Local HOD computation

J. R. Steel
July 2016

§0. Introduction

Our main goal is to prove the following.

Theorem 1. Assume $AD^+$, and let $M$ be a proper class inner model such that $RCM$, and $M = AD^+_R + V = L(P(R))$. Suppose there is an $1br$ hod pair $(P, \Sigma)$ such that $\Sigma \not\in M$; then $\text{HOD}^M$ is a least branch premouse.

This was conjectured at the end of §17. The proof involves showing that $M = \text{the 1br hod-pair order has $\le^*$ has order type $\Theta$.}$

$\le^*$ is just the natural "mouse order." We get
from the comparison process. See §5.7 of [11]. Any equivalent statement would be

\[ M \models \text{there are 1br hod pairs } (P, \varepsilon) \]

with \( \varepsilon \) of arbitrarily large Wadge degree.

Here and below we tacitly identify \( \varepsilon \) with

\[ \text{Code}(\varepsilon) = \{ x \in \mathbb{R} \mid x \text{ codes some } \varepsilon \in \text{HPC} \} \]

The equivalence holds because \( (P, \varepsilon) \leq (Q, \psi) \Rightarrow \)
\( \varepsilon \) is projective in \( P \).

Definition 1. \textbf{Hod-pair capturing} (or HPC) is the statement:
for all \( A \in \mathbb{R} \) there is an 1br hod pair \((P, \varepsilon)\) such that \( A \leq_w \text{Code}(\varepsilon) \).

The hypothesis of Theorem 1 gives us a hod pair beyond \( M \); we must "localize" to get pairs of arbitrary complexity in \( M \), i.e. to get \( M \vDash \text{HPC} \).

Remark: HPC is a version of Sargsyan's "Generation of full pointclasses".

Remark: We are tacitly assuming \( \text{AD}^+ \), here and below.
As in Definition 7.1 of LIJ, we let
$\delta^*(P, E)$ be the system of all non-dropping $E$-
iterates of $P$, and

$$M_{\omega}(P, E) = \text{dir } \lim \delta^*(P, E).$$

This makes sense for any $\text{key} \text{ hod}$ pair
$(P, E)$. In 7.3 of LIJ it was asserted
that
$$\delta(P, E) \leq^* (Q, \eta) \iff 0(M_{\omega}(P, E)) \leq
0(M_{\omega}(Q, \eta)),$$
but this is trivially false:
let $P = 0^*$ and $Q = L, \omega^J, \text{ both with}
the usual natural $\text{iso}$-order ; then $0(M_{\omega}(P, E)) > \omega$
but $M_{\omega}(Q, \eta) = Q$. The assertion

\text{b) becomes correct if we add the}
\text{hypothesis that $(P, E)$ is full, in the}
\text{sense defined just before claim 3 in the}
\text{proof of 7.4 in LIJ.}
Remark. Note that $(P, \Sigma) \equiv^* (Q, \Psi)$ iff $\text{M} \circ (P, \Sigma) = \text{M} \circ (Q, \Psi)$. No fullness hypothesis is needed. Thus $\text{M} \circ (P, \Sigma) \in \text{HOD}$ for all $\text{HOD}$ pairs. However, without fullness we cannot conclude that $\text{M} \circ (P, \Sigma)$ is an initial segment of the $\text{HOD}$ hierarchy of $\text{HOD}$. With it, we can.

Assuming $\text{AD}_{\text{Re}} + \text{HPC}$, we shall show that for any $\kappa$, $\text{HOD} \cap \Theta_{\kappa+1} = \text{M} \circ (P, \Sigma)$ for some full $(P, \Sigma)$. In fact, we get some information on the first order form of $P$ as well.

Definition 2. For $P$ an $\text{HOD}$, let

$$\eta^P = \sup \{ \eta(E) + 1 \mid E \text{ is on } \Theta, \text{ P-} \text{sequence} \},$$

(So $\eta^P = \circ^*(P) + 1$ iff $P$ is active.) Let

$$0(K)^P = \sup \{ \eta(E) + 1 \mid E \text{ is on the P-} \text{sequence} \}$$

and $\text{crit} (E) = k^P$.

We say that $P$ has a top block iff $\exists k < \eta^P (0(K)^P = \eta^P)$. Otherwise, we say $P$ has limit block-type.
Definition 3. Let $P$ be an $1$-pm, and suppose $P$ has a top block. Then:

(a) If $\eta^P$ is a limit cardinal, then

$$K^P = \text{least } k \text{ such that } \sigma(k)^P = \eta^P.$$ 

(b) If $\eta^P = \gamma + 1$, then let $F^\gamma$ be the extended indexed at $\gamma$ in $P$; and we let

$$K^P = \text{least } k \text{ such that } \sigma(k)^P = \text{crit}(F),$$

or

$$K = \text{crit}(F).$$

In either case, we say that $K^P$ begins the top block of $P$.

We shall show

Theorem 4. Assume $\text{AD}_R + \text{HPC}$, and let $\Theta_{\omega_1} = \Theta$ be a successor point of the Solovay sequence; then there is a full $1$-br hod pair $(P, \mathcal{Z})$ such that

1. $\eta^P$ is the largest cardinal of $P$, and $P = \eta^P$ is Woodin,

2. $\prod_{\omega_0}^2 (\gamma^P_\omega) = \Theta_{\omega_1}$, and

$$\text{HOD} \Theta_{\omega_1} = \prod_{\omega_0} (\mathcal{Z}, \mathcal{P}) \upharpoonright \Theta_{\omega_1},$$

and

3. $\prod_{\omega_0}^\Sigma (K^P)$ is the largest Suslin cardinal $< \Theta_{\omega_1}$.

Remark. By (8), $\eta^P$ is a limit cardinal, so 13(a) applies. $K^P$ is then the least cardinal $< \eta^P$ strong in $P$ to $\eta^P$.  


Remark By (1) \( \eta^p \) is a regular cuspoint of \( P \), and thus \( \mathcal{M}_\alpha(P \eta^p, \Sigma P \eta^p) = \Pi_{\beta_0}(P \eta^p) = \text{HOD} \Theta_{\alpha+1} \).

We get at once

Corollary 5 Assume \( AD_{\Theta} + HPC \); then \( \text{HOD} \Theta \) is an Ipm.

Corollary 6 Assume \( AD_{\Theta} + \text{HPC} \); and let \( \Theta_{\alpha+1} < \Theta \). Then there is no extendable \( E \) on the \( \text{HOD} \Theta \) sequence such that \( \text{core}(E) \leq \Theta_{\alpha+1} < \theta E \).

Proof By the "HOD \Theta sequence", we mean the union of the sequences of \( \mathcal{M}_\alpha(P, \Sigma) \), for \( (P, \Sigma) \) full. By the proof of cor. 5, each such \( \mathcal{M}_\alpha(P, \Sigma) \) is a cardinal cuopoint, initial segment of \( \bigcup \mathcal{M}_\alpha(P, \Sigma) = \text{HOD} \Theta \), (P, \Sigma) full.

So by theorem 4, \( \text{HOD} \Theta_{\alpha+1} \) is such an initial segment.
Remark Woodin has already proved some results in this direction assuming only AD$^+$, and not using the direct-limit-system analysis of HOD. For example, he showed no limit $\Theta$ is Woodin in HOD. (Unpublished.)

In order to prove the results above, we must connect the Suslin cardinals to the cardinals of the form $|M_\alpha (P, \mathcal{E})|$. In 7.4 of [17] it was stated that $\text{Code} (\mathcal{E})$ is $0(M_\alpha (P, \mathcal{E}) - \text{Suslin})$, but this is clearly wrong. It
"All sets are countable", then it cannot be iterated without dropping, so \( \mathcal{M}_\infty(P, \Sigma) = P \), and \( \omega(P) \) is countable. However, \( \Sigma \) on dropping trees could be quite complicated. The correct statement requires that we isolate the part of \( \Sigma \) relevant to forming \( \mathcal{M}_\infty(P, \Sigma) \).

**Definition 7**  
For \( P \) an \( \text{Ipm} \), and \( \alpha \in \text{ORD}^P \), 
\[ \rho^P(\alpha) = \sup \{ \beta \mid \beta + 1 \in \text{crit}(\mathcal{E}^P_\eta) = \alpha \} \] . We call \( \beta \) a strong endpoint of \( P \) iff \( \forall \eta < \beta \, (\rho^P(\alpha) < \beta) \).

**Definition 8**  
Let \((P, \Sigma)\) be an \( \text{Ibr} \) model pair, and let \( T \) be a normal iteration tree on \( P \). We say that \( T \) is \( \mathcal{M}_\infty(P, \Sigma) \)-relevant iff \( T \) is by \( \Sigma \), and there is a normal \( \mathcal{I} \) by \( \Sigma \) such that \( \mathcal{I} \) extends \( T \), \( \mathcal{I} \) has a last model \( Q \), and the branch \( P \to Q \) of \( \mathcal{I} \) does not drop (in model or degree).

Otherwise, \( T \) is \( \mathcal{M}_\infty(P, \Sigma) \)-irrelevant.
The following proposition gives a more concrete characterization.

**Proposition 9** Let \((P, \Sigma)\) be an Hbr hOd pair, and \(I\) a normal iteration tree on \(P\) such that \(I\) is by \(\Sigma\). Let \(k = k(P)\). Equivalent are:

1. \(I\) is \(M_{\Sigma}(P, \Sigma)\) - irrelevant,

2. Either
   
   (a) there is an \(\eta + 1 < \text{lh}(I)\) and a strong cutpoint \(K\) of \(M_{\eta}^\Sigma\) such that
      
      (i) \(K \leq \lambda(E_{\eta})\); equivalently, \(K \leq \text{crit}(E_{\eta})\)
      
      and
      
      (ii) either \([0, \eta]_I\) drops, and \(p(M_{\eta}^\Sigma) \leq K\),
      
      or \([0, \eta]_I\) does not drop, and
      
      \(p_k(M_{\eta}^\Sigma) \leq K\).

   Or

   (b) \(I\) has limit length, and for \(b = \Sigma(I)\),

   \(b\) drops, and either \(s(I)\) is a cutpoint of

   \[a \leq s(I) (0(K)_{M_{\eta}^\Sigma} \leq s(I))\]
Proof (Sketch)

(2) ⇒ (1): If $2(b)$ holds, then $2(a)$ must hold of any one-model extension of $T * b$, with $\eta = 1b(2)$ and $K = 8(3)+1$ being the witnesses. So it is enough if we assume $2(a)$ holds.

Letting $\eta$ be the least witness, and $K$ the associated strong cutpoint of $M^\eta$, one can see that any $T$ extending $T$ factors as $T | (\eta+1) U$, where $U$ is above $K$. Thus every $T$ applied to $M^\eta$ causes a drop.

(1) ⇒ (2): It is enough to see that if $T$ has a last model, and $2(a)$ fails, then there is a normal $\xi \geq T$ by $2$ whose branch to the last model does not drop. In fact, we can take $\xi = T$ or $\xi = T^n \langle E \rangle$ for some extender $E$. 
We leave the proof to the reader.

Remark. We don't actually need the proposition.

**Definition 10.** Let \((P, \Sigma)\) be an lbr hod pair (i.e., a normal stack) and let \(S = \langle I_0, \ldots, I_n \rangle\) be a stack of normal topoi on \(P\). Let \(Q\) be the last model of \(I_n\). We say that \(S\) is \(\text{M}\text{o}\text{o}(P, \Sigma)\)-relevant if \(S\) is by \(\Sigma\), \(P \rightarrow Q\) in \(S\) does not drop, and \(I_n\) is \(\text{M}\text{o}\text{o}(Q, \Sigma_{\text{ess}, Q})\)-relevant.

**Definition 11.** Let \((P, \Sigma)\) be an lbr hod pair; then \(\Sigma_{\text{rel}}\) is the restriction of \(\Sigma\) to all \(\text{M}\text{o}\text{o}(P, \Sigma)\)-relevant normal stacks.

We shall prove

**Theorem 12.** Assume \(\text{AD}^+\) and let \((P, \Sigma)\) be an lbr hod pair; then for \(K = \|\text{M}\text{o}\text{o}(P, \Sigma)\|\), \(\Sigma^\text{rel}\) is \(K\)-Suslin, but not \(\alpha\)-Suslin for any \(\alpha < K\).
We shall see later that there is another source for Suslin cardinals besides that given in Theorem 62. Namely, they can be of the form \( \Pi^2_{\beta_\emptyset}(K) \), where \( K \) begins in the top block of \( P^* \). We conjecture that every Suslin cardinal strictly below the supremum of the Suslin cardinals is at one of these two forms, if \( AD^+ \) and \( HPC \) hold.

Kunen-Martin easily implies that \( \Sigma^1_1 \) is not \( \omega \)-Suslin. For any \( \omega < \text{cof}(\text{Moo}(P, \Sigma)) \). If \( \Sigma \) has branch condensation, then it is easy to see that \( \Sigma^\omega \) is \( \text{cof}(\text{Moo}(P, \Sigma)) \)-Suslin. (We verify that \( \Sigma^\omega(\emptyset) = \emptyset \) by looking for a \( \sigma \) such that \( \Pi^2_{\beta_\emptyset} = \sigma \circ \Delta_b \).)
In the general case, we proceed as follows. We show that if \((P, \mathcal{E})\) is an 1br hod pair, then \(\mathcal{E}\) fully normalizes well. This implies that there is a normal tree \(U\) on \(P\) such that \(U\) has least model \(\text{Mo}(P, \mathcal{E})\), and club many countable hulls of \(U\) are by \(\mathcal{E}\). We then get

\[ \hat{\mathcal{E}} \text{ is by } \mathcal{E}\text{-rel  iff } \hat{\mathcal{E}}(\text{rel}) \text{ is nicely embedded into } \text{rel}(U). \]

Here a "nice embedding" is like a pseudo-hull embedding, except that the condition on preserving exit extenders has been weakened. The equivalence displayed shows that \(\mathcal{E}\)-rel is \(K\)-Suslin, where \(K = |U| = |\text{Mo}(P, \mathcal{E})|\). This yields Theorem 12.
The proof also gives a Suslin representation of the short-tree component of \( \Sigma_{rel} \).

**Definition 13** Let \((P, \Sigma)\) be an lbr hod pair, and \(a^P_b\) a normal tree on \(P\) with last model \(Q \models M^P_b\) such that \(a^P_b\) is by \(\Sigma_{rel}\). Then 
\( \Sigma \) is short iff \( P \to Q \) drops, or \( P \to Q \) does not drop, and 
\( \Pi(nP) > \delta(T) \), where \( \Pi : P \to Q \) is the canonical embedding.

\( \Sigma_{ste} \) is the restriction of \( \Sigma \) to short normal trees. We call it the short-tree component of \( \Sigma \). \( \Sigma_{ste, rel} \) is the further restriction to relevant trees.

**Theorem 14** Let \((P, \Sigma)\) be an lbr hod pair such that \(P\) has a top block, and let 
\[ K = \Pi_{P, \omega_0}(K^P) \]
then \( \Sigma_{ste, rel} \) is \(|K|\)-Suslin, but not \(d\)-Suslin for any \(\alpha < |K|\).
Remark. If $P \models \kappa^+ \text{ is a limit of Woodin cardinals}$
then $\Pi_{\kappa^+}^E(\kappa^+)$ is a cardinal.

The proof of Theorem 14 is like that of Theorem 12.

Let $U$ be normal on $P$ with last model $M_{\xi}(P, \Sigma)$ and such that club many countable hulls of $U$ are by $\mathcal{E}_{\kappa^+}$. We let

$$U_0 = U \upharpoonright (\Pi_{\kappa^+}^E(\kappa^+) + 1).$$

So $U = U_0 \upharpoonright U_0$, where $U_0$ is a normal tree on $M_{\xi}$ that has all critical points $> \kappa$ for $\kappa = \Pi_{\kappa^+}^E(\kappa^+)$. We then get

$\mathfrak{a}$ is by $\Sigma_{\kappa^+}$ iff $\mathfrak{a}$ nicely embeds into $U_0$.

This gives the desired Sierpinski representation of $\Sigma_{\kappa^+}$.

The gap between $\Sigma$ and $\mathcal{E}_{\kappa^+}$ is somewhat awkward. Let us say that the $M_{\xi}$-irrelevant fragment of $\Sigma$ is $\Gamma$-bounded iff whenever $\Sigma$ is by $\mathcal{E}_{\kappa^+}$ and $\Sigma_{\xi}$ is $M_{\xi}$-irrelevant, then the tail strategy $Z_{\xi} \cap \mathcal{M}_{\xi} \in \Gamma$, where $b = \Sigma_{\xi} (\xi)$. 


Continuing with our sketch of the proofs of Theorems 1 and 4, the following is the main additional ingredient.

Theorem 15 Assume AD$^+$, and let $(P; \mathcal{E})$ be an Ibrag mid pair that has a top block, and in each that $\eta^P$ is a limit ordinal. Let $K_{\omega\omega} = \prod_{\omega\omega} (\kappa^P)$ and $\eta_{\omega\omega} = \prod_{\omega\omega} (\eta^P)$. Suppose that $\varepsilon^{\eta_{\omega\omega}}$ is $K_{\omega\omega}$-Suslin; then there are no Suslin cardinals $\mu$ s.t. $K_{\omega\omega} < \mu < |\eta_{\omega\omega}|$.

Proof $K_{\omega\omega}$ has uncountable cofinality, and is a Suslin cardinal by Thm. 14. Let

$\Gamma_0 = S_{K_{\omega\omega}}$

so (see Jackson's handbook article 23J, §3)
so $\Gamma_0$ is "$\mathbb{Z}_2^1$-like", i.e. good, has the scale property, and closed under $\mathcal{J}^R$. Let $\Gamma_1$ be an inductive like point class with the scale property such that $\Gamma_0 \supseteq \Delta_1$, and $(P, E) \in \Delta_1$.

Let $(N^+, E^+, S^+)$ be a coarse $\Gamma_1$-Woodin model that captures a universal $\Gamma_1$ set.

Now let $S_0$ be the first $\Gamma_0$-Woodin of $N^+$. We can make sense of $S_0$ because $N^+$ captured $\Gamma_1$. Let

$$C = \text{the hod pair construction of } N^+ / S_0,$$

with models $(M^0_{\beta,k}, \Omega^\alpha_{\beta,k})$. This is an initial segment of the construction of $N^+$, and since $(N^+, E^+, S^+)$ captured $(P, E)$, we have that $\mathcal{J}^R \mathcal{K}^R$ for each worldly $(P, E)$ it iterates to some $(M^0_{\beta,k}, \Omega^\alpha_{\beta,k})$ for $(\alpha^+, k^+) \leq \text{lex} (\alpha, k)$, or $(P, E)$ iterates
strictly past \((\mathcal{M}^c_{\exists k}, \mathcal{L}^c_{\varphi_0})\), with no iterations in question belonging to \(\mathbb{N}^\varphi_{\mathcal{F}_0}\).

We claim that \((P, \mathcal{E})\) does not iterate strictly past \((\mathcal{M}^c_{\exists 0}, \mathcal{L}^c_{\varphi_0})\).

For otherwise it does so via a tree which is short, i.e. via \(\mathcal{T}^b\), which is by \(\Sigma^1_2\). But then \(\mathcal{T}^b \in \text{LET}_{\mathcal{F}_0}, \mathbb{N}^\varphi_{\mathcal{F}_0}\), where \(\mathcal{T}^b\) is the top of a scale on \(\mathcal{F}_0\) and \(\mathcal{F}_0\) is Woodin in \(\text{LET}_{\mathcal{F}_0}, \mathbb{N}^\varphi_{\mathcal{F}_0}\).

Contradiction.

So \((P, \mathcal{E})\) iterates to some \((\mathcal{M}^c_{\exists k}, \mathcal{L}^c_{\varphi_0})\) with \(<\varphi, k> \leq <\mathcal{F}_0, 0>\).

The worst case is \(<\xi, k> = <\mathcal{F}_0, 0>\), so
assume that. The tree has the form $\bar{b}^0 \bar{b}$, where every proper initial segment is by $\mathcal{Z}$, so that $\bar{b} \in \text{LIT}^N \setminus N^* \cup \emptyset \bar{b}$, but $b \notin \text{LIT}^N \setminus N^* \cup \emptyset \bar{b}$, and $\bar{b}^0 \bar{b}$ itself is not short. In particular, $b$ does not drop. But then we have (in $\mathcal{Z}$, because of the correctness of $N^*$)

$$\mathcal{Z} = \left( \mathcal{Z}_{\bar{b}^0 \bar{b}, N^*} \right)_{\bar{b}^0 \bar{b}} = \left( \mathcal{A}_{\bar{b}^0 \bar{b}} \right)_{\bar{b}^0 \bar{b}}.$$ 

But $\mathcal{A}_{\bar{b}^0 \bar{b}}$ is definable as "choose the unique branch moving an $e_j$ for $\bar{b}$ correctly." So it is $\mu$-Suslin, where $\mu$ is the least Suslin $> K_\infty$. So $\mathcal{Z}$ is $\mu$-Suslin, and hence $|K_\infty| \leq \mu$ by Kunen–Mehrten.
Remark. It should be possible to show that \( \kappa_0 \) is the least Suslin cardinal \( \geq \kappa_0 \).

Note that \( \text{cof}(\kappa_0) = \omega > 1 \) as would have to be the case. For \( \alpha < \kappa_0 \), let
\[
\Psi_{\alpha} = \Sigma^\text{src} \cup \{ (\mathbb{I}, b) | \mathbb{I} \text{ is by } \Sigma^\text{src} \}
\]
\( \Sigma^b \) does not drop, and for \( c = \Sigma(\mathbb{I}) \)
\[
1^{\beta+1}_c \searrow 1^{\beta+1}_c \searrow 1^{\beta+1}_c \searrow 1^{\beta+1}_c
\]
One needs to show that each \( \Psi_{\alpha} \in \text{Env}(S_{\kappa_0}) \).

Putting these ingredients together, we get

Proof sketch to Theorem 1. Let \( M = AD^\omega \)
and \( (\mathbb{P}, \Sigma) \) be mouse-least such that
\( \Sigma \in M \).

Claim: \( \Sigma^\text{rel, src} \notin M \).

Proof. Let \( \Psi_0 = \Sigma^\text{rel, src} \), and \( \Psi_0 \in M \).
Working in $M$, we define by induction on a strategy $\psi_a$ and show that $\psi_a = \Sigma^\infty_{\text{str}} \in M$ this way. Given $\psi_a$, we look for all $M_\alpha$ (irrelevant terms) that are by $\Sigma^\infty_{\text{str}}$. Note that if $j$ is such, and $b = 2(j)$, then $M_b^j$ has a cutpoint $k$, and $M_b^j$ is a dropping iterate, and $M_b^j$ projects to $k$. Suppose we have that $\psi_a$ is total on all cut-free trees on $M_b^j$. The strategy of $M_b^j$ above $k$ is in $M$, because it is strictly below $(P, \Sigma)$. It is $OD(M_b^j, \psi_a)$, uniformly. Call it $\Phi_a$. Then $\psi_{a+1} = \bigcup_{\text{such } \Phi_a} \psi_a$.

So $\Sigma^\infty_{\text{str}} \in M$, and in fact it is $OD^M(\Sigma^\infty_{\text{str}})$. But $M \models$ all reals are Suslin, so by Theorem 15, $\eta \in \Theta^M$, and thus by Theorem 12, $\Sigma \in M$, contradiction.

Claim
There are now two cases. Suppose \( P \) has limit type. Then \( \Theta^M = \Pi^P_{\alpha_0}(\eta^P) \), because each non-dropping iterate \( Q \) of \( P \) has limit type and each \( \Sigma^2, \alpha \eta \in M \) for \( \eta < \eta^Q \).

So \( \Theta^M = \Pi^P_{\alpha_0}(\eta^P) \), as otherwise \( \Sigma \in M \). So \( M = HPC \), the witnesses being pairs \( (S, \Psi) \), where \( S \) is a cusp point in an iterate of \( P \), and \( \Psi \) is the tail of \( S \). It is easy to see that if \( P \rightarrow_0 Q \) does not drop, and \( S = Q(\delta^Q + 1)^\eta \) for some \( \delta \), then \( M = (S, \Psi) \) is fullness preserving \( (\Psi \text{ iterates carry } Q \text{ along on top.}) \).

Since we have HPC via fullness-preserving strategies, \( \text{HOD}^M = \bigcup M^\alpha (S, \Psi) \). (S, \Psi)
Suppose next $P$ has a top block.

Then $\Pi_{P,\omega}(k^P) \leq \Theta^M$ by the argument above so $\Pi_{P,\omega}(k^P) = \Theta^M$ as otherwise $\exists \tau \in M$. So again, $\mathcal{M} = HPC$ via fullness-preserving pairs, so $\text{HOD}^\mathcal{M} = \bigcup M_{\alpha}(S, \Psi)$. In this case $(S, \Psi)$

the pairs are even $\langle Q, \pi, \tau, Q \rangle$, where $P - \tau - Q$ does not drop $\Psi$ is the tail of $\tau$, and $\pi < \Pi_{P,\omega}(k^P)$.

Sketch of Thm 4. 

Remark. Theorem 4 is proved using the same methods.

Remark. Nam Trang and the author have worked out a version of this for the case $M$ has a largest Mitchell cardinal.
Remark. In [27], the author has shown that the existence of certain lbr had pairs gives rise to models of LSA and stronger theories; e.g., "HOD ( wθ ) ∩ P ( M ) = P_k ( R )", for k the largest Suslin cardinal. That paper also gives a converse to Corollary 6: assuming ADK + HPC, every Woodin cardinal endpoint of HOD is a θ_{κ+1}.

The LSA result was first proved by Sargsyan.

In section 1, we describe the full normalization \( X(\mathcal{F}, U) \) of a stack \( \langle S, U \rangle \) of normal trees, and relate it to the embedding normalization \( W(\mathcal{F}, U) \). In section 2, we describe the "nice embedding" of \( \mathcal{F} \) into \( X(\mathcal{F}, U) \) that we get in abstract terms. The embedding of \( \mathcal{F} \) into \( W(\mathcal{F}, U) \) is a
pseudo-hull embedding. (See [17, def. 3.3].) Lacking all inspiration, we shall call embeddings like that from \( I \) to \( X(I, U) \) "weak hull embeddings." A strategy that condenses to itself under weak hull embeddings has "very strong hull condensation:"

In §3, we show that if \((P, E)\) is an \(1br\) hod pair, then \( E \) fully normalizes well and has very strong hull condensation. Those do not seem to be properties that one can get directly from a background construction in a model with UBH, as was done in [17] for normalizing well and strong hull condensation. The arguments of §3 involve phalanx comparisons like those at the end of §1.

In §4, we use the results of §3 to show that the strategies of \(1br\) hod pairs are positional. In §5, we tie things up by filling out the sketches in §0 a bit more.
§1. Full normalization

We outline some basic facts, and establish some notation. $L_2^1$ has a more complete account.

We begin with this atomic step.

Let $\mathcal{A}$ be a normal tree on the premouse $M$. Here $M$ can be an $\Pi^1_1$-model, or a Jensen or $\Pi^1_1$-pure extendible premouse. Let $F$ be an extendible on the sequence of $M^\mathcal{A}$, with $\lambda$ least such that this is true of $F$. Let $\beta \leq \lambda$ be such that $\text{crit}(F) < \lambda(E^\mathcal{A}_\beta)$, or $\beta = \lambda$ if no such $\eta$ exists. Suppose $\mathcal{A}$ is another normal tree on $M$ such that $\mathcal{A}^{\beta+1} = \Delta^{\beta+1}$. In this situation, $L_2^1$ defines the embedding normalization.

Assume also that if $\beta+1 < \lambda^+$, then $\text{dom} F = \lambda(E^\mathcal{A}_\beta)$.

$\Box$
\[ W(\beta, F) = \Delta^{\alpha+1}\langle F \rangle^\beta \] 

Here we define the full normalization

\[ X(\beta, F) = \Delta^{\alpha+1}\langle F \rangle^\beta \]

The difference between \( i_F \) and \( \bar{i}_F \) in the formulas above (which are only heuristic!) has to do with what functions are used in various ultrapowers.

Let \( K_F = \text{crit}(F) \). If \( \beta + 1 = \text{lh} \bar{F} \), then

\[ X(\beta, F) = W(\beta, F) \]

\[ = \Delta^{\alpha+1}\langle F \rangle \]

That is, we extend \( \Delta^{\alpha+1} \) by adding the longest \( \text{Ult}(P, F) \), where \( P \subseteq M^\beta \) is the proper possible. Similarly, if \( \beta = \alpha \) then

\[ X(\beta, F) = W(\beta, F) = \Delta^{\alpha+1}\langle F \rangle \]

So suppose \( \beta < \alpha \) and \( \beta + 1 < \text{lh} \bar{F} \); equivalently, \( E^\beta \) exists and \( E^{\Delta^{\alpha+1}} \) exists. They may not
Remark In the definition of $X_{\theta}(\mathfrak{U},u)$, we shall have $J = X_{\theta}$ and $A = X_{\theta}$, whose $\nu < u$. Here $X_{\theta} = X(R_{u}U_{1}^{+1})$ is the normal tree with last model $M_{\theta}^{u}$. We shall have $F = E_{u}^{\mathfrak{U}}$.

If both $E_{\beta}^{\mathfrak{U}}$ and $E_{\beta}^{\mathfrak{D}}$ exist, then we shall have $\lambda(E_{\beta}^{\mathfrak{U}}) \geq \lambda(E_{\beta}^{\mathfrak{D}})$, this is because $\mathfrak{D}$ and $\mathfrak{U}$ use the same extenders $G$ such that $\text{lh}(G) < \lambda(E_{\beta}^{\mathfrak{U}})$, and $\text{dom}(F) < \lambda(E_{\beta}^{\mathfrak{U}})$.

So it $\lambda(E_{\beta}^{\mathfrak{U}}) \leq \lambda(E_{\beta}^{\mathfrak{D}})$ then $E_{\beta}^{\mathfrak{D}} = E_{\beta}^{\mathfrak{U}}$.

$\lambda(E_{\beta}^{\mathfrak{U}}) = \lambda(E_{\beta}^{\mathfrak{D}})$ is impossible, because $E_{\beta}^{\mathfrak{U}}$ is the sequence of $M_{\nu}^{u} = \text{last model of } \mathfrak{U}$. If $\lambda(E_{\beta}^{\mathfrak{D}}) > \lambda(E_{\beta}^{\mathfrak{U}})$, then $E_{\beta}^{\mathfrak{U}} = E_{\beta}^{\mathfrak{D}}$, so $\text{dom}(F) < \lambda(E_{\beta}^{\mathfrak{D}})$. This implies that our assumption above holds.
If $F$ is not total over $M^2_\beta \lambda(E^2_\beta)$, then again
\[
X(\delta, F) = \Delta^\delta(\alpha+1) \wedge < F >
\]
\[
= \Delta^\delta(\alpha+1) \wedge \Omega^\delta(P, F),
\]
where $P \triangleleft M^2_\beta \lambda(E^2_\beta)$ is the first level such that $p(P) = \mathcal{K}_F$.

Now suppose $F$ is total over $M^2_\beta \lambda(E^2_\beta)$, and hence total over all $M^2_\gamma$ for $\gamma > \beta$.

Let $P$ be the first level of $M^2_\beta$ such that $p(P) = \mathcal{K}_F$ or $P = M^2_\beta$ if there is no such level. Then for $X = X(\delta, F)$, we let
\[
M^X_{\alpha+1} = \Omega^\delta(P, F).
\]
(And again, $X^\delta(\alpha+1) = \Delta^\delta(\alpha+1)$.) Let
\[
\varphi(\delta) = \begin{cases} 
\delta & \text{if } \delta < \beta \\
(\alpha+1) + (\delta - \beta) & \text{if } \beta \leq \delta < \lambda^\delta
\end{cases}
\]
We shall have \( \text{lh}(X) = (\alpha + 1) + (\text{lh}(\xi) - \beta) = \sup \{ \gamma : \gamma < \text{lh}(\xi) \} \). For \( \gamma \leq \beta \), we have defined \( M_{\gamma}(\xi) \) already. For \( \gamma > \beta \), we let \( M_{\gamma}(\xi) = \mathcal{U}_F(M_{\beta}^{\gamma}) \), and let

\[ \gamma : M_{\delta} \rightarrow M_{\gamma}(\xi) \]

be the canonical embedding. \( F \) is total over all \( M_{\delta}^{\gamma} \) for \( \gamma > \beta \), so this makes sense. For \( \gamma < \beta \), let \( \gamma : \text{id} : M_{\gamma}(\xi) \rightarrow M_{\gamma}(\xi) \), and let \( \gamma : \mathcal{P} \rightarrow \mathcal{U}_F(\mathcal{P}^F) = M_{\gamma}(\xi) \) be the canonical embedding.

We note Proposition 1. Let \( U \) be a normal stationary tree, and \( \xi + 1 < \text{lh}(U) \), and \( \mu = \text{lh}(E^U) \). Then if \( \xi < \Theta \) and \( \xi < \text{lh}(U) \), then \( M_{\Theta}^\mu = \mu \) is a successor cardinal, and for \( k = k(M_{\Theta}^\mu) \), \( \mu < P_k(M_{\Theta}^\mu) \).

From this we get \( k = \text{lh}(E^U) \), and for \( \eta \) such that \( \xi + 1 \leq \eta < \text{lh}(U) \),

\[ \gamma : (\mu + 1) \rightarrow \eta \ \rightarrow (\mu + 1) \]

and

\[ M_{\Theta}(\xi + 1)(\mu) = M_{\Theta}(\xi)(\mu + 1) \]
We do not have that \( \tau_{q+1}^* \text{lhE}_q^Z = \tau_{q+1} \text{lhE}_q^Z \) in general. What we have is the diagram

\[
\begin{array}{c}
\text{M}_q^\omega \\
\downarrow
\end{array}
\quad \xrightarrow{\tau_q^*} 
\begin{array}{c}
\text{M}_q \\
\downarrow \phi (q)
\end{array}
\quad \xrightarrow{\text{Ult}(M_q^\omega, F)} 
\begin{array}{c}
\text{M}_q^\omega (\phi (q)) \\
\downarrow
\end{array}
\quad \xrightarrow{\tau_q} 
\begin{array}{c}
\tau_q^* (M_q^\omega, \mu)
\end{array}
\quad \xrightarrow{\rho_q^*} 
\begin{array}{c}
\text{Ult}(M_q^\omega, \mu, F) = \tau_{q+1}^* (M_q^\omega, \mu)
\end{array}
\]

\( \tau_{q+1}^* (M_q^\omega, \mu) \) is the ultrapower computed using functions in \( M_q^\omega, \mu \) and \( \tau_q^* (M_q^\omega, \mu) \) is the ultrapower computed using all functions in \( M_q^\omega \). \( \rho_q^* \) is the natural factor map. ("\( \tau_q^* \)" is meant to suggest "resurrection"). From the Prop. 1, we get

Claim: For any \( n < \omega \), \( \rho_{n+1}^* \text{lh}(E^X) + 1 = \text{Identity} \). Also, \( \rho_{n+1}^* \text{lh}(E) + 1 = \text{Identity} \).

Proof: Clear.

So for any \( \Theta \geq q+1 \), \( \tau_{\Theta}^* \text{lhE}_q^Z = \tau_{q+1} \text{lhE}_q^Z \).

We must now find extenders \( E_\Theta^X \) which make \( X \) into an iteration tree.
Let
\[ E^x = \begin{cases} \epsilon \delta & \text{if } \chi < \alpha \\ F & \text{if } \chi = \alpha \end{cases} \]

Now let \( \gamma > \chi \) so \( \gamma = \varphi(\xi) \) for \( \xi \geq \beta \). Assume \( \xi > \beta \); the argument when \( \xi = \beta \) is similar, but \( M^x_\beta \) gets replaced possibly by \( B \in M^x_\beta \) s.t. \( \Ult(P, F) = M^x_\gamma(\beta) \).

Let \( \mu = \langle h(E^x_\xi) \rangle \). We have the diagram

\[
\begin{array}{ccc}
M^x_\gamma & \xrightarrow{\xi} & M^x_{\varphi(\xi)} = \Ult(M^x_\beta, F) \\
\downarrow \quad & & \downarrow \quad \\
M^x_{\varphi(\xi)} \uparrow & \xrightarrow{T_\xi} & T_{\xi}(M^x_\beta, \mu) \\
\downarrow \quad & & \uparrow \quad \\
\tilde{M}^x_{\varphi(\xi) + 1} & \xrightarrow{r_{\xi}} & \Ult(M^x_{\varphi(\xi) + 1}, F) \\
\end{array}
\]

The only difference with the preceding diagram is that \( M^x_{\varphi(\xi) + 1} \) has a predicate symbol \( F \).
for $E^\alpha_{\bar{y}}$, while $M^\alpha_{\bar{y}}$ is passive.

But we can add this predicate, and the maps remain elementary.

Claim 2 \[ \text{Ult} (M^\alpha_{\bar{y}} | \mu, F) \triangleleft M^\chi_{\bar{y}}(g) \,.
\]

Proof This is shown in ET $\S$ 81.1.

The proof uses Condensation. It also shows that $\tau_{\bar{y}}$ can be obtained by inverting a sequence of collapsing maps corresponding to hulls of the form $\text{Hull}^N (\alpha, \beta, \gamma)$, where $N \triangleleft M^\chi_{\bar{y}}(g)$.

We set

\[ E^X_{\eta(3)} = \overset{\ast}{\text{Ult}} (M^\alpha_{\bar{x}} | \mu, F) \]

\[ = \text{last extender of Ult} (M^\alpha_{\bar{x}} | \mu, F) \]

\[ = \bigcup_{\alpha < \mu} \tau_{\bar{x} + 1} (E^\alpha_{\bar{x}} \cap M^\alpha_{\bar{x}} | \alpha). \]
We may sometimes write
\[ E^X \rho(q) = Y_{q+1}(E^\varnothing_q), \]
though literally \( E^\varnothing_q \preceq \pi_{q+1}^q = \text{dom } T_{q+1}. \)
Let us write
\[ G = E^\varnothing_q, \]
\[ H = \tau_q(G), \]
\[ \bar{H} = \tau_{q+1}(G) = E^X \rho(q). \]

Claim 3
(a) For any \( \delta < \varnothing \), \( \text{lht } E^X \rho(\varnothing) < \text{lht } \bar{H} \)
(b) \( \text{lht } (F) < \lambda(\bar{H}) \)
(c) For any \( \delta < \varnothing \), \( \text{crit } (G) < \lambda(E^\varnothing_q) \)
if \( \text{crit } (H) < \lambda(E^X \rho(\varnothing)) \) if \( \text{crit } (\bar{H}) < \lambda(E^\varnothing_q) \).
(d) If \( \text{crit } (G) < \lambda(E^\varnothing_q) \), then \( \text{crit } (H) = \text{crit } (\bar{H}). \)

Proof
In fact, \( \text{lht } (E^\varnothing_q) = \text{lht } (E^X \rho(\varnothing)) \).

Note that \( \text{lht } (E^\varnothing_q) \in \text{dom } T_{q+1} \) is literally true, and \( \tau_{q+1}(\text{lht } E^\varnothing_q) = \text{lht } E^X \rho(\varnothing). \)
For (a), let $s < s'$. Then

$$\lambda (E^g_s) < \lambda (E^g_{s'})$$

so

$$\lambda (E^x_{\gamma(s)}) = \gamma_{s+1} (\lambda E^g_s) = \gamma_{s+1} (\lambda E^g_s)$$

$$< \gamma_{s+1} (\lambda E^g_{s'}) = \lambda (E^x_{\gamma(s')})$$

using claim 1.

For (b), \(\text{crit} (F)^+ < \lambda (E^g_{s'})\), so

$$\lambda (\text{crit}(F)^+) = \lambda F < \Lambda (\text{crit}(E^g_{s'})) = \lambda (E^x_{\gamma(s')})$$

For (c), let $K = \text{crit}(G) = \text{crit}(E^g_{s'})$. Thus $\gamma_{s+1}(K) = \text{crit}(H)$ and $\gamma_{s+1}(K) = \text{crit}(H)$.

Then for $s < s'$

$$K < \lambda (E^g_{s'})$$

iff $\gamma_{s+1}(K) < \lambda (E^x_{\gamma(s)})$

iff $\gamma_{s}(K) < \lambda (E^x_{\gamma(s)})$

(since $\gamma_{s}$ and $\gamma_{s+1}$ agree on $\lambda (E^g_{s'})^+1$) iff $\gamma_{s}(K) < \lambda (E^x_{\gamma(s)})$

(since $\gamma_{s+1}$ agrees with them on $\lambda (E^g_{s'})^+1$).

(d) is clear.
By claim (3), setting $E^x\phi(p) = \overline{H}$ preserves the length-increasing condition on $X$.

Let

$$\delta = T \text{- pred } (\delta+1).$$

By (3)(b), $\phi(\delta) = X \text{- pred } (\phi(\delta)+1)$ in a normal continuation of $X^\alpha (\phi(\delta)+1)$.

We now break into cases.

Case 1 $\text{crit } (G) < \text{crit } (F)$.

In this case, since $\text{crit } (F) < \lambda (E^\alpha)$, $\delta \leq \beta$. If $\delta < \beta$, so $\phi(\delta) = \delta$, then by (3)(b), $\overline{H}$ must be applied in $X$ to the same $Q \in \mathcal{M}_\delta$ that $G$ was applied to. In fact, $\text{crit } (\overline{H}) = \text{crit } (G) = \text{crit } (H)$.

The picture is:
We then have the commutative diagram

\[
\begin{array}{ccc}
M_{\mathfrak{A}} & \xrightarrow{T_{\mathfrak{A}+1}} & M_{\mathfrak{A}}^X \\
\downarrow G & & \downarrow H \\
M_{\mathfrak{B}} & = & M_{\mathfrak{B}}^X \\
\end{array}
\]

It is shown in [IT, 31.1], that the two ultrapowers are identical, and the diagram commutes. (See the calculation in Claim 5.)

The situation when \( S = \mathfrak{B} \) is the same:

\( X \)-pred \((\mathfrak{B} + 1) = \mathfrak{B} \), and \( H \) is applied to the same \( Q \) that \( G \) was. Note that \( \varphi(\mathfrak{B}) \neq \mathfrak{B} \), so \( \varphi \) does not preserve tree order, just as with embedding normalization. (It does induce a map on extenders-preserving \( \mathfrak{B} \) and \( H \).)
Case 2 \( \text{crit}(F) \leq \text{crit}(G) \).

In this case, \( \delta \geq \beta \). Also, \( \lambda(F) \leq \text{crit}(\overline{H}) \), so \( \overline{H} \) is applied in \( X \) to some \( Q \in M^x_\tau \), where \( \tau \geq \alpha+1 \). Thus \( \tau = \text{ran}(\gamma) \), and by \( 3(b) \), \( \gamma = \gamma(\delta) \).

That is, \( X = \text{prod}(\gamma(\tau+1)) = \gamma(\delta) \).

Let \( k = \text{crit}(G) \), and let \( P \in M^x_\alpha \) be least such that \( p(P) \leq k \). Thus \( M^x_{\tau+1} = \text{Ult}(P, G) \).

We claim that \( \overline{H} \) is applied to \( \text{Ult}(P, F) \) in \( X \). For we have the diagram

Claim 4 \( \overline{H} \) is applied to \( \text{Ult}(P, F) \) in \( X \).

Proof We have the diagram
So $\mathcal{M}_{\mathfrak{g}}(\mathfrak{g}\mathfrak{m}_{\mathfrak{e}}^+)$ is a local $GR(e)$-module. Thus $\text{Crit}(G)$ is a local $GR(e)$-module. The natural factor maps $\text{Crit}(G) \to \text{Crit}(G)$ are the natural factor maps $\text{Crit}(G) \to \text{Crit}(G)$. See §2 for more on this.

$\text{Crit}(G) = \mathcal{M}_{\mathfrak{g}}(\mathfrak{g}\mathfrak{m}_{\mathfrak{e}}^+)$.
Note that for \( k = \text{crit} (G) \),
\[
\rho \left( \text{Ult} \left( P, F \right) \right) \leq i^P_F (k),
\]
because \( \text{Ult} \left( P, F \right) \) is generated by \( i^P_F (p(P)) \)
and \( i^P_F \left( k \cup 1 \cup F \right) \), and \( 1 \cup F < i^P_F (k) \),
But for \( k = \text{crit} (P) \), \( f_k \left( P \right) \cong (k+1)^P \),
so \( \rho_k \left( \text{Ult} \left( P, F \right) \right) \geq i^P_F (k+1) \). \( \text{It follows that} \) \( H = \text{Ult} \left( \text{Ult} \left( P, F \right) \right) \) whose domain
is \( \tau_{<k+1} (\text{dom}(G)) = \tau_{<k+1} (\text{dom}(G)) = \)
\( i^P_F (\text{dom}(G)) \), is applied to \( \text{Ult} \left( P, F \right) \) in \( X \).

Claim 4. \( \square \)

Claim 5. \( \text{Ult} \left( \text{Ult} \left( P, F \right), \overline{H} \right) = M^X_{\psi (\xi + 1)} = \left( \text{Ult} \left( \text{Ult} \left( P, G \right), F \right) \right) \).
Proof This is shown in \( \text{EJ} \), §1.1, but
we repeat the calculations here.
Set \( N = \text{Ult}(P, G) \) and \( Q = \text{Ult}(P, F) \). We have the diagram

\[
\begin{array}{ccc}
\text{Ult}(N, F) & \rightarrow & \text{Ult}(Q, H) \\
\downarrow & & \downarrow \\
\text{Ult}(N) & \rightarrow & \text{Ult}(Q)
\end{array}
\]

Let \( E \) be an extender of \( \text{Ult}(N, F) \). Then \( \nu(E) = \sup \lambda(G) \), and for \( a \in \text{crit}(E) \), \( E_a \) concentrates on \( N \) if \( \text{crit}(G) \).

Let \( K \) be the extender of \( \text{Ult}(N, F) \), concentrated so \( \nu(K) = 1 \), \( K = \sup \lambda(G) \), and \( \text{crit}(G) \).

Let \( A = 2^{b_0, g} \), where \( g \in N \) \( \nu(A) = M^G_\xi(A) \) and \( A \in \text{crit}(G) \).

Let \( \Theta \in \text{Ult}(N, F) \) be a typical element of \( [\sup \lambda(G)]^\omega \) and \( A \in \text{crit}(G) \).

\( (a, A) \in E \iff [b, g]_F \in \text{Ult}(N, F) \) \( \Theta \in \text{Ult}(N, F) \) and \( (a, A) \in \text{Ult}(N, F) \) \( \Theta \in \text{Ult}(N, F) \)
\[(a, A) \in E \iff \exists b, g J_F \subseteq \lambda^*_F \circ \lambda^*_G (A)
\]
\[\text{iff for } \neq b \text{ a.e. } u, \ (g(u), A) \in G
\]
\[\text{iff } (\exists b, g J_F \subseteq \lambda^*_F (A)) \subseteq \lambda^*_H
\]
\[\text{iff } (\exists b, g J_F \subseteq \lambda^*_F (A)) \subseteq \lambda^*_H
\]
\[\text{for } \neq b \text{ a.e. } u, \ (g(u), A) \in G
\]

(since \(\exists b, g J_F \subseteq \lambda^*_F (A)\), and \(\lambda^*_F (A) = \lambda^*_F (A)\))

\[\text{iff } (a, A) \in \mathcal{K}_F
\]

So \(E = \mathcal{K}_F\) and \(\text{Ult}(N, F) = \text{Ult}(P, F)\).

We have

\[
\begin{array}{ccc}
N & \xrightarrow{\lambda^*_F} & \text{Ult}(N, F) \\
\downarrow{\lambda^*_G} & & \uparrow{\lambda^*_H} \\
\emptyset & \xrightarrow{\lambda^*_F} & \text{Ult}(P, F)
\end{array}
\]

Since \(E = \mathcal{K}_F\) the diagram commutes.

Claim 5. (ii)
So claim 5 gives us the diagram

\[ \begin{array}{ccc}
M_{8+1} & \xrightarrow{T_{8+1}} & M_{Q(3+1)} \\
M_{9} & \xrightarrow{T_{9}} & M_{Q(3)} \\
M_{9} & \xrightarrow{\phi} & M_{Q(3)} \uparrow \\
M_{\phi} & \xrightarrow{P} & U(P, F) \\
\end{array} \]

Which finishes the defn. of \( E_8 \) in Case 2.

That finishes the proof: the definition of \( E_8 \) in general, and we have that

**Summary** For \( X = X(\Delta, F) \), we have

(i) \( X_{k+1} = \Delta_{k+1} \), where \( \Delta \) is normal and at least \( k \cdot \tau \).

(ii) \( M_{k+1} = U(P, F) \), for \( P \subseteq M_p \).

(iii) \( M_{Q(\phi)} = U(P, F) \).

(iv) \( T_{Q(\phi)} = \tau_{T_{Q(\phi)}} \).

\[ E_{Q(\phi)} = T_{Q(\phi)}(E_{\phi}) \]
(v) If $(8) \not\in J_T$ does not drop, then

\[ M_s^x \xrightarrow{T_i} M_{y(x)} \]

\[ \downarrow \quad \downarrow \]

\[ \uparrow \quad \uparrow \]

\[ i^x \quad i^x \quad i^y \quad i^y \]

\[ M_s^y \xrightarrow{T_o} M_{y(s)} \]

commutes, provided $s \neq \beta$. (If $s = \beta$, we may need to replace $M_{y(s)}^x$ by $M_{y(s)}^x$.)

Now we want to describe the natural embedding of $X(\mathbb{A}, F)$ into $W(\mathbb{A}, F)$. Going back to the definition of $E_{y(\varphi)}^x$, we had $G = E_{y(\varphi)}^x$,

$H = Y_\varphi(G)$, and $\overline{H} = \tau_{G}^{-1}(\overline{G})$.

Let $\delta = T$-pred $(\varphi + 1)$. 
Let $W = XV(I, F)$. Suppose that we have been defining by induction

$$\psi_\eta : M^\eta \to M^\eta$$

such that

$$(*) \quad \tau_\eta^W = \psi_\eta \circ \tau_\eta.$$ 

Here $\tau_\eta^W : M^\eta \to M^\eta(\eta)$ is the map given by embedding normalization. Thus

$$E^W_{\psi(\eta)} = \tau_\eta^W(E^\eta_\eta) = \psi_\eta(H).$$

We have $\psi_\eta = \text{id}$ for all $\eta \leq \omega + 1$. We have by induction the agreement

1. $\forall \xi \geq \beta \quad \psi_\eta^{(\xi)} \mid \mid hF = \text{identity}$

2. If $\beta \leq \eta < \xi$, then

$$\psi_\eta \mid \mid hE^\eta_{\psi(\eta)} = \psi_\eta^{(\xi)} \circ \tau_\eta^{(\xi)} \mid \mid hE^\eta_{\psi(\eta)}.$$
Case 1 \( \text{crit}(G) < \text{crit}(F) \)

In this case, \( \text{dom}(\overline{H}) = \text{dom}(\overline{H}) \).

Suppose \( \overline{H} \) is applied to \( \mathcal{P} \) in \( \mathcal{X} \).

Then \( \mathcal{H} \) would also be applied to \( \mathcal{P} \), if we had seen \( E \varepsilon(\mathcal{E}) = \mathcal{H} \).

\( \overline{H} \) is a subobject of \( \mathcal{H} \) under \( \mathcal{R}_\mathcal{E} : (a, A) \in \overline{H} \text{ iff } (\mathcal{R}_\mathcal{E}(a), A) \in \mathcal{H} \).

We have the diagram

\[
\begin{array}{ccc}
M_{4+1} & \xrightarrow{\tau_{4+1}} & M_4 \\
\gamma & \downarrow & \downarrow \\
P & \xrightarrow{\mathcal{H}} & \overline{\mathcal{H}}(\mathcal{P}, \mathcal{H}) \\
\end{array}
\]

\( e \) is given by

\[ e(\tau_\mathcal{E}, \mathcal{F}_{\mathcal{H}}) = L_{\mathcal{R}_\mathcal{E}}(A), f \mathcal{F}_{\mathcal{H}} \]

So \( e \uparrow \mathcal{L}_{\mathcal{H}} \mathcal{H} = \mathcal{R}_\mathcal{E} \uparrow \mathcal{L}_{\mathcal{H}} \mathcal{H} \).
So in the present case, as \( 8 \leq p \equiv 1 \), we have \( \psi \), \( \psi \) is also applied to \( P \) in \( W \), and we have the diagram

\[
\begin{array}{c}
M_{8} \xrightarrow{\psi_{8}} M_{8} \\
G \xrightarrow{H} \xrightarrow{\psi_{8}(H)} M_{8}
\end{array}
\]

\[\psi_{8+1} \xrightarrow{\psi_{8}} \xrightarrow{\theta} M_{8} \xrightarrow{\psi_{8}(H)} \]

\[M_{8} = M_{8} = M_{8} = M_{8}
\]

\( \theta \) is given by

\[
\theta(2a, f J_{H}) = [\psi_{8}(a), f J_{\psi_{8}(H)}]
\]

So \( \theta \) agrees with \( \psi_{8} \) on \( 1 \times H \). We set

\[\psi_{8+1} = \theta \circ \psi_{8+1},\]
so
\[ \Psi_{H+1} \cup \Psi_H = \Psi_{H+1} \circ \Psi_H \]
as required in agreement hypothesis (2).

It's easy to see that \( \Psi_{H+1} \circ \Psi_H = \Psi_{H+1} \circ \Psi_H \).

**Case 2**: \( \text{crit}(F) \leq \text{crit}(G) \).

Suppose first that \( \delta < \delta \). This yields \( \Psi_\gamma \cup \Psi_E \leq \Psi_{\delta+1} \cup \Psi_E \), so \( \Psi_\gamma \) and \( \Psi_{\delta+1} \) agree on \( \text{dom}(G) \), so \( \text{dom}(\bar{H}) = \text{dom}(H) \).

Moreover, \( \forall \delta \in \text{dom}(\bar{H}) = \text{identity} \). Let \( P \in \Pi_\delta \) be what \( G \) is applied to in \( \beta \).

We have
\[ \Psi^{\beta} \cap \text{dom} G = \Psi_{\delta+1} \cap \text{dom} G = \Psi_{\delta} \cap \text{dom} G = \Psi_{\delta+1} \cap \text{dom} G \]

We have the diagram
\[ E_{\psi(\xi)}^w = \pi_\xi(E_\xi) = \psi_{\psi(\xi)}(E_\xi) = \psi_{\psi(\xi)}(H) \]

The definition of \( W(3, F) \) tells us that \( E_{\psi(\xi)}^w \) is applied to \( \pi_\xi(P) \) in \( W \).

We have
\[ \Lambda \upharpoonright \text{dom}(G) = \tau_{\xi_0} \upharpoonright \text{dom}(G) \]

[Recall \( \text{Ult}(M_{\xi_0} \upharpoonright H, E_{\xi_0}) \upharpoonright F) \xrightarrow{k} \text{Ult}(P, F) \xrightarrow{\ell} \psi_{\xi_0}(P) \),

with \( \Lambda \upharpoonright \text{dom}(G) = \text{id} \). So \( \tau_{\xi_0} \upharpoonright \text{dom}(G) = \Lambda \upharpoonright \text{dom}(G) \).

Thus,
\[ \psi_{\psi(\xi)} \upharpoonright \text{dom}(G) = \psi_{\tau_{\xi_0} \upharpoonright \text{dom}(G)} = \psi_{\psi(\xi)} \circ \Lambda \upharpoonright \text{dom}(G) \]
So by the Lefschetz lemma, we can define

$$\Theta(\mathcal{L}_a, \mathcal{F})^{\text{tr}(P, F)} = \left[ \psi_{q(3)}(a), \psi_{q(5)} \circ \chi(t) \right]_{\text{tr}(P)}^{\psi_{q(3)}(H)},$$

The diagram above then commutes. We set

$$\psi_{q(3+1)} = \Theta \circ \mathcal{E}.$$

Since \( E \cdot \text{tr}(\mathcal{H}) = \psi_{q(3+1)} \text{tr}(\mathcal{H}) \) and \( \Theta \cdot \text{tr}(\mathcal{H}) = \psi_{q(3)} \text{tr}(\mathcal{H}) \), we get

$$\psi_{q(3+1)} \text{tr}(\mathcal{H}) = \psi_{q(3)} \circ \mathcal{E} \text{tr}(\mathcal{H}),$$

as in agreement with hypothesis (2). We must also show that \( \Pi_{3+1} = \psi_{q(3+1)} \circ \mathcal{T}_{3+1} \). Note first that the two sides agree on \( \text{ran} \chi \).

For letting \( j : \Pi_3(P) \rightarrow M_{q(3+1)} \) and \( j^* = \psi_{q(3+1)}, \psi_{q(3+1)} : \)

\[
\Theta \circ \mathcal{F} \circ \mathcal{T}_{3+1} \circ \alpha = j^* \circ \psi_{q(3+1)} \circ \mathcal{T}_3 = j^* \circ \Pi_3
\]

using the commutativity in embedding normalization.
But $M_{q+1}$ is generated by $\tau(\xi) \cup \lambda(G)$, so it is enough to see $\Theta \Xi \tau_{p+1}$ agrees with $\tau_{p+1}$ on $\lambda(G)$. Since $\tau_{p+1}$ agrees with $\tau_{q}$ on $\lambda(G)$, we get

$$
\tau_{p+1} \cap \lambda(G) = \tau_{q} \cap \lambda(G)
$$

$$
= \Psi_{p} \circ R_{q} \cap \lambda(G)
$$

$$
= \Psi_{p} \circ (R_{q} \circ \tau_{p+1}) \cap \lambda(G)
$$

$$
= (\Psi_{p} \circ R_{q}) \circ \tau_{p+1} \cap \lambda(G)
$$

$$
= \Psi_{p} \left( \tau_{p+1} \cap \lambda(G) \right).
$$

So $\tau_{p+1} = \Psi_{p}(\xi) \cap \lambda(G)$ as desired.

This finishes the definition of $\Psi_{p}(\xi)$ when $\xi < \xi$. The case $\xi = \xi$ is not different in any important way. In that case,
we may have \( \text{crit}(H) < \text{crit}(H') \).

The relevant diagram is the same. We omit further detail.

It is clear that if \( (\mathcal{S}, \mathcal{F} + 1) \)

is not a drop, then

\[
\psi_{\mathcal{S}(3+1)} \circ \chi_{\mathcal{S}(3), \mathcal{S}(3+1)} = \chi_{\mathcal{S}(3), \mathcal{S}(3+1)} \circ \psi_{\mathcal{S}(3)}
\]

(When \( \mathcal{S} \neq \beta \). If \( \mathcal{S} = \beta \), we may now so replace \( \psi(S) \) by \( \beta \).)

This lets us define \( \psi_\lambda(\lambda) \) when \( \lambda \) is a limit. We omit the details. Our inductive hypotheses are preserved.

This completes the definition of \( X(\mathcal{S}, F) \) and its embedding into \( W(\mathcal{S}, F) \).

Let us write \( \beta_{\mathcal{S}, F} \), \( \alpha_{\mathcal{S}, F} \), \( \psi_{\mathcal{S}, F} \).

Remark: Given all trees by a fixed \( \mathcal{S} \), \( \alpha \) is determined by \( F \).
and \(\phi_{\lambda}^{J, F} : M_\lambda^J \to M_{\lambda}^{\mathcal{X}(J, F)}\) for the objects we defined above. \(\beta^{J, F}, \Delta^{J, F}\) and \(\gamma^{J, F}\) are the same as the corresponding named objects associated to \(\mathcal{W}(J, F)\).

\(\psi^{J, F} : \mathcal{L}(J) \to \mathcal{L}(\mathcal{X}(J, F))\), but it may be not total or partial. When it is not total, it has not total domain \(\beta^{J, F} + 1\), and \(F\) is partial on the last model of \(J\).

Let also \(\psi_{\eta}^{J, F} : M_\eta^\mathcal{X}(J, F) \to M_{\eta}^{\mathcal{W}(J, F)}\) be the \(\psi\)-map we defined.

Now let \(J\) be a normal tree on some premouse \(\mathcal{M}\) and \(U\) a normal tree on the last model of \(J\), which we assume exists. We define \(\mathcal{X}(J, U)\), and maps relating it to \(\mathcal{W}(J, U)\).
Associated to $W(\mathcal{F}, W)$ we have normal trees

$$W_\delta = W(\mathcal{F}, W\upharpoonright \delta + 1)$$

with last models

$$R_\delta = M^W_{\mathcal{F}(\delta)}$$

and

$$\sigma_\delta: M^W_\delta \to R_\delta.$$ 

For $\alpha < \delta$ we have a partial

$$\varphi_{\alpha, \delta}: \operatorname{lh} W_\alpha \to \operatorname{lh} W_\delta$$

and maps

$$\Pi^W_{\alpha, \delta}: M^W_{\alpha, \delta} \to M^W_{\varphi_{\alpha, \delta}(\alpha)}$$

for $\alpha \in \operatorname{dom} \varphi_{\alpha, \delta}$. We set $F_\delta = \sigma_\delta(E^W_\delta)$, and for $\gamma = \mathcal{U}\operatorname{-prod}(\delta + 1)$, we have

$$W_{\delta + 1} = W(W_\delta, F_\delta).$$

Associated to $X(\mathcal{F}, W)$ we have normal trees

$$X_\delta = X(\mathcal{F}, W\upharpoonright (\delta + 1)).$$
such that \( M_{X^6} \) has the same top order as \( MW \), and last model

\[
M_{X^6}^{2(6)} = M_{X^6}^{2(6)}.
\]

The endomaps \( y_{0,\alpha} \) are the same, and we have

\[
y_{0,\alpha} : \alpha \to M_{X^6}^{X^6}
\]

for \( \alpha \in \text{dom } y_{0,\alpha} \). The \( X^6 \) and \( T_{\alpha}^{2+1} \)
are defined by induction:

\[
X_0 = \varnothing
\]

and

\[
X_{\alpha+1} = X \left( X^6, E_{X^6}^{\alpha} \right)
\]

where \( \alpha = \text{u-pos of } (X^6) \). We need to show \( X^{E_{X^6}^{\alpha}} = \alpha \cup F_8 \), \( \beta^{X^6, E_{X^6}^{\alpha}} = \beta F_8 \), \( \gamma^{X^6, E_{X^6}^{\alpha}} = \gamma F_8 \), \( \alpha^{X^6, E_{X^6}^{\alpha}} = \alpha^{X^6, E_{X^6}^{\alpha}} \), \( \alpha^{X^6, E_{X^6}^{\alpha}} = \alpha^{X^6, E_{X^6}^{\alpha}} \), \( \alpha^{X^6, E_{X^6}^{\alpha}} = \alpha^{X^6, E_{X^6}^{\alpha}} \)

for \( \alpha \in \text{dom } y_{0,\alpha} \) with
and for \( \eta < \nu \) we let \( \eta^{x+1}_\alpha \rightarrow \eta^{x+1}_{\lambda(\alpha)} \) and such that

\[
\eta^{x+1}_\alpha = \eta^{x+1}_{\lambda(\alpha)} \circ \eta^{x+1}_\alpha
\]

where \( \alpha \) is in \( \text{dom } \eta^{x+1}_{\lambda(\alpha)} \).

Everything fits together properly so we can define \( \chi \) for \( \lambda \) as limit, and we define maps \( \psi^x_\alpha \) for \( \nu < \eta \) and \( \alpha \) in \( \text{dom } \psi^x_\alpha \).

We also get maps \( \psi^x_\alpha : M^\chi_\alpha \rightarrow M^\lambda_\alpha \) relating \( \chi \) to \( \lambda \) defined for \( \alpha \leq \lambda(\gamma) = \chi X_\gamma = 1 \), We have the diagram whenever \( \nu < \eta \):

\[
\begin{array}{ccc}
M^\chi_\alpha & \xrightarrow{\psi^x_\alpha} & M^\lambda_\alpha \\
\downarrow^{\psi^x_\alpha} & & \downarrow^{\psi^x_\alpha} \\
M^\nu_\alpha & \xrightarrow{T^{x,y}_\alpha} & M^\nu_{\psi^x_\alpha(\alpha)} \\
\end{array}
\]
For $\varepsilon < 1 h X_h$, pick $\varepsilon < u_1$

such that $\bar{s} = \gamma_{\bar{s}, \beta}(\alpha)$ for some $\alpha$.

Then

$$M_{\eta} = \text{direct limit of } M_{X_{\beta}} \gamma_{\beta_0, \beta}(\alpha), \quad \text{for } \alpha < u \beta < u_1$$

under the

$$\gamma_{\beta_0, \beta_1} : M_{X_{\beta_0}} \rightarrow M_{X_{\beta_1}} \gamma_{\beta_0, \beta_1}(\alpha).$$

$\gamma_{\beta_1}$ for $\beta < u_1$ is the direct limit map.

There is one point here: the maps $\gamma_{\beta_0, \beta_1}$ do not preserve exit extenders in general, so what are the exit extenders $E_{X_{\beta}}$? For this, note that the exit extenders are going down in the direct limit, i.e.,

$$E_{X_{\beta_1}} \gamma_{\beta_0, \beta_1}(\alpha) \leq \gamma_{\beta_0, \beta_1}(\alpha) \left( E_{X_{\beta_0}} \gamma_{\beta_0, \beta_1}(\alpha) \right).$$
in the order given by the $
abla^x_{\eta}$ sequence. Thus they eventually stabilize. (Assuming all $M_{\eta}$ are wellfounded, as we do—otherwise the construction of $X(\xi, \alpha)$ halts.)

So we can get

$$E_{x_{\eta}}^{X_{\eta}} = \text{common value of } \eta_{\beta, \alpha} \left( E_{x_{\beta}}^{X_{\beta}} \right)$$

for $\beta < \alpha$ sufficiently large.

This makes $X_{\eta}$ into an iteration tree.
We shall also have that
\[
\Psi_{\xi} = \sigma_{\xi}.
\]
The maps \(\Psi_{\xi}\) are defined as follows. Suppose we have defined \(\Psi_{\eta}\) for all \(\eta \leq \xi\) and \(\xi \in \Xi(G)\). Let \(\iota = \mathcal{U} \cdot \text{prod}(8 + 1)\). We have an embedding
\[
X_{8+1} = X(\mathcal{X}_0, E_8^n) \hookrightarrow W(X_0, E_8^n) = \overline{W}_{8+1}
\]
with maps
\[
\Psi_{X_0, E_8^n} : M_{\iota} \rightarrow \overline{M}_{\iota}
\]
defined above. Our embeddings of \(X_0\) into \(W_8\) and \(X_8\) into \(W_8\), together with the fact that
\[
\Psi_{\xi} (E_8^n) = \sigma_{\xi} (E_8^n) = F_8
\]
yield an embedding
\[
\overline{W}_{8+1} = W(X_0, E_8^n) \rightarrow W(W_0, F_8) = W_{8+1}
\]
with maps

$$\overline{\Psi}_{\xi}^{\delta+1} : M^\xi_{\overline{\delta}+1} \longrightarrow M^\xi_{\overline{\delta}}$$

We then set

$$\Psi_{\xi}^{\delta+1} = \overline{\Psi}_{\xi}^{\delta+1} \circ \Psi_{\xi}^{\delta+1} E^\xi_{\xi}$$

The maps $\Psi_{\xi}^{\delta}$ for $\lambda$ a limit are defined by commutativity. Fixing $\lambda$ and $\xi$, let $\xi = \Psi_{\lambda,\xi}(\overline{\xi})$ where $\lambda < \lambda_{\xi}$.

So

$$M^{\lambda}_{\lambda,\xi} = \text{direct limit of } M^{\lambda,\lambda_{\lambda,\xi}}_{\lambda,\xi}$$

for $\beta < \lambda_{\xi}$, under the maps

$$\Psi_{\lambda,\lambda_{\lambda,\xi}}^{\beta,\lambda_{\lambda,\xi}} : M^{\lambda,\lambda_{\lambda,\xi}}_{\lambda,\xi} \longrightarrow M^{\lambda,\lambda_{\lambda,\xi}}_{\lambda,\xi}$$

and
\[ M^{W_{\lambda}} = \text{direct limit of } M^{W_{\beta}} \]

for \( \beta < \lambda \) under the maps

\[ \prod_{\beta \neq \lambda} : M^{W_{\beta}} \to M^{W_{\lambda}} \]

\[ \psi_{\lambda}(\bar{\bar{x}}) \]

Because the \( \psi \)'s lift to \( T \)'s into the \( \Pi \)'s, i.e. we have the commutativity stated above, we can set

\[ \psi_{\lambda}(\bar{\bar{x}}) = \prod_{\beta < \lambda} \psi_{\beta}(\bar{\bar{x}}) \]

for \( \alpha < \beta < \lambda \), and this works.

To do this carefully, it would probably be easiest to define the \( \psi_{\lambda} \) by induction on \( \lambda \), with a subinduction on \( \beta \), and verify the necessary commutativity and agreement properties as we go. We shall not do this here.
§2. Weak hull embeddings

For $T$ and $U$ normal trees on a premouse $M$, a pseudo-hull embedding of $T$ into $U$ is a triple
\[ (u, \langle t^0, p^0 \rangle, \langle t^1, p^1 \rangle, \langle \langle \eta, f \rangle, p \rangle), \]
where $u$ maps $lhT$ into $lhU$, not quite order-preservingly. $t^0$ maps an initial segment of $\eta^T$ into $M_{u(p)}$, $t^1 = i_{u(p)}, u(p) \circ t^0$. The whole system is determined by certain rules, and the map $p: \text{Ext}(T) \to \text{Ext}(U)$ mapping extenders used in $T$ to extenders used in $U$. The key equation is
\[ p(E^\eta_\alpha) = t^1_\alpha (E^\eta_\alpha) = E^U_{u(\alpha)}. \]
We get a pseudo-hull embedding of $T$ into $W(\Sigma^F)$ as follows: $u = \gamma_{\Sigma^F}^T$, $t^0_\eta = \pi_{\eta, \Sigma^F}$ for $\eta + 1 < lhT$, $p(E^\eta) = E^W_{\gamma_{\eta}(\eta)}$, $t^1_\eta = t^1_\eta$ if $\eta \neq \beta$. \]
and \( t_\eta = \text{identity if } \eta = \beta^0 F \).

(So \( v(\eta) = \psi(\eta) \) if \( \eta \neq \beta^0 F \), and \( v(\beta^0 F) = \beta^0 F \).

More generally, if \( W_\xi = W(I, U^\xi, \xi+1) \)

and \( W_\eta = W(I, U^\eta, \eta+1) \) and \( \psi \subseteq \eta \) and \( \psi \subseteq \xi \) does not drop, then we have a natural pseudo-hull embedding of \( W_\zeta \)

into \( W_\eta \).

We wish to weaken the condition

\[ \theta^1_\eta(E^\xi_\nu) = E^\xi_\nu \]  

because the natural embedding of \( \text{I} \) into \( \text{X}(I, F) \) does not preserve exit extenders. Basically we shall just require that \( \theta^1_\eta(E^\xi_\nu) \) be isolated to \( E^\xi_\nu\) inside \( \text{I} \) by the condensation process we used to define \( \text{I} \).

We will use the condensation process we used to define \( \text{I} \) from \( \text{I} \) in the last section. The result is the notion of a weak pseudo-hull embedding. It will turn out that
The natural embeddings of $X_0$ into $X_j$, when $j < n$, and the natural embedding of $X(j; W)$ into $W(j; W)$, are weak hull embeddings.

Let's look at the process by which we derived $i_{MG}^F(G)$ from $i_F^M(G)$. Let $M$ be any premouse, then $\lambda \leq M$. We define $A_k$ and $\gamma_k$ by induction:

$$A_0 = M(<\lambda, 0>)$$
$$\gamma_0 = \lambda$$

$$A_{k+1} = M(<\eta, k+1>), \text{ where } <\eta, k> \text{ is}$$
$$k \text{-least such that}$$
$$\rho(M(<\eta, k>)) < \gamma_k$$
$$\gamma_{k+1} = \rho(M(<\eta, k>)), \text{ for this } <\eta, k>.$$ 

The $\gamma$'s are strictly decreasing, so there is a largest $m$ s.t. $A_m$ and $\gamma_m$ are defined.

As we have set it up, $\gamma_i = \rho_k(A_i)(A_i)$, possibly $\rho(A_i) < \gamma_i$. 
If $A_m = M$, then we set $n(M, \lambda) = M$ and stop. If $A_m \neq M$, then we set $n(M, \lambda) = m + 1$ and $A_{m + 1}(M, \lambda) = M$. So in either case \[ A_n(M, \lambda) = M. \]

Notation: For $M$ a pronounce,

\[ p^{-}(M) = p_{k(M)}^{-}(M). \] (Recall that $p(M) = p_{k(M) + 1}(M)$.) If $k(M) > 0$, then $M^{-} = M \{ M \leq \delta(M), k(M) - 1 \}$. 

If we reach $M$ in a normal trip $T$, and the exit extender from $M$ has length $\lambda$, then the $A_k(M, \lambda)$ are the initial segments of $M$ we might apply some later $E$ to in a normal continuation of $T$. If $\text{crit}(E) = \mu$, it would be applied to $A_k(M, \lambda)$, where $k$ is largest such that $\mu < \delta_k(M, \lambda)$.
Definition 2.0: \( \langle A_k(M,\lambda) \mid k \leq n(M,\lambda) \rangle \) is the \( \lambda \)-drop-down sequence of \( M \).

Remark: \( A_0(M,\lambda) = M \langle \lambda \rangle_0 \). If \( \lambda \models E \) for some \( E \) on \( \Theta \), \( M \)-sequence \( \langle \lambda_i \rangle \), and \( \langle \lambda, i \rangle \models \lambda(M) \), then \( A_i(M,\lambda) = M \langle \lambda \rangle_i \).

Prop 2.1: Let \( n = m(M,\lambda) \) and \( A_i = A_i(M,\lambda) \) and \( \gamma_i = \gamma_i(M,\lambda) \) for \( i \leq n \). Then for \( i \leq n \):

1. \( n(A_i,\lambda) = \lambda \), and \( A_k(A_i,\lambda) = A_k \) and \( \gamma_k(A_i,\lambda) = \gamma_k \) for all \( k < i \).

2. If \( A_i \cap B \rightarrow A_{i+1} \), then \( n(B,\lambda) = i+1 \), \( A_k(B,\lambda) = A_k \) for all \( k \leq i \), and \( \gamma_k(B,\lambda) = \gamma_k \) for all \( k \leq i \).

Proof: Easy.

Preservation of drop-down sequences under ultrapowers is given by:
Lemma 2.2. Let $F$ be an extender over $M$, with $\text{crit}(F) < \delta_\lambda(M, \lambda)$.

Let $N = \text{Ult}(M, F)$, and

$$i_F^M : M \rightarrow N$$

be the canonical embedding. Let

$$\lambda^+ = i_F^M(\lambda).$$

Then

(a) $n(M, \lambda) = n(N, \lambda^+)$,

(b) for all $k \leq n(M, \lambda)$

$$A_k(N, \lambda^+) = i_F^M(A_k(M, \lambda)).$$

(c) for all $k < n(M, \lambda)$

$$\delta_k(N, \lambda^+) = i_F^M(\delta_k(M, \lambda)).$$

(d) If $\delta_\mu(M, \lambda) = \delta_\mu^-(M)$, then

$$\delta_\mu(N, \lambda) = \sup i_F^M(\delta_\mu(M, \lambda)).$$

Otherwise,

$$\delta_\mu(N^+, \lambda) = i_F^M(\delta_\mu(M, \lambda)).$$

Proof. Elementary.
The embedding of $\mathcal{I}$ into $X(\mathcal{S}, F)$ is given by the $\tau_{\xi} = \tau_{\xi}^* F$'s. The embedding of $X(\mathcal{S}, F)$ into $W(\mathcal{S}, F)$ is given by $\phi_0 \psi_{\xi} = \psi_{\xi}^* F$'s. These satisfy the agreement formulas

$$\tau_{\xi} = r S_{\xi} \circ \tau_{\xi+1} \text{ on } \text{lh } E_{\xi},$$

$$\psi_{\xi+1} = r S_{\xi}^* \circ \psi_{\xi} \text{ on } \text{lh } E_{\xi}^x.$$  

[In the last section, we wrote $\psi_{\xi+1} = \psi_{\xi} \circ r S_{\xi}$ on $\text{lh } E_{\xi}^x$. $r S_{\xi}$ "resurrected" $H$ to $H$ inside $M_{\lambda}^x$. Here we are re-writing using $r S_{\xi}^* = \psi_{\xi}^* (r S_{\xi})$, which resurrects $\psi_{\xi}(H)$ to $\psi_{\xi}(H)$ inside $M_{\lambda}^x$. Doing it this way helps unify the two cases, the "X-case" and the "W-case", of weak pseudo hull embeddings. One important property]
Definition 2.3 Let $\sigma : \text{ML} \eta \to \text{ML} \lambda$
be elementary. We say that $\sigma$ respects drops $(\text{our}, (M, \eta, \lambda))$ iff

(a) $n(M, \eta) = n(M, \lambda)$, and $\sigma \circ \delta_n(M, \eta) = \text{id}$,

(b) For each $i < n(M, \eta)$, there is an elementary

$$\Pi_i : A_i(M, \eta) \to A_i(M, \lambda)$$

such that $\lambda \in \text{ran}(\Pi_i)$, and

$$\Pi_i \circ \rho^{-1}(A_i(M, \eta)) = \sigma \circ \rho^{-1}(A_i(M, \eta)).$$

Remark $A_i(M, \eta)$ is $k(A_i(M, \eta))$-sound, so $\Pi_i$ is uniquely determined by $\sigma, M, \eta, \lambda$.

Remark If $M \eta \xrightarrow{\sigma} \text{ML} \lambda$, and $\sigma$ respects drops over $(M, \eta, \lambda)$ and $(M, \lambda, \lambda)$ resp., then $\sigma \circ \sigma$ respects drops over $(M, \eta, \lambda)$.
Remark. Let $\sigma, M, \eta, \xi$, and the $\pi_i$'s be as in 2.8. So, for $k < n$,

$$
\sigma_k = \pi^{-1}_{k+1} \circ \pi_k \quad ,
$$

where we are setting $\pi_n = \text{identity}$, $\pi_n : M \rightarrow M$.

(This is consistent with (6), because $\sigma \circ \pi_n (m, \eta) = \text{id}$.)

Let

$$
\eta_k = \pi_k (\eta) ,
$$

so $\eta_0 = \eta$ and $\eta_n = \lambda$. One can see that

$$
\sigma_k : A_k (M, \eta_k^1) \rightarrow A_k (M, \eta_{k+1}^1) ,
$$

(and $A_k (M, \eta_k^0) = A_k (M, \eta_0)$). The picture is

$$
\begin{array}{ccc}
A_{k+1} (M, \eta_0) & = & A_{k+1} (M, \eta_{k+1}) \\
A_k (M, \eta_0) & \Rightarrow & A_k (M, \eta_k) \\
A_k (M, \eta_1) & \Rightarrow & A_k (M, \eta_{k+1}) \\
\end{array}
$$
The factor maps $\pi_k$ described above respect drops:

**Lemma 2.6** Let $M$ be a premouse and $n = n(M, \lambda)$, and $F$ an extender over $M$ with $\text{crit}(F) < \gamma_n(M, \lambda)$. Let

$$\pi_S : Ul_t(M_1), F) \to Ul_t(M, F) = N$$

be the natural embedding, and

$$\eta_0 = i_F^M(\lambda) = o(Ul_t(M_1), F))$$

and

$$\eta_n = i_F^M(\lambda).$$

Then $\pi_S$ respects drops over $(N, \eta_0, \eta_n)$.

**Remark** In particular, $Ul_t(M_1), F) \subseteq N$. This lemma 2.6 is the full statement of the condensation result we mentioned in defining $X(\mathfrak{g}, F)$.

**Proof** Let for $k \leq n$

$$\pi_k = A_k(M, \lambda),$$

$$\eta_k = \gamma_k(M, \lambda).$$
Let \( \lambda^A_k : \text{Ult}(A_k, F) \) be the canonical embedding, and

\[
\eta_k = \lambda^A_k(\eta_0).
\]

We have by 2.0.2 that \( n(\text{Ult}(A_k, F)) = k \), and setting

\[
B^k = \lambda^A_k(A_i)
\]

for \( i \leq k \), and \( B^k = \text{Ult}(A_k, F) \),

\[
B^k = A_i(B^k, \eta_k)
\]

for all \( i \leq k \). We shall show that for all \( k \leq n \)

1. \( n(N, \eta_k) = n \)

2. \( A_i(N, \eta_k) = B^k_i \) for \( i \leq k \)

3. \( A_i(N, \eta_k) = A_i(N, \eta_{k+1}) \) for all \( i \geq k + 2 \)

Let us write \( B^k = A_i(N, \eta_k) \) for all \( i \leq n \)

Set \( \chi^k_i = \chi_i(N, \eta_k) \). So for all \( k \),
$B^k_0 = N \langle \eta_{k,0} \rangle$ and $B^k_n = N$, we shall show

(2) If $k + 2 \leq n$, then for all $i \geq k + 2$,

$$B^k_i = B^k_{i+1}.$$ 

Let

$$\psi_k : B^k_k \rightarrow \odot \mathcal{F} (A^k_k), \text{ for } k < n,$$

be the natural factor map. $A_k$ and $A_{k+1}$ have the same bounded subset of $Y_{k+1}$. So

$\psi_k$ is the identity on $\sup \mathcal{F} A_k = Y_{k+1} = \sup \mathcal{F} Y_{k+1} = Y_{k+1}$. We shall show $Y_{k+1} = Y_{k+1}$ so in fact then by (2)

(3) $Y_i^k = Y_{i+1}^k$ for all $i \geq k + 1$.

To do this, we need to factor $\psi_k$. Let

$$C_k = \odot \mathcal{F} (A_{k+1}^k, F)$$

and

$$O_k : B^k_k \rightarrow \mathcal{D} \subseteq C_k$$

$$T_k : C_k \rightarrow \odot \mathcal{F} (B_{k+1}^k, F) = \odot \mathcal{F} (A_{k+1}^k)$$

be the natural maps.
\[ B^k_r = \text{UT}(A_{k_r}, F) \]
\[ C_{k_r} = \text{UT}(A_{k_{r+1}}, F) \]
\[ B^{k+1}_r = \text{UT}(A_{k_{r+1}}, F) \]
\[ \eta^* = \sigma_k(\eta_k), \quad \rho = \rho(C_{k_r}) \]
Let’s look first at $\mathcal{T}_k$. $\mathcal{T}_k$ is the natural embedding from $\text{Ult}_e(A_{k+1}, F)$ into $\text{Ult}_{e+1}(A_{k+1}, F)$, where $e = k(A_{k+1})$ and with $\text{perf}(A_{k+1}) = \delta_{k+1}$.

The typical case ($e = 0$) is the natural embedding $\pi : \text{Ult}_0(Q, F) \to \text{Ult}_1(Q, F)$, where $Q$ is 1-sound, and $\text{crit}(F) < \varphi_1(Q)$.

We have the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\pi} & \text{Ult}_1(Q, F) \\
\downarrow \pi & \nearrow \text{Ult}_0(Q, F) \\
& \downarrow & \\
& \downarrow & \\
\end{array}
\]

The ultrapowers use the same bounded functions into $Q \varphi_1^Q$. So if $\varphi_1^Q = \varphi_1^0$, and $\pi \Gamma \sup i'' \varphi_1^Q = \text{identity}$.

Also,

\[
\varphi_1(\text{Ult}_0(Q, F)) = \sup i'' \varphi_1^Q = \sup i'' \varphi_1^Q = \varphi_1(\text{Ult}_1(Q, F)).
\]
Both ultrapowers are $1$-sound, and

$$\Pi^*(\rho_1(UL_{\nu},(Q,F))) = \Pi^*(\rho(Q))$$

$$= \lambda(Q)$$

$$= \rho_1(UL_{\lambda},(Q,F)) .$$

So we can apply Condensation to conclude $UL_{\nu}(Q,F) \subseteq UL_{\lambda},(Q,F) .${}

The two are equal iff \( \lambda \) is continuous at \( o(Q) \). If \( \lambda \) is discontinuous at \( o(Q) \), $UL_{\nu}(Q,F) \not\subseteq UL_{\lambda},(Q,F)$.

where \( \rho = \rho_1(UL_{\nu},(Q,F)) = \rho_1(UL_{\lambda},(Q,F)) .\)

Applying this with $Q = A_{k+1}$ and $e+1 = k(A_{k+1})$, so that

$$\rho_{e+1}(A_{k+1}) = \gamma_{k+1}$$

and

$$\nu_k : UL_{\nu}(A_{k+1},F) \to UL_{\nu+1}(A_{k+1},F)$$

the natural embedding, we get
$$y_{k+1}^* = P_{e+1} \left( B_{k+1}^{k+1} \right)$$
$$= P_{e+1} \left( \cup_{e+1}^n \left( A_{k+1}^e, F \right) \right)$$
$$= P_{e+1} \left( \cup_{e+1}^n \left( A_{k+1}^e, F \right) \right)$$
$$= P_{e+1} \left( C_k \right) = P \left( C_k \right),$$

and

$$y^* \cap P_{e+1}(C_k) = \text{identity}$$

and

$$A_{k+1}^e \cap y_{k+1}^* = A_{k+1}^e \cap y_{k+1}^*$$

and

$$P_{e+1} \left( C_k \right) = \sup \left\{ \forall F \cap y_{k+1}^* \right\}$$

Applying Lemma 2.2, we see that the $$\eta^*$$ dropdown sequence of $$C_k$$ is given by

$$A_x^c \left( C_k, \eta^* \right) = P_{e+1}^c \left( A_x^c \right) \text{ for } i \leq k.$$
\[
\tau_k (A_i (C_k, \eta^*)) = \delta_{k+1}^{A_{k+1}} (A_i)
= A_i (B_{k+1}, \eta_{k+1}) = \xi_{k+1}^{i+1}
\]
for \(i \leq k\). It also follows that

\[
A_{k+1}^+ (N, \eta^*) = C_k^+
\]

where \(C_k^+\) is \(C_k\) with \(k(C_k)\) changed from \(e\) to \(e+1\). Also

\[
\delta_{k+1}^{+} (N, \eta^*) = \delta_{k+1}^{k+1}
\]

For \(i \geq k+2\), we then have

\[
A_i (N, \eta^*) = B_i^{k+1} \quad (i \geq k+2)
\]

For example, \(A_{k+2} (N, \eta^*)\) is the first
Level of \( N \) with projectum

\[ \delta_{k+1}(N, \eta^*) = \delta_{k+1} \]

and such that \( N / \delta_{k+1} \leq Q \). But that is \( B_{k+1} \), as well.

The argument involving \( \sigma_k \) is similar.

Set

\[ D = \Lambda_{k+1}(N, \eta^*) \]

so that \( D \in C_k \). \( D = C_k \) is possible;

this holds when \( A_{k+1} = A_k \). We get

\[ D = A_k(N, \eta^*) \]

from Lemma 2.8. We have \( \sigma_k / \delta_k = \text{identity} \), and

\[ \delta_k \leq \delta_k(N, \eta^*) \leq \delta_k(\delta_k) \]
(We can't argue \( \delta_k(N, \eta^*) = \delta_k \) as we did for \( T_k \) and \( \delta_k \), because \( A_{k+1} \) may be more than one quantifier above \( A_k \).) It follows that

\[
A^k_{k+1} = C_k
\]

since each is the first level of \( N \) past \( \delta_k \) with projectum \( < \delta_k \). From this we get

\[
B^k_i = A_i(N, \eta^*) = B^k_{i+1} = B^k_i
\]

for all \( i \geq k+2 \).
Now let us show that $\Gamma$ respects drops. We want for each $k\geq 1$ an embedding \( \pi_k : B_k^o \rightarrow B_k^n \). Using the notation above, notice that we showed for $k \geq 0$

\[
B_k^o = B_k^1 = \cdots = B_k^{k-1} = C_k^+.
\]

(If, case $k=0$ is the same. We have $A_0 = A_1$, then, so $B_0^o = (B_0^1)^-$, so $A_0$ is the identity) and $C_0 = B_0^o$, so $C_0 = B_1^o$. ) Let

\[
\pi_k = \psi_{k+1} \circ \psi_{k-2} \circ \cdots \circ \psi_{k+1} \circ \psi_k,
\]

so $\pi_k$ maps $C_k^+ = B_k^o$ into $B_k^n$, and $\pi_0 = \Gamma$.

But then

\[
\Gamma \circ \Gamma = \pi_k \circ (\pi_{k-1} \circ \cdots \circ \pi_0).
\]

Also

\[
B_k^o = Y_k^1 = \cdots = Y_k^k.
\]
and by the arguments above,

$$\sigma_k^0(\psi_k, \cdots, \psi_0) \uparrow \psi_k = \text{identity}.$$ 

It follows that

$$\Pi_k \uparrow \psi_k = \gamma \uparrow \psi_k,$$

and since $\psi_k = \rho^{-1}(B_k^0) = \rho^{-1}(A_k(N, \eta_0))$,

this is what we want.

Lemma 2.4  □
Definition 2.5 Let \( T \) and \( U \) be normal iteration trees on a premouse \( M \). A weak hull embedding of \( T \) into \( U \) is a system

\[
\langle U, \langle t_\beta \mid \beta \in \text{lh}(T) \rangle, \langle e_\beta, r_\beta, s_\beta \mid \beta + 1 \in \text{lh}(T) \rangle, P \rangle
\]

such that

(a) \( U : \exists \lambda \, 1 \cdot d + 1 < \text{lh}(U) \rightarrow \exists a \mid d + 1 < \text{lh}(U), \; \lambda < \beta \Rightarrow U(\lambda) < U(\beta) \),

and \( \lambda \) is a limit iff \( U(\lambda) \) is a limit.

(b) \( p : \text{Ext}(T) \rightarrow \text{Ext}(U) \) is such that \( E \) is used before \( F \) on the same branch of \( T \) iff \( p(E) \) is used before \( p(F) \) on the same branch of \( U \). Thus \( p \) induces \( \hat{p} : \text{Ext} \rightarrow \text{Ext} \) as in [17].

(c) Let \( v : 1 + 2 \rightarrow 1 + 2 \) be given by

\[
S_v(\beta) = \hat{p}(s_\beta),
\]

Then

\[
t^o_\beta : M^2_p \rightarrow M^U_{v(\beta)}
\]
is total and elementary. Moreover, for 
\[ \alpha < \beta \]
\[ t^0_\beta \circ t^0_\alpha = \Lambda_{u(\beta), u(\alpha)} \circ t^0_\alpha. \]

In particular, the two sides have the same domain.

(d) For \( \alpha + 1 < \beta \) and \( v(\alpha) \leq u(\alpha) \), and
\[ t^!_\alpha = \Lambda_{u(\alpha), u(\alpha)} \circ t^0_\alpha, \]
and
\[ P(E^\Sigma_\alpha) = E^u_{u(\alpha)}. \]

(e) Let \( \eta = \text{lh}(t^!_\alpha(E^\Sigma_\alpha)) \) and \( \lambda = \text{lh}(E^u_{u(\alpha)}). \)

(i) If \( \eta = \lambda \), then \( \text{rs}_\alpha \) is identity.

(ii) If \( \eta < \lambda \), then \( \text{rs}_\alpha \) respects drop
\[ \text{over } (M^u_{u(\alpha)}, \eta, \lambda), \]
and \( \text{rs}_\alpha(t^!_\alpha(E^\Sigma_\alpha)) = E^u_{u(\alpha)}. \)

We call this the \( W \)-case at \( \alpha \).

(iii) If \( \lambda < \eta \), then \( \text{rs}_\alpha \) respects drop
\[ \text{over } (M^u_{u(\alpha)}, \lambda, \eta), \]
and \( \text{rs}_\alpha(E^u_{u(\alpha)}) = t^!_\alpha(E^\Sigma_\alpha). \) We call this
the \( X \)-case at \( \alpha \).
\[ t_\alpha^2 = \begin{cases} 
 t_\alpha^1 \downarrow \text{H}^\alpha & \text{if } \eta = \lambda \\
 r_s^\alpha \circ t_\alpha^1 \downarrow \text{H}^\alpha & \text{in the W-case} \\
r_s^\alpha \circ t_\alpha^1 \downarrow \text{H}^\alpha & \text{in the X-case}, 
\end{cases} \]

(9) For \( \beta < \alpha \), \( r_s^\alpha \downarrow \text{H}^\alpha \downarrow \text{E}^\alpha_{\nu(\beta)} = \text{id} \) \( \nu(\beta) \).
(Thus \( t_\alpha^2 \) agrees with \( t_\alpha^1 \) on \( \text{H}^\alpha \downarrow \text{E}^\alpha_{\nu(\beta)} \).
Thus \( t_\alpha^2 \) agrees with \( t_\alpha^1 \) on \( \text{H}^\alpha \downarrow \text{E}^\alpha \).

Moreover, for \( \gamma > \alpha \),

\[ t_\gamma^\alpha \downarrow \text{H}^\alpha + 1 = t_\alpha^\gamma \downarrow \text{H}^\alpha + 1. \]

(9) If \( \beta = \text{T-pred}(\alpha + 1) \), then \( U_{\text{pred}}(u(\alpha) + 1) \in [\nu(\beta), \nu(\beta)\downarrow u] \) and

setting \( P^* = U_{\text{pred}}(u(\alpha) + 1) \)

\[ t_{\alpha + 1}(\mathcal{E}_\alpha, f) = t_\alpha^\alpha(\mathcal{E}_\alpha, f) \cdot U_{\nu(\beta)} \circ t_\alpha^\alpha(\mathcal{E}_\alpha, f) \cdot P^*. \]

where, \( P \preceq M^\beta \) and \( P^* \preceq \mathcal{M}_{\beta^*}^\alpha \), are what \( E^\alpha \downarrow \text{E}^\alpha \) and \( E_{\nu(\beta)}^\alpha \) are applied to.
Remark. We may as well call the case $(e)(i)$ both the X-case and the W-case.

One can show that the map we got from $X$ into $X(T,F)$ is a weak hull embedding in which the X-case occurs at all $\alpha$. Similarly for the maps of $X_\alpha$ into $X_\beta$ when $\beta < \omega_1$.

The embedding of $X(T,U)$ into $W(T,U)$ we produced is a weak hull embedding in which the W-case occurs at all $\alpha$.

In the case of $X_\alpha \to X_\beta$, the weak hull embedding is given by $\varphi$ (using our notation from §2): $u = \varphi u$, and $P(E_{\alpha,\beta}) = E_{\alpha,\beta}$, $\varphi (u) = \sup \{ u(\xi) + 1 | \xi + 1 \leq \beta \}$. $t_\alpha^\beta : M^\beta \to M_u$ is the normal map, $t_\alpha^\beta = t_\alpha^\beta$. The map $t_\alpha^\beta$ was described in §2.

In this case $X(T,U) \to W(T,U)$, the...
Weak hull embedding is given by:
\[ v = V = \text{identity}, \quad t^0 = t^1 = \psi_d, \]
where \( \psi_d \) is as described in §2.
\( r_{S_a} \) was described in §2, some of which is:
\[ r_{S_a} = \psi_d (\overline{r_{S_a}}), \]
where \( \overline{r_{S_a}} \) is what was described in §2.
§3. **The Stronger Hull Condensation**

We outline a proof of

**Theorem 3.0** Assume AD\(^+\), and let \((P,\mathcal{E})\) be an lbw hod pair. Let \(I\) and \(U\) be normal trees on \(P\), with \(U\) being by \(\mathcal{E}\) and suppose there is a weak hull embedding of \(I\) into \(U\); then \(I\) is by \(\mathcal{E}\).

**Corollary 3.1** Assume AD\(^+\), and let \((P,\mathcal{E})\) be an lbw hod pair. Suppose \((I, U)\) is a stack by \(\mathcal{E}\); then \(X(I, U)\) is by \(\mathcal{E}\).

**Proof** \(W(I, U)\) is by \(\mathcal{E}\) because \(\mathcal{E}\) normalizes well. There is a weak hull embedding of \(X(I, U)\) into \(W(I, U)\), so by 3.0, \(X(I, U)\) is by \(\mathcal{E}\).
Proof of theorem 3.0 (Sketch)

If not, then we have a tree $\mathfrak{g}$ on $\mathcal{P}$ by $\Sigma$, with distinct cofinal branches $b$ and $c$ such that $e = \mathfrak{g}(\mathfrak{g})$, and a weak hull embedding of $\mathfrak{g} \upharpoonright b$ into some normal $\mathcal{U}$ by $\Sigma$.

As usual, we compare $\Phi(\mathfrak{g} \upharpoonright b)$ with $\Phi(\mathfrak{g} \upharpoonright c)$. We do this comparison as in the proof that $\text{UBH}$ holds in hod mice, Theorem 6.3 of [12]. That is, we let $N^*$ be a coarse $\Gamma^+$-Woodin model, where $\Gamma^+$ is well beyond $\Sigma$, and such that $\Phi(\mathfrak{g} \upharpoonright b)$ and $\Phi(\mathfrak{g} \upharpoonright c)$ are countable in $N^*$. We then simultaneously compare $\Phi(\mathfrak{g} \upharpoonright b)$ and $\Phi(\mathfrak{g} \upharpoonright c)$ with each $(\lambda^e_{\mu_k}, \lambda^e_{\nu_k})$, where $\mathcal{E}$ is the $\lambda\mu$-construction of $N^*$.

This involves treating the forcing $\Sigma$ with...
moving tails of the phalanxes up at various stages, with associated stability declarations.

The strategy by which we iterate \( \Phi(I^c) \) is \( \Sigma \). The strategy for \( \Phi(I^b) \) is obtained as the pullback of \( \Sigma \) under our weak hull embedding of \( I^b \) into \( I \).

Let us call this latter strategy \( \Psi \). For each \( v, \ell \), we have the \( (\Psi, \Sigma, M_{v, \ell}) \)-coiteration of \( \Phi(I^b) \) with \( \Phi(I^c) \), defined exactly as in the proof of 6.3 in \( \Sigma I J \). This is a pair \( (W_{v, \ell}, Y_{v, \ell}) \) of pseudo trees according to \( \Psi \) and \( \Sigma \) respectively, obtained by iterating away least disagreements with \( M_{v, \ell} \), and making stability declarations (which move up phalanxes) according to certain rules.
given in $\Sigma^1$. No strategy disagreements with $\Sigma^1_0$ show up when we do this,
by arguments of $\Sigma^1$. 

Let $(R^0_2, e, \Phi^0_2, e)$ be the last model of $W^{0,2}_e$, and let $(S^0_3, N^0_3, e)$ be the last model of $U^{0,3}_e$. There is a $\pi$ corresponding to a completed comparison,
that is, a $\pi$ such that either

(a) $P \rightarrow R^0_2$ in $W^{0,2}_e$ does not drop,
and $(R^0_2, e, \Phi^0_2, e) = (M^0_2, e, N^0_2, e)$
and $(M^0_2, e, N^0_2, e) \leq (S^0_3, N^0_3, e)$

or

(b) $P \rightarrow S^0_3$ in $U^{0,3}_e$ does not drop,
and $(S^0_3, e, \Lambda^0_3, e) = (M^0_3, e, N^0_3, e)$,
and $(M^0_3, e, N^0_3, e) \leq (R^0_2, e, \Phi^0_2, e)$. 

Fix such a \( \eta \) and write
\[
(R, \Phi) = (R_{z,e}, \Phi_{z,e}) \quad \text{and} \\
(S, A) = (S_{z,e}, A_{z,e}) 
\]
Let
\[
W = W_{z,e} \quad \text{and} \quad V = V_{z,e} 
\]
Let \( W^* \) be the lift of \( W \) to a true on \( \overline{\Phi}(U) \) that is missed via our weak hull embedding, and let \( R^* \) be the last model of \( W^* \).

Claim: \( W^* \) is normal.

Proof: This is one reason we have so much structure recorded in a weak hull embedding. We defer the proof to an appendix.

Let \( \sigma : R \to R^* \) be the map we get from our lifting process.
Case 1 (a) above occurs; that is \((R, \Phi) = (M_{\omega, \omega}, \alpha_{\omega, \omega}) \triangleleft (S, \Lambda)\), and \(P \to \tau \to R\) in \(W\) does not drop.

Let \(\pi : P \to R\) be the embedding from \(W\). Let \(\pi^* : P^* \to R^*\) be the embedding of \(W^*\). We have the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\pi} & R \\
\downarrow{\pi^*} & & \uparrow{\sigma} \\
R^* & \to & R \\
\end{array}
\]

Claim 2 \(S = R\) (and thus \(\Lambda = \Phi\)).

Proof If not, then \((R, \Phi) \triangleleft (S, \Lambda)\).
\( \Lambda \) is a tail of \( \Sigma \), i.e.
\[ \Lambda = \Sigma \circ \eta \circ \delta. \]
We have a contradiction using with Dodd-Jensen if we can show that \( \Phi^\Pi = \Sigma \).

But \( \Phi = (\Sigma_{w^+, R^+})^\sigma \) by definition. Moreover, \( \Sigma = (\Sigma_{w^+, R^+})^{\pi^*} \) because \( \Sigma \) is pullback consissent, and \( w^+ \) is normal and by \( \Sigma \).

So
\[
\Phi^\Pi = (\Sigma_{w^+, R^+})^{\sigma \circ \pi^*}
\]
\[
= (\Sigma_{w^+, R^+})^{\pi^*}
\]
\[
= (\Sigma_{w^+, R^+})^{\pi^*}
\]
\[
= \Sigma
\]
as desired. Dodd-Jensen now gives us a contradiction.
Claim 3. The branch $P \rightarrow S$ of $\mathcal{V}$ does not drop.

Proof. The proof in claim 2 works.

Let $\psi: P \rightarrow S$ be the embedding of $\mathcal{V}$.

Claim 4. $\Pi = \psi$.

Proof. Suppose $\Pi(\eta) < \psi(\eta)$. Then since $\Phi^{\Pi} = \Sigma$, we contradict Dodd-Jensen, for $\psi$ is an iteration map by $\Sigma$.

Suppose $\psi(\eta) < \Pi(\eta)$. Then $\sigma(\psi(\eta)) < \sigma(\Pi(\eta)) = \Pi^+(\eta)$. But $\Pi^+$ is an iteration map by $\Sigma$, and

$$\left(\Sigma^{\Pi^+(\eta)}\right)^{\sigma(\psi)} = \Phi^{\psi}$$

$$= \Lambda^\psi$$

$$= \Sigma,$$

because $\Sigma$ is pullback consistent. So again, we contradict Dodd-Jensen.
But now, using the P-hull and definability properties, we see that the branches $P \rightarrow R$ of $W$ and $P \rightarrow S$ of $V$ use the same extenders.

It follows from the construction of $W$ and $V$ that $R$ and $S$ are unstable nodes and therefore they cannot be the last models of their trees, contradiction.

Case 2 (b) above occurs; that is, 

$$(S, \Delta) = (M_{\Delta_x}, E_{\Delta_x}) \rightarrow (R, \Phi),$$

and $P \rightarrow S$ in $V$ does not drop.

Let $\psi : P \rightarrow S$ be the embedding given by $V$. We have the diagram
Claim 5: \( R = S \), and \( P \rightarrow R \) in \( \kappa \) does not drop.

Proof: \( \sigma \circ \varphi : P \rightarrow R^* \), and \( R^* \) is a \( \Sigma \)-iterable of \( P \) and

\[
(\Sigma_{\mu^+}, R^*)^{\sigma \circ \varphi} = (\Phi_S)^{\psi} \\
= \Lambda^{\psi} \\
= \Sigma.
\]

So Dodd-Jensen tells us that \( P \rightarrow R^* \) does not drop in \( \kappa \), and \( \sigma(S) = R^* \). This yields the claim.
The rest of the proof in case 2 is the same as in case 1.

So in either case, we have a contradiction.

This completes our proof of 3.0.

 Infinite stacks

Let \( \langle U_i : i < \omega \rangle \) be an infinite stack of normal trees on \( M = M_0 \).

Setting

\[ W_0 = U_0 \]

and

\[ W_{n+1} = W(W_n, \pi U_{n+1}) \], where

- \( \pi \) is the natural embedding,
- \( \pi : \text{last model of } U \rightarrow \text{last model of } W_n \).
we can let

\[ W(\langle U_i \mid i \leq w \rangle) = \lim_n W_n, \]

where the limit is taken in the natural way. \( W(\langle U_i \mid i \leq w \rangle) \) is the embedding normalization of \( \langle U_i \mid i \leq w \rangle \).

\textbf{Caution:} It should be made part of the definition of the had pair \((P, E)\) that if \( \langle U_i \mid i \leq w \rangle \) is by \( E \), then \( W(\langle U_i \mid i \leq w \rangle) \) is by \( E \). Some constructions do produce pairs \((M_{q,k}, h, E)\) with this property.

In a similar fashion, one can define \( X(\langle U_i \mid i \leq w \rangle) \) for \( \langle U_i \mid i \leq w \rangle \) an infinite stack of normal trees on \( M \). (In taking the limit of the \( X_n \)’s, it is important that exit extenders only change
finiely often. The last model of $X(\overline{u}')$ is equal to the direct limit of the last models of the $U_i$. There is a weak hull embedding of $X(\overline{u}')$ into $W(\overline{u})$. There is a weak hull embedding of $U_o$ into $X(\overline{u}')$.

Theorem 3.2 Assume AD$^+$, and let $(P, 2)$ be an Ibr hod pair in the sense cautioned above. Then $\mathbb{E}$ fully normalizes well for infinite stacks, i.e., if $\langle U_i | i \in \mathbb{N} \rangle$ is by $\mathbb{E}$, then $X(\langle U_i | i \in \mathbb{N} \rangle)$ is by $\mathbb{E}$.

Proof $X(\overline{u}')$ is a weak hull of $W(\overline{u})$. \qed
§ 4. Sketches pseudo-completed.

Proof of theorem 12

We can add a bit more detail
to our sketch in § 0.

We are given an (br tood pair
such that $\mathcal{E}$ embedding-normalizes
well for infinite stacks. Let $G$ be
$V$-generic over $Coll(\omega, \mathcal{P})$. Then in V2G, we have an infinite stack $\langle U_i | liv \rangle$ such
that each $U_i \in V$, and $\langle U_i | liv \rangle$ is
by $\mathcal{E}$, and

$$M_{oo}(P, \mathcal{E}) = \text{last model of } \langle U_i | liv \rangle,$$

Let us assume for purposes of the sketch
that $(P, \mathcal{E})$ extends to an (br tood pair
in V2G. Then we have

$$M_{oo}(P, \mathcal{E}) = \text{last model of } K(\langle U_i | liv \rangle),$$

and $X = K(\langle U_i | liv \rangle)$ is a normal tree by $Z$. 
Proof comes from Theorem 3.0.

Assume now $I$ is by $\Sigma$. By the argument above, we have a normal $Y$ on $Q$ with last model $M_{\infty}(P, Z) = M_{\infty}(Q, \Sigma_{\infty}, Q)$, and such that all countable submodels of $Y$ are by $\Sigma$. But then

$$X(\Sigma, Y) = X,$$

so there is a weak hull embedding of $I$ into $X$.

But note $A$ is by $\Sigma_{\infty}$ if

$$I \models (A \subseteq I \text{ and } I \text{ is by } \Sigma \text{ and } \text{p-to-last model of } J \text{ does not drop})$$.
Combining this with the claim, we see that $\Sigma_{\text{rel}}$ is $\mathcal{M}(\mathcal{P}, \mathcal{E})$-Sushik. $\Sigma_{\text{rel}} = p \mathcal{L}_{TJ}$, where for each $\downarrow$, it searches for a weak hull embedding of some $\Delta \in \downarrow$ into $X$. 

$\mathbb{C}$
References

