for $\gamma \geq 1$. We set $\lambda_0^u = K_F^+$ by fiat. The rules for \mathcal{U} are

$$\mathcal{U}\text{-pred}(\gamma+1) = \text{least } \beta \text{ s.t. } \text{dom}(F_\gamma) \subseteq Q_\beta \upharpoonright \lambda_\beta^u.$$

There are no exceptions, and F_γ always gets applied to the longest possible initial segment of Q_β , for $\beta = \mathcal{U}\text{-pred}(\gamma+1)$. Note $\lambda_1^u > \alpha$. So if $\text{crit}(F_\gamma) = K_F$, and F_γ is short, then $\mathcal{U}\text{-pred}(\gamma+1) = 0$, and

$$Q_{\gamma+1} = \text{Ult}_0(M, F_\gamma),$$

while if $\text{crit}(F_\gamma) = K_F$ and F_γ is long, then $\mathcal{U}\text{-pred}(\gamma+1) = 1$, and

$$Q_{\gamma+1} = \text{Ult}_0(Q_1, F_\gamma).$$

Claim 2b Let $\eta \geq 1$; then no \mathbb{Q}_η / δ adds \overline{G} .

Proof $\overline{G} \notin \mathbb{Q}_\eta$ for $\eta \geq 1$, because $\mathbb{Q}_\eta \models \alpha$ is a cardinal, when $\eta \geq 1$.

It follows that if \mathbb{Q}_η / δ adds \overline{G} , and $\eta \geq 1$, then $\mathbb{Q}_\eta / \delta = \mathbb{Q}_\eta$, and the branch of $\mathbb{Q} \cup \delta$ ending at \mathbb{Q}_η does not drop. So assume this. Let

$$F^* = (E_{\dot{\eta}})^{\mathbb{Q}_\eta}$$

be the stretching extender involved in adding \overline{G} . Thus $K_{F^*} = K_F$. Now let

$$j = \begin{cases} i_{0\eta}^{\dot{\eta}} & \text{if } 0 \cup \eta, \\ i_{1\eta}^{\dot{\eta}} \circ i_{\overline{G}} & \text{if } 1 \cup \eta \text{ or } \dot{\eta} = \eta. \end{cases}$$

Thus $j: M \rightarrow \mathbb{Q}_\eta$. Since $K_{F^*} \in \text{ran}(j)$, and $j(\text{crit}(j)) > \alpha$, we get

$K_F < \text{crit}(j^*)$. But then $\text{crit}(j) > \alpha$, while the rules of \mathcal{U} guarantee $\text{crit}(j) \leq K_F$, contradiction.



Claim 2b implies that if $P_{\eta+1} = (\text{Ultra}(M(\alpha, E_\eta), K))$ as in claim 1b, then $E_{\eta+1} = K$, because $P_{\eta+1}$ does add \bar{G} , so it is not lined up with ^{the current} a model of \mathcal{U} .

We choose branches for \mathcal{I} and \mathcal{U} at limit stages by lifting them to trees on M , and using Σ . For \mathcal{I} , let \mathcal{I}^* be the lift of \mathcal{I} under (π_0, π_1) , where $\pi_0 = \text{id}$ and $\pi_1 : \mathcal{H} \rightarrow M$ is the uncollapse. $\pi_0 \upharpoonright \lambda_0^{\mathcal{I}} = \pi_1 \upharpoonright \lambda_0^{\mathcal{U}}$, which is what we need. Let $P_\eta^* = M_{\eta}^{\mathcal{I}^*}$, and $\pi_\eta : P_\eta \rightarrow P_\eta^*$

be the natural map, but with the following exception related to our special case in the definition of \mathcal{I} .

Namely, suppose $\text{crit}(E_\eta) = K_F$, and E_η is long. By claim 1b, E_η has a unique long generator ν . We have set $P_{\eta+1} = (\text{Ult}_0(\mathcal{M} \parallel \alpha, E_\eta), K)$, where K is the extender of $i_{E_\eta} \circ i_G$ over $P_{\eta+1}$. By 1b, $P_{\eta+1}$ is type \mathcal{Z}_1 , and $\nu = \dot{\nu}^{P_{\eta+1}}$. Let $E = E_\eta$ and $E^* = \pi_\eta(E)$. Note that $\text{dom}(E^*) = \pi_\eta(\mathcal{H} \parallel \alpha) = \pi_\eta(\mathcal{H} \parallel \alpha) = \mathcal{M} \parallel \pi_\eta(\alpha)$, so E^* is an extender over all of \mathcal{M} . We set

$$P_{\eta+1}^* = \text{Ult}_0(\mathcal{M}, E^*)$$

and more importantly

$$G^* = i_{E^*}(G).$$

Since $K_G^{++M} < K_F$, $K_{G^*} = K_G$, and G^* is total

on M . We shall leave $\pi_{\eta+1}$ undefined. (686)

All we need is

Subclaim 3b.0 There is a γ such that

$$K = G^* \uparrow (\pi_\eta'' \lambda_E \circ \gamma).$$

Remark Note $\lambda_E = \lambda_K$. A better way of saying it might be: $K \uparrow (Q_K \circ \gamma) = G^* \uparrow (\pi_\eta'' \lambda_E \circ \gamma)$.

The subclaim is enough to go on and get $\pi_{\eta+2}: P_{\eta+2} \rightarrow P_{\eta+2}^*$. We don't need $\pi_{\eta+1}$ as a map on all of $P_{\eta+1}$, because we are never going to take an ultrapower of $P_{\eta+1}$ in forming \mathcal{I} .

Proof of 3b.0 Let

$$\lambda_E^{\mathcal{M}|\xi} : \mathcal{M}|\xi \rightarrow \text{Ult}_{\mathcal{U}}(\mathcal{M}|\xi, E) = R$$

be the canonical embedding. $\lambda_E^{\mathcal{M}|\xi}$ is discontinuous at K_F^+ , so R is not a plus-one premouse. (E is not close to $\mathcal{M}|\xi$.) Let

$$F_1 = i_E^{ml\xi}(F), \text{ and}$$

$$\nu^* = i_{F_1}(\nu).$$

We claim that

$$K \cap (\lambda_K \cup \{\nu\}) = i_E^{ml\xi}(G) \cap (\lambda_K \cup \{\nu^*\}).$$

Both extenders have space K_G^{+M} . Let $b \in [\lambda_K]^{<\omega}$ and $A \in MI_{K_G^{+M}}$; what we need to see that $(b, \nu) \in i_K(A)$ iff $(b, \nu^*) \in i_{G_1}^{ml\xi}(A)$, where $G_1 = i_E^{ml\xi}(G)$. But note that $i_F(\bar{G}) \subseteq G$, so

$$(MI_{\xi, G}) \equiv \forall u \in [K_E]^{<\omega} \forall \xi < K_E^+$$

$$(u, \xi) \in i_{\bar{G}}(A) \iff (u, i_F(\xi)) \in i_G(A).$$

The formula on the right is of the form

$\Psi(K_E, i_{\bar{G}}(A), F, A)$. That is, those are the parameters. Applying $i_E^{ml\xi}$, we get

$$\text{Ult}_0((\text{mll}_\xi, G), E) \neq \forall u \in \mathbb{Z} \lambda_E J^{\ll \omega} \forall \xi < \lambda_E^+$$

$$(u, \xi) \in i_E^-(i_G^-(A)) \iff (u, i_{F_1}^-(\xi)) \in i_E^-(i_G^-(A)).$$

But $i_E^-(i_G^-(A)) = i_K^-(A)$, $i_E^-(i_G^-(A)) = i_{G_1}^-(i_E^-(A)) = i_{G_1}^-(A)$, and $b \in \mathbb{Z} \lambda_E J^{\ll \omega}$ and $\nu < \lambda_E^+$. So

$$(b, \nu) \in i_K^-(A) \iff (b, \nu^*) \in i_{G_1}^-(A),$$

as desired. Thus $K \cap (\lambda_K \circ \xi \circ \nu^*) = i_E^-(G) \cap (\lambda_K \circ \xi \circ \nu^*)$.

But now let $\sigma: \text{Ult}_0((\text{mll}_\xi, G), E) \rightarrow j_{E^*}((\text{mll}_\xi, G))$

be the natural map, given by the shift lemma:

$$\sigma(\mathbb{Z} \lambda_E J^{\text{mll}_\xi}) = \mathbb{Z} \pi_\eta(a)$$

$$\sigma(\mathbb{Z} a \circ \xi \circ \nu^*, f J_E^{\text{mll}_\xi}) = \mathbb{Z} \pi_\eta(a) \circ \xi \circ \nu^*, f J_{E^*}^m$$

for $a \in \lambda_E$. Then $\sigma \upharpoonright \lambda_E = \pi_\eta \upharpoonright \lambda_E$, and we define $\sigma(\nu^*) = \gamma$. Then $i_E^-(G)$ is a subextender of $i_{E^*}^-(G)$ under σ , finishing the proof of 3b.0. □

(68e)

We can now define

$$P_{\eta+2}^* = \text{Ult}_0(\mathcal{M}, G^*).$$

Recalling that $P_{\eta+2} = \text{Ult}_0(\mathcal{M}, K)$, we set

$$\pi_{\eta+2}(\langle a_0, \dots, a_n \rangle, f \upharpoonright_K^m) = \langle a_0, \dots, a_n \rangle, f \upharpoonright_{G^*}^m,$$

where γ is as in subclaim 3b.0. We also have $\pi_{\eta+2} \upharpoonright \lambda_K = \pi_\gamma \upharpoonright \lambda_K$, as desired.

We have shown

Claim 3b (\mathcal{M}, H, α) is iterable by the rules described; moreover \mathcal{T} is lifted to a tree \mathcal{T}^* according to Σ .

We turn now to the iterability of $(\mathcal{M}, \text{Ult}_0(\mathcal{M}, \bar{G}), \kappa_F^+)$.

be the natural map.

(69)

We lift \mathcal{U} to a tree \mathcal{U}^* on \mathcal{M} as follows. The first two models of \mathcal{U}^* are

$$Q_0^* = \mathcal{M}$$

and

$$Q_1^* = \text{Ult}_0(\mathcal{M}, G).$$

(Note $K_G^{++} < K_F$ is a cardinal of \mathcal{M} .) We define maps

$$\sigma_\gamma : Q_\gamma \longrightarrow Q_\gamma^*,$$

with

$$\sigma_0 = \text{id}$$

and

$$\sigma_1([a, f]_{\bar{G}}^{\mathcal{M}}) = [i_F(a), f]_G^{\mathcal{M}},$$

for $a \in \mathcal{D}(\bar{G}) = K_F^+$ finite. Since \bar{G} is a subextender of G under i_F , this makes sense.

Suppose now we are given $\sigma_\gamma : Q_\gamma \rightarrow Q_\gamma^*$,

and suppose by induction that we have

(†) for $0 < \gamma \leq \eta$, Q_γ^* agrees with Q_η^* below ~~λ_γ^*~~ λ_γ^u , and

$$\sigma_\gamma \upharpoonright \lambda_\gamma^u = \sigma_\eta \upharpoonright \lambda_\gamma^u.$$

Note that we don't have this for $\gamma = 0$, because σ_0 and σ_1 only agree up to K_F , while $\lambda_0^u = K_F^+$. If $U\text{-pred}(\eta+1) \neq 0$,

or $\text{crit}(F_{\eta+1}) \neq K_F$, then we get $Q_{\eta+1}^*$ and $F_{\eta+1}^*$ and $\sigma_{\eta+1}$ by the Shift Lemma, in the usual way:

$$F_{\eta+1}^* = \sigma_\eta(F_\eta),$$

$$Q_{\eta+1}^* = \text{Ult}_k(\mathcal{P}, F_\eta^*),$$

where letting $\delta = U\text{-pred}(\eta+1)$, $\mathcal{P} = \sigma_\delta(M_{\eta+1}^{*u})$, and $k = \text{deg}^u(\eta+1)$, and

$$\sigma_{\eta+1}(\langle a, f \rangle_{F_\eta}^{M_{\eta+1}^{*u}}) = [\sigma_\eta(a), \sigma_\delta(f)]_{F_\eta^*}^{\mathcal{P}}.$$

One can easily check that (*) remains true.

Remark In fact, for $0 < \delta < \gamma$, Q_γ^+ agrees with Q_δ^+ below $\text{lh } F_\delta^+$, and $\sigma_\delta \upharpoonright \text{lh } F_\delta = \sigma_\gamma \upharpoonright \text{lh } F_\gamma$. But we only use agreement up to λ_{F_δ} , because our trees are (generally) formed with short extender rules, as they must be because our background extenders in a plus-one construction are short.

We tend to record the agreement between models and lifting maps in an iteration using the λ_{F_δ} rather than the $\text{lh } F_\delta$. This ~~is~~ reminds us ~~again~~ that trying to use the additional agreement we might have to apply an extender E with $\text{crit}(E) = \lambda_{F_\delta}$ to Q_δ (say) ~~might~~ ^{could} lead to problems with iterability.

Now suppose $U\text{-pred}(\gamma+1) = 0$, $\text{crit}(F_\gamma) = K_F$, and F_γ is short. So

$$Q_{\gamma+1} = \text{Ult}_0(M, F_\gamma).$$

We cannot set $Q_{\gamma+1}^* = \text{Ult}_{\sigma_0}(Q_0^*, \sigma_\gamma(F_\gamma))$ now, because although the latter ultrapower makes sense (by the agreement of Q_0^* with Q_γ^* up to $\text{lh}(G)$), σ_γ and σ_0 do not agree far enough that we could define $\sigma_{\gamma+1}$ properly.

Instead, let

$$j: Q_\gamma^* \longrightarrow \text{Ult}_{\sigma_0}(Q_\gamma^*, \sigma_\gamma(F_\gamma))$$

be the canonical embedding. Note that F is on the Q_1^* sequence (though not on the Q_1 sequence), and hence F is on the Q_γ^* -sequence.

We have

$$\text{crit}(j) = \sigma_\gamma(K_F) = \sigma_1(K_F) = \lambda_F.$$

Set

$$Q_{\gamma+1}^* = \text{Ult}_{\sigma_0}(Q_\gamma^*, j(F)),$$

and

$$\sigma_{\gamma+1}([\![a, f]\!]_{F_\gamma}^{Q_0}) = [\![\dot{a}, f]\!]_{j(F)}^{Q_0}.$$

(Recall that $Q_0^* = Q_0 = M$.) Since $K_F = K_{j(F)}$,



$Q_{\eta+1}^*$ makes sense. Moreover,

$$lh F < (\lambda_F^+)^{Q_\eta^*}, \text{ so } lh(j(F)) < j(\lambda_F^+)^{Q_\eta^*},$$

so $j(F)$ is on the $Q_{\eta+1}^*$ -sequence by coherence of $\sigma_\eta(F_\eta)$ with that sequence. Also,

$$K_{j(F)} = K_F < \lambda_G. \text{ Thus } Q_{\eta+1}^* \text{ is a legitimate}$$

next model for \mathcal{U}^* .

We must see that $\sigma_{\eta+1}$ is well-defined

and elementary. Let $E = F_\eta$. E is

short, and $K_E = K_F = K_{j(F)}$. Let

$a \in \lambda_E$ be finite and $X \subseteq [K_E]^{<a>}$ with $X \in M$.

It is enough to show that

$$(a, X) \in E \text{ iff } (\sigma_\eta(a), X) \in j(F).$$

But we have

$$(a, X) \in E \text{ iff } a \in i_E(X)$$

$$\text{iff } \sigma_\gamma(a) \in \sigma_\gamma(i_E(X))$$

$$\text{iff } \sigma_\gamma(a) \in \sigma_\gamma(i_E)(\sigma_\gamma(X))$$

$$\text{iff } \sigma_\gamma(a) \in j(i_F(X))$$

$$\text{iff } \sigma_\gamma(a) \in j(i_F)(X)$$

$$\text{iff } \sigma_\gamma(a) \in i_{j(F)}(X)$$

$$\text{iff } (\sigma_\gamma(a), X) \in j(F)$$

For the fourth line, notice that $\sigma_\gamma(X) = \sigma_1(X) = i_F(X)$, because $X \in [K_F]^{|\alpha|}$.

For the fifth line, note $j(X) = X$.

$$\text{Clearly } \sigma_{\gamma+1} \upharpoonright \lambda_{E_\gamma} = \sigma_\gamma \upharpoonright \lambda_{E_\gamma}$$

since $\sigma_{\gamma+1}([?, ?], \text{id } J_{E_\gamma}^M) = [? \sigma_\gamma(?), \text{id } J_{j(F)}^M]$. Also,

$$\sigma_{\gamma+1} \circ \lambda_{0, \gamma+1}^u = \lambda_{0, \gamma+1}^{u^*} \circ \sigma_0 = \lambda_{0, \gamma+1}^{u^*}, \text{ since}$$

$$\sigma_{\gamma+1}([?, c_z]_{E_\gamma}^M) = [?, c_z]_{j(F)}^M.$$

This completes the successor step in the formation of \mathcal{U}^* . At limit steps, we use Σ to choose a branch of \mathcal{U}^* , and then choose the same branch for \mathcal{U} . We have shown

Claim 3b.1 $(\mathcal{M}, \text{Ult}(\mathcal{M}, \bar{G}), K_F^+)$ is iterable via the strategy described above.

Claim 4b The comparison of $(\mathcal{M}, \mathcal{H}, \alpha)$ with $(\mathcal{M}, \text{Ult}(\mathcal{M}, \bar{G}), K_F^+)$ terminates.

Proof. As before.



Now let $P = P_\gamma$ and $Q = Q_\delta$ be the last models on the two sides.

Claim 5b It is not the case that both P and Q are above M in their respective trees.

Proof. Suppose they were, i.e., suppose $0T\gamma$ and $0U\delta$.

Subclaim 5b.1 $P=Q$, neither $\Sigma_{0,\gamma}J_T$ nor $\Sigma_{0,\delta}J_U$ drops, and

$$1_{0,\gamma}^J = 1_{0,\delta}^U.$$

Proof We use J^* , U^* , and the weak Dodd-Jensen property of Σ . □

Now let K and L be the first extenders used in $\Sigma_{0,\gamma}J_T$ and $\Sigma_{0,\delta}J_U$, and let K^* and L^* be their stretchers by the short parts of their respective branch tails. We may assume both

K^* and L^* are long. Both K and L had the Jensen ISC, so this gives $K=L$. That is a contradiction.

Remark There is the case that K is the extension of $i_{E_\eta} \circ i_{\bar{G}}$. But then K adds \bar{G} , and we showed that no F_η adds \bar{G} , so $K \neq L$.



Claim 6b It is not the case that P is above M and Q is above $\text{Ult}(M, \bar{G})$.

Proof Suppose they were, i.e. 0T8 and 1U8.

Subclaim 6b.1 $P=Q$, neither

$[0, \delta]_{\mathcal{J}}$ nor $[1, \delta]_{\mathcal{U}}$ drops, and

$$\lambda_{0, \delta}^{\mathcal{J}} = \lambda_{1, \delta}^{\mathcal{U}} \circ \lambda_{\bar{G}}.$$

Proof Again, we use \mathcal{J}^* , \mathcal{U}^* , and the weak Dodd-Tensen property of \mathcal{E} .



Subclaim 6b.2 $\text{crit}(\lambda_{1, \delta}^{\mathcal{U}}) = \lambda_{\bar{G}}$.

Proof If not, then

$$\begin{aligned} \lambda_{0, \delta}^{\mathcal{J}}(K_G) &= \lambda_{0, \delta}^{\mathcal{U}}(K_G) = \lambda_{1, \delta}^{\mathcal{U}}(\lambda_{\bar{G}}(K_G)) \\ &= \lambda_{\bar{G}}(K_G) = \lambda_{\bar{G}}. \end{aligned}$$

But it is easy to see that $\lambda_{\eta}^{\mathcal{J}} > \alpha$ for all $\eta \geq 1$. So $\lambda_{0, \delta}^{\mathcal{J}}(K_G) > \alpha > \lambda_{\bar{G}}$.



Now let $K = E_\gamma$ be the first extender used in $[0, \gamma]_T$, and let

$$(P^*, K^*) = \text{Ult}_0((P_\gamma \parallel hK, K), W_0)$$

and

$$(Q^*, G^*) = \text{Ult}_0((M \parallel \bar{G}), W_1)$$

where W_0 is the short part of the extender of $i_{\gamma+1, \delta}^I$, and W_1 is the short part of the extender of $i_{1, \delta}^{i, u}$. As in comparison, we have that P^* and Q^* are initial segments of $P_\delta = Q_\delta$, below $i_{\delta}^I (K_\alpha)^+ P_\delta$.

Moreover, K^* and G^* are initial segments of the extender of $i_{\delta}^I = i_{\delta}^{i, u} \circ i_{\bar{G}}$.

Note $K^* \notin P_\delta$, because ~~(P^*, K^*) is a plus-one premouse~~ $K \notin P_\delta$. Every proper initial segment of G^* is in Q^* , so K^* is not an initial segment of G^* . $K^* \neq G^*$, because (P^*, K^*) is a plus-one premouse, and G^* has

no largest generator. It follows that $G^* = K^* \uparrow v^*$, where v^* is the largest generator of K^* . Thus (P^*, K^*) is type Z_1 , with stretching extender

$$F = \overset{\circ}{E}_{v^*}^{P^*} = \overset{\circ}{E}_{v^*}^{P_8} = \overset{\circ}{E}_{v^*}^{Q_8},$$

and satisfying

$$K^* \uparrow v^* = G^*.$$

Now let $n+1$ be least in $(1, 8]u$, so that $\text{crit}(F_T) = \lambda_{\bar{a}}$ by 6b.2, and F_T is long by our rules for \mathcal{U} . Let

$H =$ least long initial segment of F_T on the Q_T sequence.

(We shall show shortly that $H = F_T$.) Let

$$(R^*, H^*) = \cup_{T_0} ((Q_{T_0} \parallel H, H), W_2),$$

where W_2 is the short part of the branch-tail

extender of $i_{\gamma+1, \delta}^{i_{\gamma, \delta}}$. Let also

$$L^* = \text{extender of } i_{H^*} \circ i_{\bar{G}} \uparrow (M \setminus \alpha).$$

Subclaim 6b.3

(1) $K^* = L^*$,

(2) $E_{\nu^*}^{P_2} = H^* \uparrow \nu^* = W_1$, and

(3) (P^*, K^*) adds \bar{G} .

Proof The extender of $i_{H^*} \circ i_{\bar{G}}$ is just the extender of $i_{1, \delta}^{i_{1, \delta}} \circ i_{\bar{G}}$, restricted to the first of its generators that is $\geq \sup i_{1, \delta}^{i_{1, \delta}} \nu(\bar{G})$, plus one. But

$$\sup i_{1, \delta}^{i_{1, \delta}} \nu(\bar{G}) = i_{W_1}(\nu(\bar{G})) = \nu^*.$$

So

$$\begin{aligned} L^* &= \text{trivial completion of } E_{i_{1, \delta}^{i_{1, \delta}} \circ i_{\bar{G}}} \uparrow (\nu^* + 1) \\ &= \text{trivial completion of } K^* \uparrow (\nu^* + 1) \\ &= K^*. \end{aligned}$$

The branch tail W_2 stretches $H \Gamma \lambda_H$ into W_1 , by calculations we have done before. Moreover, $\sup i_{1,8}'' \rightarrow (\bar{G})$ is where the superstrong part of H^* is indexed in $\text{Ult}_0(\text{Ult}_0(\mathcal{M}, \bar{G}), H^*)$. This gives us (2).

Finally, $H^* \uparrow \nu^*$ stretches \bar{G} into G^* , by the way we have defined G^* . This gives (3).



Recall that $K = E_\gamma^{\nu}$, where $T_{\text{pred}}(\gamma+1) = 0$. We also have $\lambda_K > \alpha$. Since (P^*, K^*) adds \bar{G} , we must then have our special case

$$K = \text{extender of } i_{E_\sigma} \circ i_{\bar{G}} \uparrow (\mathcal{M} \upharpoonright \alpha),$$

where $\text{crit}(E_\sigma) = \lambda_{\bar{G}}$ and E_σ has exactly one long generator.

Subclaim 6b.4 $E_\sigma = H$.

Proof Let

$$E_\sigma^* = \text{last extender of } \text{Ult}_\sigma(P_\gamma, W_0)$$

$$= \text{extender with exactly one long generator determined by } i_{\gamma+1, \delta} \circ i_{E_\sigma} \upharpoonright (M/\alpha).$$

It is not hard to show that $E_\sigma^* = H^*$. For (Roughly) both have critical point $\lambda_{\bar{G}}$, and measure subsets of $\nu(\bar{G}) = \lambda_{\bar{G}}^{+M}$ that belong to M/α , or equivalently, belong to $\text{Ult}(M, \bar{G})$. Let

$$X = i_{\bar{G}}(f)(\bar{a}) \in M/\alpha.$$

We then have that

$$E_\sigma^* \upharpoonright \nu^* = E_{\nu^*}^{P_\delta} = H^* \upharpoonright \nu^*.$$

The second equality we have already

we have already shown, and the second comes from $E_0 \wedge (\sup_{E_0} i_{E_0} \cap \nu(\bar{G}))$ being the stretching extension in the type Z_1 structure P_7 , and the fact that i_{w_0} preserves this.
 So

$$i_{E_0^*} \wedge \nu(\bar{G}) = i_{H^*} \wedge \nu(\bar{G}).$$

But then

$$\begin{aligned} i_{E_0^*}(X) \cap (\nu^{*+1}) &= i_{E_0^*}(i_{\bar{G}}(f)(a)) \cap (\nu^{*+1}) \\ &= i_{E_0^*}(i_{\bar{G}}(f))(i_{E_0^*}(a)) \cap (\nu^{*+1}) \\ &= i_{\sigma+1, \delta}^J(i_{E_0}(i_{\bar{G}}(f)))(i_{E_0^*}(a)) \cap (\nu^{*+1}) \\ &= i_{\sigma \delta}^J(f)(i_{E_0^*}(a)) \cap (\nu^{*+1}) \\ &= i_{\delta}^{21} \circ i_{\bar{G}}(f)(i_{H^*}(a)) \cap (\nu^{*+1}) \\ &= i_{H^*}(\text{ADV}(\text{Adv } i_{\bar{G}}(f)))(i_{H^*}(a)) \cap (\nu^{*+1}) \\ &= i_{H^*}(i_{\bar{G}}(f)(a)) \cap (\nu^{*+1}) \\ &= i_{H^*}(X) \cap (\nu^{*+1}) \end{aligned}$$

The third equality from the bottom comes from $i_{\bar{G}}^*(t)$ being essentially a subset of $v(\bar{G})$, and for $Z \subseteq v(\bar{G})$ in $M1d$,

$$i_{1,8}^{2u}(Z) = i_{r+1,8}(i_{1,r+1}(Z))$$

so that

$$\begin{aligned} i_{1,8}^{2u}(Z) \cap (v^{*+1}) &= i_{r+1,8} \circ i_{1,r+1}(Z) \cap (v^{*+1}) \\ &= i_{H^*}(Z) \cap (v^{*+1}). \end{aligned}$$

Thus $E_{\sigma}^* = H^*$. The Jensen ISC then leads to $E_{\sigma} = H$.

Subclaim 6b.4 \square

Subclaim 6b.4 contradicts the fact that we were hitting disagreements. This proves claim 6b.

\square

By claims 5b and 6b, $P = P_\delta$ is above \mathcal{H} in \mathcal{I} .

Claim 7b. $Q = Q_\delta$ is above \mathcal{M} in \mathcal{U} ; that is, it is not the case that $\mathbb{I} \cup \delta$ or $\delta = 1$.

Proof Suppose otherwise. Let $\bar{r} = \pi_\delta^{-1}(r)$.

Subclaim 7b.1

- (a) The branch $\mathbb{I}, \delta \mathbb{I}_u$ does not drop.
- (b) $P = Q$.

Proof If $\mathbb{I}, \delta \mathbb{I}_u$ drops, then $\mathbb{I}, \delta \mathbb{I}_T$ does not drop. Also $\text{crit}(i_{1, \delta}^{\mathbb{I}}) > \alpha$, because $\alpha = (K_F^{++})^{\mathcal{H}}$, and the rules for \mathcal{I} are such that any extender L with $K_L = K_F$ is applied to \mathcal{M} , not \mathcal{H} . So

$$\mathcal{H}^{\mathcal{H}}(\alpha \cup \bar{r}) = \mathcal{H}^P(\alpha \cup i_{1, \delta}^{\mathbb{I}}(\bar{r})).$$

But standard arguments show that if $\mathbb{I}, \delta \mathbb{I}_u$ drops,

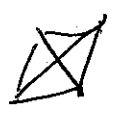
then $Th_1^P(\alpha \circ i_{1,8}^{\mathbb{Z}}(\bar{r})) \in Q_1 = \mathcal{O}l_T(\mathcal{M}, \bar{G})$.

Since $\alpha = \vartheta(\bar{G})^{+Q_1}$, we can pull back again to get

$$Th_1^H(\alpha \circ \bar{r}) \in \mathcal{M}.$$

(Note $G \in \mathcal{M}$, so $\bar{G} \in \mathcal{M}$.)

So $[1, 8]_u$ does not drop. If $Q \triangleleft P$, then $\pi_{\delta}^* \circ i_{1,8}^{\mathbb{Z}} \circ i_{\bar{G}}$ maps \mathcal{M} to a proper initial segment of P_{δ}^* , contrary to weak Dodd-Jensen. But if $P \triangleleft Q$, then again $Th_1^P(\alpha \circ i_{1,8}^{\mathbb{Z}}(\bar{r})) \in \mathcal{M}$. Thus $P = Q$.



Subclaim 7b.2 $[1, 8]_T$ does not drop.

Proof By weak Dodd-Jensen.



Subclaim 7b.4 $i_{1,8}^{\mathbb{Z}}(\bar{r}) = i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r)$.

Proof As before. We have

$$\pi_{\mathbb{Z}}(i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r)) \geq_{\text{lex}} \pi_{\mathbb{Z}}(i_{1,8}^{\mathbb{Z}}(\bar{r})) = i_{1,8}^{\mathbb{Z}}(r)$$

by weak Dodd Jensen, so $i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r) \geq_{\text{lex}} i_{1,8}^{\mathbb{Z}}(\bar{r})$.

But $i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r) \leq_{\text{lex}} i_{1,8}^{\mathbb{Z}}(\bar{r})$ because

$i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r)$ is solid over P , whereas

$$\text{Th}^P(\alpha \cup i_{1,8}^{\mathbb{Z}}(\bar{r})) \notin P.$$



$$\text{Let } S = i_{1,8}^{\mathbb{Z}}(\bar{r}) = i_{1,8}^{\mathbb{Z}} \circ i_{\mathbb{G}}^{\mathbb{Z}}(r).$$

Subclaim 7b.5 $\rho_1(M) = K_{\mathbb{G}}^{++M}$.

Proof We have

$$\rho_1(M) < \rho_1(\mathbb{H}) \leq \rho_1(P) \leq \alpha$$

as before.

If $\rho_1(M) \leq K_G = \text{crit}(i_{1,8}^u \circ i_G^-)$, then

$\rho_1(M) = \rho_1(Q)$ ~~and~~, contrary to

$\rho_1(M) < \rho_1(H) \leq \rho_1(P)$. If

$K_G^{++M} < \rho_1(M)$, then 7b.3 gives

$\alpha < \rho_1(Q_1)$, and thus $\alpha < \rho_1(Q) \leq \alpha$,

contradiction. But $\rho_1(M) = K_G^{++M}$ is ruled out by projectum-free spaces. Thus

$\rho_1(M) = K_G^{++M}$.



Subclaim 7b.6 $\rho_1(Q_1) = \rho_1(Q) = \rho_1(P) = \alpha$.

Moreover, $\text{crit}(i_{1,8}^u) > \alpha$.

Proof $\alpha = i_G^-(K_G^{++M}) = \sup i_G^- \text{ " } K_G^{++M}$

so $\alpha \leq \rho_1(Q_1)$ by 7b.4. But $\rho_1(Q_1) \leq$

$\rho_1(Q) \leq \alpha$. So $\rho_1(Q_1) = \rho_1(Q) = \alpha$. Moreover,

7b.4 tells us that if $\text{crit}(i_{1,8}^u) < \alpha$,

then $\alpha < \rho_1(Q)$. Thus $\text{crit}(i_{1,8}^{\alpha}) > \alpha$. (90)



Now notice that $Th_1^M(K_G^{++} \cup r) \in M$,
because r was not the whole of $p_1(M)$.

As usual, this gives $Th_1^Q(\alpha \cup i_{\bar{G}}(r)) \in Q$.

Since $\text{crit}(i_{1,8}^{\alpha}) > \alpha$, we then get

$Th_1^Q(\alpha \cup i_{1,8}^{\alpha} \circ i_{\bar{G}}(r)) \in Q$. That implies

$Th_1^P(\alpha \cup i_{1,8}^{\alpha}(r)) \in P$, a contradiction.

This proves claim 7b.



Now we know P is above H and

Q is above M in \mathcal{I} and \mathcal{U} respectively.

The remainder of the proof of Theorem 10
^{to non-}in anomalous case ~~(2)~~^{applies}. (That is the part
of the proof from claim 9 onward.) This
proves Theorem 10 in anomalous case (2),
when $k=0$.



Proof of theorem 10 in anomalous case 3.

Again, we have that $\mu_0 \in p_1(\mathcal{M})$,
that $\nu = p_1(\mathcal{M}) - (\mu_0 + 1)$ is solid, and that
 $\alpha \leq \mu_0$ is least such that $\mathcal{H}_1^{\mathcal{M}}(\alpha \cup \nu) \notin \mathcal{M}$.
We set $\mathcal{H} = \mathcal{H}_1^{\mathcal{M}}(\alpha \cup \nu)$, and α is a cardinal
of \mathcal{H} . In anomalous case 3, we have
 $\langle i, k \rangle$ least such that $\alpha \leq \xi$ and
 $\rho_k(\mathcal{M} \upharpoonright \xi) < \alpha$, with $\alpha < \xi$. Moreover,
 $\rho_k(\mathcal{M} \upharpoonright \xi)$ is of type \mathcal{E}_p .

We assume for simplicity that $k=i$. Let
 $\gamma_0 =$ least element of $p_1(\mathcal{M} \upharpoonright \xi)$,
 $t_0 = p_1(\mathcal{M} \upharpoonright \xi) - (\gamma_0 + 1)$,

$$F = \overset{\circ}{E}_{\gamma_0}^{\mathcal{M}}$$

$$\mathcal{N} = \mathcal{H}_1^{\mathcal{M} \upharpoonright \xi} (i_F'' K_F^+ \cup t_0).$$

So F is the stretching extender, and \mathcal{N}
is the generalized core of $\mathcal{M} \upharpoonright \xi$. We
have that $p_1(\mathcal{N}) = \sigma^{-1}(t_0)$, where $\sigma: \mathcal{N} \rightarrow \mathcal{M} \upharpoonright \xi$

is the uncollapse, and
know that $\alpha = (K_F^{++})^{M \mid \xi} = (K_F^{++})^H =$

$(K_F^{++})^N$, moreover $M \mid \alpha = N \mid \alpha = H \mid \alpha$.

Our goal is to reach a contradiction.

In this case we compare (m, n, K_F^+) .

(m, H, α) with ~~(m, n, K_F^+)~~

Let \mathcal{I} be the tree on (m, H, α) and

\mathcal{U} the tree on ~~(m, n, K_F^+)~~ produced.

The rules for forming \mathcal{I} are similar to those in the other anomalous cases.

Let $P_\eta = M \mathcal{I}^\eta$, with $P_0 = M$ and

$P_1 = H$. Let $E_\eta = E \mathcal{I}^\eta$, and $\lambda_0^\mathcal{I} = \alpha$.

Again

$T\text{-pred}(\eta+1) = \text{least } \beta \text{ s.t.}$

$$\text{dom}(E_\eta) \subseteq P_\beta \mid \lambda_\beta^\mathcal{I}$$

Again, E_η then gets applied to the longest initial segments of P_β possible, except

when $\text{crit}(E_\eta) = K_F$, and

E_η is long. Here $\beta = 0$, and $P_\beta = \mathcal{M}$.

Again, we can show E_η has exactly one long generator in this case. We then set

$$P_{\eta+1} = \text{Ult}_0(\mathcal{N}, E_\eta).$$

Claim 1c Let $\text{crit}(E_\eta) = K_F$, and suppose E_η is long; then

- (i) E_η has exactly one long generator,
- (ii) $\text{Ult}_0(\mathcal{N}, E_\eta)$ is a plus-one premouse of type Z_p , with 1st standard parameter $i_{E_\eta}^*(p_1(\mathcal{N}) \cup \{v\})$, where $v = v^{(P_\eta | \text{lh} E_\eta)}$, and stretching extender $E_\eta \upharpoonright v$, and
- (iii) \mathcal{N} is the generalized core of $\text{Ult}_0(\mathcal{N}, E_\eta)$.

Remark The solidity of $p_1(\mathcal{N})$ is used in proving (ii).

We omit the proof of claim 1c.

In contrast to our first two anomalous cases, we cannot predict what $P_{\gamma+2}$ will be. Moreover, $P_{\gamma+1}$ is not dead, in that \mathcal{I} may have models about $P_{\gamma+1}$.

We choose branches for \mathcal{I} by lifting it to \mathcal{I}^* on M , with models $P_\beta^* = M_\beta^{\mathcal{I}^*}$. We have copy maps

$$\pi_\beta: P_\beta \rightarrow P_\beta^*$$

with $\pi_0 = \text{id}$, and $\pi_\gamma: \mathcal{H} \rightarrow M$ the uncollapse. We have $\pi_\beta \uparrow \lambda_\beta^\alpha = \pi_\gamma \uparrow \lambda_\beta^\alpha$ for all $\gamma \geq \beta$. Again, we must take some care in our special case.

So suppose $\text{crit}(E_\gamma) = K_F$, and E_γ has $\vec{v} = \vec{v}(P_\gamma / \text{th} E)$ as its unique long generator. Let $E = E_\gamma$, and $E^* = \pi_\gamma(E)$.

We have that $\lambda_1^{\mathcal{I}} > \alpha$, and $\pi_{\eta} \upharpoonright \lambda_1^{\mathcal{I}} = \pi_1 \upharpoonright \lambda_1^{\mathcal{I}}$, so $\text{dom}(E^*) = \mathcal{M} \upharpoonright \pi_1(\alpha)$, so E^* is an extender over all of \mathcal{M} .

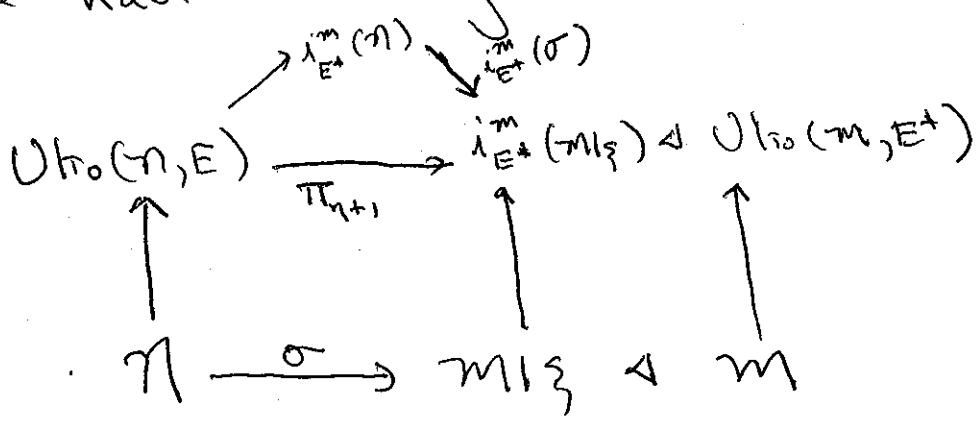
Let

$$i_{E^*}^{\mathcal{M}} : \mathcal{M} \longrightarrow \text{Ult}_0(\mathcal{M}, E^*)$$

be the canonical embedding, and set

$$P_{\eta+1}^* = i_{E^*}^{\mathcal{M}}(\mathcal{M} \upharpoonright \xi).$$

We have the diagram



where

$$\pi_{\eta+1}([a, f]_{E}^{\mathcal{N}}) = i_{E^*}^{\mathcal{M}}(\sigma)([\pi_{\eta}(a), f]_{E^*}^{\mathcal{M}}),$$

for $a \in [\lambda_E \cup \{\omega\}]^{<\omega}$ and $f : [K_E^+]^{(a)} \rightarrow \mathcal{N}$

with $f \in \mathcal{N}$. Since $\text{crit}(\sigma) = K_E = K_{E^*}$, $\text{crit}(i_{E^*}^{\mathcal{M}}(\sigma)) = \lambda_{E^*}$, and thus

$\pi_{\gamma+1} \upharpoonright \lambda_E = \pi_\gamma \upharpoonright \lambda_E$. So we have the agreement of copy maps required to continue.

This completes our proof-sketch for

Claim 2c (M, \mathcal{H}, α) is iterable by the rules described; moreover \mathcal{I} is lifted to a tree \mathcal{I}^* according to Σ .

We turn to \mathcal{U} . Set

$$Q_\eta = M_\eta^{\mathcal{U}},$$

with $Q_0 = M$ and $Q_1 = \mathcal{N}$. Let

$$\lambda_0^{\mathcal{U}} = K_F^+$$

and

$$\lambda_\eta^{\mathcal{U}} = \lambda_{E_\eta^{\mathcal{U}}}$$

for $\eta \geq 1$. The rules for \mathcal{U} are

$$\mathcal{U}\text{-pred}(\eta+1) = \text{least } \beta \text{ s.t. } \text{dom}(E_\eta^{\mathcal{U}}) \subseteq Q_\beta \upharpoonright \lambda_\beta^{\mathcal{U}}.$$

So short extenders with critical point K_F

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get applied to \mathcal{M} , while long extenders with critical point κ_F get applied to \mathcal{N} .
 (The latter because $\lambda_1^{\mathcal{U}} > \alpha = (\kappa_F^{++})^{Q_\eta}$,
 for all $\eta \geq 1$.)

We lift \mathcal{U} to a tree \mathcal{U}^* on \mathcal{M} as follows. The first two models of \mathcal{U}^* are

$$Q_0^* = \mathcal{M}$$

and

$$Q_1^* = \mathcal{M} \upharpoonright \xi.$$

We define maps

$$\sigma_\eta : Q_\eta \longrightarrow Q_\eta^*$$

with

$$\sigma_0 = \text{id}$$

and

$$\sigma_1 = \sigma,$$

where $\sigma : \mathcal{N} \rightarrow \mathcal{M} \upharpoonright \xi$ is the uncoining map.
 Suppose now by induction that we have

$$\sigma_\gamma : Q_\gamma \rightarrow Q_\gamma^*, \text{ and}$$

(†) for $0 < \gamma \leq \eta$, Q_γ^* agrees with Q_η^* below $\lambda_\gamma^{u^*}$, and $\sigma_\gamma \upharpoonright \lambda_\gamma^u = \sigma_\eta \upharpoonright \lambda_\gamma^u$.

Again, we do not have this for $\gamma = 0$, because σ_0 and σ_1 agree only to K_F , and $\lambda_0^u = K_F^+$. We then define $Q_{\eta+1}$ and $\sigma_{\eta+1}$ so that (†) remains true.

This is done using the shift lemma, except when, for $F_\eta = E_\eta^u$, $\text{crit}(F_\eta) = K_F$ and F_η is short.

So assume that. We have $\text{U-pred}(F_{\eta+1}) = 0$, and

$$Q_{\eta+1} = \text{Ult}_0(M, F_\eta).$$

Let

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$$j: Q_\gamma^* \rightarrow \text{Ult}_0(Q_\gamma^*, \sigma_\gamma(F_\gamma))$$

be the canonical embedding. Note that the stretching extender F for $\rho_1(m|\xi)$ is on the sequence of $Q_1^* = m|\xi$, so that F is on the Q_γ^* sequence (because $\alpha < \lambda_1^{\aleph_1}$, so that $\sigma_1(\alpha) = \lambda_F^{\aleph_1} < \lambda_1^{\aleph_1^*}$).

We have

$$\text{crit}(j) = \sigma_\gamma(K_F) = \sigma_1(K_F) = \lambda_F.$$

Set

$$Q_{\gamma+1}^* = \text{Ult}_0(m, j(F))$$

and

$$\sigma_{\gamma+1}(\langle a, f \rangle_{F_\gamma}^m) = \langle \sigma_\gamma(a), f \rangle_{j(F)}^m.$$

Since $K_F = K_{j(F)}$, $Q_{\gamma+1}^*$ makes sense. One can show this works by repeating the calculations in the proof of claim 3b. I. We omit further detail.

This leads to

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Claim 3c. $(\mathcal{M}, \mathcal{H}, K_F^+)$ is iterable via the strategy just described.

Claim 4c The comparison of $(\mathcal{M}, \mathcal{H}, \alpha)$ with $(\mathcal{M}, \mathcal{H}, K_F^+)$ terminates.

Now let $P = P_\delta$ and $Q = Q_\delta$ be the last models on the two sides.

Claim 5c It is not the case that both P and Q are above \mathcal{M} in their respective trees.

Proof Suppose they were.

Subclaim 5c.1 It cannot be that both $\Sigma_{0, \delta} \upharpoonright_T$ and $\Sigma_{0, \delta} \upharpoonright_U$ drop.

~~[Proof in the above, then P.A.A.]~~

This takes a little more argument than usual, because the branch to P might have dropped to \mathcal{N} . But note

Subclaim 5c.2 There is no η such that $0 \cup \eta$ end \mathcal{N} is a generalized cone of some Q_η / γ .

Proof \mathcal{N} collapses α definably to K_F^+ , so we would have $\{0, \eta\}$ does not drop, and $Q_\eta \upharpoonright \gamma = Q_\eta$. Let F^* be the stretching extender at Q_η , so that $K_F = K_{F^*}$ and $F^* \in \text{ran } i_{0, \eta}^u$.

Then $\text{crit}(i_{0, \eta}^u) \leq K_F$, and $i_{0, \eta}^u(\text{crit}(i_{0, \eta}^u)) > \alpha$, so $K_{F^*} \notin \text{ran}(i_{0, \eta}^u)$, contradiction.



Proof of 5C.1 Suppose not. Let R be the last model to which we drop in model or degree along $[0, \gamma]_T$, so that we have

$$R = M_{\tau+1}^{*\tau}$$

with

$$j = i_{\tau+1, \gamma}^{*\tau} \circ i_{\tau+1}^{*\tau} : R \longrightarrow P$$

being an n embedding with $\text{crit}(j) \geq \rho_{n+1}^R$, for some n , such that R is m -sound. Similarly, let S be the last model to which we drop in model or degree along $[0, \delta]_U$, so that

$$S = M_{\theta+1}^{*\tau}$$

with

$$l = i_{\theta+1, \delta}^{*\tau} \circ i_{\theta+1}^{*\tau} : S \longrightarrow Q$$

being an m -embedding, with $\text{crit}(l) \geq \rho_{m+1}^S$,

and S being $(m+1)$ -sound.

We have that Q is not $(m+1)$ -sound, so $P \triangleleft Q$. By SC.2, \mathcal{N} is not a generalized cone of Q , so $\mathcal{N} \neq R$, so P is not $(n+1)$ -sound. Thus $P = Q$, and $m = n$. Moreover,

$$R = S = C_{n+1}(P) = C_{n+1}(Q),$$

and

$j' = \mathcal{J} =$ uncurving embedding.

We want to show that $E_{\eta}^{\mathbb{I}} = E_{\theta}^{\mathbb{U}}$. Let $E = E_{\eta}^{\mathbb{I}}$ and $G = E_{\theta}^{\mathbb{U}}$. We assume that both E and G are long, and leave the other cases to the reader. Let

$$\nu = \nu(E) - 1$$

$$\mu = \nu(G) - 1$$

be the ~~new~~ largest generators. Let

$$K = \text{crit}(j) = K_E = K_G$$

and

$$\lambda = j'(K).$$

So

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$$i_{\gamma+1, \delta}^{j, \sigma}(\lambda_E) = i_{\theta+1, \delta}^{j, \tau}(\lambda_G) = \lambda.$$

Let $W_0 = E_{i_{\gamma+1, \delta}^{j, \sigma}} \uparrow \lambda$ and $W_1 = E_{i_{\theta+1, \delta}^{j, \tau}} \uparrow \lambda$ be

the two short parts of the branch tails.

Let

$$(P \parallel \eta_0, E^*) = \text{Ult}_0((P \parallel \uparrow h E, E), W_0)$$

and

$$(P \parallel \eta_1, G^*) = \text{Ult}_0((Q \parallel \uparrow h G, G), W_1).$$

As in the comparison argument, E^* and G^* are initial segments of the extenders of j , so by our initial segment conditions, $\eta_0 = \eta_1$ and $E^* = G^*$. But then $E = G$, because these are the first whole initial segments of $E^* = G^*$ that do not belong to $P = Q$.



Subclaim 5C.3 Neither $\Sigma_{0,\gamma} J_{\mathcal{J}}$ nor $\Sigma_{0,\delta} J_{\mathcal{U}}$ drops; moreover $P = Q$ and $i_{0,\gamma}^{\mathcal{J}} = i_{0,\delta}^{\mathcal{U}}$.

Proof If $\Sigma_{0,\delta} J_{\mathcal{U}}$ drops, then Q is unsound, so $P \not\leq Q$ and $\Sigma_{0,\gamma} J_{\mathcal{J}}$ does not drop. But then $\sigma_{\delta} \circ i_{0,\gamma}^{\mathcal{J}}$ maps \mathcal{M} to Q_{δ}^* , a dropping iterate of \mathcal{M} via Σ , contrary to weak Dodd-Jensen.

If $\Sigma_{0,\gamma} J_{\mathcal{J}}$ drops, then $Q \leq P$. This is clear if P is unsound. The alternative is that we have $T\text{-pred}(\eta+1) = 0$ with $\eta+1 \in \Sigma_{0,\gamma} J_{\mathcal{T}}$ and E_{η} long and $\text{crit}(E_{\eta}) = \kappa_F$, so that $P_{\eta+1} = \text{Ult}_{\sigma_0}(\pi, E_{\eta})$ is sound, and there is no further dropping on $\Sigma_{0,\gamma} J_{\mathcal{T}}$. But then it

M is a generalized core of P ,

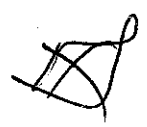
so $P \neq Q$ by 5c.2.

So if $[0, \gamma]_T$ drops, then $[0, \delta]_U$ does not, and $Q \subseteq P$. But then

$\pi_\gamma \circ j_{0, \delta}^{1, 2} : M \rightarrow P_\gamma^*$ contradicts weak

Dodd-Jensen.

The proof that $P = Q$ and $j_{0, \gamma}^{1, 5} = j_{0, \delta}^{1, 2}$ is similar,



We can now prove 5c in just the same way that we finished the proof of 5b.



Claim 6c It is not the case that P is above \mathcal{M} and Q is above \mathcal{N} .

Proof. Suppose otherwise, i.e. $\mathcal{O}T_\gamma$ and $1U_\delta$.

Subclaim 6c.1 $\{1, \delta\} \cup$ does not drop, in model or degree.

Proof Otherwise $P \trianglelefteq Q$ and $\{0, \gamma\} \cup$ does not drop. (Note here that if $\eta+1 \in \{0, \gamma\} \cup$, $\text{crit}(E_\gamma) = \kappa_F$, and thus $P_{\eta+1} = \text{Ult}_\kappa(\mathcal{M}, E_\gamma)$, and this is the only drop on $\{0, \gamma\} \cup$, then $P \not\trianglelefteq Q$. For then $\pi \in Q_\beta$, where for some $\beta \geq 1$, because π -to- Q dropped.) But then

$\pi_\delta \circ \iota_{0, \delta}^{\text{IV}} : \mathcal{M} \rightarrow Q_\delta^*$ contradicts the weak Dodd-Jensen property of Σ . □

Subclaim 6c.2 $P = \mathbb{Q}$,

Proof. If $P \triangleleft \mathbb{Q}$, then $\Sigma_{0, \delta} J_T$ does not drop. This is because P would be unsound, unless there were exactly one "special drop" to \mathcal{N} on $\Sigma_{0, \delta} J_T$, and $P = \text{Ult}_0(\mathcal{N}, E_{\delta-1})$.

But then $\mathcal{N} \in \mathbb{Q}$, whereas \mathcal{N} definably collapses α . Thus $\mathbb{Q} \triangleleft P$, by weak Dodd-Jensen.

If $\mathbb{Q} \triangleleft P$, then $\text{Th}_1^\mathcal{N}(K_F^+ \cup p_1(\mathcal{N}))$ belongs to P . This is clear if $\text{crit}(i_{1, \delta}^\mathcal{N}) > \alpha$, and it holds if $\text{crit}(i_{1, \delta}^\mathcal{N}) \leq \alpha$, for then the first extender used in $\Sigma_{1, \delta} J_U$ is long, with critical point K_F , and we can use its super-strong part as usual to define ~~and~~ $\text{Th}_1^\mathcal{N}(K_F^+ \cup p_1(\mathcal{N}))$ over the ultrapower.

But $\text{Th}_1^\mathcal{N}(K_F^+ \cup p_1(\mathcal{N})) \notin P$, because it collapses α .



Now let $\eta+1 \in [0, \gamma]_T$ be such that $T\text{-pred}(\eta+1) = 0$.

Subclaim 6c.3

$$(a) \rho_1(Q) = K_F^+$$

$$(b) \text{crit}(E_\eta) = K_F$$

Proof We already argued (a) in the proof of 6c.2. For (b), we otherwise have

$\text{crit}(E_\eta) < K_F$. ~~We cannot have $[\eta+1, \gamma]_T$ drop, as otherwise $\eta \in P_\beta$ for some $\beta \geq 1$.~~

But then $\rho_1(P) = \rho_1(Q) = \alpha$ is in the interval $(\text{crit}(E_\eta), 1_{E_\eta}^{\text{crit}(E_\eta)})$, which is impossible.

Subclaim 6c.4 $[\eta+1, \gamma]_T$ does not drop.

Proof Otherwise $\eta \in P_\beta$ for some $\beta \geq 1$.



Since $\rho_1(P) = K_F^+$ and $\text{crit}(E_\eta) \in K_F$,

it must be that $\text{crit}(E_\eta) = K_F$ and E_η is long; moreover E_η has exactly one long generator because P has no longer extenders with space K_F^+ on its sequence. Thus P is of type Z_p , and the first core of P is

$$C_1(P) = P_{\eta+1}$$

and $P_{\eta+1}$ is of type Z_p . The generalized core of $P_{\eta+1}$ is \mathcal{N} , because the branch \mathcal{N} -to- \mathcal{Q} did not drop, so that \mathcal{N} is the generalized core of \mathcal{Q} . So

$$P_{\eta+1} = \text{Olr}_0(\mathcal{N}, E_\eta),$$

and for $r+1$ least in $\{1, \delta J_u\}$,

$$Q_{r+1} = \text{Olr}_0(\mathcal{N}, F_r).$$

Again, F_r has exactly one long

generator. We have

$$\lambda_{\tau+1, \gamma}^{\mathcal{E}} \circ \lambda_{\mathcal{E}_\gamma}^{\mathcal{U}} = \lambda_{\tau+1, \gamma}^{\mathcal{U}} \circ \lambda_{\mathcal{F}_\tau}^{\mathcal{E}}$$

because both are the uncoring map from \mathcal{N} into $\mathcal{P} = \mathcal{Q}$. We then proceed as in the comparison-terminates argument to show that $\mathcal{E}_\gamma = \mathcal{F}_\gamma$. This is a contradiction.



By 5c and 6c, \mathcal{P} is above \mathcal{H} in \mathcal{J} .

Claim 7c \mathcal{Q} is above \mathcal{M} in \mathcal{U} .

Proof Suppose \mathcal{Q} is above \mathcal{M} .

Subclaim 7c.1 $[\tau+1, \gamma]_{\mathcal{J}_\tau}$ does not drop.

Proof ~~Let~~ Let $\tau+1$ be largest in $D^{\mathcal{E}} \cap [\tau+1, \gamma]_{\mathcal{J}_\tau}$. Then every

Σ_1^P subset of α is definable over $M_{\gamma+1}^{*2}$, by the proof of claim 11 (p. 43) in the non-anomalous case. Thus every Σ_1^P subset of α belongs to P_β , where $\beta = T\text{-pred}(\gamma+1)$, so that $\beta \geq 1$.

Since $\Sigma_1, \delta J_T$ drops, $\Sigma_1, \delta J_u$ does not drop. Either $\text{crit}(u_{1,\delta}^n) > \alpha$, or the first extender used in $\Sigma_1, \delta J_u$ is long, and in either case $Th, \eta(\alpha)$ is Σ_1^Q . Thus $Th, \eta(\alpha) \in P_\beta$ for $\beta \geq 1$. But α is a cardinal of P_β for $\beta \geq 1$, and $Th, \eta(\alpha)$ collapses α , contradiction.

Subclaim 7C.2 $P \triangleleft Q$.

Proof If $Q \triangleleft P$, then $\Sigma_1, \delta J_u$ does not drop, so again $Th, \eta(\alpha)$ is Σ_1^Q . That gives $Th, \eta(\alpha) \in P$, contrary to α being

a cardinal of P .

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So $\mathcal{T}_{h, \mathcal{H}}^{\mathcal{H}}(\alpha \cup \bar{F})$ is $\sum_{i=1}^Q$, because
it is equal to $\mathcal{T}_{h, \mathcal{H}}^P(\alpha \cup \mathcal{I}_{1, \mathcal{H}}^{\mathcal{H}}(\bar{F}))$. By the
proof of claim 11 on p. 43,
 $\mathcal{T}_{h, \mathcal{H}}^{\mathcal{H}}(\alpha \cup \bar{F})$ is $\sum_{i=1}^m$. However, $\mathcal{M} \in \mathcal{M}$,
so $\mathcal{T}_{h, \mathcal{H}}^{\mathcal{H}}(\alpha \cup \bar{F}) = \mathcal{T}_{h, \mathcal{M}}^m(\alpha \cup \bar{F}) \in \mathcal{M}$, a
contradiction.

claim 7c.

So P is above \mathcal{H} and Q is above \mathcal{M} .
We proceed to a contradiction exactly as in
anomalous case 2.

This completes the proof of theorem 20 when
 $k=0$, in anomalous case 3.



Proof of Theorem 10 in anomalous case 4

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We have an $\eta < o(M)$ such that
 $\alpha = (K^{++})^{M \upharpoonright \eta}$, $\rho_1(M \upharpoonright \eta) \leq (K^+)^{M \upharpoonright \eta} = K^{+M}$,
and $M \upharpoonright \eta$ is active with short last
extender F satisfying $K_F = K$.

Claim 3d. $cf(\alpha) = K^+$ in M .

Proof Let $f: K^+ \xrightarrow{\text{onto}} \alpha$ be partial
and \sum_{α_1} over $M \upharpoonright \eta$. We have that
 $M \upharpoonright \eta = (J_{\uparrow}^{\vec{E}}, i_F \upharpoonright (M \upharpoonright K^+))$ where $\uparrow =$
 $\sup i_F'' K^+$. Let for $\xi < K^+$

$$P_\xi = (J_{\sup i_F'' \xi}^{\vec{E}}, i_F \upharpoonright (M \upharpoonright \xi)).$$

Then we have

$$f = \bigcup_{\xi < K^+} f_\xi$$

where

$$f_\xi \text{ is } \sum_{\alpha_1} P_\xi.$$

Since $\rho_1(M) < \alpha$, $\rho_1(M) \leq \kappa^+$,

and hence $\rho_1(M) = \rho_1(S)$. (In fact, $\rho_1(M) \leq \kappa$ by projectum-free spaces for M , but we don't need this.) From W we can define the new \sum_{γ}^S subset of $\rho_1(S)$ from $i_G(\rho_1(M)) \cup \{\gamma\}$, where $\gamma = \gamma(G) - 1$, so that $i_G(\kappa) < \gamma < i_G(\kappa^+) < \alpha_S$. This

gives

Claim 2d $i_G(r)$ is an initial segment of $\rho_1(S)$, and there is a largest $\mu \in \rho_1(S) - (i_G(r) + 1)$, and for this μ , $\alpha_S \leq \mu$.

Proof $\rho_1(S) \leq_{\text{lex}} \left(i_G(\rho_1(M)) \cup \{\gamma\} \right) \stackrel{\text{def}}{=} \tau$

But $i_G(r)$ is solid over S , so $\rho_1(S)$ must agree with τ up through $i_G(r)$. We

cannot have $p_1(S) = i_G(r)$, because $Th_1^S(p_1(S) \cup i_G(r))$ can be computed from $i_G(Th_1^M(p_1(M) \cup r))$. So let μ be largest in $p_1(S) - (i_G(r) + 1)$. If $\mu < \alpha_S$, then $\mu < i_G(\xi)$ for some $\xi < \alpha$. But $Th_1^M(\xi \cup r) \in M$ by our choice of α , so $i_G(Th_1^M(\xi \cup r)) \in S$, and from it we can compute $Th_1^{\overline{M}}(\mu + 1 \cup i_G(r))$, contradiction.



Claim 3d α_S is the least ξ such that $Th_1^S(\xi \cup i_G(r)) \notin S$.

Proof

Claim 3d $Th_1^S(\alpha_S \cup i_G(r)) \in S$.

Proof Suppose not. Then by the proof of 2d,

$$\alpha_S = \text{least } \xi \text{ such that } \xi \leq \mu$$

$$\text{and } \text{Th}_1^S(\xi \cup i_G(r)) \notin S,$$

where μ is as in 2d. We now just apply the arguments we have given for $M, r,$ and α to $S, i_G(r),$ and $\alpha_S.$

Note anomalous case 4 does not apply to $S, i_G(r),$ and $\alpha_S.$ This is because $S \models \text{cf}(\alpha_S) = \kappa^+$, and in S, κ^+ is not the space of a total long extender on the sequence.

Note also $p_1(S) < \alpha_S.$ And finally, we cannot have that $p_1(S) = i_G(r) \cup \{\alpha_S\}$ and is of type $\mathbb{Z}_p.$ That is because in S, α_S is not the successor of a limit cardinal.

All other cases led to a contradiction.



But note

$$i_G \wedge \alpha_{\mathbb{R}} \in S,$$

because if we let $f: K^+ \xrightarrow{\text{onto}} \alpha$ with $f \in \mathcal{M}$, then $i_G(f(\xi)) = i_G(f)(i_G(\xi))$,

and $i_G \wedge K^+ \in S$. So from 3d, we get

$$Th^M(\alpha \vee r) \in S,$$

But $P(\alpha)^S = P(\alpha)^M$, so

$$Th^M(\alpha \vee r) \in \mathcal{M},$$

a contradiction.

This proves Theorem 10 when $k=0$ in anomalous case 4.



This proves Theorem 10 when $k=0$.



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