Local HOD computation

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§0. Introduction

Our main goal is to prove the following.

__Theorem 1.__ Assume AD^+, and let M be a proper class inner model such that $\mathbb{R} \subseteq M$, and $M \models AD_{\mathbb{R}} + V = L(P(\mathbb{R}))$. Suppose there is an lbr hod pair (P, Σ) such that $\Sigma \notin M$; then HOD^M is a least branch premouse.

This was conjectured at the end of [1]. The proof involves showing that

$M \models$ the lbr hod-pair order $\leq^*$ has order type Θ.

$\leq^*$ is just the natural "mouse order" we get from the comparison process. See §5.7 of [1]. An easily equivalent statement would be

$M \models$ there are lbr hod pairs (P, Σ) with Σ of arbitrarily large Wadge degree.

Here and below we tacitly identify Σ with $Code(\Sigma) = \{x \in \mathbb{R} \mid x$ codes some $I \in HC$ that is by $\Sigma\}$. The equivalence holds because $(P, \Sigma) \leq^* (Q, \Psi) \Rightarrow \Sigma$ is projective in Ψ.

<u>Definition 1</u> <u>Hod-pair capturing</u> (or HPC) is the statement: for all $A \subseteq \mathbb{R}$ there is an lbr hod pair (P, Σ) such that $A \leq_w Code(\Sigma)$.

The hypothesis of Theorem I gives us a hod pair beyond M; we must "localize" to get pairs of arbitrary complexity in M, i.e. to get $M \models$ HPC.

<u>Remark</u> HPC is a version of Sargsyan's "Generation of full pointclasses".

<u>Remark</u> We are tacitly assuming AD^+, here and below.

As in Definition 7.1 of [1], we let $\mathcal{F}(P,\Sigma)$ be the system of all non-dropping Σ-iterates of P, and

$$M_\infty(P,\Sigma) = \text{dir lim } \mathcal{F}(P,\Sigma).$$

This makes sense for any lbr hod pair (P,Σ). In 7.3 of [1] it was asserted that $(P,\Sigma) \leq^* (Q,\Psi)$ iff $o(M_\infty(P,\Sigma)) \leq o(M_\infty(Q,\Psi))$, but this is trivially false: let $P = 0^\#$ and $Q = L_1[0^\#]$, both with there natural strategies; then $o(M_\infty(P,\Sigma)) > \omega_1$ but $M_\infty(Q,\Psi) = Q$. The assertion ~~[illegible]~~ becomes correct if we add the hypothesis that (P,Σ) is <u>full</u>, in the sense defined just before claim 3 in the proof of 7.4 in [1].

<u>Remark</u> Note that $(P,\Sigma) \equiv^* (Q,\Psi)$ iff $M_\infty(P,\Sigma) = M_\infty(Q,\Psi)$. No fullness hypothesis is needed. Thus $M_\infty(P,\Sigma) \in \mathrm{HOD}$, for all lbr hod pairs. However, without fullness we cannot conclude that $M_\infty(P,\Sigma)$ is an initial segment of the lpm hierarchy of HOD. With it, we can.

Assuming $AD_{\mathbb{R}}$ + HPC, we shall show that for any α, $\mathrm{HOD} \restriction \Theta_{\alpha+1} = M_\infty(P,\Sigma)$ for some full (P,Σ). In fact, we get some information on the first order form of P as well.

<u>Definition 2</u> For P an lpm, let

$$\eta^P = \sup\{lh(E)+1 \mid E \text{ is on the } P\text{-sequence}\}.$$

(So $\eta^P = o(P)+1$ iff P is active.) Let

$$o(\kappa)^P = \sup\{lh(E)+1 \mid E \text{ is on the } P\text{-sequence and } crit(E) = \kappa\}.$$

We say that P <u>has a top block</u> iff $\exists \kappa < \eta^P \, (o(\kappa)^P = \eta^P)$. Otherwise we say P has <u>limit block-type</u>.

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<u>Definition 3</u> Let P be an lpm, and suppose P has a top block. Then

(a) If η^P is a limit ordinal, then

$$\kappa^P = \text{least } \kappa \text{ such that } o(\kappa)^P = \eta^P.$$

(b) If $\eta^P = \gamma+1$, then let F be the extender indexed at γ in P; we let

$$\kappa^P = \text{least } \kappa \text{ such that } o(\kappa)^P \geq \text{crit}(F),$$

$$\text{or } \kappa = \text{crit}(F).$$

In either case, we say that κ^P <u>begins the top block</u> of P.

We shall show

<u>Theorem 4</u> Assume $AD_{\mathbb{R}}$ + HPC, and let $\Theta_{\alpha+1} < \Theta$ be a successor point of the Solovay sequence; then there is a full lbr hod pair (P, Σ) such that

(1) η^P is the largest cardinal of P, and $P \models \eta^P$ is Woodin,

(2) $\pi^{\Sigma}_{P,\infty}(\eta^P) = \Theta_{\alpha+1}$, and $HOD|\Theta_{\alpha+1} = M_\infty(P,\Sigma)|\Theta_{\alpha+1}$,

and

(3) $\pi^{\Sigma}_{P,\infty}(\kappa^P)$ is the largest Suslin cardinal $< \Theta_{\alpha+1}$.

<u>Remark</u> By (1), η^P is a limit ordinal, so Def. 3(a) applies. κ^P is then the least cardinal $< \eta^P$ strong in P to η^P.

<u>Remark</u> By (1), η^P is a regular cutpoint of P, and thus $M_\infty(P|\eta^P, \Sigma_{P|\eta^P}) = \pi^{\Sigma}_{P,\infty}(P|\eta^P) = HOD|\theta_{\alpha+1}$.

We get at once

<u>Corollary 5</u> Assume $AD_{\mathbb{R}}$ + HPC; then $HOD|\Theta$ is an lpm.

<u>Corollary 6</u> Assume $AD_{\mathbb{R}}$ + ~~HPC~~ HPC, and let $\Theta_{\alpha+1} < \Theta$. Then there is no extender E on the $HOD|\Theta$ - sequence such that $\text{crit}(E) \leq \Theta_{\alpha+1} < \text{lh}\, E$.

<u>Proof</u> By the "$HOD|\Theta$ - sequence", we mean the union of the sequences of the $M_\infty(P,\Sigma)$, for (P,Σ) full. By the proof of cor. 5, each such $M_\infty(P,\Sigma)$ is a cardinal cutpoint initial segment of $\bigcup_{(P,\Sigma) \text{ full}} M_\infty(P,\Sigma) = HOD|\Theta$.

So by theorem 4, $HOD|\Theta_{\alpha+1}$ is such an initial segment. ☒

<u>Remark</u> Woodin has already proved some results in this direction assuming only AD^+, and not using the direct-limit-system analysis of HOD. For example, he showed no limit θ_α is Woodin in HOD. (Unpublished.)

In order to prove the results above, we must connect the Suslin cardinals to the ~~ordinals~~ cardinals of the form $|M_\infty(P,\Sigma)|$. In 7.4 of [1] it was stated that Code(Σ) is $o(M_\infty(P,\Sigma))$-Suslin, but this is clearly wrong. If

$P \models$ "All sets are countable", then it cannot be iterated without dropping, so $M_\infty(P,\Sigma) = P$, and $o(P)$ is countable. However, Σ on dropping trees could be quite complicated. The correct statement requires that we isolate the part of Σ relevant to forming $M_\infty(P,\Sigma)$.

<u>Definition 7</u> For P an lpm, and $\alpha \in ORD^P$, $o^P(\alpha) = \sup\{\eta+1 \mid crit(\dot{E}^P_\eta) = \alpha\}$. We call β a strong cutpoint of P iff $\forall \alpha < \beta\, (o^P(\alpha) < \beta)$.

<u>Definition 8</u> Let (P,Σ) be an lbr hod pair, and let $\mathcal{T}$ be a normal iteration tree on P. We say that $\mathcal{T}$ is <u>$M_\infty(P,\Sigma)$-relevant</u> iff $\mathcal{T}$ is by Σ, and there is a normal $\mathcal{U}$ by Σ such that $\mathcal{U}$ extends $\mathcal{T}$, $\mathcal{U}$ has a last model Q, and the branch P-to-Q of $\mathcal{U}$ does not drop (in model or degree). Otherwise, $\mathcal{T}$ is <u>$M_\infty(P,\Sigma)$-irrelevant.</u>

The following proposition gives a more concrete characterization.

<u>Proposition 9</u> Let (P, Σ) be an lbr hod pair, and $\mathcal{T}$ a normal iteration tree on P such that $\mathcal{T}$ is by Σ. Let $k = k(P)$. Equivalent are:

(1) $\mathcal{T}$ is $M_\infty(P, \Sigma)$-irrelevant,

(2) Either

(a) there is an $\eta + 1 < lh(\mathcal{T})$ and a strong cutpoint κ of $M_\eta^{\mathcal{T}}$ such that

(i) $\kappa \leq \lambda(E_\eta^{\mathcal{T}})$; equivalently, $\kappa \leq crit(E_\eta^{\mathcal{T}})$

and

(ii) either $[0, \eta]_T$ drops, and $\rho(M_\eta^{\mathcal{T}}) \leq \kappa$, or $[0, \eta]_T$ does not drop, and $\rho_k(M_\eta^{\mathcal{T}}) \leq \kappa$.

Or

(b) $\mathcal{T}$ has limit length, and for $b = \Sigma(\mathcal{T})$, b drops, and ~~either~~ $\delta(\mathcal{T})$ is a cutpoint of ~~(i) $\delta(\mathcal{T})$ is a strong~~ $M_b^{\mathcal{T}}$, i.e.

$\forall \alpha < \delta(\mathcal{T})\ (o(\alpha)^{M_b^{\mathcal{T}}} \leq \delta(\mathcal{T}))$.

Proof (Sketch)

(2) ⇒ (1). If 2(b) holds, then 2(a) must hold of any one-model extension of $\mathcal{T}^\frown b$, with $\eta = lh(\mathcal{T})$ and $\kappa = \delta(\mathcal{T})+1$ being the witnesses. So it is enough if we assume 2(a) holds. Letting η be the least witness, and κ the associated strong cutpoint of $\mathcal{M}_\eta^{\mathcal{T}}$, one can see that any $\mathcal{S}$ extending $\mathcal{T}$ factors as $\mathcal{T} \upharpoonright (\eta+1)^\frown \mathcal{U}$, where $\mathcal{U}$ is above κ. Any extender of $\mathcal{U}$ applied to $\mathcal{M}_\eta^{\mathcal{T}}$ causes a drop.

(1) → (2) It is enough to see that if $\mathcal{T}$ has a last model, and 2(a) fails, then there is a normal $\mathcal{S} \supseteq \mathcal{T}$ by Σ whose branch to the last model does not drop. In fact, we can take $\mathcal{S} = \mathcal{T}$, or $\mathcal{S} = \mathcal{T}^\frown \langle E \rangle$ for some extender E.

We leave the proof to the reader.

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<u>Remark</u> We don't actually need the proposition.

<u>Definition 10</u> Let (P, Σ) be an lbr hod pair, and let $s = \langle \mathcal{T}_0, \ldots \mathcal{T}_n \rangle$ be a stack of normal trees (i.e. a normal stack) on P. Let Q be the last model of $\mathcal{T}_{n-1}$. We say that s is <u>$M_\infty(P,\Sigma)$-relevant</u> iff s is by Σ, P-to-Q in s does not drop, and $\mathcal{T}_n$ is $M_\infty(Q, \Sigma_{s \upharpoonright n, Q})$-relevant.

<u>Definition 11</u> Let (P,Σ) be an lbr hod pair; then Σ^{rel} is the restriction of Σ to all $M_\infty(P,\Sigma)$-relevant normal stacks.

We shall prove

<u>Theorem 12</u> Assume AD^+, and let (P, Σ) be an lbr hod pair; then for $\kappa = |o(M_\infty(P,\Sigma))|$, Σ^{rel} is κ-Suslin, but not α-Suslin for any $\alpha < \kappa$.

We shall see later that there is another source for Suslin cardinals besides that given in Theorem 2. Namely, they can be of the form $\pi^{\Sigma}_{P,\infty}(\kappa)$, where κ begins the top block of P. We conjecture that every Suslin cardinal strictly below the sup of the Suslin cardinals is of one of these two forms, if AD^+ and HPC hold.

Kunen-Martin easily implies that Σ^{rel} is not α-Suslin, for any $\alpha < |o(M_\infty(P,\Sigma))|$. If Σ has branch condensation, the it is easy to see that Σ^{rel} is $|o(M_\infty(P,\Sigma))|$-Suslin. (We verify that $\Sigma^{rel}(\mathcal{T}) = b$ by looking for a σ such that $\pi^{\Sigma}_{P,\infty} = \sigma \circ i^{\mathcal{T}}_b$.)

In the general case, we proceed as follows. We show that if (P, Σ) is an lbr hod pair, then Σ <u>fully</u> normalizes well. This implies that there is a normal tree $\mathcal{U}$ on P such that $\mathcal{U}$ has last model $M_\infty(P, \Sigma)$, and club many countable hulls of $\mathcal{U}$ are by Σ. We then get

$\mathcal{T}$ is by Σ^{rel} iff $\mathcal{T}$ is nicely embedded into $\mathcal{U}$.

Here a "nice embedding" is like a pseudo-hull embedding, except that the condition on preserving exit extenders has been weakened. The equivalence displayed shows that Σ^{rel} is κ-Suslin, where $\kappa = |\mathcal{U}| = |M_\infty(P, \Sigma)|$. This yields Theorem 12.

The proof also gives a Suslin-representation of the short-tree component of Σ^{rel}.

<u>Definition 13</u> Let (P, Σ) be an lbr hod pair, such that P has a top block, and $\vec{T}^\frown b$ a normal tree on P with last model $Q = M_b^{\vec{T}}$ such that $\vec{T}^\frown b$ is by Σ^{rel}. Then $\vec{T}$ is <u>short</u> iff P-to-Q drops, or P-to-Q does not drop, and $\pi(\eta^P) > \delta(\vec{T})$, where $\pi : P \to Q$ is the canonical embedding.

Σ^{stc} is the restriction of Σ to short normal trees. We call it the <u>short-tree component</u> of Σ. $\Sigma^{stc,rel}$ is the further restriction to relevant trees.

<u>Theorem 14</u> Let (P, Σ) be an lbr hod pair such that P has a top block, and let

$$K = \pi^{\Sigma}_{P,\infty}(K^P);$$

then $\Sigma^{stc,rel}$ is $|K|$-Suslin, but not α-Suslin for any $\alpha < |K|$.

<u>Remark</u> If $P \models \kappa^P$ is a limit of Woodin cardinals, then $\pi_{P,\infty}(\kappa^P)$ is a cardinal. ∎

The proof of Theorem 14 is like that of Theorem 12. Let $\mathcal{U}$ be normal on P with last model $M_\infty(P,\Sigma)$ and such that club many countable hulls of $\mathcal{U}$ are by Σ^{rel}. We let

$$\mathcal{U}_0 = \mathcal{U} \restriction (\pi^{\Sigma}_{P,\infty}(\kappa^P)+1).$$

So $\mathcal{U} = \mathcal{U}_0 {}^\frown \mathcal{U}_1$, where $\mathcal{U}_1$ is a normal tree on $M^{\mathcal{U}_0}_\kappa$ that has all critical points $> \kappa$, for $\kappa = \pi^{\Sigma}_{P,\infty}(\kappa^P)$. We then get

$\mathcal{T}$ is by Σ^{stc} iff $\mathcal{T}$ nicely embeds into $\mathcal{U}_0$.

This gives the desired Suslin representation of Σ^{stc}.

The gap between Σ and Σ^{rel} is somewhat awkward. Let us say that <u>the M_∞-irrelevant fragment of Σ is Γ-bounded</u> iff whenever $\mathcal{T}$ is by Σ^{rel} and $\mathcal{T}$ is M_∞-irrelevant, then the tail strategy $\Sigma_{\mathcal{T}^\frown b, M^{\mathcal{T}}_b} \in \Gamma$, where $b = \Sigma(\mathcal{T})$.

Continuing with our sketch of the proofs of Theorems 1 and 4, the following is the main additional ingredient.

<u>Theorem 15</u> Assume AD^+, and let (P, Σ) be an lbr hod pair that has a top block, and is such that η^P is a limit ordinal. Let $\kappa_\infty = \pi_{P,\infty}(\kappa^P)$ and $\eta_\infty = \pi_{P,\infty}(\eta^P)$. Suppose that Σ^{STC} is κ_∞-Suslin; then there are no Suslin cardinals μ s.t. $\kappa_\infty < \mu < |\eta_\infty|$.

<u>Proof</u> κ_∞ has uncountable cofinality, and is a Suslin cardinal by Thm. 14. Let

$$\Gamma_0 = S_{\kappa_\infty},$$

so (see Jackson's handbook article [3], § 3)

So Γ_0 is "$\underset{\sim}{\Sigma}^1_2$-like", i.e. good, has the scale property, and closed under $\exists^{\mathbb{R}}$. Let Γ_1 be an inductive like pointclass with the scale property such that $\Gamma_0 \subseteq \Delta_1$, and $(P, \Sigma) \in \Delta_1$. Let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ_1-Woodin model that captures a universal Γ_1 set.

Now let δ_0 be the first Γ_0-Woodin of N^*. We can make sense of that because N^* captured Γ_1. Let

$$\mathbb{C} = \text{lbr hod pair construction of } N^* | \delta_0,$$

with models $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$. This is an initial segment of the construction of N^*, and since $(N^*, \Sigma^*, \delta^*)$ captured (P, Σ), we have that ~~no $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ for each ν, k~~, either (P, Σ) iterates to some $(M_{\nu',k'}, \Omega_{\nu',k'})$ for $\langle \nu', k' \rangle \le_{lex} \langle \nu, k \rangle$, or (P, Σ) iterates strictly past $(M^{\phi}_{\nu,k}, \Omega^{\phi}_{\nu,k})$, with the iterations in question belonging to $N^* | \delta_0$.

We claim that (P, Σ) does not iterate strictly past $(M^{\phi}_{\delta_0,0}, \Omega^{\phi}_{\delta_0,0})$. For otherwise it does so via a tree which is <u>short</u>, i.e. via $\mathcal{T}^\frown b$ which is by Σ^{stc}. But then $\mathcal{T}^\frown b \in L[T_{\Gamma_0}, N^*|\delta_0]$, where T_{Γ_0} is the tree of a scale on a univ. Γ_0 set. But $\mathcal{T}^\frown b$ kills the Woodinness of δ_0, and δ_0 is Woodin in $L[T_{\Gamma_0}, N^*|\delta_0]$, contradiction.

So (P, Σ) iterates to some $(M^{\phi}_{\nu,k}, \Omega^{\phi}_{\nu,k})$ with $\langle \nu, k \rangle \leq \langle \delta_0, 0 \rangle$. The worst case is $\langle \nu, k \rangle = \langle \delta_0, 0 \rangle$, so assume that. The tree has the form $\mathcal{I}^\frown b$, where every proper initial segment is by Σ^{stc}, so that $\mathcal{I} \in L[T_{\Gamma_0}, N^*|\delta_0]$. But $b \notin L[T_{\Gamma_0}, N^*|\delta_0]$, and $\mathcal{I}$ itself is not short. In particular, b does not drop. But then we have (in V, because of the correctness of N^*)

$$\Sigma = \left(\Sigma_{\mathcal{I}^\frown b, M_b^{\mathcal{I}}}\right)^{i_b^{\mathcal{I}}} = \left(\Omega^{\phi}_{\delta_0, 0}\right)^{i_b^{\mathcal{I}}}.$$

But $\Omega^{\phi}_{\delta_0, 0}$ is definable as "choose the unique branch moving an sjs for $\check{\Gamma}_0$ correctly". So it is μ-Suslin, where μ is the least Suslin $> \kappa_\infty$. So Σ is μ-Suslin, and hence $|\eta_\infty| \leq \mu$ by Kunen-Martin.

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<u>Remark</u> It should be possible to show that η_∞ is equal to the least Suslin cardinal $> \kappa_\infty$.

Note that $\mathrm{cof}(\eta_\infty) = \omega$, as would have to be the case. For $\alpha < \eta^P$, let

$$\Psi_\alpha = \Sigma^{stc} \cup \{(\mathcal{T}, b) \mid \mathcal{T} \text{ is by } \Sigma^{stc},$$

$\mathcal{T}^\frown b$ does not drop, and for $c = \Sigma(\mathcal{T})$,

$i_b^{\mathcal{T}} \restriction \alpha = i_c^{\mathcal{T}} \restriction \alpha \}$. One needs to show that each $\Psi_\alpha \in \mathrm{Env}(S_{\kappa_\infty})$.

Putting these ingredients together, we get

<u>Proof sketch for Theorem 1</u>. Let $M \models AD_{\mathbb{R}}$ and (P, Σ) be mouse-least such that $\Sigma \notin M$.

<u>Claim</u> $\Sigma^{rel,stc} \notin M$.

<u>Proof</u> Let $\Psi_0 = \Sigma^{rel,stc}$, and $\Psi_0 \in M$.

Working in M, we define by induction on α strategies Ψ_α, and show that $\bigcup_\alpha \Psi_\alpha = \Sigma^{stc} \in M$ this way. Given Ψ_α, we look for all M_∞-irrelevant trees $\vec{\mathcal{T}}$ that are by $\Sigma^{rel,stc}$. Note that if $\mathcal{T}$ is such, and $b = \Sigma(\mathcal{T})$, then $M_b^{\mathcal{T}}$ has a cutpoint κ, and $M_b^{\mathcal{T}} | \kappa$ is a dropping iterate, and $M_b^{\mathcal{T}}$ projects to κ. Suppose we have that Ψ_α is total on all ~~[illegible]~~ trees on $M_b^{\mathcal{T}} | \kappa$. The Σ-tail strategy of $M_b^{\mathcal{T}}$ above κ is in M, because it is strictly mouse-below (P, Σ). It is $OD(M_b^{\mathcal{T}} | \kappa, \Psi_\alpha)^M$, uniformly. Call it $\Phi_{\mathcal{T}}$. Then $\Psi_{\alpha+1} = \left(\bigcup_{\text{such } \mathcal{T}} \Phi_{\mathcal{T}}\right) \cup \Psi_\alpha$.

So $\Sigma^{stc} \in M$, and in fact it is $OD^M(\Sigma^{rel,stc})$. But $M \models$ all sets are Suslin, so by theorem 15, $\eta_\infty < \Theta^M$, and thus by theorem 12, $\Sigma \in M$, contradiction.

Claim ☒

There are now two cases. Suppose P has limit type. Then $\Theta^M \geq \pi_{P,\infty}(\eta^P)$, because each nondropping iterate Q of P has limit type, and each $\Sigma_{\mathcal{I}, Q|\eta} \in M$ for $\eta < \eta^Q$.

So $\Theta^M = \pi_{P,\infty}(\eta^P)$, as otherwise $\Sigma \in M$. So $M \models$ HPC, the witnesses being pairs (S, Ψ), where S is a cutpoint is an iterate of P, and Ψ is the tail of Σ. It is easy to see that if $P \to Q$ does not drop, and $S = Q|(\gamma^+)^Q$ for some γ, then $M \models (S, \Psi)$ is fullness preserving. (Ψ iterates carry Q along on top.)

Since we have HPC via fullness-preserving strategies, $HOD^M = \bigcup_{(S,\Psi)} M_\infty(S, \Psi)$.

Suppose next P has a top block. Then $\pi_{P,\infty}(\kappa^P) \leq \theta^M$ by the argument above, so $\pi_{P,\infty}(\kappa^P) = \theta^M$, as otherwise $\Sigma^{stc} \in M$. So again, $M \models HPC$ via fullness-preserving pairs, so

$$HOD^M = \bigcup_{(S,\psi)} M_\infty(S,\psi).$$

In this case the pairs are ~~ann~~ $(Q|\gamma^{+Q}, \psi)$, where P-to-Q does not drop, ψ is the tail of Σ, and $\gamma < \pi_{P,Q}(\kappa^P)$.

Sketch of Thm 4. ☒

<u>Remark</u>. Theorem 4 is proved using the same methods.

<u>Remark</u> Nam Trang and the author have worked out a version of thm. 4 in the case M has a largest Suslin cardinal,

<u>Remark</u> In [2], the author has shown that the existence of certain lbr hod pairs gives rise to models of LSA, and stronger theories, e.g. "$HOD({}^{\omega}\theta) \cap P(\mathbb{R}) = P_\kappa(\mathbb{R})$, for κ the largest Suslin cardinal". That paper also gives a converse to Corollary 6: assuming $AD_{\mathbb{R}}$ + HPC, every Woodin cardinal cutpoint of HOD is a $\theta_{\alpha+1}$. The LSA result was first proved by Sargsyan.

In section 1, we describe the full normalization $X(\mathcal{T}, \mathcal{U})$ of a stack $\langle \mathcal{T}, \mathcal{U} \rangle$ of normal trees, and relate it to the embedding normalization $W(\mathcal{T}, \mathcal{U})$. In section 2, we describe the "nice embedding" of $\mathcal{T}$ into $X(\mathcal{T}, \mathcal{U})$ that we get in abstract terms. The embedding of $\mathcal{T}$ into $W(\mathcal{T}, \mathcal{U})$ is a pseudo-hull embedding. (See [1], def. 3.3.) Lacking all inspiration, we shall call embeddings like that from $\mathcal{I}$ to $X(\mathcal{I}, \mathcal{U})$ "weak ~~pseudo~~ hull embeddings". A strategy that condenses to itself under weak ~~pseudo~~ hull embeddings has "very strong hull condensation":

In §3, we show that if (P, Σ) is an lbr hod pair, then Σ fully normalizes well and has very strong hull condensation. These do not seem to be properties that one can get directly from a background construction in a model with UBH, as was done in [1] for normalizing well and strong hull condensation. The arguments of §3 involve phalanx comparisons like those at the end of [1].

In §4, we use the results of §3 to show that the strategies of lbr hod pairs are positional. In §5, we tie things up by filling out the sketches in §0 a bit more.

§1. Full normalization

We outline some basic facts, and establish some notation. [2] has a more complete account.

We begin with the atomic step. Let $\mathcal{S}$ be a normal tree on the premouse M. Here M can be an lpm, or a Jensen or ms-pure extender premouse. (For definiteness, we use Jensen indexing.) Let F be an extender on the sequence of $M_\alpha^{\mathcal{S}}$, with α least such that this is true of F. Let $\beta \leq \alpha$ be ~~such that~~ the least η such that $\text{crit}(F) < \lambda(E_\eta^{\mathcal{S}})$, or $\beta = \alpha$ if no such η exists. Suppose $\mathcal{T}$ is another normal tree on M such that $\mathcal{T} \restriction \beta+1 = \mathcal{S} \restriction \beta+1$(*). In this situation, [1] defines the embedding normalization

(*) Assume also that if $\beta+1 < \text{lh}\,\mathcal{T}$, then $\text{dom}\,F \not< \lambda(E_\beta^{\mathcal{T}})$.

$$W(\mathcal{T}, F) = \mathcal{S}\restriction\alpha+1 {}^\frown \langle F \rangle {}^\frown i_F{}'' \mathcal{T}^{\geq\beta}.$$

Here we define the full normalization

$$X(\mathcal{T}, F) = \mathcal{S}\restriction\alpha+1 {}^\frown \langle F \rangle {}^\frown \bar{i}_F{}'' \mathcal{T}^{\geq\beta}.$$

The difference between i_F and $\bar{i}_F$ in the formulae above (which are only heuristic!) has to do with what functions are used in various ultrapowers.

Let $\kappa_F = \text{crit}(F)$. If $\beta+1 = \text{lh}\,\mathcal{T}$, then

$$X(\mathcal{T}, F) = W(\mathcal{T}, F)$$
$$= \mathcal{S}\restriction\alpha+1 {}^\frown \langle F \rangle.$$

That is, we extend $\mathcal{S}\restriction\alpha+1$ by adding $\text{Ult}(P, F)$, where $P \trianglelefteq M_\beta^{\mathcal{T}}$ is the longest possible. Similarly, if $\beta = \alpha$ then

$$X(\mathcal{T}, F) = W(\mathcal{T}, F) = \mathcal{S}\restriction\alpha+1 {}^\frown \langle F \rangle.$$

So suppose $\beta < \alpha$ and $\beta+1 < \text{lh}\,\mathcal{T}$; equivalently, $E_\beta^{\mathcal{T}}$ exists and $E_\beta^{\mathcal{S}\restriction\alpha+1}$ exists. They may not be equal.

Remark In the definition of $X(R, \mathcal{U})$, we shall have $\mathcal{I} = X_\nu$ and $\mathcal{J} = X_\gamma$, where $\nu <_{\mathcal{U}} \gamma$. Here $X_\xi = X(R, \mathcal{U}|\xi+1)$ is the normal tree with last model $M^{\mathcal{U}}_\xi$. We shall have $F = E^{\mathcal{U}}_\gamma$.

If both $E^{\mathcal{I}}_\beta$ and $E^{\mathcal{J}}_\beta$ exist, then we shall have $\lambda(E^{\mathcal{I}}_\beta) \geq \lambda(E^{\mathcal{J}}_\beta)$. This is because $\mathcal{J}$ and $\mathcal{I}$ use the same extenders G such that $lh(G) < \lambda(E^{\mathcal{U}}_\nu)$, ~~and dom(F) < $\lambda(E^{\mathcal{U}}_\nu)$~~ so if $\lambda(E^{\mathcal{J}}_\beta) < \lambda(E^{\mathcal{U}}_\nu)$ then $E^{\mathcal{J}}_\beta = E^{\mathcal{I}}_\beta$. $\lambda(E^{\mathcal{J}}_\beta) = \lambda(E^{\mathcal{U}}_\nu)$ is impossible, because $E^{\mathcal{U}}_\nu$ is on the sequence of $M^{\mathcal{U}}_\nu$ = last model of $\mathcal{I}$. If $\lambda(E^{\mathcal{J}}_\beta) > \lambda(E^{\mathcal{U}}_\nu)$, then $E^{\mathcal{U}}_\nu = E^{\mathcal{I}}_\beta$, so $\lambda(E^{\mathcal{J}}_\beta) \leq \lambda(E^{\mathcal{I}}_\beta)$.

This implies that our assumption (*) above holds.

If F is not total over $M^{\vec{\gamma}}_\beta \mid \lambda(E^{\vec{\gamma}}_\beta)$, then again

$$X(\vec{\gamma}, F) = \vec{s}\restriction(\alpha+1) {}^\frown \langle F \rangle$$

$$= \vec{s}\restriction(\alpha+1) {}^\frown \mathrm{Ult}(P, F),$$

where $P \trianglelefteq M^{\vec{\gamma}}_\beta \mid \lambda(E^{\vec{\gamma}}_\beta)$ is the first level such that $\rho(P) = \kappa_F$.

Now suppose F is total over $M^{\vec{\gamma}}_\beta \mid \lambda(E^{\vec{\gamma}}_\beta)$, and hence total over all $M^{\vec{\gamma}}_\xi$ for $\xi > \beta$. Let P be the first level of $M^{\vec{\gamma}}_\beta$ such that $\rho(P) = \kappa_F$, or $P = M^{\vec{\gamma}}_\beta$ if there is no such level. Then for $X = X(\vec{\gamma}, F)$, we let

$$M^X_{\alpha+1} = \mathrm{Ult}(P, F).$$

(And again, $X \restriction \alpha+1 = \vec{s}\restriction \alpha+1$.) Let

$$\varphi(\xi) = \begin{cases} \xi & \text{if } \xi < \beta \\ (\alpha+1)+(\xi-\beta) & \text{if } \beta \leq \xi < \mathrm{lh}\,\vec{\gamma} \end{cases}$$

We shall have $\mathrm{lh}(\mathcal{X}) = (\alpha+1) + (\mathrm{lh}\mathcal{T} - \beta) =$
$\sup\{\varphi(\xi)+1 \mid \xi < \mathrm{lh}(\mathcal{T})\}$. For $\xi \le \beta$, we have defined $M^{\mathcal{X}}_{\varphi(\xi)}$ already. For $\xi > \beta$, we let

$$M^{\mathcal{X}}_{\varphi(\xi)} = \mathrm{Ult}(M^{\mathcal{T}}_{\xi}, F),$$

and let

$$\tau_\xi : M^{\mathcal{T}}_\xi \longrightarrow M^{\mathcal{X}}_{\varphi(\xi)}$$

be the canonical embedding. F is total over all $M^{\mathcal{T}}_\xi$ for $\xi > \beta$, so this makes sense. For $\xi < \beta$, let τ_ξ = identity, and let $\tau_\beta : P \longrightarrow \mathrm{Ult}(P, F) = M^{\mathcal{X}}_{\varphi(\beta)}$ be the canonical embedding.

We note

Proposition 1 Let $\mathcal{U}$ be a normal iteration tree, and $\xi+1 < \mathrm{lh}(\mathcal{U})$, and $\mu = \mathrm{lh}\, E^{\mathcal{U}}_\xi$. Then if $\xi < \theta < \mathrm{lh}(\mathcal{U})$, then $M^{\mathcal{U}}_\theta \models \mu$ is a successor cardinal, and for $k = k(M^{\mathcal{U}}_\theta)$, $\mu < \rho_k(M^{\mathcal{U}}_\theta)$.

From this we get that for $\mu = \mathrm{lh}(E^{\mathcal{T}}_\xi)$, and $\xi + 1 \le \eta < \mathrm{lh}(\mathcal{T})$

$$\tau_{\xi+1} \restriction (\mu+1) = \tau_\eta \restriction (\mu+1)$$

and

$$M^{\mathcal{X}}_{\varphi(\xi+1)} \restriction \tau_{\xi+1}(\mu) = M^{\mathcal{X}}_{\varphi(\eta)} \restriction \tau_{\xi+1}(\mu).$$

We do not have that $\tau_\xi \restriction \mathrm{lh} E_\xi^{\mathcal{T}} = \tau_{\xi+1} \restriction \mathrm{lh} E_\xi^{\mathcal{T}}$ in general. What we have is the diagram

$$
\begin{array}{ccc}
M_\xi^{\mathcal{T}} & \xrightarrow{\ \tau_\xi\ } & M^X_{\varphi(\xi)} = \mathrm{Ult}(M_\xi^{\mathcal{T}}, F) \\
\triangledown & & \triangledown \\
M_\xi^{\mathcal{T}} \| \mu & \xrightarrow{\ \tau_\xi\ } & \tau_\xi(M_\xi \| \mu) \\
& & \uparrow rs_\xi \\
M_\xi^{\mathcal{T}} \| \mu & \xrightarrow{\ \tau_{\xi+1}\ } & \mathrm{Ult}(M_\xi \| \mu, F) = \tau_{\xi+1}(M_\xi \| \mu)
\end{array}
$$

$\tau_{\xi+1}(M_\xi \| \mu)$ is the ultrapower computed using functions in $M_\xi \| \mu$, and $\tau_\xi(M_\xi \| \mu)$ is the ultrapower computed using all functions in M_ξ. rs_ξ is the natural factor map. ("rs" is meant to suggest "resurrection".) From ~~the~~ Prop. 1, we get

<u>Claim 1</u> For any $\eta < \xi$, $rs_\xi \restriction \mathrm{lh}(E^X_{\varphi(\eta)}) + 1 = \text{identity}$. Also, $rs_\xi \restriction \mathrm{lh}(F) + 1 = \text{identity}$.

<u>Prf.</u> Clear. ☒

So for any $\theta \geq \xi + 1$, $\tau_\theta \restriction \mathrm{lh} E_\xi^{\mathcal{T}} = \tau_{\xi+1} \restriction \mathrm{lh} E_\xi^{\mathcal{T}}$.

We must now find extenders E_γ^X which make X into an iteration tree. ~~For $\gamma \leq \alpha$, let~~

Let

$$E^{X}_{\gamma} = \begin{cases} E^{\mathcal{T}}_{\gamma} & \text{if } \gamma < \alpha \\ F & \text{if } \gamma = \alpha \end{cases}.$$

Now let $\gamma > \alpha$, so $\gamma = \varphi(\xi)$ for $\xi \geq \beta$. Assume $\xi > \beta$; the argument when $\xi = \beta$ is similar, but $M^{\mathcal{T}}_{\beta}$ gets replaced ~~possibly~~ by $P \trianglelefteq M^{\mathcal{T}}_{\beta}$ s.t. $\mathrm{Ult}(P, F) = M_{\varphi(\beta)}$.

Let $\mu = \mathrm{lh}(E^{\mathcal{T}}_{\xi})$. We have the diagram

$$\begin{array}{ccc} M^{\mathcal{T}}_{\xi} & \xrightarrow{\ \iota_{\xi}\ } & M^{X}_{\varphi(\xi)} = \mathrm{Ult}(M^{\mathcal{T}}_{\xi}, F) \\ \trianglerighteq & & \trianglerighteq \\ M^{\mathcal{T}}_{\xi}|\mu & \xrightarrow{\ \tau_{\xi}\ } & \tau_{\xi}(M^{\mathcal{T}}_{\xi}|\mu) \\ & \searrow^{\Upsilon_{\xi+1}} & \uparrow rs_{\xi} \\ & & \mathrm{Ult}(M^{\mathcal{T}}_{\xi}|\mu, F) \end{array}.$$

The only difference with the preceding diagram is that $M^{\mathcal{T}}_{\xi}|\mu$ has a predicate symbol $\dot{F}$

for $E_\xi^{\mathcal{T}}$, while $M_\xi^{\mathcal{T}} \| \mu$ is passive.
But we can add this predicate, and the maps remain elementary.

<u>Claim 2</u> $\mathrm{Ult}(M_\xi^{\mathcal{T}} | \mu, F) \trianglelefteq M^X_{\varphi(\xi)}$.

<u>Proof</u> This is shown in [1], §1.1. The proof uses Condensation. It also shows that σ_ξ can be obtained by inverting a sequence of collapsing maps corresponding to hulls of the form ~~$\mathrm{Hull}_n^{M_{\varphi(\xi)}}(\alpha \cup \{p\})$~~ $\mathrm{Hull}^N(\alpha \cup \{p\})$, where $N \trianglelefteq M^X_{\varphi(\xi)}$.

$\boxtimes$

We set

$$E^X_{\varphi(\xi)} = \dot{F}^{\mathrm{Ult}(M_\xi^{\mathcal{T}} | \mu, F)}$$

$$= \text{last extender of } \mathrm{Ult}(M_\xi^{\mathcal{T}} | \mu, F)$$

$$= \bigcup_{\alpha < \mu} \pi_{\xi+1}(E_\xi^{\mathcal{T}} \cap M_\xi^{\mathcal{T}} | \alpha).$$

We may sometimes write

$$E^X_{\varphi(\xi)} = \tau_{\xi+1}(E^T_\xi),$$

though literally $E^T_\xi \notin M^T_{\xi+1} = \text{dom } \tau_{\xi+1}$.

Let us write

$$G = E^T_\xi,$$

$$H = \tau_\xi(G),$$

$$\bar{H} = \tau_{\xi+1}(G) = E^X_{\varphi(\xi)}.$$

Claim 3

(a) For any $\delta < \xi$, $\text{lh } E^X_{\varphi(\delta)} < \text{lh } \bar{H}$

(b) $\text{lh}(F) < \lambda(\bar{H})$

(c) For any $\delta < \xi$, $\text{crit}(G) < \lambda(E^T_\delta)$ iff $\text{crit}(H) < \lambda(E^X_{\varphi(\delta)})$ iff $\text{crit}(\bar{H}) < \lambda(E^X_{\varphi(\delta)})$.

(d) If $\text{crit}(G) < \lambda(E^T_\delta)$, then $\text{crit}(H) = \text{crit}(\bar{H})$. In fact, $H \restriction \text{lh}(E^X_{\varphi(\delta)}) = \bar{H} \restriction \text{lh}(E^X_{\varphi(\delta)})$.

Proof Note that $\text{lh}(E^T_\eta) \in \text{dom } \tau_{\eta+1}$ is literally true, and $\tau_{\eta+1}(\text{lh } E^T_\eta) = \text{lh } E^X_{\varphi(\eta)}$.

For (a), let $\delta < \xi$. Then $\mathrm{lh}(E_\delta^{\mathcal{T}}) < \mathrm{lh}(E_\xi^{\mathcal{T}})$, so

$$\mathrm{lh}(E_{\varphi(\delta)}^{\mathcal{X}}) = \tau_{\delta+1}(\mathrm{lh}\, E_\delta^{\mathcal{T}}) = \tau_{\xi+1}(\mathrm{lh}\, E_\delta^{\mathcal{T}})$$
$$< \tau_{\xi+1}(\mathrm{lh}\, E_\xi^{\mathcal{T}}) = \mathrm{lh}(E_{\varphi(\xi)}^{\mathcal{X}}),$$

using claim I.

For (b), $\mathrm{crit}(F)^+ < \lambda(E_\xi^{\mathcal{T}})$, so
$i_F^{M_\xi^{\mathcal{T}} | \mathrm{lh} E_\xi^{\mathcal{T}}}(\mathrm{crit}(F)^+) = \mathrm{lh}\, F < i_F^{M_\xi^{\mathcal{T}} | \mathrm{lh} E_\xi^{\mathcal{T}}}(\lambda(E_\xi^{\mathcal{T}})) = \lambda(E_{\varphi(\xi)}^{\mathcal{X}})$.

For (c), let $\kappa = \mathrm{crit}(G) = \mathrm{crit}(E_\xi^{\mathcal{T}})$. Thus $\tau_\xi(\kappa) = \mathrm{crit}(H)$, and $\tau_{\xi+1}(\kappa) = \mathrm{crit}(\bar{H})$.

Then for $\delta < \xi$

$$\kappa < \lambda(E_\delta^{\mathcal{T}}) \text{ iff } \tau_{\delta+1}(\kappa) < \lambda(E_{\varphi(\delta)}^{\mathcal{X}})$$
$$\text{iff } \tau_\xi(\kappa) < \lambda(E_{\varphi(\delta)}^{\mathcal{X}})$$

(since τ_ξ and $\tau_{\delta+1}$ agree on $\mathrm{lh}(E_\delta^{\mathcal{T}})+1$)

$$\text{iff } \tau_{\xi+1}(\kappa) < \lambda(E_{\varphi(\delta)}^{\mathcal{X}})$$

(since $\tau_{\xi+1}$ agrees with them on $\mathrm{lh}(E_\delta^{\mathcal{T}})+1$).

(d) is clear. ☒

By claim (3), setting $E^X_{\varphi(\xi)} = \bar{H}$ preserves the length-increasing condition on X.

Let
$$\delta = T\text{-pred}(\xi+1).$$

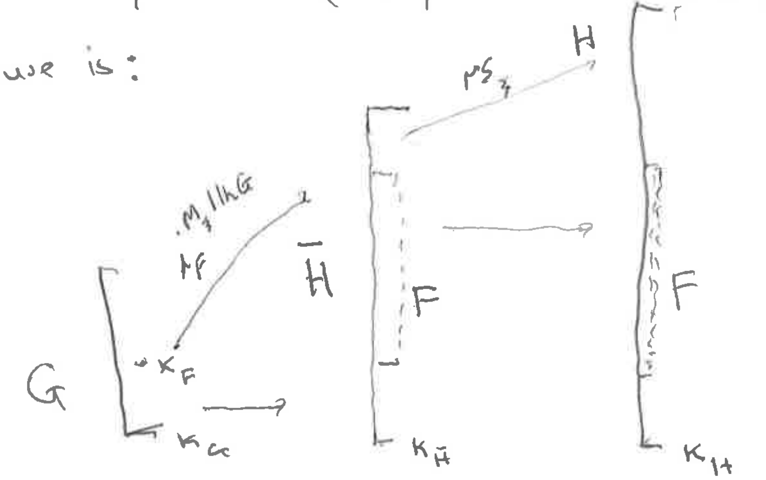
By (3)(b), $\varphi(\delta) = X\text{-pred}(\varphi(\xi)+1)$ in a normal continuation of $X \upharpoonright (\varphi(\xi)+1)$.

We now break into cases.

<u>Case 1</u> $\text{crit}(G) < \text{crit}(F)$.

In this case, since $\text{crit}(F) < \lambda(E^{\hat{\alpha}}_\beta)$, $\delta \leq \beta$. If $\delta < \beta$, so $\varphi(\delta) = \delta$, then by 3(b), $\bar{H}$ must be applied in X to the same $Q \trianglelefteq M^{\hat{\alpha}}_\delta$ that G was applied to. In fact, $\text{crit}(\bar{H}) = \text{crit}(G) = \text{crit}(H)$.

The picture is:



We then have the commutative diagram

$$\begin{array}{ccl} M^{\tilde{\mathcal{T}}}_{\xi+1} & \xrightarrow{\ \tau_{\xi+1}\ } & M^{\mathcal{X}}_{\varphi(\xi+1)} = \mathrm{Ult}(M^{\tilde{\mathcal{T}}}_{\xi+1}, F) = \mathrm{Ult}(Q, \bar{H}) . \\ \uparrow G & \nearrow \bar{H} & \\ Q & & \\ \trianglelefteq & & \\ M^{\mathcal{T}}_{\delta} = M^{\mathcal{X}}_{\delta} & & \end{array}$$

It is shown in [1], §1.1, that the two ultrapowers are identical, and the diagram commutes. (See the calculations in Claims, case 2 below.)

The situation when $\delta = \beta$ is the same: $\bar{H}$ is applied to $\mathcal{X}\text{-pred}(\xi+1) = \beta$, and $\bar{H}$ is applied to the same Q that G was. Note that $\varphi(\beta) \neq \beta$, so φ does not preserve tree order, just as with embedding normalization. (It does induce a map on extender-trees preserving $\subseteq$ and $\perp$.)

<u>Case 2</u> $\mathrm{crit}(F) \leq \mathrm{crit}(G)$.

In this case, $\delta \geq \beta$. Also, $\lambda(F) \leq \mathrm{crit}(\bar{H})$, so $\bar{H}$ is applied in X to some $Q \trianglelefteq M^X_\tau$, where $\tau \geq \alpha + 1$. Thus $\tau \in \mathrm{ran}(\varphi)$, and by 3(b), $\tau = \varphi(\delta)$. That is, $X\text{-pred}(\varphi(\xi+1)) = \varphi(\delta)$.

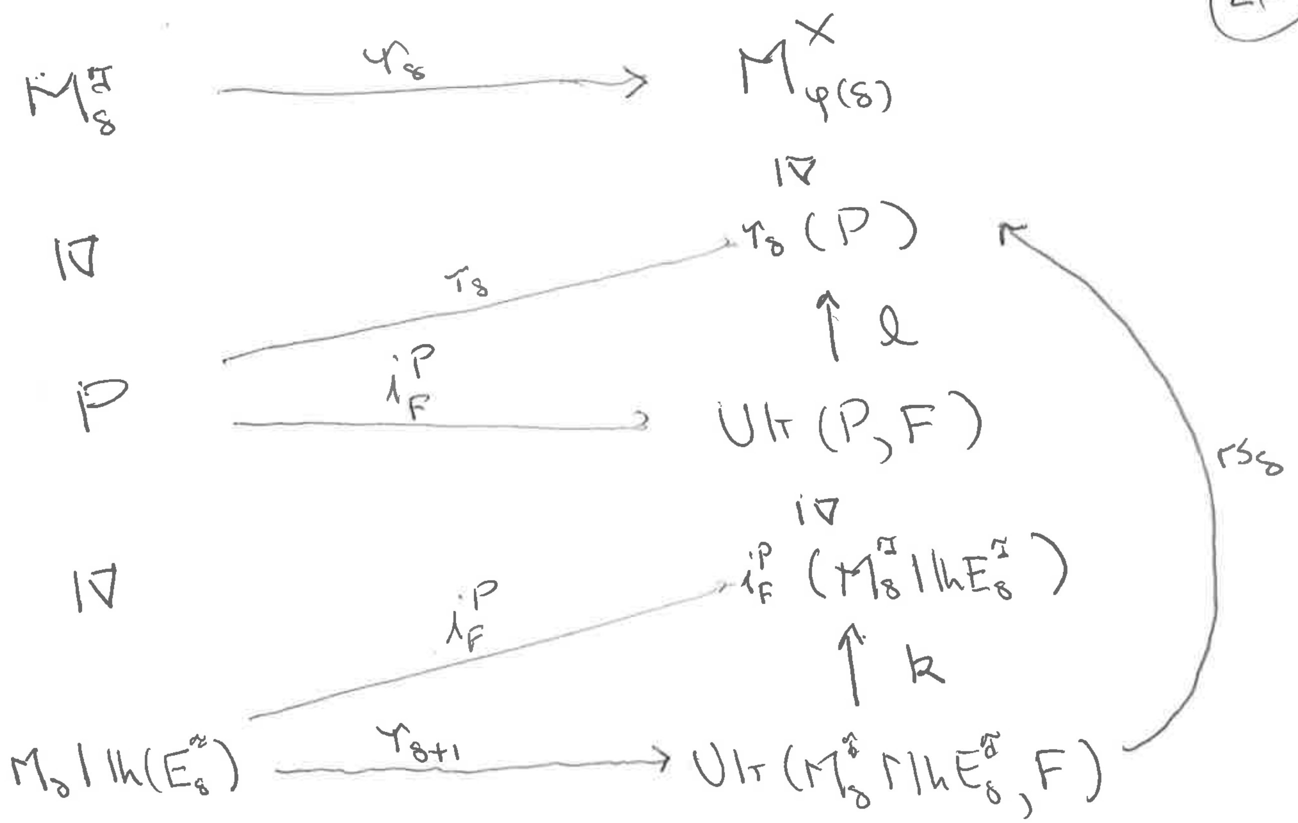
Let $\kappa = \mathrm{crit}(G)$, and let $P \trianglelefteq M^{\mathcal{T}}_\delta$ be least such that $\rho(P) \leq \kappa$. Thus

$$M^{\mathcal{T}}_{\xi+1} = \mathrm{Ult}(P, G).$$

~~We claim that $\bar{H}$ is applied to $\mathrm{Ult}(P,F)$ in X. For we have the diagram~~

<u>Claim 4</u> $\bar{H}$ is applied to $\mathrm{Ult}(P,F)$ in X.

<u>Proof</u> We have the diagram



k and l are the natural factor maps, and $rs_\delta = l \circ k$. Note that

$$k \upharpoonright \tau_{\delta+1}\left(\text{crit}(G)^{+M_\delta \| lh E_\delta^{\mathcal{T}}}\right) = \text{identity}$$

because P was least, so $\rho_{k(P)}(P) > \text{crit}(G)$.

So

$$\tau_{\delta+1}(\text{dom}(G)) = i_F^P(\text{dom}\, G)$$

$$\tau_{\delta+1} \upharpoonright \text{crit}(G)^{+P} = i_F^P \upharpoonright \text{crit}(G)^{+P}.$$

P is a level of the $lh E_\delta^{\mathcal{T}}$-dropdown sequence of $M_\delta^{\mathcal{T}}$, and [1], §1.1 shows then that

$$\text{Ult}(P,F) \trianglelefteq \tau_\delta(P).$$

See §2 for more on this.

(That's a step towards $\text{Ult}(M_\delta^{\mathcal{T}} \| lh E_\delta^{\mathcal{T}}, F) \trianglelefteq \tau_\delta(M_\delta^{\mathcal{T}} \| lh E_\delta^{\mathcal{T}})$.)

Note that for $\kappa = \text{crit}(G)$,

$$\rho(\text{Ult}(P,F)) \leq i_F^P(\kappa),$$

because $\text{Ult}(P,F)$ is generated by $i_F^P(p(P))$ $\cup\, i_F^P{}``\kappa \cup \text{lh}F$, and $\text{lh}F < i_F^P(\kappa)$.

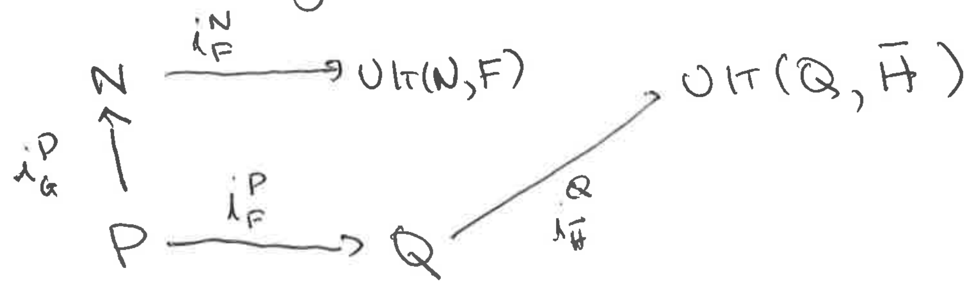
But for $k = k(P)$, $\rho_k(P) \geq (\kappa^+)^P$, so $\rho_k(\text{Ult}(P,F)) \geq i_F^P(\kappa^{+P})$. It follows that $\overline{H}$, whose domain is $\varphi_{\xi+1}(\text{dom}(G)) = \tau_{\delta+1}(\text{dom}(G)) = i_F^P(\text{dom}(G))$, is applied to $\text{Ult}(P,F)$ in $\mathcal{X}$.

Claim 4. ☒

<u>Claim 5</u> $\text{Ult}(\text{Ult}(P,F), \overline{H}) = M^{\mathcal{X}}_{\varphi(\xi+1)} = \text{Ult}(\text{Ult}(P,G), F)$.

<u>Proof</u> This is shown in [IT], §1.1, but we repeat the calculations here.

Set $N = \mathrm{Ult}(P, G)$ and $Q = \mathrm{Ult}(P, F)$. We have the diagram



Let E be the extender of $i^N_F \circ i^P_G$. Then $\nu(E) \le \sup i^N_F{}'' \lambda(G)$, and for $a \in [\nu(E)]^{<\omega}$, E_a concentrates on $N \| lh(G) = M^J_\xi \| lh(G)$. $\mathrm{crit}(G)^{|a|}$. Let K be the extender of $i^Q_{\bar H} \circ i^P_F$, so $\nu(K) \le lh\,\bar H = \sup i^N_F{}'' \lambda(G)$, and each K_a concentrates on $\mathrm{crit}(G)^{|a|}$.

Let $a = [b, g]^N_F$, where $g \in N | \lambda(G) = M^J_\xi | \lambda(G)$ be a typical element of $[\sup i^N_F{}'' \lambda(G)]^{<\omega}$, and $A \subseteq \mathrm{crit}(G)^{|a|}$; then

$(a, A) \in E$ iff $[b, g]^N_F \in i^N_F \circ i^P_G(A)$

iff for F_b a.e. u, $g(u) \in i^P_G(A)$

iff for F_b a.e. u, $(g(u), A) \in G$

iff $([b, g]^P_F, i^P_F(A)) \in H$

iff $[b, g]^P_F \in i_{\bar H}(i^P_F(A))$

iff $(a, A) \in K$.

$(a,A) \in E$ iff $[b,g]^N_F \in i^N_F \circ i^P_G(A)$

iff for F_b a.e. u, $g(u) \in i^P_G(A)$

iff for F_b a.e. u, $(g(u), A) \in G$

iff $\left([b,g]^{M_\xi \| hG}_F,\ i^{M_\xi \| hG}_F(A)\right) \in \overline{H}$

iff $\left([b,g]^N_F,\ i^P_F(A)\right) \in \overline{H}$

(since $[b,g]^N_F = [b,g]^{M_\xi \| hG}_F$, and $i^{M_\xi \| hG}_F(A) = i^P_F(A)$)

iff $[b,g]^N_F \in i^Q_{\overline{H}} \circ i^P_F(A)$

iff $(a,A) \in K$.

So $E = K$, and $\mathrm{Ult}(N,F) = \mathrm{Ult}(Q,\overline{H})$.

We have

$$\begin{array}{ccc} N & \xrightarrow{\ i^N_F\ } & \mathrm{Ult}(N,F) \\ \uparrow{\scriptstyle i_G} & & \uparrow{\scriptstyle i_{\overline{H}}} \\ P & \xrightarrow{\ i^P_F\ } & \mathrm{Ult}(P,F) \end{array}.$$

Since $E = K$, the diagram commutes.

Claim 5. ☒

So claim 5 gives us the diagram

$$\begin{array}{ccc}
M^{\mathcal{T}}_{\xi+1} & \xrightarrow{\tau_{\xi+1}} & M^{X}_{\varphi(\xi+1)} \\
\uparrow \hat{\lambda}^{\mathcal{T}}_{\delta,\xi+1} & & \uparrow \tilde{\lambda}^{X}_{\varphi(\delta),\varphi(\xi+1)} \\
P & \xrightarrow{i^P_F} & \mathrm{Ult}(P,F) \\
\trianglelefteq & & \trianglelefteq \\
M^{\mathcal{T}}_{\delta} & \xrightarrow{\tau_\delta} & M^{X}_{\varphi(\delta)}
\end{array}$$

Which finishes ~~the successor step~~ the defn. of E^X_γ in Case 2.

That finishes the ~~proof the~~ definition of E^X_γ in general, and we have that

<u>Summary</u> For $X = X(\mathcal{T}, F)$, we have

(i) $X \restriction \alpha+1 = \mathcal{T} \restriction \alpha+1$, where $\mathcal{T}$ is normal and α least s.t. F is on the $M^{\mathcal{T}}_\alpha$-sequence

(ii) $M^X_{\alpha+1} = \mathrm{Ult}(P, F)$, for $P \trianglelefteq M^{\mathcal{T}}_\beta$

(iii) $M^X_{\varphi(\xi)} = \mathrm{Ult}(M^{\mathcal{T}}_\xi, F)$

$\tau_\xi : M^{\mathcal{T}}_\xi \to M^X_{\varphi(\xi)}$ is the canon. emb

(iv) ~~$\tau_{\xi+1}$~~ $\tau_\xi = rs_\xi \circ \tau_{\xi+1}$

$$E^X_{\varphi(\xi)} = \tau_{\xi+1}(E^{\mathcal{T}}_\xi)$$

(v) If $(\delta, \xi]_T$ does not drop, then

$$\begin{array}{ccc}
M^{\mathcal{T}}_{\xi} & \xrightarrow{\ \tau_\xi\ } & M^{X}_{\varphi(\xi)} \\
\uparrow {\scriptstyle i^{\mathcal{T}}_{\delta,\xi}} & & \uparrow {\scriptstyle i^{X}_{\varphi(\delta),\varphi(\xi)}} \\
M^{\mathcal{T}}_{\delta} & \xrightarrow{\ \tau_\delta\ } & M^{X}_{\varphi(\delta)}
\end{array}$$

commutes, provided $\delta \neq \beta$. (If $\delta = \beta$, we may need to replace $M^X_{\varphi(\delta)}$ by M^X_β.)

Now we want to describe the natural embedding of $X(\mathcal{T}, F)$ into $W(\mathcal{T}, F)$. Going back to the definition of $E^X_{\varphi(\xi)}$, we had $G = E^{\mathcal{T}}_\xi$, $H = \tau_\xi(G)$, and $\bar{H} = \tau_{\xi+1}(G)$.

Let $\delta = T\text{-pred}(\xi+1)$.

Let $\mathcal{W} = \gamma\mathcal{V}(\mathcal{T}, F)$. Suppose that we have been defining by induction

$$\Psi_\alpha : M_\alpha^{\mathcal{X}} \longrightarrow M_\alpha^{\mathcal{W}}$$

such that

$$(\circledcirc) \qquad \pi_\alpha^{\mathcal{W}} = \Psi_\alpha \circ \tau_\alpha .$$

Here $\pi_\eta^{\mathcal{W}} : M_\eta^{\mathcal{T}} \longrightarrow M_{\varphi(\eta)}^{\mathcal{W}}$ is the map given by embedding normalization. Thus

$$E_{\varphi(\xi)}^{\mathcal{W}} = \pi_\xi^{\mathcal{W}}(E_\xi^{\mathcal{T}}) = \Psi_\xi(H).$$

We have $\Psi_\eta = \text{id}$ for all $\eta \leq \alpha+1$. We have by induction the agreement

(1) $\forall \xi \geq \beta \quad \Psi_{\varphi(\xi)} \restriction \text{lh} F = \text{identity}$

(2) if $\beta \leq \eta < \xi$, then

$$\Psi_{\varphi(\xi)} \restriction \text{lh} E_{\varphi(\eta)}^{\mathcal{X}} = \Psi_{\varphi(\eta)} \circ \sigma_\eta \restriction \text{lh} E_{\varphi(\eta)}^{\mathcal{X}} .$$

$\Psi_{\varphi(\xi)}$

$\Psi_{\varphi(\eta)}$

<u>Case 1</u> $\operatorname{crit}(G) < \operatorname{crit}(F)$

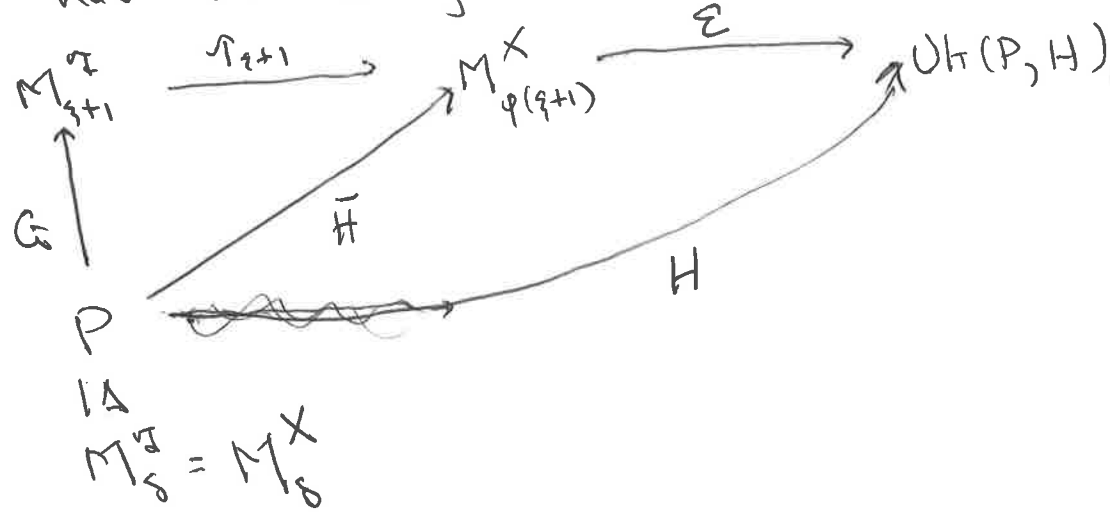
In this case, $\operatorname{dom}(\bar{H}) = \operatorname{dom}(\overset{\circ}{H})$.

Suppose $\bar{H}$ is applied to P in X. Then H would also be applied to P, if we had set $E_{\varphi(\xi)} = H$.

$\bar{H}$ is a subextender of H under rs_ξ:

$$(a, A) \in \bar{H} \quad \text{iff} \quad (rs_\xi(a), A) \in H.$$

We have the diagram



ε is given by

$$\varepsilon([a, f]^P_{\bar{H}}) = [rs_\xi(a), f]^P_H.$$

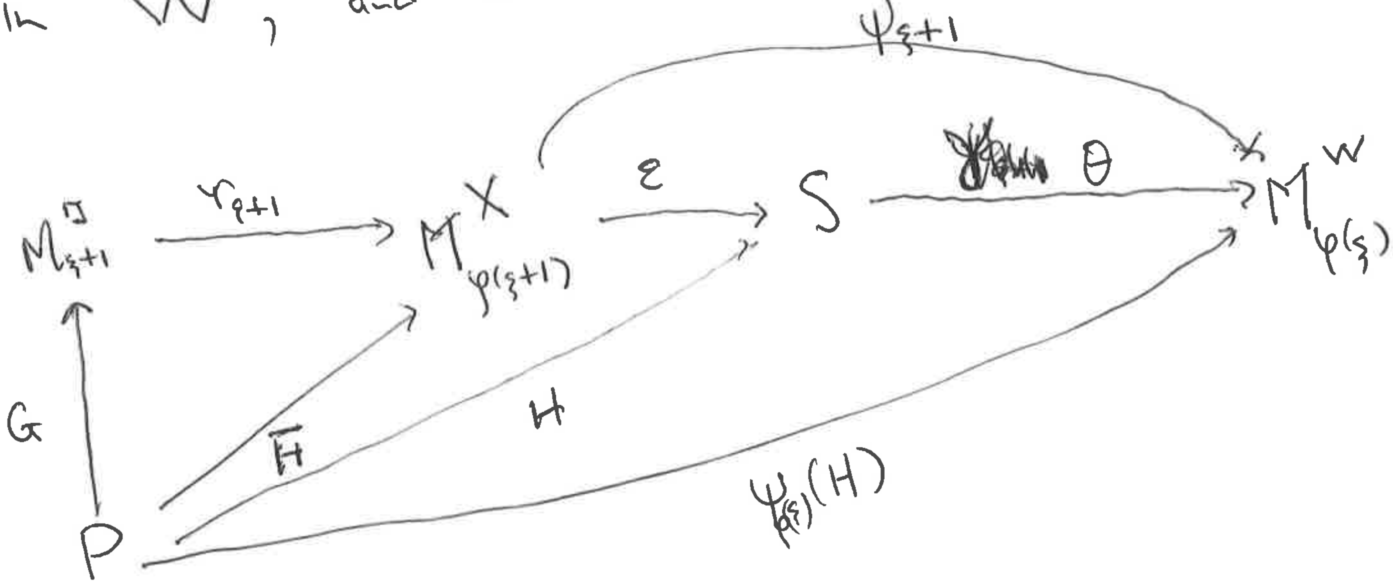
So $\varepsilon \restriction \operatorname{lh} \bar{H} = rs_\xi \restriction \operatorname{lh} \bar{H}$.

So in the present case, as $\delta \le \beta$,

we have $\psi_\xi \restriction \mathrm{dom}(H) = \text{identity}$,

so $\psi_\xi(H)$ is ~~also~~ applied to P

in W, and we have the diagram



in

$$M^{\mathcal{I}}_\delta = M^X_\delta = M^W_\delta$$

θ is given by

$$\theta([a,f]^P_H) = [\psi_{\phi(\xi)}(a), f]^P_{\psi_{\phi(\xi)}(H)}$$

So θ agrees with $\psi_{\phi(\xi)}$ on $\mathrm{lh}\,H$. We set

$$\psi_{\phi(\xi+1)} = \theta \circ \varepsilon,$$

so

$$\Psi_{\varphi(\xi+1)} \restriction \mathrm{lh}\,\bar{H} = \Psi_{\varphi(\xi)} \circ rs_\xi \restriction \mathrm{lh}\,\bar{H},$$

as required in agreement hypothesis (2).

It's easy to see that $\pi^{W}_{\xi+1} = \Psi_{\varphi(\xi+1)} \circ \tau_{\xi+1}$.

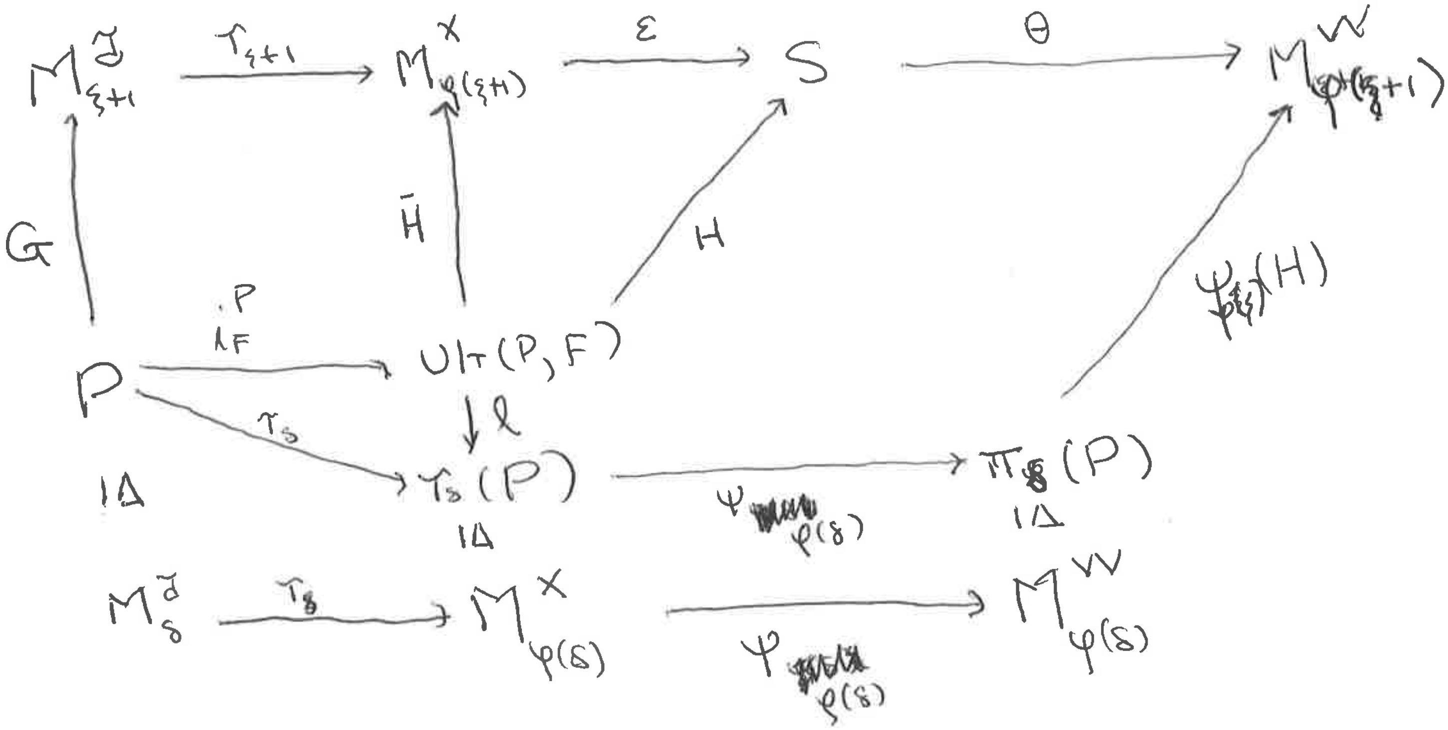
<u>Case 2</u> $\mathrm{crit}(F) \le \mathrm{crit}(G)$.

Suppose first that $\delta < \xi$. This yields $\tau_\xi \restriction \mathrm{lh}\,E^{\mathcal{T}}_\delta = \tau_{\xi+1} \restriction \mathrm{lh}\,E^{\mathcal{T}}_\delta$, so τ_ξ and $\tau_{\xi+1}$ agree on $\mathrm{dom}(G)$, so $\mathrm{dom}(\bar{H}) = \mathrm{dom}(H)$. Moreover, $rs_\xi \restriction \mathrm{dom}(\bar{H}) = \text{identity}$. Let $P \trianglelefteq M^{\mathcal{T}}_\delta$ be what G is applied to in $\mathcal{T}$.

We have

$$i^{P}_{F} \restriction \mathrm{dom}\,G = \tau_{\delta+1} \restriction \mathrm{dom}\,G = \tau_\xi \restriction \mathrm{dom}\,G = \tau_{\xi+1} \restriction \mathrm{dom}\,G.$$

We have the diagram



Again, $E^{\mathcal{W}}_{\varphi(\xi)} = \pi_\xi(E^{\mathcal{T}}_\xi) = \psi_{\varphi(\xi)}(\tau_\xi(E^{\mathcal{T}}_\xi)) = \psi_{\varphi(\xi)}(H)$.
The definition of $\mathcal{W}(\mathcal{T},F)$ tells us that $E^{\mathcal{W}}_{\varphi(\xi)}$ is applied to $\pi_\delta(P)$ in $\mathcal{W}$.

We have
$$l \upharpoonright \mathrm{dom}(G) = rs_\delta \upharpoonright \mathrm{dom}(G).$$

{Recall $\mathrm{Ult}(M_\delta \| \mathrm{lh} E^{\mathcal{T}}_\delta, F) \xrightarrow{k} \mathrm{Ult}(P,F) \xrightarrow{l} \tau_\delta(P)$, with $k \upharpoonright \mathrm{dom}(G) = \mathrm{id}$. So $rs_\delta \upharpoonright \mathrm{dom}(G) = l \circ k \upharpoonright \mathrm{dom}(G) = l \upharpoonright \mathrm{dom}(G)$.] Thus

$$\psi_{\varphi(\xi)} \upharpoonright \mathrm{dom}(G) = \psi_{\varphi(\delta)} \circ rs_\delta \upharpoonright \mathrm{dom}(G) = \psi_{\varphi(\delta)} \circ l \upharpoonright \mathrm{dom}(G)$$

So by the shift lemma, we can define

$$\Theta([a,f]^{\mathrm{Ult}(P,F)}_H) = [\psi_{\varphi(\xi)}(a), \psi_{\varphi(\delta)} \circ l(f)]^{\pi_\delta(P)}_{\psi_{\varphi(\xi)}(H)}.$$

The diagram above then commutes. We set

$$\psi_{\varphi(\xi+1)} = \Theta \circ \Sigma.$$

Since $\Sigma \restriction \mathrm{lh}\,\bar{H} = rs_\xi \restriction \mathrm{lh}\,\bar{H}$ and $\Theta \restriction \mathrm{lh}\,H = \psi_{\varphi(\xi)} \restriction \mathrm{lh}\,H$, we get

$$\psi_{\varphi(\xi+1)} \restriction \mathrm{lh}\,\bar{H} = \psi_{\varphi(\xi)} \circ rs_\xi \restriction \mathrm{lh}\,\bar{H},$$

as in agreement hypothesis (2). We must also show that $\pi_{\xi+1} = \psi_{\varphi(\xi+1)} \circ \tau_{\xi+1}$. Note first that the two sides agree on $\mathrm{ran}\, i^P_G$. For letting $j : \pi_\delta(P) \to M^W_{\varphi(\xi+1)}$, $j = \hat{i}^W_{\varphi(\delta),\varphi(\xi+1)}$:

$$\Theta \circ \Sigma \circ \tau_{\xi+1} \circ i^P_G = j \circ \psi_{\varphi(\delta)} \circ \tau_\delta = j \circ \pi_\delta$$
$$= \pi_{\xi+1} \circ i^P_G,$$

using the commutativity in embedding normalization.

But $M^{\mathbb{I}}_{\xi+1}$ is generated by $\operatorname{ran} i^{P}_{G} \cup \lambda(G)$, so it is enough to see $\Theta \circ \Sigma \circ \Upsilon_{\xi+1}$ agrees with $\Pi_{\xi+1}$ on $\lambda(G)$. Since $\Pi_{\xi+1}$ agrees with Π_{ξ} on $\lambda(G)$, we get

$$\begin{aligned}
\Pi_{\xi+1} \restriction \lambda(G) &= \Pi_{\xi} \restriction \lambda(G) \\
&= \Psi_{\varphi(\xi)} \circ \Upsilon_{\xi} \restriction \lambda(G) \\
&= \Psi_{\varphi(\xi)} \circ (rs_{\xi} \circ \Upsilon_{\xi+1}) \restriction \lambda(G) \\
&= (\Psi_{\varphi(\xi)} \circ rs_{\xi}) \circ \Upsilon_{\xi+1} \restriction \lambda(G) \\
&= \Psi_{\varphi(\xi+1)} \circ \Upsilon_{\xi+1} \restriction \lambda(G).
\end{aligned}$$

So $\Pi_{\xi+1} = \Psi_{\varphi(\xi+1)} \circ \Upsilon_{\xi+1}$, as desired.

This finishes the definition of $\Psi_{\varphi(\xi+1)}$ when $\delta < \xi$. The case $\delta = \xi$ is not different in any important way. In that case, we may have $\text{crit}(\bar{H}) < \text{crit}(H)$. The relevant diagram is the same. We omit further detail.

It is clear that if $(\delta, \xi+1]_{\mathcal{J}}$ is not a drop, then

$$\psi_{\varphi(\xi+1)} \circ i^{X}_{\varphi(\delta),\varphi(\xi+1)} = i^{W}_{\varphi(\delta),\varphi(\xi+1)} \circ \psi_{\varphi(\delta)}$$

(when $\delta \neq \beta$. If $\delta = \beta$, we may need to replace $\varphi(\delta)$ by β.) This lets us define $\psi_{\varphi(\lambda)}$ when λ is a limit. We omit the details. Our induction hypotheses are preserved.

This completes the definition of $X(\mathcal{J}, F)$, and its embedding into $W(\mathcal{J}, F)$.

Let us write $\beta^{\mathcal{J},F}$, $\alpha^{\mathcal{J},F}$, $\varphi^{\mathcal{J},F}$,

Remark Given all trees by a fixed Σ, α is determined by F.

and $\varphi_\alpha^{\mathcal{T},F} : M_\alpha^{\mathcal{T}} \longrightarrow M_{\varphi^{\mathcal{T},F}(\alpha)}^{X(\mathcal{T},F)}$ for the objects we defined above. $\beta^{\mathcal{T},F}$, $\alpha^{\mathcal{T},F}$, and $\varphi^{\mathcal{T},F}$ are the same as the correspondingly named objects associated to $W(\mathcal{T},F)$. $\varphi^{\mathcal{T},F} : \mathrm{lh}(\mathcal{T}) \longrightarrow \mathrm{lh}(X(\mathcal{T},F))$, but it may be ~~be partial~~ not total. When it is ~~partial~~ not total, it has domain $\beta^{\mathcal{T},F}+1$, and F is ~~partial~~ not total on the last model of $\mathcal{T}$.

Let also $\psi_\eta^{\mathcal{T},F} : M_\eta^{X(\mathcal{T},F)} \longrightarrow M_\eta^{W(\mathcal{T},F)}$ be the ψ-map we defined.

Now let $\mathcal{T}$ be a normal tree on some premouse, and $\mathcal{U}$ a normal tree on the last model of $\mathcal{T}$, which we assume exists. We define $X(\mathcal{T},\mathcal{U})$, and maps relating it to $W(\mathcal{T},\mathcal{U})$.

Associated to $\mathcal{W}(\mathcal{T}, \mathcal{U})$ we have normal trees

$$\mathcal{W}_\gamma = \mathcal{W}(\mathcal{T}, \mathcal{U} \upharpoonright \gamma+1)$$

with last models

$$R_\gamma = M^{\mathcal{W}_\gamma}_{z(\gamma)}$$

and

$$\sigma_\gamma : M^{\mathcal{U}}_\gamma \longrightarrow R_\gamma .$$

For $\nu <_U \gamma$ we have a partial

$$\varphi_{\nu,\gamma} : \mathrm{lh}\, \mathcal{W}_\nu \longrightarrow \mathrm{lh}\, \mathcal{W}_\gamma$$

and maps

$$\pi^{\nu,\gamma}_\alpha : M^{\mathcal{W}_\nu}_\alpha \longrightarrow M^{\mathcal{W}_\gamma}_{\varphi_{\nu,\gamma}(\alpha)}$$

for $\alpha \in \mathrm{dom}\, \varphi_{\nu,\gamma}$. We set $F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, and for $\nu = U\text{-pred}(\gamma+1)$, we have

$$\mathcal{W}_{\gamma+1} = \mathcal{W}(\mathcal{W}_\nu, F_\gamma).$$

Associated to $\mathcal{X}(\mathcal{T}, \mathcal{U})$ we have normal trees

$$\mathcal{X}_\gamma = \mathcal{X}(\mathcal{T}, \mathcal{U} \upharpoonright (\gamma+1))$$

such that X_γ has the same tree order as W_γ, and last model

$$M^{X_\gamma}_{z(\gamma)} = M^{\mathcal{U}}_\gamma .$$

The ~~extender~~ maps $\varphi_{\nu,\gamma}$ are the same, and we have

$$\pi^{\nu,\gamma}_\alpha : M^{X_\nu}_\alpha \longrightarrow M^{X_\gamma}_\alpha$$

for $\alpha \in \operatorname{dom} \varphi_{\nu,\gamma}$. The X_γ and $\pi^{\nu,\gamma}_\alpha$ are defined by induction:

$$X_0 = \mathcal{I}$$

and

$$X_{\gamma+1} = X(X_\nu, E^{\mathcal{U}}_\gamma),$$

where $\nu = \mathcal{U}\text{-pred}(\gamma+1)$. We need to show $\alpha^{E^{\mathcal{U}}_\gamma} = \alpha^{F_\gamma}$, $\beta^{X_\nu, E^{\mathcal{U}}_\gamma} = \beta^{W_\nu, F_\gamma}$, and $\varphi^{X_\nu, E^{\mathcal{U}}_\gamma} = \varphi^{W_\nu, F_\gamma}$. We then have

$$\pi^{\nu,\gamma+1}_\alpha = \pi^{X_\nu, E^{\mathcal{U}}_\gamma}_\alpha$$

for $\alpha \in \operatorname{dom} \varphi_{\nu,\gamma+1}$, with

$$\tau_\alpha^{\nu,\gamma+1} : M_\alpha^{X_\nu} \longrightarrow M^{X_{\gamma+1}}_{\varphi_{\nu,\gamma+1}(\alpha)}$$

and for $\xi <_u \nu$ ~~in dom $\varphi_{\xi,\nu}$ and such that~~ we let $\tau_\alpha^{\xi,\gamma+1} = \tau^{\nu,\gamma+1}_{\varphi_{\nu,\xi}(\alpha)} \circ \tau_\alpha^{\xi,\nu}$ whenever

$\alpha \in \mathrm{dom}\, \varphi_{\xi,\gamma+1}$.

Everything fits together properly, so we can define X_λ for λ a limit, along with maps $\tau_\alpha^{\nu,\lambda}$ for $\nu <_u \lambda$ and $\alpha \in \mathrm{dom}\, \varphi_{\nu,\lambda}$. [Insert 43a,b]

We also get maps $\psi_\alpha^\gamma : M_\alpha^{X_\gamma} \longrightarrow M_\alpha^{W_\gamma}$ relating X_γ to W_γ, defined for $\alpha \le z(\gamma) = \mathrm{lh}\, X_\gamma - 1$. We have the diagram, whenever $\nu <_u \eta$:

$$\begin{array}{ccc}
M_\alpha^{W_\nu} & \xrightarrow{\pi_\alpha^{\nu,\eta}} & M^{W_\eta}_{\varphi_{\nu,\eta}(\alpha)} \\
\uparrow \psi_\alpha^\nu & & \uparrow \psi^\eta_{\varphi_{\nu,\eta}(\alpha)} \\
M_\alpha^{X_\nu} & \xrightarrow{\tau_\alpha^{\nu,\eta}} & M^{X_\eta}_{\varphi_{\nu,\eta}(\alpha)}
\end{array}$$

For $\xi < \mathrm{lh}\, X_\lambda$, pick $\nu <_u \lambda$ such that $\xi = \varphi_{\nu,\lambda}(\alpha)$ for some α.

Then

$$M_\xi^{X_\lambda} = \text{direct limit of } M^{X_\beta}_{\varphi_{\nu,\beta}(\alpha)}, \text{ for}$$

$\alpha <_u \beta <_u \lambda$, under the

$$\pi^{\beta_0,\beta_1}_{\varphi_{\nu,\beta_0}(\alpha)} : M^{X_{\beta_0}}_{\varphi_{\nu,\beta_0}(\alpha)} \longrightarrow M^{X_{\beta_1}}_{\varphi_{\nu,\beta_1}(\alpha)}.$$

$\pi^{\beta,\lambda}_{\varphi_{\nu,\beta}(\alpha)}$ for $\beta <_u \lambda$ is the direct limit map.

There is one point here: the maps $\pi^{\beta_0,\beta_1}_{\eta}$ do not preserve exit extenders in general, so what are the exit extenders $E^{X_\lambda}_\xi$? For this, note that the exit extenders are going down in the direct limit, i.e.

$$E^{X_{\beta_1}}_{\varphi_{\nu,\beta_1}(\alpha)} \trianglelefteq \pi^{\beta_0,\beta_1}_{\varphi_{\nu,\beta_0}(\alpha)}\left(E^{X_{\beta_0}}_{\varphi_{\nu,\beta_0}(\alpha)}\right)$$

in the order given by the $M^{X_{\beta\rho}}_{\varphi_{\nu,\beta,}(\alpha)}$ - sequence. Thus they eventually stabilize. (Assuming all $M^{X_\lambda}_\eta$ are wellfounded, as we do - otherwise the construction of $X(\tilde{J}, \mathcal{U})$ halts.)

So we can set

$$E^{X_\lambda}_{\varphi_{\nu,\lambda}(\alpha)} = \text{common value of } \pi^{\beta,\lambda}_{\varphi_{\nu,\beta}(\alpha)}\left(E^{X_\beta}_{\varphi_{\nu,\beta}(\alpha)}\right)$$

for $\beta <_U \lambda$ sufficiently large.

This makes X_λ into an iteration tree.

We shall also have that

$$\psi^{\gamma}_{z(\gamma)} = \sigma_{\gamma} .$$

The maps ψ^{γ}_{α} are defined as follows. Suppose we have defined ψ^{η}_{ξ} for all $\eta \leq \gamma$ and $\xi \leq z(\eta)$. Let $\nu = \mathcal{U}\text{-prod}(\gamma+1)$.

We have an embedding

$$X_{\gamma+1} = X(X_{\nu}, E^{\mathcal{U}}_{\gamma}) \longrightarrow W(X_{\nu}, E^{\mathcal{U}}_{\gamma}) = \overline{W}_{\gamma+1}$$

with maps

$$\psi^{X_{\nu}, E^{\mathcal{U}}_{\gamma}}_{\xi} : M^{X_{\gamma+1}}_{\xi} \longrightarrow M^{\overline{W}_{\gamma+1}}_{\xi}$$

defined above. Our embeddings of X_{ν} into W_{ν} and X_{γ} into W_{γ}, together with the fact that

$$\psi^{\gamma}_{z(\gamma)}(E^{\mathcal{U}}_{\gamma}) = \sigma_{\gamma}(E^{\mathcal{U}}_{\gamma}) = F_{\gamma}$$

yield an embedding

$$\overline{W}_{\gamma+1} = W(X_{\nu}, E^{\mathcal{U}}_{\gamma}) \longrightarrow W(W_{\nu}, F_{\gamma}) = W_{\gamma+1}$$

with maps

$$\overline{\psi}_{\xi}^{\gamma+1} : M_{\xi}^{\overline{W}_{\gamma+1}} \longrightarrow M_{\xi}^{W_{\gamma+1}} .$$

We then set

$$\psi_{\xi}^{\gamma+1} = \overline{\psi}_{\xi}^{\gamma+1} \circ \psi_{\xi}^{X_{\gamma}, E_{\gamma}^{u}} .$$

The maps ψ_{ξ}^{λ} for λ a limit are defined by commutativity. Fixing λ and ξ, let $\xi = \varphi_{\alpha,\lambda}(\bar{\xi})$ where $\alpha <_u \lambda$.

So

$$M_{\xi}^{X_\lambda} = \text{direct limit of } M^{X_\beta}_{\varphi_{\alpha,\beta}(\bar{\xi})} ,$$

for $\beta <_u \lambda$, under the maps

$$\varphi^{\beta_0,\beta_1}_{\varphi_{\alpha,\beta_0}(\bar{\xi})} : M^{X_{\beta_0}}_{\varphi_{\alpha,\beta_0}(\bar{\xi})} \longrightarrow M^{X_{\beta_1}}_{\varphi_{\alpha,\beta_1}(\bar{\xi})}$$

and

$$M_{\xi}^{W_\lambda} = \text{direct limit of } M_{\varphi_{\alpha,\beta}(\bar\xi)}^{W_\beta}, \text{ for}$$

$\beta <_u \lambda$, under the maps

$$\pi_{\varphi_{\alpha,\beta_0}(\bar\xi)}^{\beta_0,\beta_1} : M_{\varphi_{\alpha,\beta_0}(\bar\xi)}^{W_{\beta_0}} \longrightarrow M_{\varphi_{\alpha,\beta_1}(\bar\xi)}^{W_{\beta_1}}.$$

Because the ψ's lift the τ's into the π's, i.e. we have the commutativity stated above, we can set

$$\psi_\xi^\lambda\left(\tau_{\varphi_{\alpha,\beta}(\bar\xi)}^{\beta,\lambda}(x)\right) = \pi_{\varphi_{\alpha,\beta}(\bar\xi)}^{\beta,\lambda}\left(\psi_{\varphi_{\alpha,\beta}(\bar\xi)}^{\beta}(x)\right)$$

for $\alpha <_u \beta <_u \lambda$, and this works.

To do this carefully, it would probably be easiest to define the ψ_ξ^γ by induction on γ, with a subinduction on ξ, and verify the necessary commutativity and agreement properties as we go. We shall not do this here.

§2. Weak ~~pseudo~~ hull embeddings

For $\mathcal{T}$ and $\mathcal{U}$ normal trees on a premouse M, a psuedo-hull embedding of $\mathcal{T}$ into $\mathcal{U}$ is a triple

$\langle u, \langle t^0_\beta \mid \beta < \mathrm{lh}\,\mathcal{T}\rangle, \langle t^1_\beta \mid \beta+1 < \mathrm{lh}\,\mathcal{T}\rangle, p\rangle$.

u maps $\mathrm{lh}\,\mathcal{T}$ into $\mathrm{lh}\,\mathcal{U}$, not quite order-preservingly. t^1_β maps an initial segment of $M^{\mathcal{T}}_\beta$ into $M^{\mathcal{U}}_{u(\beta)}$. $t^1_\beta = \hat{\imath}^{\mathcal{U}}_{v(\beta),u(\beta)} \circ t^0_\beta$. The whole system is determined by certain rules, and the map $p : \mathrm{Ext}(\mathcal{T}) \to \mathrm{Ext}(\mathcal{U})$ mapping extenders used in $\mathcal{T}$ to extenders used in $\mathcal{U}$. The key equation is

$$p(E^{\mathcal{T}}_\alpha) = t^1_\alpha(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}.$$

We get a psuedo-hull embeding of $\mathcal{T}$ into $W(\mathcal{T}, F)$ in the case $\mathrm{dom}\,\varphi^{\mathcal{T},F} = \mathrm{lh}\,\mathcal{T}$ as follows: $u = \varphi^{\mathcal{T},F}$, $t^1_\eta = \pi^{\mathcal{T},F}_\eta$ for $\eta+1 < \mathrm{lh}\,\mathcal{T}$, $p(E^{\mathcal{T}}_\eta) = E^{W(\mathcal{T},F)}_{\varphi(\eta)}$, $t^0_\eta = t^1_\eta$ if $\eta \neq \beta$,

and t^0_η = identity if $\eta = \beta^{\bar{J},F}$.
(So $v(\eta) = \varphi(\eta)$ if $\eta \neq \beta^{\bar{J},F}$, and $v(\beta^{\bar{J},F}) = \beta^{\bar{J},F}$.)

More generally, if $W_\nu = W(\bar{J}, \mathcal{U} \restriction \nu+1)$ and $W_\gamma = W(\bar{J}, \mathcal{U} \restriction \gamma+1)$ and $\nu <_U \gamma$ and $[\nu, \gamma]_U$ does not drop, then we have a natural pseudo-hull embedding of W_ν into W_γ.

We wish to weaken the condition $t^1_\alpha(E^{\bar{J}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}$, because the natural embedding of $\bar{J}$ into $X(\bar{J}, F)$ does not preserve exit extenders. Basically, we shall just require that $t^1_\alpha(E^{\bar{J}}_\alpha)$ be related to $E^{\mathcal{U}}_{u(\alpha)}$ inside $M^{\mathcal{U}}_{u(\alpha)}$ by the condensation process we used to relate $i^{M \| G}_F(G)$ and $i^M_F(G)$ in the last section. The result is the notion of a weak pseudo-hull embedding. It will turn out that the natural embeddings of X_ν into X_γ, when $\nu <_u \gamma$, and the natural embedding of $X(\mathcal{I}, \mathcal{U})$ into $W(\mathcal{I}, \mathcal{U})$, are weak ~~proper~~ hull embeddings.

Let's look at the process by which we derived $i_F^{M \| lh G}(G)$ from $i_F^M(G)$.

Let M be any premouse, and $\lambda \leq o(M)$. We define $A_k \trianglelefteq M$ and γ_k by induction:

$$A_0 = M | \langle \lambda, 0 \rangle$$
$$\gamma_0 = \lambda$$

$$A_{i+1} = M | \langle \eta, k+1 \rangle, \text{ where } \langle \eta, k \rangle \text{ is lex-least such that } \rho(M | \langle \eta, k \rangle) < \gamma_i$$

$$\gamma_{i+1} = \rho(M | \langle \eta, k \rangle), \text{ for this } \langle \eta, k \rangle.$$

The γ's are strictly decreasing, so there is a largest m s.t. A_m and γ_m are defined.

Rmk As we have set it up, $\gamma_i = \rho_{k(A_i)}(A_i)$. Possibly $\rho(A_i) < \gamma_i$.

If $A_m = M$, then we set $n(M,\lambda) = M$ and stop. If $A_m \lhd M$, then we set $n(M,\lambda) = m+1$ and $A_{m+1}(M,\lambda) = M$ (and $\gamma_{m+1}(M,\lambda) = \gamma_m(M,\lambda)$.) So in either case

$$A_n(M,\lambda) = M.$$

<u>Notation</u> For M a premouse, $\rho^-(M) = \rho_{k(M)}(M)$. (Recall that $\rho(M) = \rho_{k(M)+1}(M)$.) If $k(M) > 0$, then $M^- = M \mid \langle \hat{o}(M), k(M)-1 \rangle$.

If we reach M in a normal tree $\mathcal{T}$, and the exit extender from M has length λ, then the $A_k(M,\lambda)$ are the initial segments of M we might apply some later E to in a normal continuation of $\mathcal{T}$. If $\text{crit}(E) = \mu$, it would be applied to $A_k(M,\lambda)$, where k is largest such that $\mu < \gamma_k(M,\lambda)$.

__Definition 2.0__ $\langle A_k(M,\lambda) \mid k \le n(M,\lambda) \rangle$ is the λ-dropdown sequence of M.

__Remark__ $A_0(M,\lambda) = M|\langle \lambda, 0 \rangle$. If $\lambda = \ln E$ for some E on the M-sequence, and $\langle \lambda, 1 \rangle \le l(M)$, then $A_1(M,\lambda) = M|\langle \lambda, 1 \rangle$.

__Prop 2.1__ Let $n = n(M,\lambda)$ and $A_i = A_i(M,\lambda)$ and $\gamma_i = \gamma_i(M,\lambda)$ for $i \le n$. Then for $i \le n$

(1) $n(A_i, \lambda) = i$, and $A_k(A_i,\lambda) = A_k$ and $\gamma_k(A_i,\lambda) = \gamma_k$ for all $k < i$.

(2) If $A_i \triangleleft B \trianglelefteq A_{i+1}$, then $n(B,\lambda) = i+1$, $A_k(B,\lambda) = A_k$ for all $k \le i$, and $\gamma_k(B,\lambda) = \gamma_k$ for all $k \le i$.

__Proof__ Easy. ☒

Preservation of dropdown sequences under ultrapowers is given by:

Lemma 2.2 Let F be an extender over M, with $\mathrm{crit}(F) < \gamma_{n(M,\lambda)}$. Let $N = \mathrm{Ult}(M,F)$, and

$$i_F^M : M \longrightarrow N$$

be the canonical embedding. Let

$$\lambda^* = i_F^M(\lambda).$$

Then

(a) $n(M,\lambda) = n(N,\lambda^*)$,

(b) for all $k \leq n(M,\lambda)$

$$A_k(N,\lambda^*) = i_F^M(A_k(M,\lambda)),$$

(c) for all $k < n(M,\lambda)$

$$\gamma_k(N^*,\lambda^*) = i_F^M(\gamma_k(M,\lambda))$$

(d) If $\gamma_n(M,\lambda) = \rho^-(M)$, then $\gamma_n(N^*,\lambda) = \sup i_F^M{}''\gamma_n(M,\lambda)$. Otherwise, $\gamma_n(N^*,\lambda) = i_F^M(\gamma_n(M,\lambda))$.

Proof Elementary. ⊠

The embedding of $\mathcal{J}$ into $X(\mathcal{J}, F)$ is given by the $\tau_\xi = \tau_\xi^{\mathcal{J},F}$'s. The embedding of $X(\mathcal{J}, F)$ into $W(\mathcal{J}, F)$ is given by the $\psi_\xi = \psi_\xi^{\mathcal{J},F}$'s. These satisfy the agreement formulae

$$\tau_\xi = rs_\xi \circ \tau_{\xi+1} \quad \text{on } \mathrm{lh}\, E_\xi^{\mathcal{J}},$$

$$\psi_{\xi+1} = rs_\xi^* \circ \psi_\xi \quad \text{on } \mathrm{lh}\, E_\xi^X.$$

[In the last section, we wrote $\psi_{\xi+1} = \psi_\xi \circ rs_\xi$ on $\mathrm{lh}\, E_\xi^X$. rs_ξ "resurrected" $\bar{H}$ to H inside M_ξ^X. Here we are re-writing using $rs_\xi^* = \psi_\xi(rs_\xi)$, which resurrects $\psi_\xi(\bar{H})$ to $\psi_\xi(H)$ inside M_ξ^W. Doing it this way helps unify the two cases, the "X-case" and the "W-case", of weak pseudo hull embeddings.] One important property of rs_ξ and rs^*_ξ is the following.

<u>Definition 2.3</u> Let $\sigma : M|\eta \to M|\lambda$ be elementary. We say that <u>σ respects drops</u> (over (M, η, λ)) iff

(a) $n(M, \eta) = n(M, \lambda)$, and $\sigma \restriction \gamma_n(M, \eta) =$ identity,

(b) For each $i \leq n(M, \eta)$, there is an elementary

$$\pi_i : A_i(M, \eta) \to A_i(M, \lambda)$$

such that $\lambda \in \mathrm{ran}(\pi_i)$, and

$$\pi_i \restriction \rho^-(A_i(M, \eta)) = \sigma \restriction \rho^-(A_i(M, \eta)).$$

<u>Remark</u> $A_i(M, \eta)$ is $k(A_i(M, \eta))$-sound, so π_i is uniquely determined by σ, M, η, λ.

<u>Remark</u> If $M|\eta \xrightarrow{\sigma} M|\lambda \xrightarrow{\tau} M|\gamma$ and σ, τ respect drops over (M, η, λ) and (M, λ, γ) resp., then $\tau \circ \sigma$ respects drops over (M, η, γ).

<u>Remark</u> Let σ, M, η, λ, and the π_i's be as in 2.3. Set, for $k<n$

$$\sigma_k = \pi_{k+1}^{-1} \circ \pi_k ,$$

where we are setting π_n = identity, $\pi_n : M \to M$. (This is consistent with (b), because $\sigma \restriction \gamma_n(M,\eta) = id.$)

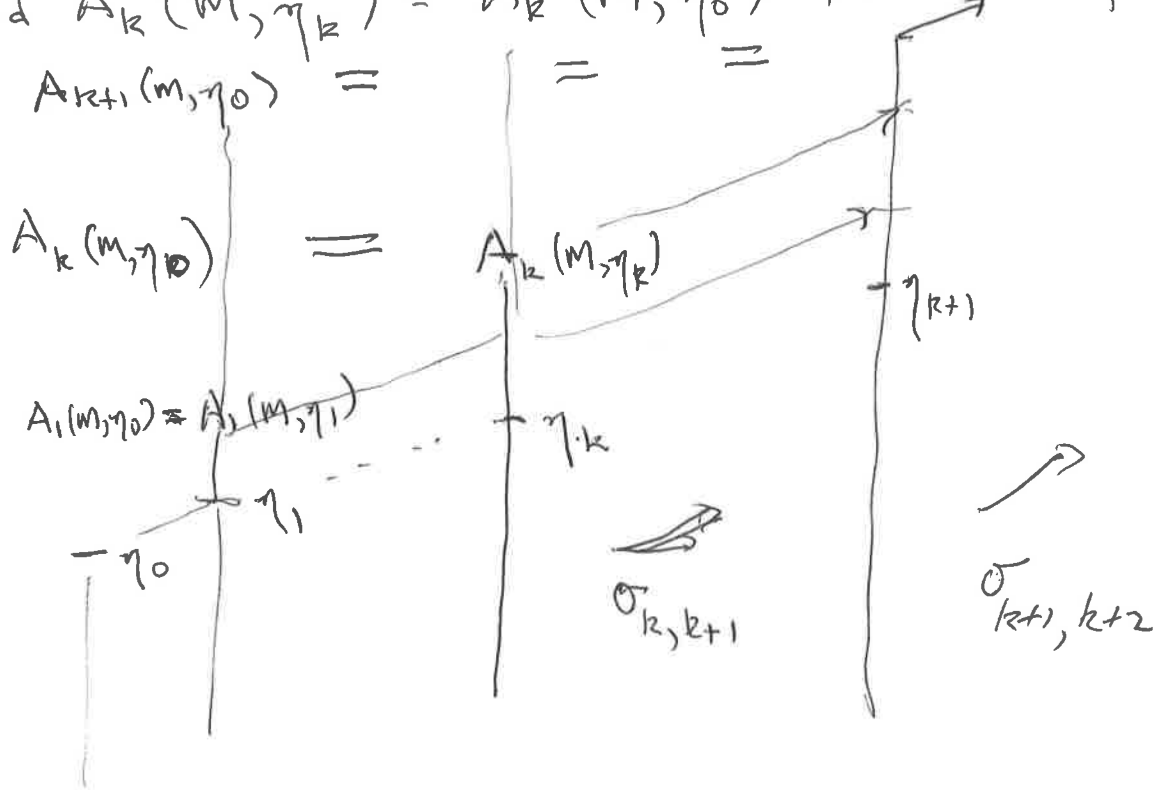
Let

$$\eta_i = \pi_i(\eta),$$

so $\eta_0 = \eta$ and $\eta_n = \lambda$. One can see that

$$\sigma_k : A_k(M, \eta_k) \to A_k(M, \eta_{k+1})$$

(and $A_k(M,\eta_k) = A_k(M,\eta_0)$). The picture is



The factor maps rs described above respect drops:

Lemma 2.~~4~~ Let M be a premouse, and $n = n(M,\lambda)$, and F an extender over M with $crit(F) < \gamma_n(M,\lambda)$. Let

$$rs : Ult(M|\lambda, F) \longrightarrow Ult(M, F) = N$$

be the natural embedding, and

$$\eta_0 = i_F^{M|\lambda}(\lambda) = o(Ult(M|\lambda, F))$$

and

$$\eta_n = i_F^M(\lambda).$$

Then rs respects drops over (N, η_0, η_n).

Remark In particular, $Ult(M|\lambda, F) \trianglelefteq N$. That lemma 2.~~4~~ is the full statement of the condensation result we mentioned in defining $X(\mathcal{T}, F)$.

Proof Let for $k \le n$

$$A_k = A_k(M,\lambda),$$
$$\gamma_k = \gamma_k(M,\lambda).$$

Let $i^{A_k}_F : \mathrm{Ult}(A_k, F)$ be the canonical embedding, and

$$\eta_k = i^{A_k}_F(\eta_0).$$

We have by 2.2 that $n(\mathrm{Ult}(A_k, F)) = k$, and setting

$$B^k_i = i^{A_k}_F(A_i)$$

for $i < k$, and $B^k_k = \mathrm{Ult}(A_k, F)$,

$$B^k_i = A_i(B^k_k, \eta_k)$$

for all $i \leq k$. We shall show that for all $k \leq n$

(1) $n(N, \eta_k) = n$

(2) $A_i(N, \eta_k) = B^k_i$ for $i \leq k$

(3) $A_i(N, \eta_k) = A_i(N, \eta_{k+1})$ for all $i \geq k+2$

Let us write $B^k_i = A_i(N, \eta_k)$ for all $i \leq n$

Set $\gamma^k_i = \gamma_i(N, \eta_k)$. So for all k,

$B^k_0 = N | \langle \eta_k, 0 \rangle$ and $B^k_n = N$. We shall show

(2) If $k+2 \le n$, then for all $i \ge k+2$,

$$B^k_i = B^{k+1}_i .$$

Let

$$\psi_k : B^k_k \longrightarrow i^{A_{k+1}}_F (A_k) = B^{k+1}_k, \text{ for } k < n,$$

be the natural factor map. A_k and A_{k+1} have the same bounded subsets of γ_{k+1}. So ψ_k is the identity on $\sup i^{A_k}_F {}'' \gamma_{k+1} =$

$$\sup i^{A_{k+1}}_F {}'' \gamma_{k+1} = \gamma^{k+1}_{k+1} .$$ We shall show $\gamma^k_{k+1} = \gamma^{k+1}_{k+1}$, so in fact then by (2)

(3) $\gamma^k_i = \gamma^{k+1}_i$ for all $i \ge k+1$.

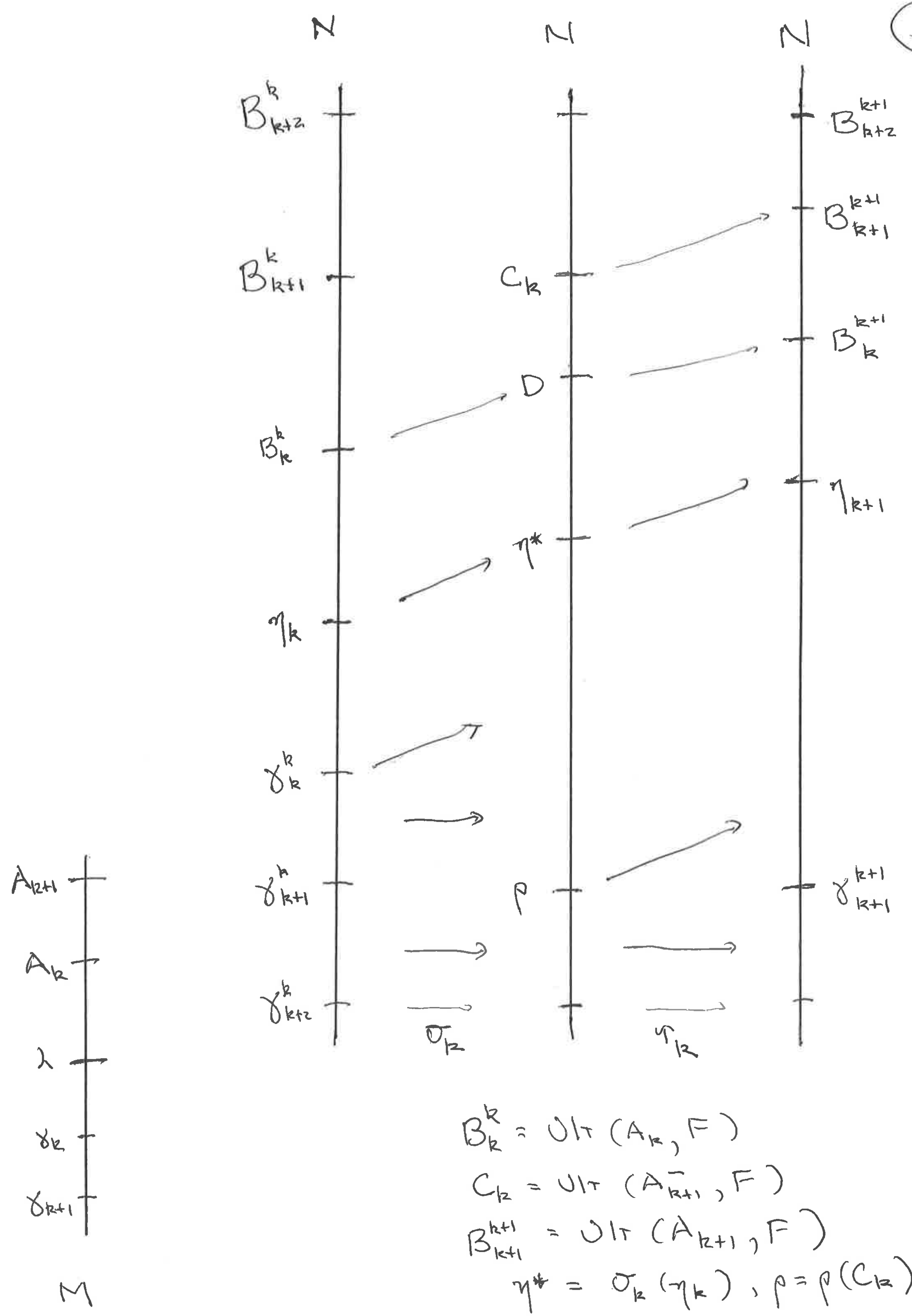
To do this, we need to factor ψ_k. Let

$$C_k = \mathrm{Ult}(A^-_{k+1}, F)$$

and

$$\sigma_k : B^k_k \longrightarrow D \trianglelefteq C_k$$
$$\tau_k : C_k \longrightarrow (B^{k+1}_{k+1})^- = i^{A_{k+1}}_F (A^-_{k+1})$$

be the natural maps.



$B^k_k = \mathrm{Ult}(A_k, F)$

$C_k = \mathrm{Ult}(A^-_{k+1}, F)$

$B^{k+1}_{k+1} = \mathrm{Ult}(A_{k+1}, F)$

$\eta^* = \sigma_k(\eta_k), \ \rho = \rho(C_k)$

Let's look first at $\tau_k \circ \psi_k$ is the natural embedding from $\mathrm{Ult}_e(A_{k+1}, F)$ into $\mathrm{Ult}_{e+1}(A_{k+1}, F)$, where $e = k(A_{k+1})$, with $\rho_{e+1}(A_{k+1}) = \gamma_{k+1}$.

The typical case ($e = 0$) is the natural embedding $\pi : \mathrm{Ult}_0(Q,F) \to \mathrm{Ult}_1(Q,F)$, where Q is 1-sound, and $\mathrm{crit}(F) < \rho_1(Q)$.

We have the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\ i\ } & \mathrm{Ult}_1(Q,F) \\ & \searrow_{j} & \uparrow \pi \\ & & \mathrm{Ult}_0(Q,F) \end{array}$$

The ultrapowers use the same bounded functions into $Q|\rho_1^Q$. So $i \restriction \rho_1^Q = j \restriction \rho_1^Q$, and $\pi \restriction \sup i''\rho_1^Q = \text{identity}$.

Also

$\rho_1(\mathrm{Ult}_0(Q,F)) = \sup j''\rho_1^Q = \sup i''\rho_1^Q = \rho_1(\mathrm{Ult}_1(Q,F))$.

Both ultrapowers are 1-sound, and

$$\pi(\rho_1(Ult_0(Q,F))) = \pi(j(\rho_1^Q))$$
$$= i(\rho_1^Q)$$
$$= \rho_1(Ult_1(Q,F)).$$

So we can apply Condensation to conclude $Ult_0(Q,F) \trianglelefteq Ult_1(Q,F)$. The two are equal iff j is continuous at $o(Q)$. If j is discontinuous at $o(Q)$, $Ult_0(Q,F) \triangleleft Ult_1(Q,F)|\rho^{+Ult_1(Q,F)}$, where $\rho = \rho_1(Ult_0(Q,F)) = \rho_1(Ult_1(Q,F))$.

Applying this with $Q = A_{k+1}$ and $e+1 = k(A_{k+1})$, so that $\rho_{e+1}(A_{k+1}) = \gamma_{k+1}$, and

$$\tau_k : Ult_e(A_{k+1}, F) \to Ult_{e+1}(A_{k+1}, F)$$

the natural embedding, we get

$$\gamma^{k+1}_{k+1} = \rho_{e+1}(B^{k+1}_{k+1})$$
$$= \rho_{e+1}(\cup \mid \tau_{e+1}(A_{k+1}, F))$$
$$= \rho_{e+1}(\cup \mid \tau_{e}(A_{k+1}, F))$$
$$= \rho_{e+1}(C_k) = \rho(C_k),$$

and

$$\eta_k \restriction \rho_{e+1}(C_k) = \text{identity}$$

and $i_F^{A_{k+1}} \restriction \gamma_{k+1} = i_F^{\bar{A}_{k+1}} \restriction \gamma_k$, and

$$\rho_{e+1}(C_k) = \sup i_F^{\bar{A}_{k+1}} {}'' \gamma_{k+1}.$$

Applying Lemma 2.2, we see that the η^* dropdown sequence of C_k is given by

$$A_i(C_k, \eta^*) = i_F^{\bar{A}_{k+1}}(A_i) \quad \text{for } i \leq k.$$

(And $A_i(C_k, \eta^*) = A_i(N, \eta^*)$ for $i \leq k$.)

So

$$\Upsilon_k\left(A_i\left(C_k, \eta^*\right)\right) = \lambda_F^{A_{k+1}}\left(A_i\right)$$

$$= A_i\left(B_{k+1}^{k+1}, \eta_{k+1}\right)$$

$$= B_i^{k+1}$$

for $i \le k$. It also follows that

$$A_{k+1}\left(N, \eta^*\right) = C_k^+$$

where C_k^+ is C_k with $k(C_k)$ changed from e to $e+1$. Also

$$\gamma_{k+1}\left(N, \eta^*\right) = \gamma_{k+1}^{k+1}.$$

For $i \ge k+2$, we then have

$$A_i\left(N, \eta^*\right) = B_i^{k+1} \qquad (i \ge k+2).$$

For example, $A_{k+2}(N, \eta^*)$ is the first level Q of N with projectum

$< \gamma_{k+1}(N, \eta^*) = \gamma^{k+1}_{k+1}$), and such

that $N | \gamma^{k+1}_{k+1} \trianglelefteq Q$. But that is

B^{k+1}_{k+2}, as well.

The argument involving σ_k is similar.

Set

$$D = i_F^{A^-_{k+1}}(A_k),$$

so that $D \trianglelefteq C_k$. $D = C_k$ is possible; this holds when $A^-_{k+1} = A_k$. We get

$$D = A_k(N, \eta^*)$$

from lemma 2.2. We have $\sigma_k \upharpoonright \gamma^k_k$ = identity, and

$$\gamma^k_k \le \gamma_k(N, \eta^*) \le \sigma_k(\gamma^k_k).$$

(We can't argue $\gamma_k(N, \eta^*) = \gamma_k^k$ as we did for τ_k and γ_{k+1}^{k+1}, because A_{k+1} may be more than one quantifier above A_k.) It follows that

$$B_{k+1}^k = C_k^+ ,$$

since each is the first level of N past γ_k^k with projectum $< \gamma_k^k$. From this we get

$$B_i^k = A_i(N, \eta^*) = B_i^{k+1}$$

for all $i \geq k+2$.

Now let us show that rs respects drops. We want for each $k \leq n$ an embedding $\pi_k : B_k^0 \longrightarrow B_k^n$. Using the notation above, notice that we showed for $k > 0$

$$B_k^0 = B_k^1 = \cdots = B_k^{k-1} = C_k^+ .$$

(The case $k=0$ is the same. We have $A_0 = A_1^-$ then, so $B_0^0 = (B_1^0)^-$. σ_0 is the identity) and $C_0 = B_0^0$, so $C_0^+ = B_1^0$.) Let

$$\pi_k = \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_{k+1} \circ \varphi_k ,$$

so π_k maps $C_k^+ = B_k^0$ into B_k^n, and $\pi_0 = rs$. But then

$$rs = \pi_k \circ (\sigma_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_0) .$$

Also $\gamma_k^0 = \gamma_k^1 = \cdots = \gamma_k^k$, and by the argument above,

$$\sigma_k \circ (\psi_{k-1} \circ \cdots \circ \psi_0) \restriction \gamma_k^0 = \text{identity}.$$

It follows that

$$\pi_k \restriction \gamma_k^0 = rs \restriction \gamma_k^0,$$

and since $\gamma_k^0 = \rho^-(B_k^0) = \rho^-(A_k(N, \eta_0))$, this is what we want.

Lemma 2.4 ☒

Definition 2.5 Let $\mathcal{T}$ and $\mathcal{U}$ be normal iteration trees on a premouse M. A <u>weak hull embedding</u> of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, \langle t^0_\beta \mid \beta < \text{lh}\,\mathcal{T}\rangle, \langle t^1_\beta, s_\beta \mid \beta+1 < \text{lh}\,\mathcal{T}\rangle, p\rangle$$

such that

(a) $u : \{\alpha \mid \alpha+1 < \text{lh}\,\mathcal{T}\} \to \{\alpha \mid \alpha+1 < \text{lh}\,\mathcal{U}\}$, $\alpha < \beta \Rightarrow u(\alpha) < u(\beta)$, and λ is a limit iff $u(\lambda)$ is a limit.

(b) $p : \text{Ext}(\mathcal{T}) \to \text{Ext}(\mathcal{U})$ is such that E is used before F on the same branch of $\mathcal{T}$ iff $p(E)$ is used before $p(F)$ on the same branch of $\mathcal{U}$. Thus p induces $\hat{p} : \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$ as in [1].

(c) Let $v : \text{lh}\,\mathcal{T} \to \text{lh}\,\mathcal{U}$ be given by

$$s^{\mathcal{U}}_{v(\beta)} = \hat{p}(s^{\tilde{\mathcal{T}}}_\beta).$$

Then

$$t^0_\beta : M^{\mathcal{T}}_\beta \longrightarrow M^{\mathcal{U}}_{v(\beta)}$$

is total and elementary. Moreover, for $\alpha <_T \beta$

$$t^0_\beta \circ \hat{\imath}^{\mathcal{T}}_{\alpha,\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),v(\beta)} \circ t^0_\alpha .$$

In particular, the two sides have the same domain.

(d) For $\alpha+1 < \mathrm{lh}\,\mathcal{T}$, $v(\alpha) \leq_U u(\alpha)$, and

$$t^1_\alpha = \imath^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ t^0_\alpha ,$$

and
$$p(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)} .$$

(e) Let $\eta = \mathrm{lh}(t^1_\alpha(E^{\mathcal{T}}_\alpha))$ and $\lambda = \mathrm{lh}(E^{\mathcal{U}}_{u(\alpha)})$.

(i) If $\eta = \lambda$, then rs_α = identity.

(ii) If $\eta < \lambda$, then rs_α respects drops over $(M^{\mathcal{U}}_{u(\alpha)}, \eta, \lambda)$, and $rs_\alpha(t^1_\alpha(E^{\mathcal{T}}_\alpha)) = E^{\mathcal{U}}_{u(\alpha)}$. We call this the <u>W-case</u> at α.

(iii) If $\lambda < \eta$, then rs_α respects drops over $(M^{\mathcal{U}}_{u(\alpha)}, \lambda, \eta)$, and $\mathrm{ran}(t^1_\alpha \restriction \mathrm{lh}\, E^{\mathcal{T}}_\alpha) \subseteq \mathrm{ran}\, rs_\alpha$ and $rs_\alpha(E^{\mathcal{U}}_{u(\alpha)}) = t^1_\alpha(E^{\mathcal{T}}_\alpha)$. We call this the <u>X-case</u> at α.

Let

$$t^2_\alpha = \begin{cases} t^1_\alpha \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha & \text{if } \eta = \lambda \\ rs_\alpha \circ t^1_\alpha \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha & \text{in the } W\text{-case} \\ rs_\alpha^{-1} \circ t^1_\alpha \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha & \text{in the } X\text{-case}. \end{cases}$$

(f) For $\beta < \alpha$, $rs_\alpha \upharpoonright \mathrm{lh}\, E^{\mathcal{U}}_{u(\beta)} = \text{identity}$.
(Thus t^2_α agrees with t^1_α on $\mathrm{lh}\, E^{\mathcal{U}}_{u(\beta)}$.)
Moreover, for $\gamma > \alpha$,

$$t^0_\gamma \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha + 1 = t^2_\alpha \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha + 1 .$$

(g) If $\beta = T\text{-pred}(\alpha+1)$, then
$U\text{-pred}(u(\alpha)+1) \in [v(\beta), u(\beta)]_U$, and
setting $\beta^* = U\text{-pred}(u(\alpha)+1)$

$$t^0_{\alpha+1}([a,f]^P_{E^{\mathcal{T}}_\alpha}) = [t^2_\alpha(a), i^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta(f)]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}},$$

where $P \trianglelefteq M^{\mathcal{T}}_\beta$ and $P^* \trianglelefteq M^{\mathcal{U}}_{\beta^*}$ are what $E^{\mathcal{T}}_\alpha$ and $E^{\mathcal{U}}_{u(\alpha)}$ are applied to.

Remark We may as well call the case (e)(i) both the X-case and the W-case.

One can show that the map we got from $\hat{\mathcal{J}}$ into $X(\mathcal{J}, F)$ is a a weak hull embedding in which the X-case occurs at all α. Similarly for the maps of X_ν into X_γ when $\nu <_u \gamma$. The embedding of $X(\mathcal{J}, \mathcal{U})$ into $W(\mathcal{J}, \mathcal{U})$ we produced is a weak hull embedding in which the W-case occurs at all α.

In the case of $X_\nu \to X_\gamma$, the weak hull embedding is given by: (using our notation from §2): $u = \varphi_{\nu,\gamma}$ and $p(E_\alpha^{X_\nu}) = E_{u(\alpha)}^{X_\gamma}$. $v(\eta) = \sup\{u(\xi)+1 \mid \xi+1 \leq_T \eta\}$, and $t_\eta^0 : M_\eta^{\mathcal{J}} \to M_{v(\eta)}^{\mathcal{U}}$ is the natural map. $t_\eta^1 = \tau_\eta^{\nu,\gamma}$. The map rs_η was described in §2.

In the case $X(\mathcal{I}, \mathcal{U}) \longrightarrow W(\mathcal{I}, \mathcal{U})$, the weak hull embedding is given by:

$u = v = \text{identity}$. $t^0_\alpha = t^1_\alpha = \psi_\alpha$,

where ψ_α is as described in §2.

rs_α was described in §2, sort of —

$rs_\alpha =$ "$\psi_\alpha(\overline{rs}_\alpha)$", where $\overline{rs}_\alpha$ is what was described in §2.

§3. ~~Very strong~~ Stronger hull condensation.

We outline a proof of

<u>Theorem 3.0</u> ~~Let~~ Assume AD^+, and let (P, Σ) be an lbr hod pair. Let $\mathcal{T}$ and $\mathcal{U}$ be normal trees on P, with $\mathcal{U}$ being by Σ, and suppose there is a weak hull embedding of $\mathcal{T}$ into $\mathcal{U}$; then $\mathcal{T}$ is by Σ.

<u>Corollary 3.1</u> Assume AD^+, and let (P, Σ) be an lbr hod pair. Suppose $\langle \mathcal{T}, \mathcal{U} \rangle$ is a stack by Σ; then $X(\mathcal{T}, \mathcal{U})$ is by Σ.

<u>Proof</u> $W(\mathcal{T}, \mathcal{U})$ is by Σ because Σ normalizes well. There is a weak hull embedding of $X(\mathcal{T}, \mathcal{U})$ into $W(\mathcal{T}, \mathcal{U})$, so by 3.0, $X(\mathcal{T}, \mathcal{U})$ is by Σ.

⊠

Proof of theorem 3.0 (Sketch)

If not, then we have a normal tree $\mathcal{T}$ on P by Σ, with distinct cofinal branches b and c, such that $c = \Sigma(\mathcal{T})$, ~~via~~ and a weak hull embedding of $\mathcal{T}^\frown b$ into some normal $\mathcal{U}$ by Σ.

As usual, we compare $\Phi(\mathcal{T}^\frown b)$ with $\Phi(\mathcal{T}^\frown c)$. We do this comparison as in the proof that UBH holds in hod mice, Theorem 6.3 of [1]. That is, we let N^* be a coarse Γ^*-Woodin model, where Γ^* is well beyond Σ, and such that $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are countable in N^*. We then simultaneously compare $\Phi(\mathcal{T}^\frown b)$ and $\Phi(\mathcal{T}^\frown c)$ with each $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, where $\mathbb{C}$ is the lpm-construction of N^*. This involves ~~moving the phalanxes up~~

moving tails of the phalanxes up at various stages, with associated stability declarations.

The strategy by which we iterate $\Phi(\mathcal{J}^\frown c)$ is Σ. The strategy for $\Phi(\mathcal{J}^\frown b)$ is obtained as the pullback of Σ under our weak hull embedding of $\mathcal{J}^\frown b$ into $\mathcal{U}$. Let us call this latter strategy Ψ. For each ν, l, we have the <u>$(\Psi, \Sigma, M^l_{\nu,l})$-coiteration</u> of $\Phi(\mathcal{J}^\frown b)$ with $\Phi(\mathcal{J}^\frown c)$, defined exactly as in the proof of 6.3 in [SIJ]. This is a pair $(\mathcal{W}_{\nu,l}, \mathcal{V}_{\nu,l})$ of pseudo trees according to Ψ and Σ respectively, obtained by iterating away least disagreements with $M^l_{\nu,l}$, and making stability declarations (which move up phalanxes) according to certain rules given in [1]. No strategy disagreements with $\Omega^{\mathbb{C}}_{\nu,l}$ show up when we do this, by arguments of [1].

Let $(R_{\nu,l}, \Phi_{\nu,l})$ be the last model of $\mathcal{W}_{\nu,l}$, and let $(S_{\nu,l}, \Lambda_{\nu,l})$ be the last model of $\mathcal{V}_{\nu,l}$. There is a ν,l corresponding to a completed comparison, that is, a ν,l such that either

(a) P-to-$R_{\nu,l}$ in $\mathcal{W}_{\nu,l}$ does not drop,

and $(R_{\nu,l}, \Phi_{\nu,l}) = (M^{\mathbb{C}}_{\nu,l}, \Omega^{\mathbb{C}}_{\nu,l})$

and $(M^{\mathbb{C}}_{\nu,l}, \Omega^{\mathbb{C}}_{\nu,l}) \trianglelefteq (S_{\nu,l}, \Lambda_{\nu,l})$

or

(b) P-to-$S_{\nu,l}$ in $\mathcal{V}_{\nu,l}$ does not drop,

and $(S_{\nu,l}, \Lambda_{\nu,l}) = (M^{\mathbb{C}}_{\nu,l}, \Omega^{\mathbb{C}}_{\nu,l})$,

and $(M^{\mathbb{C}}_{\nu,l}, \Omega^{\mathbb{C}}_{\nu,l}) \trianglelefteq (R_{\nu,l}, \Phi_{\nu,l})$.

Fix such a ν, ℓ, and write $(R, \Phi) = (R_{\nu,\ell}, \Phi_{\nu,\ell})$ and $(S, \Lambda) = (S_{\nu,\ell}, \Lambda_{\nu,\ell})$. Let $\mathcal{W} = \mathcal{W}_{\nu,\ell}$ and $\mathcal{U} = \mathcal{U}_{\nu,\ell}$. Let $\mathcal{W}^*$ be the lift of $\mathcal{W}$ to a tree on $\Phi(\mathcal{U})$ ~~that is used~~ via our weak hull embedding, and let R^* be the last model of $\mathcal{W}^*$.

<u>Claim</u> $\mathcal{W}^*$ is normal.

<u>Proof</u> This is one reason we have so much structure recorded in a weak hull embedding. We defer the proof to an appendix.

⊠

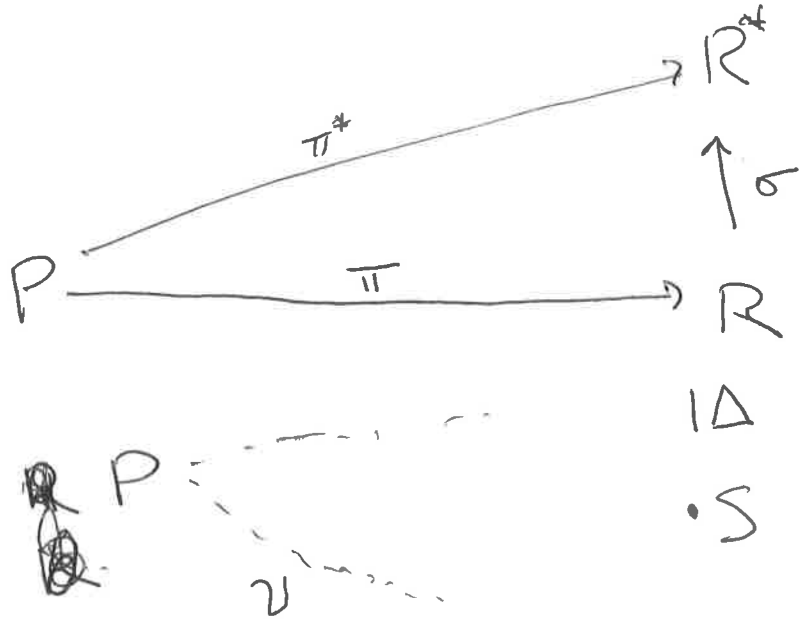
Let $\sigma : R \longrightarrow R^*$ be the map we get from our lifting process.

<u>Case 1</u> (a) above occurs; that is

$(R, \Phi) = (M^{c}_{\nu,\ell}, \Omega^{c}_{\nu,\ell}) \trianglelefteq (S, \Lambda)$,

and P-to-R in W does not drop.

Let $\pi : P \to R$ be the embedding from W. Let $\pi^* : P^* \to R^*$ be the embedding of W^*. We have the diagram



<u>Claim 2</u> $S = R$ (and thus $\Lambda = \Phi$).

<u>Proof</u> If not, then $(R, \Phi) \triangleleft (S, \Lambda)$.

Λ is a tail of Σ, i.e.

$\Lambda = \Sigma_{\mathcal{Q},s}$. We have a contradiction using ~~with~~ Dodd-Jensen if we can show that $\Phi^{\pi} = \Sigma$.

But $\Phi = (\Sigma_{W^*,R^*})^{\sigma}$ by definition. Moreover, $\Sigma = (\Sigma_{W^*,R^*})^{\pi^*}$ because Σ is pullback consistent, and W^* is normal and by Σ.

So

$$\begin{aligned}\Phi^{\pi} &= \left(\left(\Sigma_{W^*,R^*}\right)^{\sigma}\right)^{\pi} \\ &= \left(\Sigma_{W^*,R^*}\right)^{\sigma\circ\pi} \\ &= \left(\Sigma_{W^*,R^*}\right)^{\pi^*} \\ &= \Sigma,\end{aligned}$$

as desired. Dodd-Jensen now gives us a contradiction.

$\boxtimes$

Claim 3 The branch P-to-S of $\mathcal{V}$ does not drop.

Proof The proof in claim 2 works. ☒

Let $\psi : P \to S$ be the embedding of $\mathcal{V}$.

Claim 4 $\pi = \psi$.

Prf Suppose $\pi(\eta) < \psi(\eta)$. Then since $\Phi^{\pi} = \Sigma$, we contradict Dodd-Jensen, for ψ is an iteration map by Σ.

Suppose $\psi(\eta) < \pi(\eta)$. Then $\sigma(\psi(\eta)) < \sigma(\pi(\eta)) = \pi^*(\eta)$. But π^* is an iteration map by Σ, and

$$(\Sigma_{W^*, R^*})^{\sigma \circ \psi} = \Phi^{\psi}$$

$$= \Lambda^{\psi}$$

$$= \Sigma,$$

because Σ is pullback consistent. So again, we contradict Dodd-Jensen. ☒

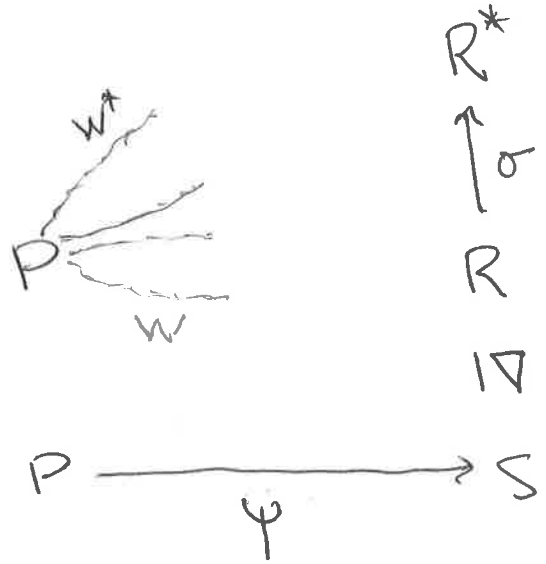
But now, using the P-hull and definability properties, we see that the branches P-to-R of $\mathcal{W}$ and P-to-S of $\mathcal{V}$ use the same extenders. It follows from the construction of $\mathcal{W}$ and $\mathcal{V}$ that R and S are unstable nodes, and therefore they cannot be the last models of their trees, contradiction.

<u>Case 2</u> (b) above occurs; that is,

$(S, \Lambda) = (M^{e}_{\nu,l}, \Omega^{e}_{\nu,l}) \trianglelefteq (R, \Phi)$,

and P-to-S in $\mathcal{V}$ does not drop.

Let $\psi : P \to S$ be the embedding given by $\mathcal{V}$. We have the diagram



<u>Claim 5</u> $R = S$, and P-to-R in W does not drop.

<u>Proof</u> $\sigma \circ \psi : P \longrightarrow R^*$, and R^* is a Σ-iterate of P, and

$$\left(\Sigma_{W^*, R^*}\right)^{\sigma \circ \psi} = \left(\Phi_S\right)^{\psi}$$
$$= \Lambda^{\psi}$$
$$= \Sigma.$$

So Dodd-Jensen tells us that P-to-R^* does not drop in W^*, and $\sigma(S) = R^*$. This yields the claim.

$\boxtimes$

The rest of the proof in case 2 is the same as in Case I.

So in either case, we have a contradiction.

This completes our proof of 3.0.

$\boxtimes$

Infinite stacks

Let $\langle \mathcal{U}_i \mid i<\omega \rangle$ be an infinite stack of normal trees on $M = M_0^{\mathcal{U}_0}$.

Setting

$$\mathcal{W}_0 = \mathcal{U}_0$$

and

$$\mathcal{W}_{n+1} = \mathcal{W}(\mathcal{W}_n, \pi\mathcal{U}_{n+1}),$$ where π : last model of $\mathcal{U} \to$ last model of $\mathcal{W}_n$ is the natural embedding,

we can let

$$W(\langle \mathcal{U}_i \mid i<\omega\rangle) = \lim_n W_n ,$$

where the limit is taken in the natural way. $W(\langle \mathcal{U}_i \mid i<\omega\rangle)$ is the embedding normalization of $\langle \mathcal{U}_i \mid i<\omega\rangle$.

Caution: It should be made part of the definition of lbr hod pair (P, Σ) that if $\langle \mathcal{U}_i \mid i<\omega\rangle$ is by Σ, then $W(\langle \mathcal{U}_i \mid i<\omega\rangle)$ is by Σ. lpm constructions do produce pairs $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ with this property.

In a similar fashion, one can define $X(\langle \mathcal{U}_i \mid i\in\omega\rangle)$ for $\langle \mathcal{U}_i \mid i\in\omega\rangle$ an infinite stack of normal trees on M. (In taking the limit of the X_n's, it is important that exit extenders only change finitely often.) The last model of $X(\vec{\mathcal{U}})$ is equal to the ~~di~~ last model of $\vec{\mathcal{U}}$, i.e. the direct limit of the last models of the $\mathcal{U}_i$. There is a weak hull embedding of $X(\vec{\mathcal{U}})$ into $W(\vec{\mathcal{U}})$. There is a weak hull embedding of $\mathcal{U}_0$ into $X(\vec{\mathcal{U}})$.

Theorem 3.2 Assume AD^+, and let (P, Σ) be an lbr hod pair in the sense cautioned above. ~~that~~ Then Σ fully normalizes well for infinite stacks; i.e. if $\langle \mathcal{U}_i \mid i < \omega \rangle$ is by Σ, then $X(\langle \mathcal{U}_i \mid i < \omega \rangle)$ is by Σ.

Proof $X(\vec{\mathcal{U}})$ is a weak hull of $W(\vec{\mathcal{U}})$.

$\boxtimes$

§4. Sketches pseudo-completed.

Proof of theorem 12

We can add a bit more detail to our sketch in §0.

We are given an lbr hod pair (P, Σ) such that Σ embedding-normalizes well for infinite stacks. Let G be V-generic over $Col(\omega, \mathbb{R})$. Then in $V[G]$ we have an infinite stack $\langle \mathcal{U}_i \mid i \in \omega \rangle$ such that each $\mathcal{U}_i \in V$, and $\langle \mathcal{U}_i \mid i \in \omega \rangle$ is by Σ, and

$$M_\infty(P, \Sigma) = \text{last model of } \langle \mathcal{U}_i \mid i \in \omega \rangle.$$

Let us assume for purposes of the sketch that (P, Σ) extends to an lbr hod pair in $V[G]$. Then we have

$$M_\infty(P, \Sigma) = \text{last model of } X(\langle \mathcal{U}_i \mid i \in \omega \rangle),$$

and $X = X(\langle \mathcal{U}_i \mid i \in \omega \rangle)$ is a normal tree by Σ^*.

But then X is determined uniquely in V by being a normal tree ~~with~~ on P with last model $M_\infty(P, \Sigma)$, and having all its countable elementary submodels by Σ. So $X \in V$, our AD^+ world. Note also X is essentially a set of ordinals, and $|X| = |M_\infty(P, \Sigma)|$.

~~**Claim** In V: $\mathcal{T}$ is by Σ^{rel} iff there is a weak hull embedding of $\mathcal{T}$ into X~~

Claim In V, let $\mathcal{T}$ be a countable normal tree on P with last model Q, and such that P-to-Q does not drop. Then $\mathcal{T}$ is by Σ iff there is a weak hull embedding of $\mathcal{T}$ into X.

Proof $\Leftarrow$ comes from theorem 3.0.

Assume now $\mathcal{T}$ is by Σ. By the argument above, we have a normal $\mathcal{Y}$ on Q with last model $M_\infty(P,\Sigma) = M_\infty(Q,\Sigma_{\mathcal{T},Q})$, and such that all countable submodels of $\mathcal{Y}$ are by Σ. But then

$$X(\mathcal{T},\mathcal{Y}) = X,$$

so there is a weak hull embedding of $\mathcal{T}$ into X.

$\boxtimes$

But note $\mathcal{S}$ is by Σ^{tol} iff $\exists\mathcal{T}$ ($\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T}$ is by Σ and p-to-last model of $\mathcal{T}$ does not drop).

Combining this with the claim, we see that Σ^{rel} is $|M_\infty(P,\Sigma)|$-Suslin. $\Sigma^{rel} = p[T]$, where for each Δ, T_Δ searches for a weak hull embedding of some $\tilde{\Delta} \supseteq \Delta$ into X.

$\boxtimes$

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