Fine structure for plus-one premice

Itay Neeman
J. R. Steel
April 2014

We develop further the theory of [IJ]. Our main results here concern solidity, universality, and condensation for iterable plus-one premice.

We begin by defining the Σ_0-code of a plus-one premouse $\mathcal{M}$. This is a structure in the language $\mathcal{L}_0$ having $\in$, unary predicate symbols $\dot{E}$ and $\dot{F}$, and a constant symbol $\dot{\nu}$. We shall usually identify $\mathcal{M}$ with its Σ_0-code. There are three types:

(1) $\mathcal{M}$ is passive. In this case, $\dot{E}^{\mathcal{M}}$ is the extender sequence from which $|\mathcal{M}|$ is constructed. $\dot{F}^{\mathcal{M}} = \emptyset$, and $\dot{\nu}^{\mathcal{M}} = 0$.

(2) $\mathcal{M}$ is active, with short last extender. Then again, $\mathring{E}^{\mathcal{M}}$ is the extender sequence from which $|\mathcal{M}|$ is constructed. $\mathring{F}^{\mathcal{M}}$ is the last extender, presented as in Jensen. See page 2a

That is, $\mathring{F}^{\mathcal{M}} = \mathring{i}_F^{\mathcal{M}}$, where

$$i_F^{\mathcal{M}} : \mathcal{M} | \kappa^{+\mathcal{M}} \longrightarrow \mathcal{M}^-, \text{ cofinal.}$$

Here we are thinking of F as the corresponding system of measures; so $\mathcal{M}^- = \mathrm{Ult}_0(\mathcal{M}|\kappa^+, F)$, where $\mathcal{M}^-$ is $\mathcal{M}$ without its last extender. We shall tend to identify F with $\mathring{F}^{\mathcal{M}}$.

(3) $\mathcal{M}$ is active, with long last extender F, and $\nu + 1 = \nu(F)$. Then $\mathring{E}^{\mathcal{M}}$ is the extender sequence from which $|\mathcal{M}|$ is constructed, and $\mathring{\nu}^{\mathcal{M}} = \nu$. We have the coherence/indexing condition

$$\mathcal{M}^- = \mathrm{Ult}_0(\mathcal{M}, F) | (\lambda_F^+)^{\mathrm{Ult}(\mathcal{M}, F)}.$$

(Recall $\kappa_F = \mathrm{crit}(F)$, $\lambda_F = i_F(\kappa_F)$.) We can

If $\dot{F}^m = F$ is short, and $\eta < \lambda_F$ is largest such that $F \restriction \eta$ is whole, then $\dot{\nu}^m = \gamma$, where $\dot{E}^M_\gamma$ is the trivial completion of $F \restriction \eta$. If $\dot{F}^m$ is short, but has no largest proper whole initial segment, then $\dot{\nu}^m = 0$.

That is, we have a name for the largest whole proper initial segment, if there is one. This is the convention of both FSIT and Jensen. Without it, it's not clear that Σ_1-hulls preserve premousehood.

When one gets to the proof of $\square$, it is important to consider the "language of coherent structures", which does not have this name. See ΣSJ.

code F as an amenable predicate as in [2]: for each $\xi < (\kappa_F^{++})^{\mathcal{M}}$,

$$F_\xi = \{(a, X) \mid a \in [\lambda_F \cup \{\nu\}]^{<\omega} \wedge X \in \mathcal{M}|\xi \wedge a \in i_F^{\mathcal{M}}(X)\}$$

is in $\mathcal{M}$. Let γ_ξ be least such that $F_\xi \in \mathcal{M}|\gamma_\xi$. The γ_ξ's are cofinal in $o(\mathcal{M}) = (\lambda_F^+)^{Ult_0(\mathcal{M}, F)}$. We set

$$\dot{F}^{\mathcal{M}} = \{(\gamma, a, X) \mid \gamma < o(\mathcal{M}) \wedge \exists \xi\, (\gamma_\xi \le \gamma \wedge (a, X) \in F_\xi\}.$$

So the three types of $\mathcal{M}$ are passive, active short, and active long.

<u>Definition I</u> Let $\mathcal{M}$ be active long; then $\mathcal{M}$ is <u>Dodd-solid</u> iff $\dot{F}^{\mathcal{M}} \restriction \dot{\nu}^{\mathcal{M}} \in |\mathcal{M}|$. If $\dot{F}^{\mathcal{M}} \restriction \dot{\nu}^{\mathcal{M}} \notin |\mathcal{M}|$, then $\mathcal{M}$ is <u>type Z_1</u>. //

By [1], if $\mathcal{M}$ is type Z_1, then ~~than~~ $H = (\dot{E}_{\dot{\gamma}})^{\mathcal{M}}$ is a short extender, and for $F = \dot{F}^{\mathcal{M}}$, we have that

$$\kappa_F < \kappa_H < \lambda_F = \lambda_H,$$

and there are cofinally many $\xi < \nu$ such that $F \restriction \xi \in \operatorname{ran}(i_H^{\mathcal{M}})$, and $(\kappa_H^+)^{\mathcal{M}}$ is not the space of an extender on $\dot{E}^{\mathcal{M}}$.

When we get to 1-cores, we shall have to divide the type Z_1 case further, according to whether $\dot{\nu}^{\mathcal{M}}$ is Σ_1-singular or not. We can defer that for now, however.

<u>Definition 2</u> Let $\mathcal{M}$ be (the Z_0-code of) a plus-one premouse; then

$\rho_1(\mathcal{M})$ = least η such that there is a $\underset{\sim}{\Sigma}_1^{\mathcal{M}}$ set $A \subseteq \eta$ such that $A \notin |\mathcal{M}|$

and

$\rho_1(\mathcal{M}) = <_{lex}$ - least $\langle \alpha_0, \dots, \alpha_k \rangle$
such that $\alpha_0 > \alpha_1 > \cdots > \alpha_k$
and there is an $A \subseteq \rho_1(\mathcal{M})$
such that $A \notin |\mathcal{M}|$, and
A is $\Sigma_1^{\mathcal{M}}(\{\alpha_0 \dots \alpha_k\})$.

<u>Definition 3</u> Let $\mathcal{M}$ be a $\mathcal{L}_0$ code, and $X \subseteq |\mathcal{M}|$; then

(i) $Th_1^{\mathcal{M}}(X) = \{(\varphi, s) \mid \varphi$ is Σ_1 in $\mathcal{L}_0$, $s \in X^{<\omega}$, and $\mathcal{M} \models \varphi[s]\}$,

(ii) $Hull_1^{\mathcal{M}}(X)$ = substructure of $\mathcal{M}$ with universe the set of all a such that a = unique b s.t. $\mathcal{M} \models \varphi[b, s]$, for some Σ_1 φ in $\mathcal{L}_0$, and $s \in X^{<\omega}$,

(iii) $\mathcal{H}_1^{\mathcal{M}}(X)$ = transitive collapse of $Hull_1^{\mathcal{M}}(X)$.

<u>Definition 4</u> Let $\mathcal{M}$ be a Σ_0-code, and $p_1(\mathcal{M}) = \langle \alpha_0 \dots \alpha_k \rangle$; then

(a) $\mathcal{M}$ is <u>1-solid at α_i</u> iff $Th_1^{\mathcal{M}}(\alpha_i \cup \{\alpha_0 \dots \alpha_{i-1}\}) \in \mathcal{M}$, and

(b) $\mathcal{M}$ is <u>1-universal</u> iff $P(\rho_1(\mathcal{M})) \cap \mathcal{M} \subseteq \mathcal{H}_1^{\mathcal{M}}(\rho_1(\mathcal{M}) \cup p_1(\mathcal{M}))$.

We often identify $p_1(\mathcal{M})$ with $\{\alpha_0 \dots \alpha_k\}$. So $\mathcal{M}$ (or $p_1(\mathcal{M})$) is 1-solid at α iff $\alpha \in p_1(\mathcal{M})$, and for $r = p_1(\mathcal{M}) - (\alpha+1)$, $Th_1^{\mathcal{M}}(\alpha \cup r) \in \mathcal{M}$.

The following example shows that we cannot expect that our $\mathcal{M}$ will always be fully 1-solid. It is a minor variant of a counterexample to condensation due to Woodin.

Example 5. Let $\mathcal{M}$ be an active long plus-one premouse reached in a plus-one construction, and not yet cored down. Let $\kappa = \mathrm{crit}(\dot{F}^{\mathcal{M}})$, and

$$\mathcal{N} = \mathcal{H}_1^{\mathcal{M}}(\kappa^{+\mathcal{M}}).$$

Assume that $\rho_1(\mathcal{M}) \geq \kappa$, so that in fact $\rho_1(\mathcal{M}) > (\kappa^+)^{\mathcal{M}}$ by amenable closure. (See [1], Cor. 10.) Then

$$\rho_1(\mathcal{N}) = (\kappa^+)^{\mathcal{N}}$$

and

$$p_1(\mathcal{N}) = \emptyset.$$

$\mathcal{M}$ was amenably closed at $(\kappa^+)^{\mathcal{M}} = (\kappa^+)^{\mathcal{N}}$, ~~by~~ but $\mathcal{N}$ is not, and indeed $\mathcal{N} \ntrianglelefteq \mathcal{M}$ by Cor. 10 of [1]. (Which is due to Woodin; this is his example of a failure of condensation.) $\mathcal{N}$ fails to have projectum-free spaces.

Now assume that no proper initial segment of $\dot{F}^{m}$ is long, so that the same is true of $\dot{F}^{n}$, and let

$$\mathcal{P} = Ult_0(\mathcal{M}, \dot{F}^{n}).$$

Let $\nu = \dot{\nu}^{n}$, so that

$$E_\nu^{\mathcal{P}} = \dot{F}^{n} \restriction \nu$$

is the "superstrong part" of $\dot{F}^{n}$. Letting

$$A = Th_1^{n}(\kappa^{+n})$$

we have that $A \notin |\mathcal{P}|$, and A is $\Sigma_1^{\mathcal{P}}(\{\nu\})$, because $i_{\dot{F}^n}^{n} \restriction (\kappa^+)^{n}$ can be computed from $E_\nu^{\mathcal{P}}$ (in fact, they are basically equal), and $(\varphi, s) \in A$ iff $(\varphi, i_F(s)) \in Th_1^{\mathcal{P}}(\nu)$. This shows that $\rho_1(\mathcal{P}) \leq (\kappa^+)^{n} = (\kappa^+)^{\mathcal{P}}$, and that $Th_1^{\mathcal{P}}(\nu) \notin |\mathcal{P}|$.

Any $\Sigma_1^{\mathcal{P}}$ subset of ~~κ~~ κ is also Σ_1^{n}, so $\rho_1(\mathcal{P}) > \kappa$. Thus $\rho_1(\mathcal{P}) = (\kappa^+)^{\mathcal{P}}$. Moreover, if $\xi < \nu$, then $Th_1^{\mathcal{P}}(\xi) \in |\mathcal{P}|$, since it is an initial segment of $i_{F_n}(Th_1^{\mathcal{N}}(\alpha))$, for some $\alpha < (\kappa^+)^{\mathcal{N}}$. Thus

$$\rho_1(\mathcal{P}) = (\kappa^+)^{\mathcal{P}}$$

and

$$p_1(\mathcal{P}) = \{\nu\}.$$

Since $Th_1^{\mathcal{P}}(\nu) \notin |\mathcal{P}|$, $p_1(\mathcal{P})$ is not solid at ν.

But $\mathcal{P}$ has projectum-free spaces, so it is the sort of thing our construction might reach. Our condensation results below will show that in fact $\mathcal{P} \trianglelefteq \mathcal{M}$.

Notice that $p_1(\mathcal{P})$ is weakly solid, in that $Th_1^{\mathcal{P}}(\xi) \in |\mathcal{P}|$ for all $\xi < \nu$. Moreover, we have a type Z_1 - like explanation for the failure of full solidity. We shall show that this is always the case for premice in a plus-one construction. That in turn will be enough to insure Σ_0 ultrapower maps preserve p_1.

One can think of $\mathcal{N}$ as a "generalized core" of $\mathcal{P}$. Our construction does not take such generalized cores. The construction of [6] does.

<u>Definition 6</u> Let $\mathcal{M}$ be a plus-one premouse, $\alpha \in p_1(\mathcal{M})$, and $r = p_1(\mathcal{M}) - (\alpha+1)$; then $\mathcal{M}$ (or $p_1(\mathcal{M})$) is <u>weakly solid at α</u> iff for all $\xi < \alpha$, $Th_1^{\mathcal{M}}(\xi \cup r) \in \mathcal{M}$.

<u>Definition 7.</u> Let $\mathcal{M}$ be a plus-one premouse; then $\mathcal{M}$ (or $p_1(\mathcal{M})$) is <u>of stretched type</u>, or <u>type Z_p</u> iff $p_1(\mathcal{M}) \neq \emptyset$, and letting α be least in $p_1(\mathcal{M})$, and $r = p_1(\mathcal{M}) - (\alpha+1)$, we have

(1) $\mathcal{M}$ is solid at all $\beta \in r$,
(2) $\mathcal{M}$ is weakly solid at α, and
(3) Letting $E = \dot{E}_\alpha^{\mathcal{M}}$ and $\kappa = \kappa_E$,
  (a) E is short
  (b) $\rho_1(\mathcal{M}) = (\kappa^+)^{\mathcal{M}}$, and
  (c) Letting $\mathcal{H} = \mathcal{H}_1^{\mathcal{M}}(i_E{}''\kappa^{+\mathcal{M}} \cup r)$, and $\pi : \mathcal{H} \to \mathcal{M}$ be the uncollapse map,
    (i) $\mathcal{M} | (\kappa^{++})^{\mathcal{M}} = \mathcal{H} | (\kappa^{++})^{\mathcal{H}}$,
    (ii) $\pi \restriction (\kappa^+)^{\mathcal{H}} = i_E \restriction (\kappa^+)^{\mathcal{M}}$, and
    (iii) $\pi^{-1}(r) = p_1(\mathcal{H})$, and $p_1(\mathcal{H})$ is solid.

Remarks

(1) "Type Z_p" stands for "Type Z for parameters".

(2) By (3)(c), $\mathcal{H} = \mathrm{Hull}_1^{\mathcal{H}}((\kappa^+)^{\mathcal{H}} \cup \pi^{-1}(r))$, so $\rho_1(\mathcal{H}) = (\kappa^+)^{\mathcal{H}}$, and $\mathcal{H}$ collapses $(\kappa^{++})^{\mathcal{M}} = (\kappa^{++})^{\mathcal{H}}$. We call $\mathcal{H}$ a generalized core of $\mathcal{M}$. Clearly, example 5 exhibits this situation. For the type Z_p premice $\mathcal{M}$ that our construction reaches, $\kappa^{\mathcal{H}}$ is the critical point of a total long extender from the $\mathcal{H}$-sequence. Thus $\mathcal{H}$ violates projectum-free spaces, and is therefore not a plus-one premouse.

The construction of Woodin's [6] will core down to $\mathcal{H}$ if it reaches a type Z_p $\mathcal{M}$. Our construction does not do that, and thus ~~[illegible]~~ seems superficially to preserve more information.

We shall show, however, that if $\mathcal{M}$ is 1-sound and of type Z_p, and $\mathcal{N}$ is its generalized core, then $\mathcal{M}$ is embeddable into an ultrapower of $\mathcal{N}$. More precisely, letting κ be the critical point of the stretching extender, there is a first long extender L on the $\mathcal{N}$-sequence with domain $\mathcal{N} | \kappa^{++\mathcal{N}}$, and

$$\mathcal{M} = \mathrm{Ult}_0(\mathcal{N}, L \restriction (\kappa^{+\mathcal{N}} \cup \{\nu\})),$$

where ν is the unique long generator of L. Thus $\mathcal{M}$ and $\mathcal{N}$ are easily interdefinable, and coring down all the way to $\mathcal{N}$ does not lose information.

(3) We shall generalize the definitions and example above to higher levels of the Levy hierarchy later.

<u>Lemma 8.</u> Let $\mathcal{M}$ be type Z_p, and let α be least in $p_1(\mathcal{M})$; then $\mathcal{M}$ is not solid at α.

<u>Proof.</u> Letting E be as in clause (3) and $\kappa = \kappa_E$, we have $i_E \restriction \kappa^+ \in \mathcal{M}$. So if $Th_1^{\mathcal{M}}(\alpha \cup r) \in \mathcal{M}$, then since $i_E``\kappa^+ \subseteq \alpha$, $Th_1^{\mathcal{H}}(\kappa^{+\mathcal{H}} \cup \pi^{-1}(r)) \in \mathcal{M}$. But then $(\kappa^{++})^{\mathcal{M}}$ is not a cardinal in $\mathcal{M}$.

$\boxtimes$

We shall prove a converse to lemma 8 for the levels of a plus-one construction. We shall need the following consequence of condensation for those levels in its proof.

<u>Lemma 9.</u> Let $\mathcal{M}$ be a level of a plus-one construction, and $\alpha < o(\mathcal{M})$. Suppose there is an η such that $\dot{F}^{\mathcal{M}|\eta}$ is a long extender with domain $\mathcal{M}|\alpha$, and let γ be the least such η; then $\rho_1(\mathcal{M}|\gamma) \le \alpha$, and $p_1(\mathcal{M}|\gamma) \subseteq \alpha$.

We defer the proof of lemma 9 until after we have proved condensation for the levels of plus-one constructions.

Definition 9a Let $\mathcal{M}$ be a plus-one premouse; then $\mathcal{M}$ has long extender condensation iff whenever $\alpha < o(\mathcal{M})$, and γ is least such that $\dot{F}^{\mathcal{M}|\gamma}$ is long with domain $\mathcal{M}|\alpha$, then $\rho_1(\mathcal{M}|\gamma) \leq \alpha$ and $p_1(\mathcal{M}|\gamma) \subseteq \alpha$.

In these notes, we shall prove the following theorem in the case $k=0$.

Theorem 10 Let $\mathcal{M}$ be a k-sound, ω_1+1-iterable plus-one premouse having long extender condensation; then

(a) $p_{k+1}(\mathcal{M})$ is either solid at all $\alpha \in p_{k+1}(\mathcal{M})$, or of type $\mathcal{Z}_\rho$, and

(b) $p_{k+1}(\mathcal{M})$ is $k+1$-universal.

Of course, we have not yet formally defined k-soundness and $p_{k+1}(\mathcal{M})$ when $k>0$. Our proof in the case $k=0$ will therefore have some gaps. These are easy to fill.

Proof of Theorem 10 when $k=0$.

We follow the solidity proof in §8 of [5], making use of the repairs and improvements of [4] and [2]. But there are new issues, and our conclusion is more complicated.

By Löwenheim-Skolem, we may assume that $\mathcal{M}$ is countable. Let $|\mathcal{M}| = \{e_i \mid i \in \omega\}$, and let Σ be a $(0, \omega_1, \omega_1+1)$-iteration strategy for $\mathcal{M}$ with the weak Dodd-Jensen property relative to $\vec{e}$.

If $\mathcal{M}$ is not solid at some element of $p_1(\mathcal{M})$, then set

$$\mu_0 = \text{largest } \mu \in p_1(\mathcal{M}) \text{ such that } \mathcal{M} \text{ is not solid at } \mu,$$

$$r = p_1(\mathcal{M}) - (\mu_0 + 1),$$

and

$$\alpha = \text{least } \xi \leq \mu_0 \text{ such that } Th_1^{\mathcal{M}}(\xi \cup r) \notin \mathcal{M}.$$

Our goal in this case is to show that $\alpha = \mu_0$, and that $p_1(\mathcal{M})$ is of type $\mathbb{Z}_p$, and 1-universal.

If $\rho_1(\mathcal{M})$ is solid at all $\mu \in \rho_1(\mathcal{M})$, then set

$$\alpha = \rho_1(\mathcal{M}),$$
$$r = p_1(\mathcal{M}).$$

Our goal in this case is to show that r is 1-universal.

Notice that in both cases, r is fully solid, and α is the least ξ such that $Th_1^{\mathcal{M}}(\xi \cup r) \notin \mathcal{M}$. In the first case, $\alpha > \rho_1(\mathcal{M})$, while in the second case, $\alpha = \rho_1(\mathcal{M})$.

Let

$$\mathcal{H} = \mathcal{H}_1^{\mathcal{M}}(\alpha \cup r),$$

and

$$\pi : \mathcal{H} \longrightarrow \mathcal{M}$$

be the collapse map. So $\mathrm{crit}(\pi) \geq \alpha$. We note that $\alpha \in \mathcal{H}$, as otherwise $\mathcal{H} \trianglelefteq \mathcal{M}$, and so $Th_1^{\mathcal{M}}(\alpha \cup r) \in \mathcal{M}$.

<u>Claim 4</u>. α is a cardinal of $\mathcal{H}$.

<u>Proof</u> Suppose not, and let $\eta = |\alpha|^{\mathcal{H}}$. Since α is not a cardinal of $\mathcal{H}$, $\mathrm{crit}(\pi) > \alpha$. Thus we can fix ξ such that $\eta < \xi < \alpha$ and $\alpha \in \mathrm{Hull}_1^{m}(\xi \cup r)$. But then there is an $f : \xi \xrightarrow{\text{onto}} \alpha$ such that $f \in \mathrm{Hull}_1^{m}(\xi \cup r)$. This implies that $Th_1^{m}(\alpha \cup r)$ can be easily computed from $Th_1^{m}(\xi \cup r)$. But $Th_1^{m}(\xi \cup r) \in \mathcal{M}$, so $Th_1^{m}(\alpha \cup r) \in \mathcal{M}$, a contradiction.

$\boxtimes$

<u>Remark</u>. It is possible that $\mathrm{crit}(\pi) > \alpha$. For example, $\mathcal{M}$ may be 1-sound already, so that $\pi = id$.

Our strategy is to compare the phalanx $(\mathcal{M}, \mathcal{H}, \alpha)$ vs. $\mathcal{M}$. There are four anomalous cases, however.

In these cases, α is not a cardinal of $\mathcal{M}$, and the level of $\mathcal{M}$ collapsing α has an awkward form.

<u>Anomalous case 1.</u> $\alpha = lh(G)$, for some G on the $\mathcal{M}$-sequence.

<u>Anomalous case 2.</u> α is the psuedo-index of some $\overline{G}$ certified by the $\mathcal{M}$-sequence. That is, there is a type Z_1 level $(\mathcal{M}|\eta, G)$ of $\mathcal{M}$, with stretching extender $F = (\dot{E}_\nu)^{\mathcal{M}|\eta}$, such that $\alpha = (\kappa_F^{++})^{\mathcal{M}|\eta}$.

<u>Anomalous case 3.</u> α is not a cardinal of $\mathcal{M}$, and letting $\langle \eta, k \rangle$ be lex-least such that $\rho_{k+1}(\mathcal{M}) < \alpha$, we have that $\rho_{k+1}(\mathcal{M}|\eta)$ is of type Z_p, and for γ least in $\rho_{k+1}(\mathcal{M})$ and $F = (\ddot{E}_\gamma)^{\mathcal{M}}$, $\alpha = (\kappa_F^{++})^{\mathcal{M}|\eta}$.

<u>Anomalous case 4.</u> α is not a cardinal of $\mathcal{M}$, and letting $\langle \eta, k \rangle$ be lex-least such that $\rho_{k+1}(\mathcal{M}) < \alpha$, we have that $F = \dot{F}^{\mathcal{M}|\eta}$ is short, $\alpha = (\kappa_F^{++})^{\mathcal{M}|\eta}$, $k = 0$, and there are total long extenders on the $\mathcal{M}$-sequence with critical point κ_F.

In each of the anomalous cases, α is not a cardinal of $\mathcal{M}$, so $\alpha > \rho_1(\mathcal{M})$. We shall show that anomalous cases 2-4 lead to a contradiction, while in anomalous case 1, $\rho_1(\mathcal{M})$ is of type Z_P, with the G such that $lh(G) = \alpha$ being the stretching extender, and $\rho_1(\mathcal{M})$ is 1-universal.

§3 of [4] dealt with an anomaly similar to cases 1-3, and we shall use the arguments of that paper here. We deal with anomaly 4 by using ideas related to amenable closure.

We shall also show that in anomalous case 1, the 1-core $\mathcal{H}_1^{\mathcal{M}}(\rho_1(\mathcal{M}) \cup p_1(\mathcal{M}))$ of $\mathcal{M}$ is embeddable into an ultrapower of the generalized core of $\mathcal{M}$.

The anomalous cases are clearly mutually exclusive.

We begin by assuming that none of the anomalous cases applies. Our desired conclusion then is that $\alpha = \rho_1(\mathcal{M})$, and $P(\alpha)^{\mathcal{M}} \subseteq \mathcal{H}_1^{\mathcal{M}}(\alpha \cup r)$.

As usual, we compare $(\mathcal{M}, \mathcal{H}, \alpha)$ with $\mathcal{M}$. Let $\mathcal{T}$ on the $(\mathcal{M}, \mathcal{H}, \alpha)$ side and $\mathcal{U}$ on the $\mathcal{M}$ side be the two iteration trees produced. Let

$$P_\xi = \mathcal{M}_\xi^{\mathcal{T}}$$

and

$$Q_\xi = \mathcal{M}_\xi^{\mathcal{U}},$$

with $P_0 = \mathcal{M}$ and $P_1 = \mathcal{H}$. Let

$$E_\xi = E_\xi^{\mathcal{T}},$$

$$F_\xi = E_\xi^{\mathcal{U}},$$

and $\lambda_\xi^{\mathcal{T}} = \lambda_{E_\xi}$ and $\lambda_\xi^{\mathcal{U}} = \lambda_{F_\xi}$. ($E_0$ is undefined.)

The rules for $\mathcal{U}$ are the usual ones; F_ξ gets applied to the longest initial segment of $\mathcal{Q}_\beta$ possible, where β is least such that $\kappa_{F_\xi} < \lambda_\beta^{\mathcal{U}}$. We use Σ to choose branches at limit stages.

The rules for $\mathcal{T}$ are as follows. Since $\mathcal{H}|(\alpha+1) \trianglelefteq \mathcal{M}$, we must have $\lambda_1^{\mathcal{T}} \geq \alpha$. Set by convention

$$\lambda_0^{\mathcal{T}} = \alpha.$$

Recall that by convention the domain of an extender E over N is $N|(\kappa_E^+)^N$ if E is short, and $N|(\kappa_E^{++})^N$ if E is long and plus-one. Then in $\mathcal{T}$

$$T\text{-pred}(\xi+1) = \text{least } \beta \text{ such that } \operatorname{dom}(E_\xi) \subseteq M_\beta | \lambda_\beta^{\mathcal{T}}.$$

This is equivalent to the critical point rule $T\text{-pred}(\xi+1) =$ least β such that $\kappa_{E_\xi} < \lambda_\beta^{\mathcal{I}}$, except when $(\kappa_{E_\xi}^+)^{P_\xi} = \lambda_0^{\mathcal{I}} = \alpha$, and E_ξ is long. In that case, the critical point rule would have us apply E_ξ to an initial segment of $P_0 = \mathcal{M}$. That does not make sense if P_0 and P_ξ do not agree up to $\mathrm{dom}(E_\xi)$.

<u>Remark</u> In an ordinary iteration tree like $\mathcal{U}$, the critical point rule is equivalent to the domain rule, which is why the critical point rule makes sense. This equivalence will not obtain if we ms-index short extenders. One can get it by moving a little closer to Jensen indexing them. We have gone all the way.

So we use the critical point rule in $\mathcal{T}$ except when $\mathrm{crit}(E_\xi^*)^{+P_\xi} = \alpha$ and E_ξ is long. In that case, we apply E_ξ to the longest possible initial segment of P_1. (Since $\lambda_1^{\mathcal{T}} > \alpha$ in this case.)

The trees $\mathcal{T}$ and $\mathcal{U}$ are fine-structural, 0-maximal trees. So they may drop, and the argument to follow relies on the proper initial segments of $\mathcal{M}$ being sound, and on the maps of a 0-maximal tree preserving the appropriate parameters and cores. We have not yet defined the higher-level parameters and cores, so parts of the proof of theorem 10 we are giving here must be filled in later.

We lift $\mathcal{T}$ to a tree $\mathcal{T}^*$ on $\mathcal{M}$ using $\pi : P_1 \to M$ and $id : M \to M$. The two maps agree up to $\lambda_0^{\mathcal{T}} = \alpha$, which is what we need. Letting

$$P_\xi^* = M_\xi^{\mathcal{T}^*}$$

we have

$$\pi_\xi : P_\xi \longrightarrow P_\xi^*$$

such that $\pi_\xi \restriction \lambda_\xi^{\mathcal{T}} = \pi_\eta \restriction \lambda_\xi^{\mathcal{T}}$ for all $\eta \geq \xi$. Here $\pi_0 = id$, and $\pi_1 = \pi$. We use Σ to choose branches for $\mathcal{T}^*$, and this gives us a branch choice for $\mathcal{T}$.

Since $\mathcal{U}$ is an ordinary 0-maximal iteration tree on $\mathcal{M}$, all extenders F_ξ are close to the models to which they are applied, and all models Q_ξ of $\mathcal{U}$ are plus-one premice. The proof of this is outlined in lemmas 4b and 4 of [IJ].

For $\mathcal{T}$ we have the problem that extenders applied to initial segments of $\mathcal{M}$ may not be close to the model to which they are applied. We do get

Claim 2.

(a) Suppose that T-pred$(\eta+1) \neq 0$; then E_η is close to $\mathcal{M}^{*\mathcal{I}}_{\eta+1}$;

(b) Suppose that P_η is above $\mathcal{H}$ in $\mathcal{I}$, i.e. that $1 T \eta$; then P_η is a plus-one premouse.

(c) Suppose $0 T_\gamma T_\eta$, and P_γ is a plus-one premouse; then P_η is a plus-one premouse.

The proof of claim 2 is like the proof that extenders in an n-maximal tree are close to the models to which they are applied. We omit it.

So we need to look at closeness and premouse-hood when $T\text{-pred}(\eta+1) = 0$.

<u>Claim 3</u> Let $T\text{-pred}(\eta+1) = 0$, and suppose $\mathrm{dom}(E_\eta) = \mathcal{M}|\beta$, where $\beta < \alpha$; then $\mathcal{M}^{*\mathcal{T}}_{\eta+1} = \mathcal{M}$, and E_η is close to $\mathcal{M}$. Thus $P_{\eta+1}$ is a plus-one premouse.

<u>Proof</u> Let $E = E^{\mathcal{T}}_\eta$. By hypothesis, β is a cardinal of $P_\eta | \mathrm{lh} E$. Thus β is a cardinal of $\mathcal{H}$, so $\pi(\beta) = \beta$ is a cardinal of $\mathcal{M}$. Thus $\mathcal{M}^{*\mathcal{T}}_{\eta+1} = \mathcal{M}$. Since E could have been applied in $\mathcal{T}$ to $\mathcal{H}$, our inductive closeness proof shows that E is close to $\mathcal{H}$. But then using $\pi : \mathcal{H} \to \mathcal{M}$, we can see that E is close to $\mathcal{M}$. We omit further detail.

$\boxtimes$

Claim 4 Let $T\text{-pred}(\eta+1) = 0$, and suppose $\mathrm{dom}(E_\eta) = \mathcal{M}|\alpha$, and α is a cardinal of $\mathcal{M}$; then $\mathcal{M}^{*\mathcal{I}}_{\eta+1} = \mathcal{M}$, E_η is close to $\mathcal{M}$, and $P_{\eta+1}$ is a plus-one premouse.

Proof. The same as claim 3.

$\boxtimes$

We now consider the case that $T\text{-pred}(\eta+1) = 0$, $\mathrm{dom}(E_\eta) = \mathcal{M}|\alpha$, and α is not a cardinal of $\mathcal{M}$. We have $\alpha < \pi(\alpha)$. Letting $E = E_\eta$ and $\kappa = \kappa_E$, it is easy to see that

$$\alpha = \kappa^{+\mathcal{H}} = (\kappa^+)^{P_\eta}$$

if E is short, and

$$\alpha = \kappa^{++\mathcal{H}} = (\kappa^{++})^{P_\eta}$$

if E is long. Let

$$\mathcal{M}^{*\mathcal{I}}_{\eta+1} = \mathcal{M}|\xi$$

be the collapsing structure for α in $\mathcal{M}$, and k be least such that

$$\rho_{k+1}(\mathcal{M}|\xi) < \alpha .$$

Claim 5. Let $T\text{-pred}(\eta+1) = 0$, $E = E_\eta$, and $\mathcal{M}^{*\mathcal{T}}_{\eta+1} = \mathcal{M}|\xi$, where $\xi < o(\mathcal{M})$. Let $k = \deg^{\mathcal{T}}(\eta+1)$ be least such that $\rho_{k+1}(\mathcal{M}|\xi) < \alpha$. Suppose E is short; then

(a) $\mathrm{Ult}_k(\mathcal{M}|\xi, E)$ is a plus-one premouse, and

(b) the canonical $i : \mathcal{M}|\xi \to \mathrm{Ult}_k(\mathcal{M}|\xi, E)$ is a k-embedding.

Proof. Note that $\xi > \alpha$ because we are not in anomalous case I. Moreover, $\xi < \pi(\alpha)$.

~~Proof We consider first the case that E is short. So assume that.~~

<u>Subclaim 5.0</u> For all finite $a \subseteq \lambda_E$, $E_a \in \mathcal{M}$.

Proof. If E is not the last extender of P_η, or the branch ending at P_η drops somewhere, then $E_a \in \mathcal{H}$, because E_a is coded by a subset of α. But then $E_a = \pi(E_a) \cap \mathcal{M}|\alpha \in \mathcal{M}$.

If $E = \dot{F}^{P_\eta}$ and the branch to η did not drop, then $1 T \eta$. (Since if $0 T \eta$, then $\mathrm{crit}(i^{\mathcal{T}}_{0\eta}) \leq \kappa_E$ because $\alpha = (\kappa_E^{++})^{\mathcal{M}|\alpha}$, so $\alpha < \kappa_E$, contradiction.) In this case, we consider $\mathcal{T}^*$. We have $E = \dot{F}^{P_\eta}$,

and $\pi_\eta(E) = \dot{F}^{P_\eta^*}$. $E_a \subseteq \mathcal{H}|\alpha$,

and $\pi_\eta \restriction \alpha = \pi_0 \restriction \alpha$, so

$$E_a = \left(\dot{F}^{P_\eta^*}\right)_{\pi_\eta(a)} \cap P_\eta^*|\alpha,$$

But $\kappa = \pi_\eta(\kappa) < \alpha < \pi_\eta(\alpha) = (\kappa^+)^{P_\eta^*}$,

so $E_a \in P_\eta^*$ by weak amenability. Thus

$E_a \in P_0^* = \mathcal{M}$. $\boxtimes$

We also have that each E_a is weakly amenable to $\mathcal{M}|\xi$. For if $\beta < \alpha$, then

$E_a \cap \mathcal{M}|\beta = E_a \cap \mathcal{H}|\beta \in \mathcal{H}$, since E is close to $\mathcal{H}$. So $E_a \cap \mathcal{M}|\beta \in \mathcal{M}|\xi$. $\boxtimes$

<u>Subclaim ~~5~~.1</u> $\mathrm{Ult}_k(\mathcal{M}|\xi, E)$ is a plus-one premouse.

<u>Proof</u> If $k > 1$, the ultrapower is sufficiently elementary. We shall worry about $k = 1$

later. So assume $k=0$. If $\mathcal{M}|\xi$ is not type Z_1, there is no problem, so assume that it is. Let $i : \mathcal{M}|\xi \to Ult_0(\mathcal{M}|\xi, E)$ be the canonical embedding. We have that $\mathcal{M}|\xi \models cf(\dot{\nu})$ is a successor cardinal, and thus i is continuous at $\dot{\nu}^{\mathcal{M}|\xi}$. It follows easily that $Ult_0(\mathcal{M}|\xi, E)$ is a type Z_1 premouse, with $\dot{\nu}^{Ult(\mathcal{M}|\xi, E)} = i(\dot{\nu}^{\mathcal{M}|\xi})$.

$\boxtimes$

<u>Subclaim 5.2</u> $\rho_{k+1}^{Ult(\mathcal{M}|\xi, E)} = \rho_{k+1}^{\mathcal{M}|\xi} = \kappa_E$.

<u>Proof</u> If A is a bounded subset of κ_E, and A is $\Sigma_{k+1}^{Ult_k(\mathcal{M}|\xi, E)}$ in the parameter $[a, f]_E^{\mathcal{M}|\xi}$, then since $E_a \in \mathcal{M}$, $A \in \mathcal{M}$. Thus $A \in \mathcal{M}|\xi$. So $\rho_{k+1}^{Ult_k(\mathcal{M}|\xi, E)} \geq \kappa$. That $\rho_{k+1}^{Ult_k(\mathcal{M}|\xi, E)} \leq \kappa$ follows from E_a being weakly amenable to $\mathcal{M}|\xi$. $\boxtimes$

<u>Subclaim 5.3</u> Let $i : \mathcal{M}|\xi \to Ult_k(\mathcal{M}|\xi, E)$ be the canonical embedding; then i is a k-embedding.

<u>Proof</u> We consider just whether $i(p_{k+1}(\mathcal{M}|\xi)) = p_{k+1}(Ult_k(\mathcal{M}|\xi, E))$. We have

$$p_{k+1}(\mathcal{M}|\xi) = \langle r_0, u \rangle,$$

where u collects the solidity witnesses for $p_{k-1}(\mathcal{M}|\xi)$. (See [3], def. 2.8.1.) i is sufficiently elementary that if b is a solidity witness for γ at level $k-1$, i.e. $b = Th_{k-1}^{\mathcal{M}|\xi}(\gamma \cup s)$, then $i(b)$ is a solidity witness for $i(\gamma)$ in $Ult_k(\mathcal{M}|\xi, E)$. So if $p_{k-1}(\mathcal{M}|\xi)$ is not of type $\mathcal{Z}_p$, then $i(p_{k-1}(\mathcal{M}|\xi)) = p_{k-1}(Ult_k(\mathcal{M}|\xi, E))$, with solidity witnesses in $i(u)$. Suppose now $p_{k-1}(\mathcal{M}|\xi)$ is type $\mathcal{Z}_p$, with least element γ -

Then let $F = (\mathring{E}_\gamma)^{\mathcal{M}|\xi}$ be the stretching extender, so that

$$(\kappa_F^+)^{\mathcal{M}|\xi} = \rho_k(\mathcal{M}|\xi).$$

We have $\kappa_E < \rho_k(\mathcal{M}|\xi)$. Moreover, there can be no $\Sigma_k^{\mathcal{M}|\xi}$ function from κ_F cofinally into $(\kappa_F^+)^{\mathcal{M}|\xi}$, because otherwise $\rho_k(\mathcal{M}|\xi) \leq \kappa_F$. Thus i is continuous at $(\kappa_F^+)^{\mathcal{M}|\xi}$. It follows that $i(F)$ still witnesses the type Z_p property for $i(\gamma)$. So again, $\rho_{k-1}(\mathrm{Ult}(\mathcal{M}|\xi, E)) = i(\rho_{k-1}(\mathcal{M}|\xi))$.

The argument that $i(r)$ is the $k+1^{st}$ standard parameter over $i(u)$ for $\mathrm{Ult}_k(\mathcal{M}|\xi, E)$ is similar. If b is a solidity witness for $\gamma \in \Gamma_0$, then some initial segment of

$i(b)$, in its natural prewellorder, is a solidity witness for $i(\gamma)$ over $Ult_k(\mathcal{M}|\xi, E)$. If r_0 is of type Z_p, with least element γ and stretching extender $F = \dot{E}_\gamma^{\mathcal{M}|\xi}$, then we have $\rho_{k+1}(\mathcal{M}|\xi) = \kappa_F^+$. But we are in the case that $\rho_{k+1}(\mathcal{M}|\xi) = \kappa_E$, which is a limit cardinal of $\mathcal{M}$, so this impossible.

$\boxtimes$

Next we consider the case E is long.

CNA/MN/91

Claim 6. Let T-pred$(\eta+1)=0$, $E=E_\eta$, and $\mathcal{M}^{*\mathcal{T}}_{\eta+1}=\mathcal{M}|\xi$, where $\xi<o(\mathcal{M})$. Suppose $k=\deg^{\mathcal{T}}(\eta+1)$ is least such that $\rho_{k+1}(\mathcal{M}|\xi)<\alpha$. Suppose E is long; then

(a) E is the first long extender on the P_η-sequence having domain $P_\eta|\alpha=\mathcal{M}|\alpha$,

(b) E has exactly one long generator, call it γ,

(c) $\mathrm{Ult}_k(\mathcal{M}|\xi, E)$ is ~~kth~~ a $k+1$-sound plus-one premouse, with

$$\rho_{k+1}^{\mathrm{Ult}_k(\mathcal{M}|\xi,E)}=\rho_{k+1}^{\mathcal{M}|\xi}$$

and

$$P_{k+1}(\mathrm{Ult}_k(\mathcal{M}|\xi,E))=i_E(P_{k+1}(\mathcal{M}|\xi))\cup\{\gamma\},$$

(d) $P_{k+1}(\mathrm{Ult}_k(\mathcal{M}|\xi,E))$ is of type Z_p.

Proof Let $\kappa = \kappa_E$. We have $\alpha = (\kappa^{++})^{\mathcal{M}|\xi}$. We write "$\kappa^+$" for $(\kappa^+)^{\mathcal{M}|\xi} = (\kappa^+)^{\mathcal{H}} = (\kappa^+)^{P_\eta}$. We have

$$\kappa^+ = \rho_{k+1}(\mathcal{M}|\xi).$$

Let

$$\nu = \nu(E) - 1 = \dot{\nu}^{P_\eta}.$$

Subclaim 6.0 For any finite $a \subseteq \lambda_E \cup \{\nu\}$, $E \restriction (\kappa^+ \cup a) \in \mathcal{M}$.

Proof. The same as in the case that E is short.

$\boxtimes$

<u>Subclaim 6.2</u> E is the first long extender on the P_η-sequence with domain $\mathcal{H}|\alpha$.

<u>Proof</u> Let $G = \dot{F}^{P_\eta|\gamma}$ be the first such extender. By lemma 9, $\rho_1(P_\eta|\gamma) = \alpha$. Assume that $G \neq E$; then $\gamma < \Theta(P_\eta)$. Thus $\gamma < (\alpha^+)^{P_\eta}$.

Suppose first $\eta \geq 2$. Then $lh E_1$ is a cardinal in P_η, and $lh E_1 > \alpha$, and $\mathcal{H}||lh E_1 = P_\eta || lh E_1$. Thus G is on the sequence of $\mathcal{H}$, with $lh G < lh E_1$. It follows that G is on the sequence of $\mathcal{M}$, and of $\mathcal{M}|\xi$. But $\rho_{k+1}(\mathcal{M}|\xi) = \kappa^+ =$ space(G), so this violates the projectum-free spaces property of $\mathcal{M}|\xi$.

If $\eta = 1$, then since $G \neq E_1$, ~~(as otherwise if E_1 is the first long extender with domain $\mathcal{H}|\alpha$ on the $\mathcal{H}$-sequence, then)~~ we again have that G is on the sequence of $\mathcal{M}|\xi$, a contradiction.

$\boxtimes$

Subclaim 6.3 $Ult_k(\mathcal{M}|\xi, E)$ is a plus-one premouse.

Proof (Sketch.) If $k > 1$, there is enough elementarity. We do for the case $k = 1$. Suppose then that $k = 0$.

The proof that works when E is close to $\mathcal{M}|\xi$ will work here. First, notice that we are not in anomalous case 2. Thus if $\mathcal{M}|\xi$ is type Z_1, then $\mathring{\nu}^{\mathcal{M}|\xi}$ has cofinality in $\mathcal{M}|\xi$ different from κ^+, and thus i_E is continuous at $\mathring{\nu}^{\mathcal{M}|\xi}$, and thus $i_E(\mathring{\nu}^{\mathcal{M}|\xi}) = \mathring{\nu}^{Ult_0(\mathcal{M}|\xi, E)}$, and $(\dot{E}_{\mathring{\nu}})^{Ult_0(\mathcal{M}|\xi, E)}$ witnesses the type Z_1-ness of $Ult_0(\mathcal{M}|\xi, E)$.

Second, notice that we are not in the situation that would lead to $Ult_0(\mathcal{M}|\xi, E)$ being merely a protomouse, by claim 5.

We omit the remaining calculations. $\boxtimes$

~~Remark It is also important that we are not in anomalous case 3. If $\mathcal{M}|\xi$ is type Z_p and its generalized cover is a collapsing structure for α, then we want to take $Ult_k(\mathcal{N}, E)$ now.~~

<u>Subclaim 6.4</u> Let $i: \mathcal{M}|\xi \to Ult_k(\mathcal{M}|\xi, E)$ be the canonical embedding. Let $\mathcal{N} =$ $Ult_k(\mathcal{M}|\xi, E)$; then $\rho_{k+1}(\mathcal{N})$ is of type $\mathbb{Z}_p$, with $\rho_{k+1}(\mathcal{N}) = i''(\rho_{k+1}(\mathcal{M}|\xi)) \cup \{\nu\}$.

<u>Proof</u> Because we are not in anomalous case 3, $p_{k+1}(\mathcal{M}|\xi)$ is solid. Let $t = p_{k+1}(\mathcal{M}|\xi)$, and

$$A = Th_{k+1}^{\mathcal{M}|\xi}(\kappa^+ \cup t),$$

recalling here that $\kappa^+ = \rho_{k+1}(\mathcal{M}|\xi)$.

Using 6.4, we have that $\kappa^+ \leq \rho_{k+1}(\mathcal{N})$, and that $A \notin \mathcal{N}$, as in the proof of 6.2. But A is $\Sigma_{k+1}^{\mathcal{N}}$ in the parameters $i(t)$ and $i \restriction \kappa^+$, and $i \restriction \kappa^+$ is essentially $\dot{E}_\nu^{\mathcal{N}}$. This gives $\rho_{k+1}(\mathcal{N}) = \kappa^+$, ~~and~~ By solidity of t, and the fact that for all $\beta < \kappa^+$,

$Th^{\mathcal{N}}_{k+1}(i(\beta) \cup i(t)) \in \mathcal{N}$, we get

that $p_{k+1}(\mathcal{N}) = i(t) \cup \{\nu\}$, as

desired. The stretching extender

witnessing the type $\mathbb{Z}_p$ property is of

course $(\mathring{E}_\nu)^{\mathcal{N}}$. The generalized core of

$\mathcal{N}$ is $\mathcal{M}|_\xi$, so its parameter is solid. $\boxtimes$

This completes our proof of claim 6. $\boxtimes$

<u>Claim 7</u>. The comparison of $(\mathcal{M}, \mathcal{H}, \alpha)$ vs. $\mathcal{M}$ terminates. Moreover, letting $\mathcal{P}$ and $\mathcal{Q}$ be the last models on the two sides

(a) if $\mathcal{P} \trianglelefteq \mathcal{Q}$, then the branch to $\mathcal{P}$ does not drop,

(b) if $\mathcal{Q} \trianglelefteq \mathcal{P}$, then the branch to $\mathcal{Q}$ does not drop,

and (c) it is not the case that both the branch to $\mathcal{P}$ and the branch to $\mathcal{Q}$ drop.

<u>Proof</u> The termination argument of 10.3, theorem 5 works, because all P_β and Q_β are plus-one premice.

Part (b) is proved as usual: if the branch to Q drops, then Q is unsound, and thus cannot be a proper initial segment of P. This argument depends on the preservation of parameters and cores by the maps of $\mathcal{U}$.

Part (a) takes a little more argument, because the maps of $\mathcal{T}$ may not always preserve cores. The potentially bad case is described in claim 6. So what we need to rule out is

$$P = Ult_k(\mathcal{M}|\xi, E),$$

where ξ is least such that $\rho_{k+1}(\mathcal{M}|\xi) < \alpha$, and $\mathrm{dom}(E) = \mathcal{M}|\alpha$, and E has exactly one long generator γ, yet $P \triangleleft Q$. To rule this out, for

$$A = Th_{k+1}^{\mathcal{M}|\xi}\left(\kappa^+ \cup p_{k+1}(\mathcal{M}|\xi)\right),$$

where $\kappa = \kappa_E$, and $p_{k+1}(\mathcal{M}|\xi) = \kappa^+ = (\kappa^+)^{\mathcal{M}|\xi}$.
A is definable over $\mathcal{P}$, but not in $\mathcal{P}$, and definable over $\mathcal{M}|\xi$, but not in $\mathcal{M}|\xi$. Since $\mathcal{P} \neq \mathcal{M}|\xi$, we have $\mathcal{Q} \neq \mathcal{M}$. So let $G = E_0^{\mathcal{U}}$ be the first extender used in $\mathcal{U}$. ~~on the branch~~ ~~$\mathcal{M}$ to $\mathcal{Q}$, $\mathrm{lh}\, G \geq \alpha$~~; Note $\lambda_G > \alpha$. If $\mathrm{lh}\, G > \xi$, then $\mathcal{M}|\xi \trianglelefteq \mathcal{Q}$, so that $\mathcal{P} \ntrianglelefteq \mathcal{Q}$. So assume $\mathrm{lh}(G) < \xi$. Then $\mathcal{M}\|\mathrm{lh}\, G \trianglelefteq \mathcal{Q}$, and $\mathrm{lh}\, G$ is a cardinal of $\mathcal{Q}$, so $A \notin \mathcal{Q}$. So $\mathcal{P} \ntrianglelefteq \mathcal{Q}$.

<u>Remark</u> One can show that $\mathcal{P} \neq \mathcal{Q}$ in this case as well. One cannot reach $Ult_k(\mathcal{M}|\xi, E)$ in an ordinary tree on $\mathcal{M}$.

Part (c) is proved as usual, but taking into account the special case just described.

$\boxtimes$

Let $P = P_\gamma$ and $Q = Q_\delta$ be the last models of $\mathcal{T}$ and $\mathcal{U}$ respectively.

<u>Claim 8</u>. $1 T \gamma$; that is, P is above $\mathcal{H}$ in $\mathcal{T}$.

<u>Proof</u> Suppose $0 T \gamma$. The standard arguments (cf. [3], §8, ~~3~~ and [2]) with the weak Dodd-Jensen property show

(a) Neither $[0, \gamma]_T$ nor $[0, \delta]_U$ drops, and

(b) $i^{\mathcal{T}}_{0\gamma} = i^{\mathcal{U}}_{0\delta}$, and $P = Q$.

We now get a contradiction just as in the proof that comparison terminates. Let $\xi + 1$ and $\eta + 1$ be least in $(0, \gamma]_T$ and $(0, \delta]_U$ respectively. Let $G = E^{\mathcal{T}}_\xi$ and $H = E^{\mathcal{U}}_\eta$. Suppose G and H are long, to take one case. Let

$$(P^*, G^*) = Ult_0((P_\xi \| lh G, G), K)$$

and

$$(Q^*, H^*) = Ult_0((Q_\eta \| lh H, H), L),$$

where K is the short part of the extender of the branch tail $i^{\mathcal{T}}_{\xi+1,\gamma}$, and L is the short part of the extender of the branch tail $i^{\mathcal{U}}_{\eta+1,\delta}$. We get

$$(P^*, G^*) = (Q^*, H^*)$$

from $i^{\mathcal{T}}_{0\gamma} = i^{\mathcal{U}}_{0\delta}$. Using the Jensen ISC, this gives $G = H$, a contradiction.

$\boxtimes$

<u>Claim 9</u> $P \trianglelefteq Q$, and $[1,\gamma]_{\mathcal{T}}$ does not drop.

<u>Proof</u> If $Q \triangleleft P$, then $[0,\delta]_{\mathcal{U}}$ does not drop, and we have

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, dotted, no head] & & P^*_\gamma \\
\mathcal{H} \arrow[u, "\pi"] \arrow[rr, dotted, no head] & & P_\gamma \arrow[u, "\pi_\gamma"'] \arrow[d, phantom, "\triangledown"] \\
\mathcal{M} \arrow[rr, "i^{\mathcal{U}}_{0\delta}"'] & & Q
\end{tikzcd}$$

where π_γ is the copy map. Then $\pi_\gamma \circ i^{\mathcal{U}}_{0\delta}$

maps $\mathcal{M}$ into a proper initial segment of P_γ, which is a Σ-iterate of $\mathcal{M}$. This contradicts the weak Dodd-Jensen property of Σ.

Similarly, $[1,\gamma]_T$ does not drop.

$\boxtimes$

<u>Claim 10</u> $Th_1^{\mathcal{M}}(\alpha \cup r)$ is Σ_1 definable over P.

<u>Proof</u> $Th_1^{\mathcal{M}}(\alpha \cup r) = Th_1^{\mathcal{H}}(\alpha \cup \pi^{-1}(r))$. If $crit(i_{1,\gamma}^{\mathcal{I}}) \geq \alpha$, then $Th_1^{\mathcal{M}}(\alpha \cup r) = Th_1^{P}(\alpha \cup i_{1,\gamma}^{\mathcal{I}}(\pi^{-1}(r)))$ is clearly Σ_1 over P in $i_{1,\gamma}^{\mathcal{I}}(\pi^{-1}(r))$.

So suppose $crit(i_{1,\gamma}^{\mathcal{I}}) < \alpha$, and let G be the first extender used in $(1,\gamma]_T$. Our rules for $\mathcal{I}$ guarantee that G is long, and $(\kappa_G^+)^{\mathcal{H}} = \alpha$. Let $G = E_\xi^{\mathcal{I}}$, and $L = G \restriction \theta$, where θ is the least long generator of G. Then L is on the sequence of $P_{\xi+1}$ by coherence. So

$L^* = i^{\mathcal{T}}_{\xi+1,\gamma}(L)$ is on the sequence of $P_\gamma = P$. But L^* is the extender of $i^{\mathcal{T}}_{1,\gamma} \upharpoonright (\mathcal{H} \| \alpha)$, so $i^{\mathcal{T}}_{1,\gamma} \upharpoonright \alpha \in P$.

But

$$(\varphi, u) \in Th_1^{\mathcal{H}}(\alpha \cup \pi^{-1}(r)) \text{ iff}$$

$$(\varphi, i_{1,\gamma}(u)) \in Th_1^{P}(i_{1,\gamma}(\alpha) \cup i_{1,\gamma}(\pi^{-1}(r))).$$

This shows $Th_1^{\mathcal{M}}(\alpha \cup r)$ is definable over P from $i^{\mathcal{T}}_{1,\gamma}(\pi^{-1}(r))$ and $i^{\mathcal{T}}_{\xi+1,\gamma}(\theta)$.

$\boxtimes$

<u>Claim 11.</u> $P = Q$, and $[0, \theta]_{\mathcal{U}}$ does not drop.

<u>Proof</u>. If $P \trianglelefteq Q$, then $Th_1^{\mathcal{M}}(\alpha \cup r) \in Q$. But all extenders used in $\mathcal{U}$ have length $> \alpha$, because we are not in anomalous case I. This implies $P(\alpha)^{Q} \subseteq P(\alpha)^{\mathcal{M}}$, so $Th_1^{\mathcal{M}}(\alpha \cup r) \in \mathcal{M}$, contradiction.

Suppose now that $[0,\delta]_{\mathcal{U}}$ drops, and let $\xi+1$ be largest in $D^{\mathcal{U}} \cap [0,\delta]_{\mathcal{U}}$. We claim that any $A \subseteq \alpha$ that is Σ_{k+1}^{Q} is also Σ_{k+1} over $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$, where $k = \deg^{\mathcal{U}}(\xi+1)$.

Let us prove this only when $k=0$. For that, let R be first on the branch $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$ ~~such~~-to-Q such that A is Σ_1^R. Clearly $R \neq \mathcal{M}_\beta^{\mathcal{U}}$ for β a limit. (All extenders on $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$-to-$Q$ have critical point $>\alpha$, except perhaps the first.) If $R \neq \mathcal{M}_{\xi+1}^{*\mathcal{U}}$, let

$$R = \mathrm{Ult}_0(S, E)$$

where E is on $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$-to-$Q$. Let

$$\beta \in A \quad \text{iff} \quad R \models \exists v\, \varphi(v, [a, f]_E^S)$$

where φ is Σ_0. If $\mathrm{crit}(i_E) > \alpha$, we have

$$\beta \in A \quad \text{iff} \quad S \models \exists g\ \exists X \in E_a\ \forall u \in X\ \varphi(g(u), f(u)),$$

where we use the fact that E_a is $\Sigma_1^{\mathcal{U}}$ to make sense of the right hand side. So A is $\Sigma_1^{\mathcal{U}}$, contradiction. So assume $\mathrm{crit}(E) = \kappa < \alpha$. We must then have $\mathcal{U}\text{-pred}(\xi+1) = 0$ and $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$ is the collapsing level for α in $\mathcal{M}$. Moreover, $\mathrm{dom}(E) = \mathcal{M}|\alpha$. E cannot be long, as then $\rho_1(\mathcal{M}_{\xi+1}^{*\mathcal{U}}) = \kappa^{+\mathcal{M}|\alpha} = \kappa^{+\mathcal{M}}$, and ~~this violates~~ since E is close to $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$, $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$ has a total long extender on its sequence with critical point κ, violating projectum-free spaces for $\mathcal{M}_{\xi+1}^{*\mathcal{U}}$.

So E is short, and $\alpha = \kappa^{+\mathcal{M}_{\xi+1}^{*\mathcal{U}}}$. But then

$$\beta \in A \iff \mathcal{M}_{\xi+1}^{*\mathcal{U}} \models \exists \text{ wellorder } W \text{ of } \kappa \ \exists g \ \exists X \in E_{a \cup \{\kappa\}} \ \forall u \in X \ \varphi(|W \cap u_0|, g(u_1 \ldots u_n), f(u_1 \ldots u_n)).$$

(where $u = \{u_0 \ldots u_n\}$, and we assume $a \cap (\kappa+1) = \emptyset$.)

Thus $Th_1^{\mathcal{M}}(\alpha \cup r)$ is definable over $\mathcal{M}^{*\mathcal{U}}_{\xi+1}$, so $Th_1^{\mathcal{M}}(\alpha \cup r) \in Q_\eta$, where $\eta = \mathcal{U}\text{-pred}(\xi+1)$. It follows that $Th_1^{\mathcal{M}}(\alpha \cup r) \in \mathcal{M}$, contradiction.

$\boxtimes$

The following general lemma is useful.

Lemma 11. Let N be a plus-one premouse, and E be a plus-one extender (possibly short) over N, and suppose $\mathrm{dom}(E) \subseteq N | \rho_1^N$; then $\sup i_E^N " \rho_1^N \leq \rho_1^{Ult_0(N,E)}$.

Proof Let $\xi < \rho_1^N$ and $A \subseteq i_E^N(\xi)$, with

$$\beta \in A \text{ iff } Ult_0(N,E) \models \varphi[\beta, i_E^N(f)(b)],$$

where $b \subseteq \nu(E)$. Then $Th_1^N(\xi \cup \{f\}) \in N$, and carries a natural Σ_1^N prewellorder $\leq^*$ such that $\leq^* \in N$. But then $Th_1^{Ult_0(N,E)}(i_E(\xi) \cup \{i_E(f)\}) \in Ult_0(N,E)$, because it is an initial segment of $i_E(\leq^*)$. But $\mathrm{space}(E) < \mathrm{dom}(E)$, so we may assume $\xi \geq \mathrm{space}(E)$. This implies $\nu(E) \leq i_E(\xi)$, so $b \in [i_E(\xi)]^{<\omega}$. That gives $A \in Ult_0(N,E)$. $\boxtimes$

Claim 12 $\alpha = \rho_1(\mathcal{M})$.

Proof If not, then we are in the failure-of-solidity case. We then have

$$\rho_1(\mathcal{M}) < \rho_1(\mathcal{H}) \leq \rho_1(\mathcal{P}) \leq \alpha.$$

For $\rho_1(\mathcal{M}) < \rho_1(\mathcal{H})$ because $\mu_0 \notin \operatorname{ran}(\pi)$. $Th_1^{\mathcal{M}}(\alpha \cup r)$ is $\Sigma_1^{\mathcal{P}}$ by claim 10, but $Th_1^{\mathcal{M}}(\alpha \cup r) \notin \mathcal{P}$ because $P(\alpha)^{\mathcal{Q}} \subseteq P(\alpha)^{\mathcal{M}}$. Thus $\rho_1(\mathcal{P}) \leq \alpha$. Finally, $\rho_1(\mathcal{H}) \leq \rho_1(\mathcal{P})$ because all extenders used in the branch $\mathcal{H}$-to-$\mathcal{P}$ are close to the models to which they are applied.

Now let F be the first extender used in $[0, \delta]_{\mathcal{U}}$. We have $\operatorname{lh} F > \alpha$ because we are not in anomalous case I. Since $[0, \delta]_{\mathcal{U}}$ does not drop, F is applied to all of $\mathcal{M}$. Now if $\mathcal{M} | \rho_1(\mathcal{M}) \subseteq \operatorname{space}(F)$, then $\rho_1(\mathcal{M}) =$

$\rho_1(\mathrm{Ult}_0(\mathcal{M}, F)) = \rho_1(Q) = \rho_1(P)$,
contradiction. Thus

$$\mathrm{dom}(F) \subseteq \mathcal{M}|\rho_1(\mathcal{M}).$$

But then lemma 11 tells us that since $\alpha < \sup i_F'' (\mathrm{dom}(F) \cap \mathrm{Ord})$, ~~wk~~ we have $\alpha < \rho_1(\mathrm{Ult}_0(\mathcal{M}, F))$, so $\alpha < \rho_1(Q)$, a contradiction.

$\boxtimes$

So we are in the universality case $\alpha = \rho_1(\mathcal{M})$. But then $\alpha = \rho_1(\mathcal{M}) \leq \rho_1(Q) \leq \alpha$, so $\alpha = \rho_1(Q)$. Let F be the first extender used in $[0, \delta]_U$, so that $\mathrm{lh} F \geq \alpha$. Since $\rho_1(Q) = \alpha$, either $\mathrm{crit}(F) > \alpha$, or F is long and $\mathrm{space}(F) = \mathcal{M}|\alpha$. In either case, we get $P(\alpha)^Q = P(\alpha)^{\mathcal{M}}$. Thus

$$P(\alpha)^{\mathcal{M}} \subseteq P(\alpha)^Q = P(\alpha)^P = P(\alpha)^{\mathcal{H}},$$

as desired for universality.

This proves theorem 10 in the non-anomalous case. $\boxtimes$

Proof of theorem 10 in anomalous case I

We have that $E_\alpha^{\mathcal{M}} \neq \emptyset$. Let $G = E_\alpha^{\mathcal{M}}$. α is not a cardinal of $\mathcal{M}$, so $\alpha > \rho_1(\mathcal{M})$. Our goal is to show that $p_1(\mathcal{M}) = r \cup \{\alpha\}$, that $p_1(\mathcal{M})$ is of type Z_p, and that $p_1(\mathcal{M})$ is 1-universal. We shall also show that the 1-core of $\mathcal{M}$ is an ultrapower of its generalized core.

Again, we compare $(\mathcal{M}, \mathcal{H}, \alpha)$ with $\mathcal{M}$, with $\mathcal{T}$ and $\mathcal{U}$ being the two trees, and $\mathcal{T}^*$ the lift of $\mathcal{T}$ under $(id, \pi) = (\pi_0, \pi_1)$. The rules for $\mathcal{T}$ and $\mathcal{U}$ are as before, and we adopt our previous notation. That is, $P_\xi = \mathcal{M}_\xi^{\mathcal{T}}$, $Q_\xi = \mathcal{M}_\xi^{\mathcal{U}}$, $P_\xi^* = \mathcal{M}_\xi^{\mathcal{T}^*}$, $\pi_\xi : P_\xi \to P_\xi^*$, etc.

We have $\alpha = (\lambda_G^+)^{\mathcal{H}}$, so if E_ξ has critical point λ_G, then if E_ξ is short, T-pred$(\xi+1) = 0$, while if E_ξ is long, then T-pred$(\xi+1) = I$. (Note $\lambda_{E_1} > \lambda_G$, since $lh\, E_1 > \alpha$.)

One new problem here is that the models of $\mathcal{T}$ may fail to satisfy the Jensen ISC. Claim 2 above still holds, so this problem only occurs when E_ξ is short and $\mathrm{crit}(E_\xi) = \lambda_G$. Our rules for $\mathcal{T}$ then require

$$P_{\xi+1} = \mathrm{Ult}_0(\mathcal{M}|\alpha, E_\xi).$$

Here $G = \dot{F}^{\mathcal{M}|\alpha}$, and $i_{E_\xi}(G) = \dot{F}^{P_{\xi+1}}$.

G is a missing whole initial segment of $i_{E_\xi}(G)$, and so $P_{\xi+1}$ does not satisfy the Jensen ISC.

The solution to this problem can be found in §3 of [4], and it works here. Notice first that $o(P_{\xi+1}) = lh\, E_\xi = lh\,(i_{E_\xi}(G))$, so $i_{E_\xi}(G)$ is going to be part of the least disagreement between $P_{\xi+1}$ and the current model of $\mathcal{U}$. (All models of $\mathcal{U}$ are plus-one premice.) So $T\text{-pred}(\xi+2) = 0$, and

$$P_{\xi+2} = \text{Ult}_0(\mathcal{M}, i_{E_\xi}(G)).$$

(Since λ_G is a cardinal of ~~$\mathcal{H}$~~ $\mathcal{H}$, it is a cardinal of $\mathcal{M}$, so $i_{E_\xi}(G)$ is an extender over $\mathcal{M}$.) Moreover

$$\lambda_\xi^{\mathcal{T}} = \lambda_{\xi+1}^{\mathcal{T}},$$

so there are no models above $P_{\xi+1}$ in $\mathcal{T}$. It is a dead node. We then have

<u>Claim 1a</u> All models of $\mathcal{T}$ except those of the form $P_{\xi+1} = \text{Ult}_0(\mathcal{M}|\alpha, E_\xi)$ with $\text{crit}(E_\xi) = \lambda_G$, E_ξ short, are plus-one premice. Moreover, except for the $i^{\mathcal{T}}_{0,\xi+1}$ of this form, the maps of $\mathcal{T}$ preserve parameters and cores.

We get that the comparison terminates as before. Let $P = P_\gamma^{\mathcal{T}}$ and $Q = Q_\delta^{\mathcal{U}}$ be the last models. We need more argument now to show

<u>Claim 26</u> $, T_\gamma$; that is, P is above $\mathcal{H}$ in $\mathcal{T}$.

<u>Proof</u> Suppose P is above $\mathcal{M}$.

<u>Subclaim 26.1</u>

(a) $P = Q$,

(b) neither $[0,\gamma]_T$ nor $[0,\delta]_U$ drops,

(c) $i^{\mathcal{T}}_{0\gamma} = i^{\mathcal{U}}_{0\delta}$.

<u>Proof</u> The same weak Dodd-Jensen argument.

$\boxtimes$

Now let $K = E^{\mathcal{T}}_\eta$ and $L = E^{\mathcal{U}}_\rho$ be the first extenders used in $[0,\gamma]_T$ and $[0,\delta]_U$ respectively. Let

$$(P^*, K^*) = \mathrm{Ult}_0(P_\eta \| \mathrm{lh} K, F_0),$$

$$(Q^*, L^*) = \mathrm{Ult}_0(Q_\rho \| \mathrm{lh} L, F_1),$$

where F_0 is the short part of the branch tail extender of $i^T_{\eta+1,\gamma}$ and F_1 the short part of the branch tail extender of $i^U_{\rho+1,\delta}$. As before, $K^* = L^*$, and both are long. Since L had the Jensen ISC, letting ν^* be the largest long generator of L^*,

$$\lambda_L = \text{least } \theta \text{ such that } L^* \upharpoonright (\theta \cup \{\nu^*\}) \text{ is whole},$$

and $\qquad L = L^* \upharpoonright (\lambda_L \cup \{\nu^*\})$.

If K had the Jensen ISC, this would give $K = L$, contradiction. So we assume K does not have the Jensen ISC. This implies

$$K = i^{\bar{U}}_{E_\xi}(G)$$

where $\mathrm{crit}(E_\xi) = \lambda_G$, and L is the least whole initial segment of K, that is, $L = G$. (Note $G = E_0^{\mathcal{U}}$.) So $\rho = 0$, and $1 \in [0, \delta]_{\mathcal{U}}$. We have

$$
\begin{aligned}
i^{\mathcal{T}}_{0,\gamma}(\kappa_K) &= i^{\mathcal{T}}_{\eta+1,\gamma}(\lambda_K) \\
&= i^{\mathcal{U}}_{0,\delta}(\kappa_G) \\
&= i^{\mathcal{U}}_{1,\delta}(\lambda_G).
\end{aligned}
$$

But $\lambda_G = \kappa_{E_\xi} < \lambda_K$, so $\mathrm{crit}(i^{\mathcal{U}}_{1,\delta}) = \lambda_G$.

Let F be the first extender used in $[1, \delta]_{\mathcal{U}}$, say $F = E^{\mathcal{U}}_\beta$, and let F^* be the stretch of F by the branch tail extender of $i^{\mathcal{U}}_{\beta+1,\delta}$ (or rather, its short part).

<u>Subclaim 26.2</u> $E_\xi \upharpoonright \lambda_{E_\xi} = F^* \upharpoonright \lambda_{E_\xi}$.

<u>Proof</u> $\lambda_{E_\xi} = \lambda_K$. Both extenders have critical point λ_G, and measure subsets of λ_G in $M_1 \| h G$. Let $A \subseteq \lambda_G$ and $A \in M_1 \| h G$. We can write

$$A = [a, f]_G = i_G(f)(a),$$

where $f : [K_G]^{|a|} \longrightarrow P(K_G)$, $f \in M \,|\, (K_G^+)^M$,

and $a \subseteq \lambda_G$. But then

$$i_{E_\xi}(A) = [a, f]_K = i_K(f)(a).$$

But also

$$\begin{aligned} i_{F^*}(A) &= i^{\mathcal{U}}_{1,\delta}\left(i_G(f)(a)\right) \\ &= i^{\mathcal{U}}_{1,\delta}(i_G(f))\left(i^{\mathcal{U}}_{1,\delta}(a)\right) \\ &= i^{\mathcal{U}}_{0,\delta}(f)(a) \\ &= i^{\mathcal{T}}_{0,\gamma}(f)(a) \\ &= i^{\mathcal{T}}_{\eta+1,\gamma}(i_K(f))(a) \\ &= i^{\mathcal{T}}_{\eta+1,\gamma}\left(i_K(f)(a)\right). \end{aligned}$$

So $i^{\mathcal{T}}_{\eta+1,\gamma}(i_{E_\xi}(A)) = i_{F^*}(A)$. Since $\operatorname{crit}(i^{\mathcal{T}}_{\eta+1,\gamma}) = \lambda_K$, we have proved the subclaim.

$\boxtimes$

But then if F is short, we have that E_ξ is the least whole initial segment of F^*, and so is F, so that $E_\xi = F$, a contradiction. If F is long, then E_ξ is the least whole initial segment of $F^* \restriction \lambda_{F^*}$, so $E_\xi \in Q_\delta$, again a contradiction.

This proves claim 2a.

$\boxtimes$

We are now where we were after having proved claim 8 in the non-anomalous case. Claims 9 - 12 go through here, ~~since~~ because the anomalously stretched extenders $i_{E_\xi}(G)$ were all applied to $\mathcal{M}$, and now we know ~~we~~ P is above $\mathcal{H}$, not $\mathcal{M}$. Repeating the~~se~~ proofs of those claims, we get

Claim 3a

(1) Neither $[0,\delta]_{\mathcal{U}}$ nor $[1,\gamma]_{\mathcal{T}}$ drops,

(2) $P = Q$,

(3) $Th_1^{\mathcal{M}}(\alpha \cup r)$ is Σ_1 definable over P, but not in P, and

(4) $\rho_1(\mathcal{M}) < \rho_1(\mathcal{H}) \leq \rho_1(P) \leq \alpha$.

There is one minor point here. We argued that $P(\alpha)^Q \subseteq P(\alpha)^{\mathcal{M}}$ because all $E_\eta^{\mathcal{U}}$ have length $> \alpha$. In the present case, the first extender used in $\mathcal{U}$ is

$$E_0^{\mathcal{U}} = G,$$

and $lh(G) = \alpha$. But note $Q_1 = Ult_0(\mathcal{M}, G)$, and $P(\alpha)^Q \subseteq P(\alpha)^{Q_1}$ because $lh(E_\eta^{\mathcal{U}}) > \alpha$ for $\eta > 0$. But $P(\alpha)^{Q_1} \subseteq \mathcal{M}$ because $G \in \mathcal{M}$.

<u>Claim 4a</u> G is short, and G is the first extender used in $[0, \delta]_{\mathcal{U}}$, and $\rho_1(\mathcal{M}) = \kappa_G^{+\mathcal{M}}$.

<u>Proof.</u> Let $E = E_\eta^{\mathcal{U}}$ be the first extender used in $[0,\delta]_{\mathcal{U}}$. Let $\rho = \rho_1(\mathcal{M})$. We have $\kappa_E < \rho$, as otherwise $\rho = \rho_1(Q)$. Thus $\mathrm{dom}(E) \subseteq \mathcal{M}|\rho$. Also, $lh(E) \geq \alpha$.

But

$$\sup i_E{}''\rho \leq \rho_1(Q) = \alpha$$

by lemma 11. If E is long, then $\kappa_E^{++\mathcal{M}} \leq \rho$, and

$$\alpha \leq i_E(\kappa_E^+) < \sup i_E{}''\rho \leq \rho_1(Q) \leq \alpha,$$

a contradiction. Thus E is short. Also

$$i_E(\kappa_E^+) = \sup i_E{}''\kappa_E^+ \leq \rho_1(Q) \leq \alpha,$$

so $lh(E) \leq \alpha$, so $lh(E) = \alpha$, so $E = G$.

Finally, if $\kappa_E^+ < \rho$, then $\alpha = i_E(\kappa_E^+) < \rho_1(Q)$, so $\kappa_E^+ = \rho$.

$\boxtimes$

So $G = E_0^{\mathcal{U}}$ is short, and $I\,\mathcal{U}\,\delta$ or $I = \delta$.

Claim 5a. $\alpha = \rho_1(Q_1) = \rho_1(Q)$.

Proof $Q_1 = Ult_0(\mathcal{M}, G)$, and $\kappa_G^{+\mathcal{M}} = \rho_1(\mathcal{M})$,

so

$$\alpha = \sup i_G'' \rho_1(\mathcal{M}) \leq \rho_1(Q_1) \leq \rho_1(Q) \leq \alpha.$$

This proves 5a.

$\boxtimes$

Claim 6a. $crit(i^{\mathcal{U}}_{1,8}) > \alpha$.

Proof. The alternative is $crit(i^{\mathcal{U}}_{1,8}) = \lambda_G$. Let K be the first extender used in $(1,8]_{\mathcal{U}}$. K cannot be short, because then $\alpha = \rho_1(Q_1) < \rho_1(Q) = \alpha$. But K cannot be long either, because K is close to Q_1, whereas $\rho_1(Q_1) = \alpha$, so that Q_1 would not have projectum-free spaces.

$\boxtimes$

Now let

L = first extender used in $[1, \gamma]_T$

be the extender applied to $\mathcal{H}$ in $\mathcal{H}$-to-P.

Claim 7a. $\kappa_L = \lambda_G$.

Proof. $\kappa_L \geq \lambda$ by our rules for $\mathcal{T}$. Suppose $\kappa_L > \lambda$, so that $\kappa_L > \alpha$. Let $\bar{r} = \pi^{-1}(r)$.

Since $\kappa_L > \lambda$,

$$\begin{aligned} Th_1^{\mathcal{M}}(\alpha \cup r) &= Th_1^{\mathcal{H}}(\alpha \cup \bar{r}) \\ &= Th_1^{P}(\alpha \cup i^{\mathcal{T}}_{1,\gamma}(\bar{r})). \end{aligned}$$

Thus $Th_1^P(\alpha \cup i^{\mathcal{T}}_{1,\gamma}(\bar{r})) \notin P$, by 3a(3). We claim that

$$i^{\mathcal{T}}_{1,\gamma}(\bar{r}) = i^{\mathcal{U}}_{0,\delta}(r).$$

For if $i^{\mathcal{T}}_{1,\gamma}(\bar{r}) <_{lex} i^{\mathcal{U}}_{0,\delta}(r)$, then since $i^{\mathcal{U}}_{0,\delta}(r)$ is solid over Q, $Th_1^P(\alpha \cup i^{\mathcal{T}}_{1,\gamma}(\bar{r})) \in P$, contradiction. On the other hand, $\pi_\gamma(i^{\mathcal{U}}_{0,\delta}(r)) \geq_{lex} \pi_\gamma(i^{\mathcal{T}}_{1,\gamma}(\bar{r})) = i^{\mathcal{T}^*}_{1,\gamma}(r)$ by the weak Dodd-Jensen property. Thus $i^{\mathcal{U}}_{0,\delta}(r) \geq_{lex} i^{\mathcal{T}}_{1,\gamma}(\bar{r})$, so $i^{\mathcal{U}}_{0\delta}(r) = i^{\mathcal{T}}_{1,\gamma}(\bar{r})$.

But $Th_1^{\mathcal{M}}(\rho_1(\mathcal{M}) \cup r) \in \mathcal{M}$, because r is

a proper initial segment of $\rho_1(\mathcal{M})$.
This implies $Th_1^{Q_1}(\alpha \cup i_{0,1}(r)) \in Q_1$ by 4a,
and then $Th_1^{Q}(\alpha \cup i_{0\delta}(r)) \in Q$ because
$crit(i_{1,\delta}^{\mathcal{U}}) > \alpha$. This is a contradiction.

$\boxtimes$

<u>Claim 8a</u> L has exactly one long generator.

<u>Proof.</u> If L were short, it would get applied to $\mathcal{M}$, rather than $\mathcal{H}$. If L has more than one long generator, then Q has a total long extender with critical point λ_G on its sequence. But $\rho_1(Q) = \alpha$, so this violates projectum-free spaces for Q.

$\boxtimes$

Let ν be the unique long generator of L.

Let

$$L^* = i_{\eta+1,\delta}^{\mathcal{T}}(L \restriction \nu),$$
$$\nu^* = i_{\eta+1,\delta}^{\mathcal{T}}(\nu).$$

Let also $\bar{r} = \pi^{-1}(r)$, and $s = i_{1,\delta}^{\mathcal{T}}(\bar{r})$.

<u>Claim 9a</u> $\rho_1(P)=\alpha$, $p_1(P) = s\cup\{\nu^*\}$, and $p_1(P)$ satisfies all the type z_p conditions, with stretching extender L^* and generalized core $\mathcal{H}$, except perhaps the solidity of $p_1(\mathcal{H})$.

<u>Proof.</u> We have shown $\rho_1(P)=\alpha$ already. Clearly $(\kappa_{L^*}^+)^P = \alpha$. Also, for $\beta<\alpha$

$$i_{1,\gamma}^{\mathcal{T}}(\beta) = i_{\gamma+1,\gamma}^{\mathcal{T}}(i_L(\beta))$$
$$= i_{L^*}(\beta),$$

by familiar calculations. Since $\mathcal{H} = \mathrm{Hull}_1^{\mathcal{H}}(\alpha\cup r)$,

$$\mathcal{H} = \mathcal{H}_1^P(i_{1,\gamma}^{\mathcal{T}}{}''\alpha \cup s)$$
$$= \mathcal{H}_1^P(i_{L^*}{}''\alpha \cup s).$$

Now $\mathcal{H}|(\alpha^+)^{\mathcal{H}} = P|(\alpha^+)^P$, so this gives us a natural partial $f : \alpha \xrightarrow{\text{onto}} P(\alpha^+)^P$ such that f is Σ_1^P in $s\cup\{i_{L^*}\restriction\alpha\}$, hence Σ_1^P in $s\cup\{\nu^*\}$. Thus $p_1(P) \leq_{lex} s\cup\{\nu^*\}$.

<u>Subclaim 9a.1</u> $s = i^{\mathcal{U}}_{0\delta}(r)$.

<u>Proof</u>. $s \leq_{lex} i^{\mathcal{U}}_{0,\delta}(r)$ follows from the weak Dodd-Jensen property of Σ as before. But $i^{\mathcal{U}}_{0\delta}(r)$ has solidity witnesses, so if $s <_{lex} i^{\mathcal{U}}_{0,\delta}(r)$, then $Th_1^P(\alpha \cup s \cup \{\nu^*\})$ belongs to P, contradiction.

$\boxtimes$

Since s has solidity witnesses, it is an initial segment of $p_1(P)$.

<u>Subclaim 9b.2</u> For any $\xi < \nu^*$, $Th_1^P(\xi \cup s) \in P$.

<u>Proof</u>. Let $\beta < \alpha$ be such that $i^{\mathcal{T}}_{1,\gamma}(\beta) > \xi$. Using the weak Dodd-Jensen property of Σ and $\mathcal{T}^*$, and noting that $\pi_\gamma(\beta) = \beta$, we get $i^{\mathcal{U}}_{0,\delta}(\beta) > \xi$. But $Th_1^{\mathcal{M}}(\beta \cup r) \in \mathcal{M}$ by our choice of α, so $Th_1^Q(i^{\mathcal{U}}_{0,\delta}(\beta) \cup s) \in Q$,

because it can be computed from $i^{\mathcal{U}}_{0,\delta}(Th_1^{\mathcal{N}}(\beta \cup r))$. This implies $Th_1^{P}(\xi \cup s) \in P$, as desired.

$\boxtimes$

The two subclaims imply that $p_1(P) = s \cup \{\nu^*\}$.

<u>Subclaim 9a.3</u> $p_1(P)$ satisfies the type $\mathcal{Z}_p$ conditions, with stretching extender L^*, generalized core $\mathcal{H}$, and $p_1(\mathcal{H}) = \bar{r}$, except perhaps the solidity of $p_1(\mathcal{H})$.

<u>Proof.</u> L^* is short, and $L^* = \dot{E}^P_{\nu^*}$. Also

$$\mathcal{H} = \text{collapse of } Hull_1^P(i_{1,\gamma}{}''\alpha \cup s)$$
$$= \text{collapse of } Hull_1^P(i_{L^*}{}''\alpha \cup s),$$

with $i^{\mathcal{U}}_{1,\gamma}$ being the uncollapse map. Also, $i^{\mathcal{U}}_{1,\gamma} \restriction \alpha = i_{L^*} \restriction \alpha$. This easily gives (3)(a)(b) and (3)(c)(i)(ii) of definition 7. To see that $p_1(\mathcal{H}) = \bar{r}$, note $Th_1^{\mathcal{H}}(\alpha \cup \bar{r}) \notin \mathcal{H}$, so $p_1(\mathcal{H}) \leq_{lex} \bar{r}$. But $t <_{lex} \bar{r} \Rightarrow Th_1^{\mathcal{H}}(\alpha \cup t) \in \mathcal{H}$, by the proof of 9a.1.

subclaim $\boxtimes$ claim $\boxtimes$

Claim 10a $\nu^* = i^{\mathcal{U}}_{0,\delta}(\alpha)$.

Proof We have $\nu^* = \sup i_{L^*}{}''\alpha = \sup i^{\mathcal{T}}_{1,\gamma}{}''\alpha$.

But for all $\beta < \alpha$

$$i^{\mathcal{T}}_{1,\gamma}(\beta) \leq i^{\mathcal{U}}_{0,\delta}(\beta),$$

because $\pi_\gamma(i^{\mathcal{T}}_{1,\gamma}(\beta)) = i^{\mathcal{T}^*}_{1,\gamma}(\pi_\gamma(\beta)) = i^{\mathcal{T}^*}_{1,\gamma}(\beta) \leq$ $\pi_\gamma(i^{\mathcal{U}}_{0,\delta}(\beta))$ by weak Dodd-Jensen. Thus

$$\nu^* \leq \sup i^{\mathcal{U}}_{0,\delta}{}''\alpha = i^{\mathcal{U}}_{0\delta}(\alpha).$$

But if $\beta < \alpha$ is such that $i^{\mathcal{U}}_{0,\delta}(\beta) > \nu^*$, then $Th_1^{Q}(\alpha \cup s \cup \{\nu^*\})$ can be computed from $i^{\mathcal{U}}_{0\delta}(Th_1^{m}(\beta \cup r))$, and hence belongs to $Q = P$, contrary to 99. So $\nu^* = i^{\mathcal{U}}_{0,\delta}(\alpha)$.

Remark. So $L^* = i^{\mathcal{U}}_{0,\delta}(G)$. $\boxtimes$

Claim 11a $Th_1^{m}(\kappa_G^+ \cup r \cup \{\alpha\}) \notin \mathcal{M}$.

Proof. If not, then $Th_1^{Q_1}(\alpha \cup i_{0,1}(r) \cup \{i_{0,1}(\alpha)\}) \in Q_1$. Since $\mathrm{crit}(i^{\mathcal{U}}_{1,\delta}) > \alpha$, this gives $Th_1^{Q}(\alpha \cup s \cup \{\nu^*\}) \in Q$.

$\boxtimes$

61a

It follows from claim 1a that $p_1(\mathcal{M}) = r \cup \{\alpha\}$. (So $\alpha = \mu_0$.) Clearly $p_1(\mathcal{M})$ is weakly solid. We claim that G witnesses clause (3) in the type Z_p conditions (definition 7). We have that $G = \dot{E}^{\mathcal{M}}_\alpha$ and G is short. Let $\kappa = \kappa_G$. We have that $p_1(\mathcal{M}) = (\kappa^+)^{\mathcal{M}}$ from claim 1a. We are left to verify (3)(c) of definition 7, concerning the behavior of the generalized core.

We have already shown that the generalized core of $\mathcal{Q}$ behaves properly, because the uncoring map is $i^{\mathcal{T}}_{1,\gamma} : \mathcal{H} \longrightarrow \mathcal{P} = \mathcal{Q}$, and $i_{E^*} \restriction (\alpha^+)^{\mathcal{Q}} = i^{\mathcal{T}}_{1,\gamma} \restriction (\alpha^+)^{\mathcal{Q}}$. We now just pull the relevant facts back to $\mathcal{M}$ using $i^{\mathcal{U}}_{0,\delta} : \mathcal{M} \longrightarrow \mathcal{Q}$.

More precisely, let

$$\mathcal{N} = \mathcal{H}_1^{m}(i_F{}'' \kappa^{+\mathcal{M}} \cup r)$$

be the generalized core of $\mathcal{M}$, and let

$$\sigma : \mathcal{N} \longrightarrow \mathcal{M}$$

be the uncollapse map. We show first that $\sigma \upharpoonright \kappa^{+\mathcal{M}} = i_G \upharpoonright (\kappa^+)^{\mathcal{M}}$. For this, we must show that $\mathrm{Hull}_1^M(i_G{}'' \kappa^{+\mathcal{M}} \cup r) \cap \alpha \subseteq i_G{}''(\kappa^+)^{\mathcal{M}}$.

But let τ be a Σ_1 Skolem term, and $\beta < \kappa^{+\mathcal{M}}$, and

$$M \models \tau(i_F(\beta), r) < \alpha.$$

Note $i^{\mathcal{U}}_{0,\delta}(G) = G^*$, $i^{\mathcal{U}}_{0\delta}(\beta) = i^{\mathcal{U}}_{1,\delta}(i_F(\beta)) = i_F(\beta)$, and $i^{\mathcal{U}}_{0,\delta}(\alpha) = \nu^*$. So

$$Q \models \tau(i_{G^*}(i_F(\beta)), s) < \nu^*.$$

But we have already shown Q is almost type $\mathbb{Z}_p$, with $p_1(Q) = s \cup \{\nu^*\}$, in claim 9a. So

$$Q \models \tau(i_{G^*}(i_F(\beta)), s) \in i_{G^*}{}''\alpha.$$

Pulling back by $i^{\mathcal{U}}_{0,\delta}$, we get

$$\mathcal{M} \models \tau(i_F(\beta), r) \in i_F{}''\kappa^+$$

as desired.

Remark. It is not true that $\sigma(\kappa^{+\mathcal{M}}) = i_G(\kappa^{+\mathcal{M}})$. For $\sigma(\kappa^{+\mathcal{M}}) = \lambda_G^{+\mathcal{M}} > \alpha$, whereas $i_G(\kappa^{+\mathcal{M}}) = \alpha$.

The argument above shows that $\mathcal{M}$ and $\mathcal{N}$ agree to $(\kappa^{+})^{\mathcal{M}} = (\kappa^{+\mathcal{N}})$, and $\sigma \restriction (\mathcal{M}|\kappa^{+\mathcal{M}}) = i_G \restriction (\mathcal{M}|\kappa^{+\mathcal{M}})$. We need to see that $\mathcal{M}|(\kappa^{++\mathcal{M}}) = \mathcal{N}|(\kappa^{++})^{\mathcal{N}}$.

Claim 12a. Let $\beta < (\kappa^{++})^{\mathcal{M}}$ be such that $\rho_\omega(\mathcal{M}|\beta) = (\kappa^{+})^{\mathcal{M}}$; then there is an $\eta < (\kappa^{++})^{\mathcal{N}}$ such that for all formulae φ and all $u \in [\kappa^{+\mathcal{M}}]^{<\omega}$

$$\mathcal{M}|i_G(\beta) \models \varphi[i_G(u)] \text{ iff } \mathcal{M}|\sigma(\eta) \models \varphi[\sigma(u)].$$

Remark The claim implies that for all $u \in [\kappa^{+\mathcal{N}}]^{<\omega}$, $\mathcal{M}|\beta \models \varphi[u]$ iff $\mathcal{N}|\eta \models \varphi[u]$. Thus $\beta = \eta$, and $\mathcal{M}|\beta = \mathcal{N}|\beta$. So $\mathcal{M}|(\kappa^{++})^{\mathcal{M}} \trianglelefteq \mathcal{N}$. But $(\kappa^{++})^{\mathcal{M}} < (\kappa^{++})^{\mathcal{N}}$ is impossible, for if $W \subseteq (\kappa^+)^m$ is in $\mathcal{N}$, then $\sigma(W) \in \mathcal{M}$, so $W = i_G^{-1}{}''\sigma(W)$ is in $\mathcal{M}$, because $i_G \upharpoonright \kappa^+ = \sigma \upharpoonright \kappa^+$. Thus $\mathcal{M} | (\kappa^{++})^{\mathcal{M}} = \mathcal{N} | (\kappa^{++})^{\mathcal{N}}$.

<u>Proof of claim 12a</u>. Let $W = Th_\omega^{\mathcal{M}|\beta}(\kappa^{+\mathcal{M}})$. Then $i_G(W) \subseteq \mathcal{M}|\alpha$, and

$$i_G(W) = Th_\omega^{Q_1 | i_G(\beta)}(\alpha)$$
$$= Th_\omega^{\mathcal{H} | i_G(\beta)}(\alpha).$$

Let τ be a Σ_1 Skolem term such that $i_G(W) = \tau^{\mathcal{H}}[c, F]$, where $c < \alpha$. We have $i_{1,\gamma}^{\mathcal{T}} : \mathcal{H} \to P = Q$, and $i_{1,\gamma}^{\mathcal{T}} \upharpoonright \alpha = i_{L^*} \upharpoonright \alpha$. So since $\mathcal{H}$ is the generalized core of P by 9a,

$Q \models \exists c < \alpha\, (\tau(i_{L^*}(c), s)$ is the theory of one of my levels, and $\tau(i_{L^*}(c), s) \cap i_{L^*}{}''\alpha = i_{L^*}(i_G(W)) \cap i_{L^*}{}''\alpha)$.

This is a statement about the parameters s, L^*, α, and $i_{L^*}(i_G(W))$.

We pull it back to $\mathcal{M}$ using $i^u_{0\delta}$. Notice $i^u_{0\delta}(\kappa^{+\mathcal{M}}) = \alpha$, and $i^u_{0\delta}(G) = L^*$. But also

$$i^u_{0\delta}(i_G(W)) = i_{L^*}(i^u_{0\delta}(W))$$

$$= i_{L^*}(i^u_{1,\delta}(i_G(W)))$$

$$= i_{L^*}(i_G(W)).$$

The third equality holds because $\operatorname{crit}(i^u_{1,\delta}) > \alpha$. Finally, $i^u_{0\delta}(r) = s$. So we get

$\mathcal{M} \models \exists c < \kappa^+ \,($ $T[i_G(c), r]$ is the theory of one of my levels, and $T(i_G(c), r) \cap i_G{}''\alpha = i_G(W) \cap i_G{}''\alpha$ $)$.

Fix a c witnessing the right hand side, and let

$$T^{\mathcal{M}}[i_G(c), r] = Th^{\mathcal{M}|\gamma}_\omega(\alpha).$$

Then $\gamma \in \mathrm{ran}(\sigma)$, and $\eta = \sigma^{-1}(\gamma)$ witnesses claim 12a.

$\boxtimes$

By claim 12a, $\mathcal{M} | (\kappa^{++})^{\mathcal{M}} = \mathcal{N} | (\kappa^{++})^{\mathcal{N}}$. To finish the proof that $p_1(\mathcal{M})$ is type $\mathcal{Z}_p$, we must show that $\sigma^{-1}(r) = p_1(\mathcal{N})$, and that $p_1(\mathcal{N})$ is solid.

Clearly, every $x \in \mathcal{N}$ is Σ_1-definable from $\sigma^{-1}(r)$ and ordinals $< (\kappa^+)^{\mathcal{N}} = \rho_1(\mathcal{N})$. Thus $p_1(\mathcal{N}) \leq_{lex} \sigma^{-1}(r)$. But now let $t <_{lex} \sigma^{-1}(r)$, and

$$B = Th_1^{\mathcal{N}}(\kappa^{+\mathcal{N}} \cup t).$$

We must see $B \in \mathcal{N}$. But $\sigma(t) <_{lex} r$, so

$$A =_{df.} Th_1^{\mathcal{M}}(\alpha \cup \sigma(t)) \in \mathcal{M},$$

because $\mathcal{M}$ has solidity witnesses for r. But for $\beta < \kappa^+$

$$(\varphi, \beta) \in B \text{ iff } (\varphi, \sigma(\beta)) \in A.$$

Since $\sigma \restriction \kappa^{+\mathcal{N}} \in \mathcal{M}$, we get $B \in \mathcal{M}$. Since $\mathcal{M} | (\kappa^{++})^{\mathcal{M}} = \mathcal{N} | (\kappa^{++})^{\mathcal{N}}$, we get $B \in \mathcal{N}$, as desired.

Our last goal is to show that $\rho_1(\mathcal{N})$ is solid. This takes a fair amount of additional work, but we shall need it. The proof also gives us a better picture of the relationship of $\mathcal{M}$ to its generalized core.

<u>Claim 13a</u> $\rho_1(\mathcal{M})$ is 1-universal.

<u>Proof</u> We have $\rho_1(\mathcal{M}) = \kappa_G^{+\mathcal{M}}$, and

$\mathcal{M} | (\kappa^{++})^{\mathcal{M}} \subseteq \mathcal{H}_1^{\mathcal{M}}(i_G'' \kappa^{+\mathcal{M}} \cup r)$. But clearly

$$Hull_1^{\mathcal{M}}(i_G'' \kappa^{+\mathcal{M}} \cup r) \subseteq Hull_1^{\mathcal{M}}(\kappa^{+\mathcal{M}} \cup r \cup \{\alpha\}),$$

because G is definable from α. Thus

$$\mathcal{M} | (\kappa^{++})^{\mathcal{M}} \subseteq \mathcal{H}_1^{\mathcal{M}}(\kappa^{+\mathcal{M}} \cup r \cup \{\alpha\}),$$

as desired.

$\boxtimes$

Now let

$$\mathcal{M}_0 = \mathcal{H}_1^m\left(K^{+m} \cup p_1(\mathcal{M})\right)$$

be the usual 1-core of $\mathcal{M}$, and

$$\psi : \mathcal{M}_0 \longrightarrow \mathcal{M}$$

the collapse map.

<u>Claim 14a</u> $\rho_1(\mathcal{M}_0) = \rho_1(\mathcal{M}) = K^{+m}$

$p_1(\mathcal{M}_0) = \psi^{-1}(p_1(\mathcal{M}))$, and $p_1(\mathcal{M}_0)$ satisfies the type Z_p conditions, with $\mathcal{N}$ being its generalized core, except perhaps for the solidity of $p_1(\mathcal{N})$.

<u>Proof.</u> That $\rho_1(\mathcal{M}) = \rho_1(\mathcal{M}_0)$ and $\psi^{-1}(p_1(\mathcal{M})) = p_1(\mathcal{M}_0)$ is an easy calculation. It uses claim 13a.

That $p_1(\mathcal{M}_0)$ satisfies the type Z_p conditions follows from the work we have just done, and are about to do in the other anomalous cases. For $\mathcal{M}_0$ satisfies the hypotheses of theorem 10. $p_1(\mathcal{M}_0)$ is not solid at all $\mu \in p_1(\mathcal{M}_0)$, since it is not solid at $\varphi^{-1}(\alpha)$. So let μ^* be largest in $p_1(\mathcal{M}_0)$ such that $p_1(\mathcal{M}_0)$ is not solid at μ^*, let $r^* = p_1(\mathcal{M}_0) - (\mu^*+1)$, and let α^* be least such that $Th_1^{\mathcal{M}_0}(\alpha^* \cup r^*) \notin \mathcal{M}_0$. So $\alpha^* > p_1(\mathcal{M}_0)$.

Claims 1-12 then imply that the level of $\mathcal{M}_0$ collapsing α^* exists, and falls under one of our anomalous cases. We shall show later that anomalous cases 2-4 lead to contradiction. That means anomalous case I applies.

But then the work we have done so far in anomalous case I implies

$$\alpha^* = \varphi^{-1}(\alpha)$$

and

$$r^* = \varphi^{-1}(r),$$

and $p_1(\mathcal{M}_0) = r^* \cup \{\alpha^*\}$ is 1-universal.

Moreover, $p_1(\mathcal{M}_0)$ satisfies the type Z_p conditions, except perhaps solidity for the parameter of its generalized core $\mathcal{N}_0$. Using ψ, we easily see that $\mathcal{N}_0 = \mathcal{N}$.

<u>Claim 15a</u> There is a long extender on the $\mathcal{N}$-sequence with domain $\mathcal{N} | (\kappa^{++})^{\mathcal{N}}$.

<u>Proof</u> Let $L = E_\eta$ be the first extender used in $\mathcal{I}_{1,\gamma}\mathcal{J}_T$, as in claims 7a and 8a. L is long, with space $(L) = \mathcal{H}|\alpha = \mathcal{P}_\eta|\alpha$. and domain $\mathcal{H}|(\alpha^+)^{\mathcal{H}} = \mathcal{P}_\eta|(\alpha^+)^{\mathcal{P}_\eta}$.

Since $\lambda_1^{\mathcal{T}} > \alpha$,

$$\pi_\eta \upharpoonright (\alpha^+ + 1)^{\mathcal{P}_\eta} = \pi_1 \upharpoonright (\alpha^+ + 1)^{\mathcal{H}}.$$

Thus $\pi_\eta(L)$ is a total extender on the $\mathcal{P}_\eta^*$ sequence with ~~dom~~

$$\begin{aligned} \mathrm{dom}(\pi_\eta(L)) &= \mathcal{P}_\eta^* | (\pi_1(\alpha)^+)^{\mathcal{P}_\eta^*} \\ &= \mathcal{M} | (\pi_1(\alpha)^+)^{\mathcal{M}}. \end{aligned}$$

If $[0,\eta]_T$ drops or $\pi_\eta(L)$ is not the top extender of P^*_η, we then get a long extender on $\dot{E}^{\mathcal{M}}$ with domain ~~$\pi_1(\alpha)$~~ $\mathcal{M} | \pi_1(\alpha)^{+\mathcal{M}}$. If $[0,\eta]_T$ does not drop and $\pi_\eta(L) = \dot{F}^{P^*_\eta}$, then $\mathrm{crit}(i^{\mathcal{T}}_{0\eta}) > \alpha$. (Otherwise $i^{\mathcal{T}}_{0\eta}(\mathrm{crit}(\dot{F}^{\mathcal{M}})) = \mathrm{crit}(\dot{F}^{P^*_\eta}) > \alpha$.) So $\dot{F}^{\mathcal{M}}$ is on the $\mathcal{M}$ sequence, with domain $\pi_1(\alpha)^{+\mathcal{M}}$.

Claim 15a now follows by using $\sigma : \mathcal{N} \to \mathcal{M}$ to pull this back to $\mathcal{N}$. Note $\sigma(\kappa^{++\mathcal{N}}) = \pi_1(\alpha)^{+\mathcal{M}}$.

$\boxtimes$

Now let

E ~~$\ldots$~~ = first long extender on the $\mathcal{N}$-sequence with domain $\mathcal{N} | (\kappa^{++})^{\mathcal{N}}$.

<u>Claim 16a</u> $\mathcal{M}_0 = \mathrm{Ult}_0(\mathcal{N}, E \restriction (\kappa^{+\mathcal{M}} \cup \{\nu\}))$, where $\nu = \nu(E) - 1$ is the unique long generator of E.

<u>Proof</u> We compare $\mathcal{M}_0$ with the phalanx $(\mathcal{M}_0, \mathrm{Ult}_0(\mathcal{N}, E \restriction (\kappa^{+\mathcal{M}} \cup \{\nu\})), \kappa^{+\mathcal{M}})$. Let $\mathcal{U}$ be the tree on $\mathcal{M}_0$, and $\mathcal{T}$ the tree on the phalanx side. Let

and

$$Q_\xi = \mathcal{M}_\xi^{\mathcal{U}},$$

$$P_\xi = \mathcal{M}_\xi^{\mathcal{T}},$$

with $P_0 = \mathcal{M}_0$ and $P_1 = \mathrm{Ult}_0(\mathcal{N}, E \restriction (\kappa^{+\mathcal{M}} \cup \{\nu\}))$. Note that $\mathcal{M}$, $\mathcal{M}_0$, $\mathcal{N}$, and $\mathrm{Ult}_0(\mathcal{N}, E \restriction (\kappa^{+\mathcal{M}} \cup \{\nu\}))$ all agree up to their common value for $\kappa^{++} + 1$. ($\mathcal{M} \mid \kappa^{++\mathcal{M}} = \mathcal{M}_0 \mid \kappa^{++\mathcal{M}}$ by universality.) Thus all extenders used in $\mathcal{T}$ or $\mathcal{U}$ have length $> \kappa^{++\mathcal{M}}$.

~~Her~~ The rules for $\mathcal{U}$ are the usual ones. We can as usual assume that $\mathcal{M}_0$ is countable, and fix an iteration strategy Σ_0 for $\mathcal{M}_0$ with the weak Dodd-Jensen property relative to some $\vec{e}$ enumerating $|\mathcal{M}_0|$. We use Σ_0 to choose branches for $\mathcal{U}$.

The rules for $\mathcal{T}$ are as before. Extenders with critical point κ get applied to $\mathcal{M}_0$ if they are short, and applied to $\mathrm{Ult}_0(\mathcal{N}, E\restriction(\kappa^{+m}\cup\{\nu\}))$ if they are long. We choose branches by lifting $\mathcal{T}$ to a tree $\mathcal{T}^*$ on $\mathcal{M}_0$, and then using Σ_0.

<u>Subclaim 16a.1</u> $(\mathcal{M}_0, \mathrm{Ult}_0(\mathcal{N}, E\restriction(\kappa^{+m}\cup\{\nu\})), \kappa^{+m})$ is iterable.

<u>Proof</u> We lift the evolving $\mathcal{T}$ on our phalanx to $\mathcal{T}^*$ on $\mathcal{M}_0$. Let $P^*_\xi = \mathcal{M}^{\mathcal{T}^*}_\xi$. We have maps $\pi_\xi : P_\xi \to P^*_\xi$ such that

(†) for $1 \le \gamma \le \eta$, $\pi_\gamma \restriction \lambda^{\mathcal{T}}_\gamma = \pi_\eta \restriction \lambda^{\mathcal{T}}_\gamma$, and P^*_γ agrees with P^*_η below $\lambda^{\mathcal{T}^*}_\gamma$.

Here $\lambda^{\mathcal{T}}_\gamma = \lambda_{E^{\mathcal{T}}_\gamma}$ and $\lambda^{\mathcal{T}^*}_\gamma = \lambda_{E^{\mathcal{T}^*}_\gamma}$, for $\gamma \ge 1$.

We set $\pi_0 = id$, and we obtain π_1 as follows. Let

$$\sigma_0 : \mathcal{N} \longrightarrow \mathcal{M}_0$$

be the map given by the fact that $\mathcal{N}$ is the generalized core of $\mathcal{M}_0$. We set

$$P_1^* = \mathrm{Ult}_0(\mathcal{M}_0, \sigma_0(E)).$$

(It is possible that $E = \dot{F}^{\mathcal{N}}$, in which case $\sigma_0(E) = \dot{F}^{\mathcal{M}_0}$.) Let us write

$$\bar{E} = E \restriction (\kappa^{+\mathcal{M}_0} \cup \{\nu\});$$

then

$$\pi_1([a,f]^{\mathcal{M}}_{\bar{E}}) = [\sigma_0(a), \sigma_0(f)]^{\mathcal{M}}_{E}.$$

The maps π_η for $\eta > 1$ can be defined using the shift lemma as usual, maintaining $(\dagger)$, except when $T\text{-pred}(\eta+1) = 0$ and $\mathrm{crit}(E^{\mathcal{T}}_\eta) = \kappa$. In this case, $E_\eta = E^{\mathcal{T}}_\eta$ must be short. Since $\pi_0 \restriction \kappa^+ \neq \pi_1 \restriction \kappa^+$, we must take some care. Let

$$j : P^*_\eta \longrightarrow \mathrm{Ult}_0(P^*_\eta, \pi_\eta(E_\eta))$$

be the canonical embedding. Let G_0 be the stretching extender associated to $p_1(\mathcal{M}_0)$, so that

$$\sigma_0 \restriction \kappa^{+\mathcal{M}_0} = i_{G_0} \restriction \kappa^{+\mathcal{M}_0},$$

and $G_0 = \dot{E}^{\mathcal{M}_0}_{\alpha_0}$, where $\alpha_0 = \sup \sigma_0 {}'' \kappa^{+\mathcal{M}_0}$.

Since all $\lambda^{\mathcal{T}}_\gamma$ are $> \kappa^{++\mathcal{M}_0}$, all $\lambda^{\mathcal{T}^*}_\gamma$ are $> \alpha_0$, so G_0 is on the P^*_η-seq.

Now set

$$P^*_{\eta+1} = \mathrm{Ult}_0(\mathcal{M}_0, j(G_0)),$$

and

$$\pi_{\eta+1}\left([a, f]^{\mathcal{M}_0}_{E_\eta}\right) = [\pi_\eta(a), f]^{\mathcal{M}_0}_{j(G_0)},$$

for $a \subseteq \lambda_{E_\eta}$ finite, and $f : [\kappa]^{|a|} \to \mathcal{M}_0$, ~~for~~ $f \in \mathcal{M}_0$.

$\pi_{\eta+1}$ is well-defined and elementary because for $a \subseteq \lambda_{E_\eta}$ finite and $X \subseteq [\kappa]^{|a|}$ in $\mathcal{M}_0$,

$$a \in i_{E_\eta}(X) \text{ iff } \pi_\eta(a) \in i_{\pi_\eta(E_\eta)}(\pi_\eta(X))$$
$$\text{iff } \pi_\eta(a) \in j(i_{G_0}(X))$$
$$\text{iff } \pi_\eta(a) \in i_{j(G_0)}(X),$$

noting that $\operatorname{crit}(j) > \kappa$ for the last line. It is clear that $\pi_{\eta+1}$ and π_η agree on λ_{E_η}. Also, $\lambda_{j(G_0)} = j(\lambda_{G_0}) = j(\operatorname{crit}(j)) = \lambda_{\pi_\eta(E_\eta)} > \lambda_\gamma^{\mathcal{T}^*}$ for $\gamma < \eta$. So $(\dagger)$ is preserved.

Remark This argument is given in somewhat more detail in the proof preceding Claim 3b.1 below. 16a.1 $\boxtimes$

The comparison terminates. Let

$P = P_\gamma$ be the last model of $\mathcal{T}$, and

$Q = Q_\delta$ be the last model of $\mathcal{U}$.

<u>Subclaim 16a.2</u> P is above $Ult_0(\mathcal{N}, \bar{E})$ in $\mathcal{T}$; that is, $\gamma = 1$ or $1 T_\gamma$.

<u>Proof</u> If not, then the usual weak Dodd-Jensen arguments involving $\mathcal{T}^*$ lead to a contradiction.

$\boxtimes$

<u>Subclaim 16a.3</u>

(a) $P = Q$, and neither $[1, \gamma]_T$ nor $[0, \delta]_U$ drops.

(b) $crit(i^{\mathcal{T}}_{1,\gamma}) > \kappa$ and $crit(i^{\mathcal{U}}_{0,\delta}) \geq \kappa$.

<u>Proof</u> If either $[1, \gamma]_T$ drops or $Q \triangleleft P$, then $[0, \delta]_U$ does not drop, and $\pi_\gamma \circ i^{\mathcal{U}}_{0,\delta}$ contradicts weak Dodd-Jensen. So

$i^{\mathcal{I}}_{1,\gamma} : P_1 \longrightarrow P \trianglelefteq Q$. We have $\mathrm{crit}(i^{\mathcal{I}}_{1,\gamma}) \geq \kappa$ by the phalanx rules. If $\mathrm{crit}(i^{\mathcal{I}}_{1,\gamma}) = \kappa$, then the first extender used in $[1,\gamma]_T$ is long, with domain $\mathcal{M} | (\kappa^{++})^{\mathcal{M}}$, by the phalanx rules. But P_1 has no extenders with this domain, by our choice of E as the first. The agreement of P_1 with P_η shows that for $\eta \geq 1$, P_η has no extender on its sequence with domain $\mathcal{M} | \kappa^{++\mathcal{M}}$. Thus $\mathrm{crit}(i^{\mathcal{I}}_{1,\gamma}) > \kappa$.

Note $Th_1^n(\kappa^{+n} \cup p_1(n))$ is definable over P_1, using

$$\bar{\nu} = [\{\nu\}, id]^n_E$$

as a parameter to define $i_E \restriction \kappa^{+n}$, as usual. It follows that if either

$P \triangleleft Q$ or $[0,\delta]_\mathcal{U}$ drops, then $\mathcal{N} \in \mathcal{M}_0$. But $\mathcal{N}$ collapses $\kappa^{++\mathcal{M}_0}$. Thus $[0,\delta]_\mathcal{U}$ does not drop, and $P = Q$.

This implies that $\rho_1(Q) = \kappa^{+\mathcal{M}_0}$. But then $\operatorname{crit}(i^{\mathcal{U}}_{0,\delta}) \geq \kappa$ by lemma 11. If $\operatorname{crit}(i^{\mathcal{U}}_{0,\delta}) = \kappa$, then by lemma 11 again, the first extender used in $[0,\delta]_\mathcal{U}$ has domain $\mathcal{M} | \kappa^{++\mathcal{M}}$. But $\mathcal{M}_0 = Q_0$ has no such extenders on its sequence, and this implies that no Q_η has such an extender on its sequence. Thus $\operatorname{crit}(i^{\mathcal{U}}_{0,\delta}) > \kappa$.

$\boxtimes$

Let now

$$p_1(\mathcal{M}_0) = r_0 \cup \{\alpha_0\}$$

where α_0 is least in $p_1(\mathcal{M}_0)$, and let

$$p_1(\mathcal{N}) = \sigma_0^{-1}(r_0).$$

Subclaim 16a.4

(a) $\quad i^{\mathcal{T}}_{1,\gamma} \circ i^{\mathcal{N}}_{\bar{E}}\left(p_1(\mathcal{N})\right) = i^{\mathcal{U}}_{0,\delta}(r_0).$

(b) $\quad i^{\mathcal{T}}_{1,\gamma}(\bar{\vartheta}) = i^{\mathcal{U}}_{0,\delta}(\alpha_0)$, where

$$\bar{\vartheta} = [\{\vartheta\}, \mathrm{id}]^{\mathcal{N}}_{\bar{E}}.$$

Proof (a) We get $i_{1,\gamma} \circ i_{\bar{E}}(p_1(\mathcal{N})) \leq_{lex} i^{\mathcal{U}}_{0,\delta}(r_0)$ by applying weak Dodd-Jensen to the diagram

$$\begin{tikzcd}
P^*_\gamma & P_\gamma \arrow[l, "\pi_\gamma"'] & = & Q_\delta \\
P^*_1 \arrow[u] & P_1 \arrow[l, "\pi_1"'] \arrow[u, "i^{\mathcal{T}}_{1,\gamma}"'] & & \\
\mathcal{M}_0 \arrow[u, "i_{\sigma_0(E)}"] & \mathcal{N} \arrow[l, "\sigma_0"] \arrow[u, "i_{\bar{E}}"'] & & \mathcal{M}_0 \arrow[uu, "i^{\mathcal{U}}_{0,\delta}"']
\end{tikzcd}$$

We have $\cancel{i_{\sigma_0(E)}}\; i^{\mathcal{T}^*}_{1,\gamma} \circ i_{\sigma_0(E)}(r_0) \leq_{lex} \pi_\gamma(i^{\mathcal{U}}_{0,\delta}(r_0))$, so $\pi_\gamma(i^{\mathcal{T}}_{1,\gamma} \circ i_{\bar{E}}(p_1(\mathcal{N}))) \leq_{lex} \pi_\gamma(i^{\mathcal{U}}_{0,\delta}(r_0))$, and this gives (a).

Let $s = i^{\mathcal{N}}_{\bar{E}}(p_1(\mathcal{N}))$, and $t = i^{\mathcal{T}}_{1,\gamma}(s)$.

So $Th_1^{P_1}(K^{+n} \cup s \cup \{\bar{\nu}\}) \notin P_1$, because from it one can recover n. So $Th_1^P(K^{+n} \cup t \cup \{i_{1,\gamma}^{\bar{x}}(\bar{\nu})\}) \notin P$. But $i_{0,\delta}^{u}(r_0)$ is solid over $Q = P$. It follows that $t \geq_{lex} i_{0,\delta}^{u}(r_0)$, as desired.

(b) It is easy to see that $s \cup \{\bar{\nu}\}$ is weakly solid at $\bar{\nu}$ over P_1. Moreover, $i_{1,\gamma}^{\bar{x}}$ is continuous at $\bar{\nu}$, so if $i_{0,\delta}^{u}(\alpha_0) < i_{1,\gamma}^{\bar{x}}(\bar{\nu})$, then $Th_1^Q(K^{+m} \cup i_{0,\delta}^{u}(r_0) \cup \{i_{0,\delta}^{u}(\alpha_0)\}) \in Q$, a contradiction. If $i_{1,\gamma}^{\bar{x}}(\bar{\nu}) < i_{0,\delta}^{u}(\alpha_0)$, we get a similar contradiction using the weak solidity of $p_1(M_0)$ at α_0.

$\boxtimes$

Putting our subclaims together, we have

$$i^{\mathcal{Q}}_{1,\delta} : P_1 \cong Hull_1^{\mathcal{Q}}(\kappa^{+\mathcal{M}} \cup i^{\mathcal{U}}_{0,\delta}(p_1(\mathcal{M}_0))),$$

and since $\mathcal{M}_0 \cong Hull_1^{\mathcal{Q}}(\kappa_\xi^{+\mathcal{M}} \cup i^{\mathcal{U}}_{0\delta}(p_1(\mathcal{M}_0)))$,

we have that $P_1 = \mathcal{M}_0$, as desired.

16a $\boxtimes$

Finally, this gives

<u>Claim 17a</u> $p_1(\mathcal{N})$ is solid.

<u>Proof</u> Let E and $\bar{E}$ be as above.

We have that $p_1(\mathcal{M}_0) = r_0 \cup \{\alpha_0\}$, where r_0 is solid over $\mathcal{M}_0$, moreover, $r_0 = i^{\mathcal{N}}_{\bar{E}}(p_1(\mathcal{N}))$. If $E \in \mathcal{N}$, then in $\mathcal{N}$ we can use the solidity witnesses for r_0 to compute solidity witnesses for $\mathcal{N}$.

If $E \notin \mathcal{N}$, then $E = \dot{F}^{\mathcal{N}}$. So $F = \dot{F}^{\mathcal{M}}$ is the first extender on the $\mathcal{M}$ sequence with domain $\pi(\alpha)^{+\mathcal{M}}$. Because

$\mathcal{M}$ has long extender condensation, $p_1(\mathcal{M}) \subseteq \pi(\alpha)^{+\mathcal{M}} = \lambda_G^{++\mathcal{M}}$. We have that $\lambda_G < \alpha < \lambda_G^{+\mathcal{M}}$, and α is least in $p_1(\mathcal{M})$. We may assume that $r \neq \emptyset$, as otherwise $p_1(\mathcal{N}) = \emptyset$, and there is nothing to prove. It follows that

$$r = \{\beta\}$$

where

$$\lambda_G^{+\mathcal{M}} \leq \beta < \lambda_G^{++\mathcal{M}},$$

and

$$p_1(\mathcal{N}) = \{\sigma^{-1}(\beta)\}.$$

Let $\bar{\beta} = \sigma^{-1}(\beta)$. For $\gamma < \bar{\beta}$ and φ Σ_1,

$$(\varphi, \gamma) \in Th_1^{\mathcal{N}}(\bar{\beta}) \text{ iff } (\varphi, \sigma(\gamma)) \in Th_1^{\mathcal{M}}(\beta).$$

But $\bar{\beta} < \kappa^{++\mathcal{M}}$, so $\sigma \restriction \bar{\beta} = i_G^{\mathcal{M}} \restriction \bar{\beta} \in \mathcal{M}$. ~~Also~~ Moreover, $Th_1^{\mathcal{M}}(\beta) \in \mathcal{M}$ because

r is solid over $\mathcal{M}$. Thus $Th_1^{\mathcal{N}}(\bar{\beta}) \in \mathcal{M}$. But $\mathcal{N} | \kappa^{++\mathcal{N}} = \mathcal{M} | \kappa^{++\mathcal{M}}$, so $Th_1^{\mathcal{N}}(\bar{\beta}) \in \mathcal{N}$, as desired.

$\boxtimes$

Claims 1a – 17a finish the proof of theorem 10, in the case $k=0$ and we are in anomalous case I,

$\boxtimes$