§11. The termination proof.

We suppose toward contradiction that $\hat{W}_\nu$ and $\hat{\mathcal{V}}_\nu$, etc., are defined for all $\nu < \omega_1$. We then use reflection arguments to show that $\hat{W}_{\omega_1,\eta}$ and $\hat{\mathcal{V}}_{\omega_1,\eta}$ are defined for all $\eta < \omega_2$. This contradicts Lemma 10.6.

Thus there is some largest $\nu < \omega_1$ such that $\hat{W}_\nu$ and $\hat{\mathcal{V}}_\nu$ are defined. This implies that letting $R = P^*_{z(\nu)}$ and $S = Q^*_{z(\nu)}$ be the last models of $W^*_{z(\nu)}$ and $\mathcal{V}^*_{z(\nu)}$, ~~and~~ ~~$\psi = \Sigma_R$ and Λ~~ then $(R, \Sigma_R) \trianglelefteq (S, \Lambda_S)$ and the branch to R in $W^*_{z(\nu)}$ does not drop, or vice-versa. So we have proved Theorem I.

~~So let us assume $\hat{W}_\nu$ and $\hat{\mathcal{V}}_\nu$ exist for all $\nu < \omega_1$. Let us consider the completion stages $\hat{W}_{\omega_1,\eta}$ and $\hat{\mathcal{V}}_{\omega_1,\eta}$. We need to show that if $(\hat{W}_{\omega_1,\eta}, \hat{\mathcal{V}}_{\omega_1,\eta})$ is defined, then Case 2 applies,~~

Let us assume that $\hat{W}_\nu$ and $\hat{\nu}_\nu$ exist for all $\nu < \omega_1$, so that $z(\omega_1)$ exists. By 10.6, $z(\omega_1) < \omega_2$. We shall obtain a contradiction by a reflection argument.

Let

$$\pi : H \to V$$

where H is countable transitive, everything relevant is in $\mathrm{ran}(\pi)$, and let

$$\nu = \mathrm{crit}(\pi) = \pi^{-1}(\omega_1).$$

We have $\pi^{-1}(\langle(\hat{W}_\alpha, \hat{\nu}_\alpha) \mid \alpha < \omega_1\rangle) = \langle(\hat{W}_\alpha, \hat{\nu}_\alpha) \mid \alpha < \nu\rangle$, and moreover, the definitions of $\hat{W}_\nu$, $\hat{\nu}_\nu$, $\hat{W}_{\omega_1}$, and $\hat{\nu}_{\omega_1}$ are absolute modulo Σ and Λ, and $\pi^{-1}(\Sigma) \subseteq \Sigma$ and $\pi^{-1}(\Lambda) \subseteq \Lambda$. So

$$(\hat{W}_\nu, \hat{\nu}_\nu) = \pi^{-1}((\hat{W}_{\omega_1}, \hat{\nu}_{\omega_1})),$$

and so on. In particular

$$\pi(z(\nu)) = z(\omega_1).$$

Notice that $\omega_1 = z_0(\omega_1)$, so $\nu = z_0(\nu)$.

Claim 1. $z(\omega_1) > \omega_1$

Proof We have $\omega_1 = h(W^-_{\omega_1}) = h(\mathcal{V}^-_{\omega_1})$

(cf. the claim in the proof of 10.6)

and

$$W^-_\nu = \pi^{-1}(W_{\omega_1}) = W^-_{\omega_1} \wedge \nu$$

and

$$\mathcal{V}^-_\nu = \pi^{-1}(\mathcal{V}_{\omega_1}) = \mathcal{V}^-_\omega \wedge \nu .$$

Let

$$b = \Sigma(W^-_{\omega_1})$$

and

$$c = \Lambda(\mathcal{V}^-_{\omega_1}).$$

So $b = b_0^{\omega_1}$ and $c = c_0^{\omega_1}$ in the notation of §10. We will show that b and c justify (ν, d), where in fact d is a common tail of e_b and e_c. This then implies that $z(\omega_1) > \omega_1$.

Note that

~~and~~ since b and c are clubs in ω_1,

$$\nu \in b \cap c.$$

This gives that $M_\nu^{W_{\omega_1}^-} = M_\nu^{W_\nu}$ and $M_\nu^{\mathcal{V}_{\omega_1}^-} = M_\nu^{\mathcal{V}_\nu}$, moreover all these models have the same $P(\nu)$.

But then as usual

$$(*) \qquad i_{\nu,b}^{W_{\omega_1}^-} \restriction P(\nu)^{M_\nu^{W_\nu}} = \pi \restriction P(\nu)^{M_\nu^{W_\nu}}$$

$$= i_{\nu,c}^{\mathcal{V}_{\omega_1}^-} \restriction P(\nu)^{M_\nu^{\mathcal{V}_\nu}}.$$

For $n < \omega_1$, let

$d(n) =$ nth extender used in $b - \nu$.

~~$d(n)$ is the first whole initial segment of the factor embedding σ: $Ult(M_\nu^{W_\nu}, d \restriction n) \to M$~~,

By induction, letting $M_{\eta_n}^{W_{\omega_1}^-} = Ult(M_\nu^{W_\nu}, d \restriction n)$, we get that $M_{\eta_n}^{W_{\omega_1}^-} = Ult(M_\nu^{W_\nu}, E_\pi \restriction \gamma_n)$ where $\gamma_n = \sup\{\lambda(d(i)) \mid i < n\}$, and $i_{\eta_n, b}^{W_{\omega_1}^-}$ is the factor embedding from $Ult(M_\nu^{W_\nu}, E_\pi \restriction \gamma_n)$ to $Ult(M_\nu^{\mathcal{V}_\nu}, E_\pi)$. Thus π determines d, as the sequence of missing whole initial segments of its factors.

Of course the same proof works on the ν-side, and thus

$$d(n) = n^{\text{th}} \text{ extender used in } c\text{-}\nu.$$

We can ~~dos~~ now see that b and c both justify (ν, d). First, $\beta_d \le \nu$ because $\mathrm{crit}(d(0)) = \mathrm{crit}(\pi) = \nu$, and no W_i for $i < \nu$ agrees with W_ν up to ν, so $\beta_d = \nu$.

We have

$$\hat{p}_d\left(e^{W_\nu}_{b\cap\nu}\right) = e^{W_\nu}_{b\cap\nu} \circledast d$$

$$= e^{W_\nu}_{b\cap\nu} {}^\frown d$$

$$= e^{W^-_{\omega_1}}_{b},$$

and similarly on the ~~$\mathcal{A}$~~ ν-side,

$$\hat{p}_d\left(e^{\nu_\nu}_{c\cap\nu}\right) = e^{\nu^-_{\omega_1}}_{c}.$$

Finally, d is cofinal in $\left(\hat{W}^{\to}_{\omega_1,0}\right)^{ext}$ and $\left(\hat{\nu}^{\to}_{\omega_1,0}\right)^{ext}$. Moreover, each $d \restriction n$ for $n<\omega$, is an A-branch, so $\in W_{\omega_1}^{\nu, ext}$ by the proof of 6.2.10. This is all we need to check.

It finishes ~~Case XA~~ Claim 7; i.e. Case 2 applied in the completion step from $(\omega_1, 0)$ to $(\omega_1, 1)$.

Remark We need to see that for $n < \omega_1$, $d_i^n \in (\hat{W}^\nu)^{ext}$; ~~that~~ and that d_i^n is an A-branch. We showed this in 6.2.10 for the ~~branch~~ $\hat{W}_\alpha^\nu$ with α finite. The general form of 6.2.10 reads the same:

Lemma 6.2.10* Let i be a limit ordinal, $\eta = \sup\{\xi + 1 \mid E_\xi^{W_i} \in \text{ran}(d_{W_i}^0)\}$, and suppose s is a tail of $e_\eta^{W_i}$ such that ~~a~~ every $E \in \text{ran}(s)$ is an A-extender; then s is a tail of $d_{W_i}^0$. Similarly on the ν-side.

This went with 6.2.9, which connected $d_{W_i}^0$ with $e_\eta^{W_i}$ in general, and which also should be part of our general induction:

Lemma 6.2.9* Let $\eta = \sup\{\xi + 1 \mid$ ~~...~~ $E_\xi^{W_i} \in \text{ran}(d_{W_i}^0)\}$; then

(1) a tail of $d_{W_i}^0$ fits cofinally into $e_\eta^{W_i}$, and

(2) if s is a branch of $\hat{W}^{0,ext}$ such that a tail of s fits cofinally into $e_\eta^{W_i}$, then $s = d_{W_i}^0$.

6.2.9* and 6.2.10* apply to our situation.

For let $n < \omega_1$, and let

$$\iota = \sup\{\theta_E + 1 \mid E \in \mathrm{ran}(d \restriction n)\}$$

and

$$\eta = \sup\{\xi + 1 \mid E_\xi^{W_{\omega_1}} \in \mathrm{ran}(d \restriction n)\}.$$

Then $W_\iota \restriction \eta+1 = W_{\omega_1} \restriction \eta+1$. ~~By 166.~~ If n is a limit ordinal, then so is ι, and by 6.2.10* we get $d \restriction n$ is a tail of $d^{w_\iota^0}$. Similarly, $d \restriction n$ is a tail of $d^{v_\iota^0}$.

This implies $d \restriction n = d^{w_\iota^\to} = d^{v_\iota^\to}$ when $\iota = \sup\{\theta_E + 1 \mid E \in \mathrm{ran}(d \restriction n)\}$.

Claim 1. $\boxtimes$

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So $\omega_1 = z_0(\omega_1) < z(\omega_1)$.

<u>Claim 2</u> If $z_\eta(\omega_1) < z(\omega_1)$, then $z_{\eta+1}(\omega_1) \leq z(\omega_1)$ and $z_{\eta+1}(\omega_1) = z_\eta(\omega_1) + \omega_1$.

<u>Proof</u> Since $z_\eta(\omega_1) < z(\omega_1)$, we have that $b_\eta^{\omega_1}$ and $c_\eta^{\omega_1}$ justify (γ, d), for some $\gamma < \omega_1$ and some cofinal branch d of $\hat{W}^\gamma_{\omega_1,\eta}$ and $\hat{\mathcal{V}}^\gamma_{\omega_1,\eta}$. Then for all $\xi \geq \gamma$ we have d^ξ such that they justify (ξ, d^ξ). For $\gamma \leq \xi < \delta < \omega_1$, $\vec{\mathcal{J}}^\xi_{d^\xi}$ is a proper initial segment of $\vec{\mathcal{J}}^\delta_{d^\delta}$, moreover

$$[z_\eta(\omega_1), z_{\eta+1}(\omega_1)) = \bigcup_{\gamma < \xi < \omega_1} \vec{\mathcal{J}}^\xi_{d^\xi}.$$

The $\vec{\mathcal{J}}^\xi_{d^\xi}$'s are all countable, so their union has order type ω_1

$\boxtimes$

We now show by induction on $\eta < \omega_1$ that $z_\eta(\vartheta) \leq z(\vartheta)$. This implies that $\omega_1 \leq z(\vartheta)$, contrary to Lemma 10.6.

We need an additional induction hypothesis that connects branches in $\hat{W}^{\vartheta}_{\omega_1,\eta}$ and $\hat{\mathcal{V}}^{\vartheta}_{\omega_1,\eta}$ to π.

Let us make a very general definition. We shall apply it with $\sigma = \pi$ and $N = H$.

<u>Definition</u> Let $\sigma : N \to V$ where N is transitive, and let P be a countable premouse in N and $\mathcal{T} \in N$ an iteration tree on P s.t. $\mathcal{T} \in N$.

Let $\mathcal{U} = \sigma(\mathcal{T})$. Then

$$\Gamma_{\sigma,\mathcal{T}} : \mathcal{T} \to \mathcal{U}$$

is the tree embedding generated by π. That is, for $\Gamma = \Gamma_{\sigma,\mathcal{T}}$:

(a) $u^\Gamma = \sigma \restriction lh(\mathcal{T})$,

(b) $p^\Gamma = \sigma \restriction Ext(\mathcal{T})$,

and (c) for all $\alpha < lh(\mathcal{T})$, $t_\alpha^\Gamma = \pi \restriction M_\alpha^{\mathcal{T}}$.

$\Gamma_{\sigma,\mathcal{T}}$ is a very nice tree embedding. For example, u^Γ preserves tree order and tree-predecessor.

Note v^Γ and s_α^Γ are determined by $v^\Gamma(\eta+1) = u^\Gamma(\eta+1)$ and $v^\Gamma(\lambda) = \sup_{\eta<\lambda} v^\Gamma(\eta)$, and $s^\sigma_{\eta+1} = t^\sigma_{\eta+1}$

and for λ a limit,

$$s^\sigma_\lambda(i^{\mathcal{T}}_{\eta\lambda}(x)) = i^{\mathcal{U}}_{u(\eta),v(\lambda)}(t_\eta(x))$$
$$= i^{\mathcal{U}}_{v(\eta),v(\lambda)}(s_\eta(x))$$

(even if $v(\eta) \neq u(\eta)$).

We call $\Gamma_{\sigma,\mathcal{T}}$ the <u>tree embedding generated by σ</u> from $\mathcal{T}$ to $\pi(\mathcal{T})$.

Our induction hypothesis is

$(\dagger)_\eta$ $\quad z_\eta(\nu) \leq z(\nu)$, and for all $\alpha < z_\eta(\nu)$ s.t. $\nu \leq \alpha$

(a) $\alpha <^\nu_A \pi(\alpha)$, ~~and~~

(b) $\Phi^\nu_{\alpha,\pi(\alpha)} = \Gamma_{\pi, W_\alpha}$, and

(c) $\Psi^\nu_{\alpha,\pi(\alpha)} = \Gamma_{\pi,\mathcal{V}_\alpha}$.

Notice that $\alpha <^\nu_A \pi(\alpha) \Rightarrow \nu \leq \alpha$ by our definitions.

Also, $\nu \leq \alpha < z(\nu) \Longrightarrow W_\alpha = W^*_\alpha$ and $\mathcal{V}_\alpha = \mathcal{V}^t_\alpha$.

Remark In general, $\Phi^{\gamma}_{\alpha,\beta}$ is only a weak tree embedding, moreover u^{Φ} need not preserve tree order, tree predecessor, or even successor ordinals. So the case $\alpha <^{\mathcal{U}} \pi(\alpha)$ has special regularities.

Proposition 14.0 Let $\sigma : N \to V$ and $\mathcal{T}$ be such that $\sigma(\mathcal{T}) = \mathcal{U}$. Let $\Gamma = \Gamma_{\sigma,\mathcal{T}}$ be defined, with $u = u^{\Gamma}$, $v = v^{\Gamma}$, $t_\alpha = t_\alpha^{\Gamma}$, $s_\alpha = s_\alpha^{\Gamma}$, where $\alpha < lh(\mathcal{T})$. Then

$$M^{\mathcal{U}}_{v(\alpha)} = Ult(M^{\mathcal{T}}_\alpha, E_\sigma \upharpoonright \varepsilon^{\mathcal{U}}_{v(\alpha)}),$$

(where $\varepsilon^{\mathcal{S}}_{\xi} = \sup\{\lambda_E \mid E \in ran(e^{\mathcal{S}}_{\xi})\}$), and s_α^{Γ} is the canonical ultrapower embedding, and $i^{\mathcal{U}}_{v(\alpha),u(\alpha)}$ is the canonical factor embedding from $Ult(M^{\mathcal{T}}_\alpha, E_\sigma \upharpoonright \varepsilon^{\mathcal{U}}_{v(\alpha)})$ into $M^{\mathcal{U}}_{u(\alpha)} = \sigma(M^{\mathcal{T}}_\alpha)$.

Proof We can assume α is a limit ordinal, and $v(\alpha) < u(\alpha)$, as otherwise $u(\alpha) = v(\alpha)$, $s_\alpha^{\Gamma} = t_\alpha^{\Gamma}$, and

$$M^{\mathcal{U}}_{v(\alpha)} = M^{\mathcal{U}}_{u(\alpha)} = \pi(M^{\mathcal{T}}_\alpha) = Ult(M^{\mathcal{T}}_\alpha, E_\sigma \upharpoonright \varepsilon^{\mathcal{U}}_{u(\alpha)}),$$

so there is nothing to show. So let α be a limit, and $v(\alpha) < u(\alpha)$.

Proof By definition, $\sigma = t_\alpha$. Let $i = i^{\mathcal{U}}_{v(\alpha),u(\alpha)}$. We have the diagram

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$$\begin{tikzcd}
 & M^{\mathcal{U}}_{u(\alpha)} & \\
M^{\mathcal{T}}_{\alpha} \arrow[ur, "t_\alpha = \sigma"] \arrow[r, "s_\alpha"] & M^{\mathcal{U}}_{v(\alpha)} \arrow[u, "i"'] & R = \mathrm{Ult}(M^{\mathcal{T}}_{\alpha}, E_\sigma \restriction \varepsilon^{\mathcal{U}}_{v(\alpha)}) \arrow[ul, "k"'] \\
M^{\mathcal{T}}_{\eta} \arrow[u, dashed] \arrow[r, "t_\eta"] & M^{\mathcal{U}}_{u(\eta)} \arrow[u, dashed] & \\
P \arrow[u, dashed, no head] & P \arrow[u, dashed, no head] &
\end{tikzcd}$$

Here k is the factor embedding. Note $k \restriction \varepsilon^{\mathcal{U}}_{v(\alpha)} = i \restriction \varepsilon^{\mathcal{U}}_{v(\alpha)} = \mathrm{id}$. ~~Thus we have $M^{\mathcal{U}}_{v(\alpha)} = R$~~

We claim that $\mathrm{ran}(i) = \mathrm{ran}(k)$.

$\mathrm{ran}(i) \subseteq \mathrm{ran}(k)$: Let $x = i(y)$. Pick $\eta < \alpha$ and z s.t. $y = i^{\mathcal{U}}_{u(\eta),v(\alpha)}(z)$. We have $g \in M^{\mathcal{T}}_{\eta}$ and $b \subseteq \varepsilon^{\mathcal{U}}_{u(\eta)}$ finite s.t.

$$z = t_\eta(g)(b)$$

Since this diagram commutes

$$\sigma(i^{\mathcal{T}}_{\eta,\alpha}(g)) = i^{\mathcal{U}}_{u(\eta),u(\alpha)}(t_\eta(g))$$

so
$$\sigma(i^{\mathcal{T}}_{\eta,\alpha}(g))(b) = i^{\mathcal{U}}_{u(\eta),u(\alpha)}(t_\eta(g))(b)$$
$$= i^{\mathcal{U}}_{u(\eta),u(\alpha)}(t_\eta(g)(b))$$
$$= x$$

But $k(b) = b$, so we get that $x \in \operatorname{ran}(k)$. because $\operatorname{ran}(\sigma) \subseteq \operatorname{ran}(k)$.

<u>$\operatorname{ran}(k) \subseteq \operatorname{ran}(i)$</u> Let $x \in \operatorname{ran}(k)$, so that $x = \sigma(f)(a)$ for $a \in [\mathcal{E}^{\mathcal{U}}_{v(\alpha)}]^{<\omega}$, and $f \in M^{\mathcal{T}}_\alpha$. Let $\eta < \alpha$ and g be such that $f = i^{\mathcal{T}}_{\eta,\alpha}(g)$. Then

$$x = \sigma(f)(a) = t_\alpha(f)(a)$$
$$= i_{u(\eta),u(\alpha)} \circ t_\eta(g)\ (a)$$
$$= i \circ i_{u(\eta),v(\alpha)} \circ t_\eta(g)\ (a)$$
$$= i(i_{u(\eta),v(\alpha)} \circ t_\eta(g)\ (a)),$$

since $i(a) = a$. Thus $x \in \operatorname{ran}(i)$, as desired.

Since $\mathrm{ran}(i) = \mathrm{ran}(k)$, $R = M^{\mathcal{U}}_{\nu(\alpha)}$,

and $i = k$. It follows that s_α is

the ultrapower embedding from $M^{\mathcal{E}}_\alpha$ to R.

(Let $j : M^{\mathcal{E}}_\alpha \to R$ be the ultrapower emb.

Then

$$k \circ j = \sigma$$
$$= t_\alpha$$
$$= i \circ s_\alpha .$$

But $k = i$, so $j = s_\alpha$.)

Prop. 14.0 $\boxtimes$

<u>Claim 3</u> If λ is a limit ordinal and $(\dagger)_\eta$ for all $\eta < \lambda$, then $(\dagger)_\lambda$.

<u>Proof</u> Trivial. $\boxtimes$

<u>Claim 3a</u> $(\dagger)_0$ holds

<u>Proof</u> This just requires $z_0(\omega_1) < z(\omega_1)$, which was Claim I. $\boxtimes$

By Claims 1–3, we will be done when we show

<u>Claim 4</u> $(\dagger)_\eta \Rightarrow (\dagger)_{\eta+1}$.

The proof of Claim 4 breaks into three cases: η is a successor ordinal, $cf^H(\eta) = \omega$, and $cf^H(\eta) = \partial$.
In the first and third cases, $cf(z_{\pi(\eta)}(\omega_1)) = \omega_1$, and the proofs are close to one another. In the second case, $cf(z_{\pi(\eta)}(\omega_1)) = \omega$, and the proof is slightly different.

<u>Proof of Claim 4:</u>

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~~Claim 4 If $(\dagger)_\eta$ then $(\dagger)_{\eta+1}$~~

~~Proof~~

<u>Subclaim 4.1</u> If η is a successor ordinal and $(\dagger)_\eta$ holds, then $(\dagger)_{\eta+1}$ holds.

<u>Proof</u> Let $\gamma+1=\eta$. We showed in Claim ... that $z_{\pi(\eta)}(\omega_1) = z_{\pi(\gamma)}(\omega_1)+\omega_1$. Thus $z_\eta(\vartheta) = z_\gamma(\vartheta)+\vartheta$. Let

$$\mathcal{W}^- = \mathcal{W}^-_{z_{\pi(\eta)}(\omega_1)}$$

and

$$\mathcal{V}^- = \mathcal{V}^-_{z_{\pi(\eta)}(\omega_1)}$$

be the common part trees. (So $\mathcal{S} \triangleleft \mathcal{W}^-$ iff $\mathcal{S} \triangleleft \mathcal{W}_\alpha$ for all sufficiently large $\alpha < z_{\pi(\eta)}(\omega_1)$.)

Let

$$b = \Sigma(\mathcal{W}^-)$$

and

$$c = \Lambda(\mathcal{V}^-).$$

Let

$$\langle \bar{\mathcal{W}}^-, \bar{\mathcal{V}}^-, \bar{b}, \bar{c}\rangle = \pi^{-1}(\langle \mathcal{W}^-, \mathcal{V}^-, b, c\rangle).$$

Thus $\bar{\mathcal{W}}^- = \mathcal{W}^-_{z_\eta(\vartheta)}$ = common part tree on $\mathcal{W}$-side at $z_\eta(\vartheta)$, $\bar{b} = \Sigma(\bar{\mathcal{W}}^-)$, etc.

Let
$$\theta = lh(\mathcal{W}^-) = lh(\mathcal{T}^-)$$
$$= \sup(b) = \sup(c)$$
and $\tilde{\theta} = \pi^{-1}(\theta)$.

Subclaim 4.1a $\quad cf(\theta) = \omega_1$.

Proof Once $\mathcal{W}_\alpha \restriction \xi$ has stabilized as $\alpha \to z_{\pi(\gamma)}(\omega_1)$, $\mathcal{W}_\alpha \restriction \xi+1$ can only change value finitely often. (Never if ξ is a limit ordinal. If $\xi = \gamma+1$, then $lh(E_\gamma^{\mathcal{W}_\alpha})$ goes down at each change.)

$\boxtimes$

Remark If $\mu = lh(\mathcal{W}^-_{z_{\pi(\gamma)}(\omega_1)})$, then
$$\mathcal{W}_{z_{\pi(\gamma)}(\omega_1)} \restriction \mu+1 = \mathcal{W}_\alpha \restriction \mu+1 \quad \text{for all } \alpha \in [z_{\pi(\gamma)}(\omega_1), z_{\pi(\eta)}(\omega_1)].$$
A little more work shows $\theta = \mu + \omega_1$. (We don't need to use this.) The reason is that for $\alpha \in [z_{\pi(\gamma)}(\omega_1), z_{\pi(\eta)}(\omega_1))$, $\mathcal{W}_\alpha = X(\mathcal{W}_i, d)$ for some d justified by the main branch of $\mathcal{W}_{z_{\pi(\gamma)}(\omega_1)} \restriction \mu+1$. Since $\mathcal{W}_i$ is countable and d fits into that main branch, $lh(\mathcal{W}_\alpha) = \mu + \xi$ for some $\xi < \omega_1$. Thus $\theta \leq \mu + \omega_1$, so $\theta = \mu + \omega_1$.

More notation: let

$$\gamma = \sup \pi''\bar{\theta}$$

and

$$\rho = z_{\pi(\eta)}(\omega_1)$$

and $\cancel{\langle \bar{\xi}, \bar{\rho} \rangle = \pi^{-1}(\langle \gamma, \rho \rangle)}$ $\bar{\rho} = \pi^{-1}(\rho) = z_\eta(\nu)$.

Let

$$\delta = \sup \pi'' z_\eta(\nu)$$
$$= \rho + \nu.$$

(Recall $z_\eta(\nu) = \bar{\rho} + \nu$.) It is not hard to see that

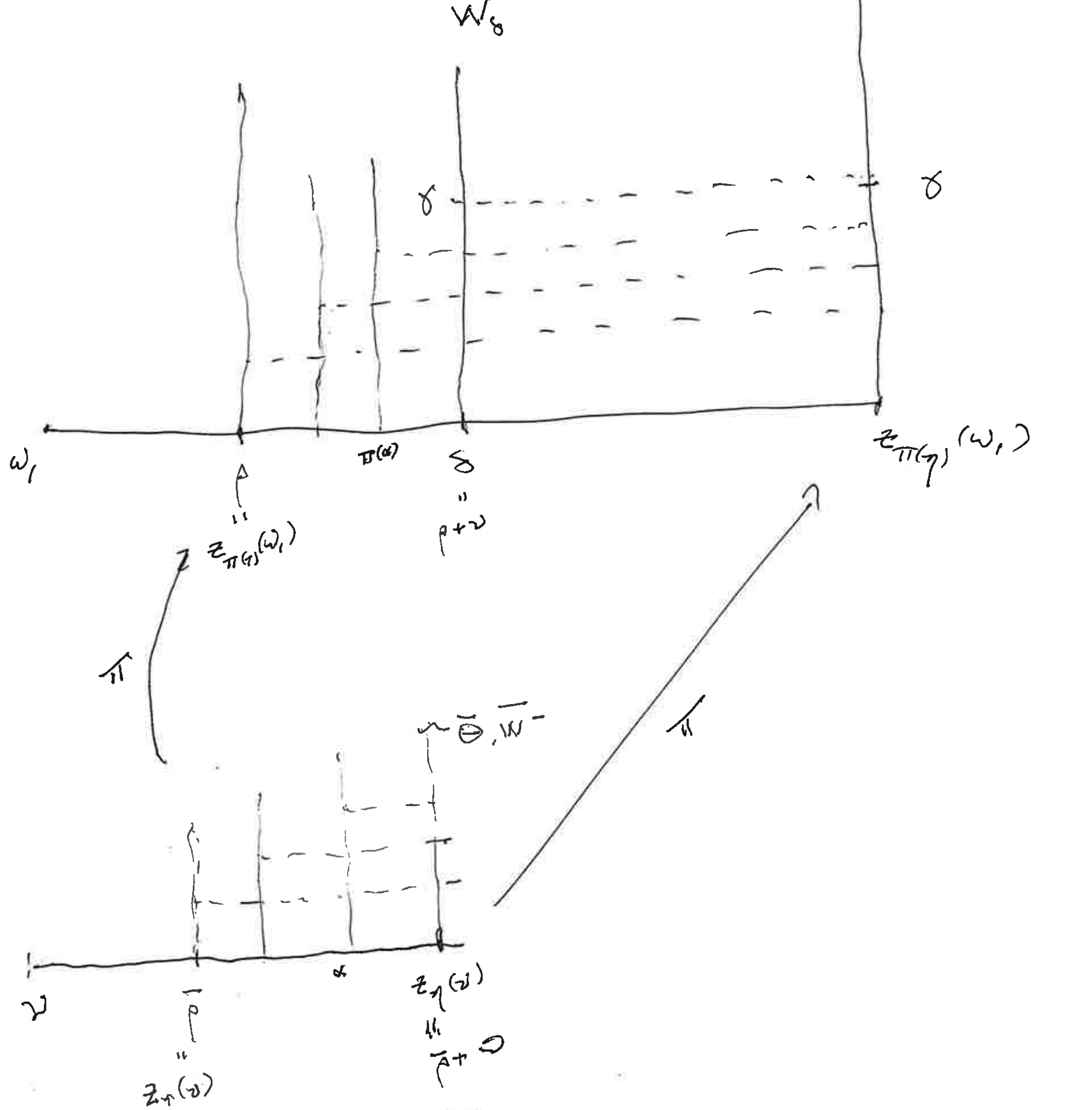
$$W_\delta \restriction \gamma+1 = W^- \restriction \gamma+1$$

and

$$\nu_\delta \restriction \gamma+1 = \nu^- \restriction \gamma+1,$$

since the stabilization point for some $W^- \restriction \xi$ with $\xi \in \mathrm{ran}(\pi)$ must also be in $\mathrm{ran}(\pi)$.

Here is a diagram



π determines associated tree embeddings

$$\Gamma_{\pi, \bar{W}^\frown \bar{b}} = \Phi_\pi : \bar{W}^{-\frown} \bar{b} \longrightarrow W^{-\frown} b$$

and

$$\Gamma_{\pi, \bar{\nu}^\frown \bar{c}} = \Psi_\pi : \bar{\nu}^{-\frown} \bar{c} \longrightarrow \nu^{-\frown} c$$

Notice that b and c are club in Θ, so

$$\gamma \in b \cap c,$$

and $b \cap \gamma = [0, \gamma)_{W^-}$, $c \cap \gamma = [0, \gamma)_{\nu^-}$.

This means that π also determines ~~that~~ the embeddings

$$\bar{\Phi}_\pi^- : \bar{W}^- \wedge \bar{b} \longrightarrow W^- \wedge \gamma+1 = W_\delta \wedge \gamma+1$$

and

$$\Psi_\pi^- : \bar{\nu}^- \wedge \bar{c} \longrightarrow \nu^- \wedge \gamma+1 = \nu_\delta \wedge \gamma+1$$

$\bar{\Phi}_\pi^-$ and Φ_π^- have the same ρ-map on extenders, the only difference is in the last t-maps:

$$t_\infty^{\Phi_\pi} = i_{\gamma b}^{\nu^-} \circ t_\infty^{\Phi_\pi^-}.$$

Similarly

$$t_\infty^{\Psi_\pi} = i_{\gamma c}^{\nu^-} \circ t_\infty^{\Psi_\pi^-}.$$

For $\alpha < z_\eta(\nu)$, we have $\alpha <_A^\nu \pi(\alpha)$ by $(\dagger)_\eta$.

Let

$$d = a(\nu, \bar{\rho}, \rho).$$

Our inflation method is such for all $i \leq \nu$,

$$d = a(\nu, \bar{\rho}+i, \rho+i).$$

Thus for $i \leq \nu$

$$W_{\rho+i} = X(W_{\bar{\rho}+i}, d).$$

Note that ~~$\Phi_{\bar{\rho}+i, \rho+i}$~~ for all $i, j \in [\bar{\rho}, \bar{\rho}+\nu)$, the ρ-maps of $\bar{\Phi}_{i,\pi(i)}^\nu$ and $\bar{\Phi}_{j,\pi(j)}$ agree on all extenders in $\mathrm{Ext}(\mathcal{W}_1) \cap \mathrm{Ext}(\mathcal{V}_1)$,

since they are obtained by inserting the same α. In fact, they agree with π by $(\dagger)_\eta$. But $\bar{\mathcal{W}}^*$ is the common part of tree at $\bar{\rho}+\nu$, so the p-map of

$\Phi^{\nu}_{\bar{\rho}+\nu, \bar{\rho}+\nu} \upharpoonright (\bar{\mathcal{W}}^{-} \frown \bar{b}) = \Phi^{\nu}_{z_\eta(\nu), \delta} \upharpoonright (\bar{\mathcal{W}}^{-} \frown \bar{b})$ is just π. Similarly, the p map of $\Psi^{\nu}_{z_\eta(\nu), \delta} \upharpoonright (\bar{\mathcal{V}}^{-} \frown \bar{c})$ is π. So in fact

$$\Phi^{\nu}_{z_\eta(\nu), \delta} \upharpoonright (\bar{\mathcal{W}}^{-} \frown \bar{b}) = \Phi^{-}_{\pi}$$

and

$$\Psi^{\nu}_{z_\eta(\nu), \delta} \upharpoonright (\bar{\mathcal{V}}^{-} \frown \bar{c}) = \Psi^{-}_{\pi}$$

are the same weak tree embeddings from $\bar{\mathcal{W}}^{-} \frown \bar{b}$ into $\mathcal{W}_\delta \upharpoonright \gamma+1$ and $\bar{\mathcal{V}}^{-} \frown \bar{c}$ into $\mathcal{V}_\delta \upharpoonright \gamma+1$.

<u>Remark</u> ~~the~~ $\Phi^{\nu}_{z_\eta(\nu), \delta}$ acts on the whole of $\mathcal{W}^*_{z_\eta(\nu)}$, by taking a dr ultrapower. $\bar{\mathcal{W}}^{-} \frown \bar{b}$ ~~[illegible]~~ is ~~the~~ a proper initial segment of $\mathcal{W}_{z_\eta(\nu)}$ iff $z_\eta(\nu) < z(\nu)$. So we need to restrict the domains of

$\Phi^{v}_{z_\eta(v),\delta}$ and $\Psi^{v}_{z_\eta(v),\delta}$ to $\bar{w}^- \cap \bar{b}$ and $\bar{v}^- \cap \bar{c}$ in the formulae above.

$\boxed{\text{Insert Remark 4.1a.1}}$

We are going to use these relationships to show that $i^{w^-}_{\gamma,b}$ and $i^{v^-}_{\gamma,c}$ determine the same branch extender sequence. Let

$$\begin{aligned} \varepsilon &= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_b)\} \\ &= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_c)\}, \end{aligned}$$

and $\bar{\varepsilon} = \pi^{-1}(\varepsilon)$. Let also

$$\begin{aligned} \varepsilon_\gamma &= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_{b\restriction\gamma})\} \\ &= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_{c\restriction\gamma})\} \\ &= \sup \pi''\bar{\varepsilon}. \end{aligned}$$

<u>Subclaim 4.1b</u> $M^{w^-}_b \mid (\varepsilon^+)^{M^{w^-}_b} = M^{v^-}_c \mid (\varepsilon^+)^{M^{v^-}_c}$.

<u>Proof</u> Suppose not. Pulling back by π, we can put bars over everything in the inequality. This gives us an $A \subseteq \bar{\varepsilon}$ such that $A \in M^{\bar{w}^-}_{\bar{b}} \,\Delta\, M^{\bar{v}^-}_{\bar{c}}$.

<u>Remark 4.1a.1</u> Since $(\dagger)_\eta$ holds, we have

$$e^{W^-}_{b\cap\gamma} = e^{\bar{W}^-}_{\bar{b}} \circledast d$$

and

$$e^{\nu^-}_{c\cap\gamma} = e^{\bar{\nu}^-}_{\bar{c}} \circledast d.$$

Since η is a successor ordinal, the insertion is not cofinal, e.g. there is a $k \in \operatorname{dom}(e_{\bar{b}})$ s.t.

$$e_{b\cap\gamma} = (e_{\bar{b}} \upharpoonright k \circledast d)^\frown \hat{p}_d(e_{\bar{b}}^{\geq k}),$$

and similarly on the ν-side. k is just the least i s.t. $\operatorname{crit}(e_{\bar{b}}(i)) > \lambda_F$ for all $F \in \operatorname{Ext}(W^-_{z_f(\nu)})$.

Notice that

$$P(\mathcal{E}_\gamma) \cap M_\gamma^{w_\delta^*} = P(\mathcal{E}_\gamma) \cap M_\gamma^{w^-}$$
$$= P(\mathcal{E}_\gamma) \cap M_\theta^{w^-}$$
$$= P(\mathcal{E}_\gamma) \cap M_\theta^{v^-}$$
$$= P(\mathcal{E}_\gamma) \cap M_\gamma^{v^-}$$
$$= P(\mathcal{E}_\gamma) \cap M_\gamma^{v_\delta}.$$

Line 1 because $M_\gamma^{w_\delta} = M_\gamma^{w^-}$, line 2 because $b-\gamma$ has no drops (as otherwise $\bar{b}$ would have infinitely many drops), line 3 because $M_\theta^{w^-} | \mathcal{E} = M_\theta^{v^-} | \mathcal{E}$, etc.

Let

$$t_0 : M_{\bar{b}}^{\bar{w}^-} \longrightarrow M_\gamma^{w_\delta}$$

and

$$t_1 : M_{\bar{c}}^{\bar{v}^-} \longrightarrow M_\gamma^{v_\delta}$$

be the t-maps of Φ_π^- and Ψ_π^-, i.e. the d-ultrapower maps. It's the same d, so

$$t_0 \upharpoonright \bar{\mathcal{E}} = t_1 \upharpoonright \bar{\mathcal{E}}.$$

But at the moment we don't know that $M_{\bar{b}}^{\bar{w}^-}$ and $M_{\bar{c}}^{\bar{v}^-}$ have the same subsets of $\bar{\varepsilon}$, so we don't know that $t_0(\bar{\varepsilon}) = t_1(\bar{\varepsilon})$. For $\alpha \leq \operatorname{dom}(d)$, let

$$R_\alpha = Ult(M_{\bar{b}}^{\bar{w}^-}, d \restriction \alpha)$$

and

$$S_\alpha = Ult(M_{\bar{c}}^{\bar{v}^-}, d \restriction \alpha).$$

Let

$$i_{\alpha\beta} : R_\alpha \longrightarrow R_\beta$$

and

$$j_{\alpha\beta} : S_\alpha \longrightarrow S_\beta$$

be the canonical embeddings. Suppose wlog $A \in M_{\bar{b}}^{\bar{w}^-} - M_{\bar{c}}^{\bar{v}^-}$, where $A \subseteq \bar{\varepsilon}$. We have by induction

$$R_\alpha | \sup i_{0\alpha}{}'' \bar{\varepsilon} = S_\alpha | \sup j_{0\alpha}{}'' \bar{\varepsilon}$$

and $i_{\alpha\beta} \restriction \sup i_{0\alpha}{}'' \bar{\varepsilon} = j_{\alpha\beta} \restriction \sup j_{0\alpha}{}'' \bar{\varepsilon}$.

~~[illegible]~~ We can then use a very useful lemma due to Schlutzenberg (see [3], Lemma 9.6.1(a)) to show by induction on α that

$$i_{0\alpha}(A) \cap \sup i_{0\alpha}{}''\bar{\mathcal{E}} \notin S_\alpha .$$

Schlutzenberg's lemma applies at successor steps because $d(\alpha)$ is weakly amenable to S_α, so by his lemma if $B \subseteq \xi$ and for all $\beta < \xi$ we have $B \cap \beta \in S_\alpha$, and $\bigcup_{\beta < \xi} j(B \cap \beta) \in S_{\alpha+1}$), then $B \in S_\alpha$. We apply this with $\xi = \sup i_{0\alpha}{}''\bar{\mathcal{E}}$ and $B = i_{0\alpha}(A) \cap \sup i_{0\alpha}{}''\bar{\mathcal{E}}$.

At limit steps, if $i_{0\lambda}(A) \cap \sup i_{0\lambda}{}''\bar{\mathcal{E}} \in S_\lambda$, then we get $j_{\alpha\lambda}(B) = i_{0\lambda}(A) \cap \sup i_{0\lambda}{}''\bar{\mathcal{E}}$ for some α, B. But $j_{\alpha\lambda} \restriction \sup i_{0\alpha}{}''\bar{\mathcal{E}} = i_{\alpha\lambda} \restriction \sup i_{0\alpha}{}''\bar{\mathcal{E}}$, so $B \cap \sup i_{0\alpha}{}''\bar{\mathcal{E}} = i_{0\alpha}(A) \cap \sup i_{0\alpha}{}''\bar{\mathcal{E}}$ by pulling back the equality at λ. So $i_{0\alpha}(A) \cap \sup i_{0\alpha}{}''\bar{\mathcal{E}} \in S_\alpha$, contradiction.

But now let $\alpha = \mathrm{dom}(d)$; then

$$\sup i_{0\alpha}{}'' \bar{\varepsilon} = \varepsilon_\gamma$$

~~and thus~~ $$= \sup j_{0\alpha}{}'' \bar{\varepsilon}$$

by the way these embeddings work.

Moreover, $R_\alpha = M_\gamma^{w_\delta}$ and $S_\alpha = M_\gamma^{\nu_\delta}$.

So $i_{0\alpha}(A) \cap \varepsilon_\gamma \in (M_\gamma^{w_\delta} - M_\gamma^{\nu_\delta})$, a contradiction.

Subclaim 4.1b $\boxtimes$

This implies at once

<u>Subclaim 4.1c</u> Let t_0 and t_1 be the t_∞-maps of Φ_π^- and Ψ_π^-; then

(i) $M_{\bar{b}}^{\bar{w}^-} \upharpoonright (\bar{\varepsilon}^+)^{M_{\bar{b}}^{\bar{w}^-}} = M_{\bar{c}}^{\bar{\nu}^-} \upharpoonright (\bar{\varepsilon}^+)^{M_{\bar{c}}^{\bar{\nu}^-}}$,

and (ii) t_0 and t_1 agree on $M_{\bar{b}}^{\bar{w}^-} \upharpoonright (\bar{\varepsilon}^+)^{M_{\bar{b}}^{\bar{w}^-}}$,

(iii) $M_\gamma^{w_\delta} \upharpoonright t_0(\bar{\varepsilon})^{+, M_\gamma^{w_\delta}} = M_\gamma^{\nu_\delta} \upharpoonright t_1(\bar{\varepsilon})^{+, M_\gamma^{\nu_\delta}}$.

Proof (a) comes from subclaim 4.1b by pulling back under π. Given (a), we get (b) because these restrictions of t_0 and t_1 come from taking the α-ultrapower of the same structure. Note here that although the measures in $d(\alpha)$ all concentrate on bounded subsets of sup $i''_{d\alpha}\bar{\varepsilon}$, $i_{d\alpha}$ may be discontinuous at $\bar{\varepsilon}$, so we really needed 4.1b here.

Part (iii) is implicit in (ii).

Again: $t_0(\bar{\varepsilon}) > \varepsilon_\gamma = \sup t_0''\bar{\varepsilon}$ is possible.

4.1(c) $\boxtimes$

Some more notation: let

$$R = M_b^{w^-} \upharpoonright \mathcal{E}^{+, M_b^{w^-}},$$
$$= M_c^{v^-} \upharpoonright \mathcal{E}^{+, M_c^{v^-}},$$

$$\bar{R} = \pi^{-1}(R),$$
$$= M_{\bar{b}}^{\bar{w}^-} \upharpoonright \bar{\mathcal{E}}^{+, M_{\bar{b}}^{\bar{w}^-}} = M_{\bar{c}}^{\bar{v}^-} \upharpoonright \bar{\mathcal{E}}^{+, M_{\bar{c}}^{\bar{v}^-}},$$

$$t = t_\infty^{\Phi_\pi} \upharpoonright \bar{R},$$
$$= t_\infty^{\Psi_\pi^-} \upharpoonright \bar{R},$$

Note $t = S_{\bar{\Theta}}^{\Phi_\pi} = S_{\bar{\Theta}}^{\Psi_\pi}$, where $\Phi_\pi = \Gamma_{\pi, \bar{w} \frown \bar{b}}$ and $\Psi_\pi = \Gamma_{\pi, \bar{v} \frown \bar{c}}$

and

$$N = M_\gamma^{w_\gamma} \mid t(\bar{\mathcal{E}})^{+, M_\gamma^{w_\gamma}}$$
$$= M_\gamma^{v_\gamma} \mid t(\bar{\mathcal{E}})^{+, M_\gamma^{v_\gamma}} = \mathrm{Ult}(\bar{R}, d).$$

Then $t : \bar{R} \longrightarrow N$ is cofinal. Recall that $M_\gamma^{w_\gamma} = M_\gamma^{w^-}$ and $M_\gamma^{v_\gamma} = M_\gamma^{v^-}$. By $(\dagger)_\eta$,

~~$t = t^{\Phi^v_{e_\eta(\gamma), \delta}} = t^{\Psi^v_{e_\eta(\gamma), \delta}}$, that is, t is the canonical embe~~

t is the restriction to $\bar{R}$ of $t_\infty^{\Phi^v_{e_\eta(\gamma), \delta}}$ and $t_\infty^{\Psi^v_{e_\eta(\gamma), \delta}}$, that is, it is the canonical embedding from $\bar{R}$ into $\mathrm{Ult}(\bar{R}, d)$.

Subclaim 4.1e

(1) $N = \mathrm{Ult}(\bar{R}, E_\pi \restriction \varepsilon_\gamma)$, and i is the canonical emb. associated to this ultrapower.

(2) Let $k : N \longrightarrow \pi(R)$ be the factor map, i.e. $k([a, f]^{\bar{R}}_{E_\pi \restriction \varepsilon_\gamma}) = \pi(f)(a)$; then

$$k = i^{\mathcal{W}^-}_{\gamma, b} \restriction N$$
$$= i^{\mathcal{V}^-}_{\gamma c} \restriction N.$$

Proof Part (1) is just Proposition ~~4~~.0.

Part (2) also follows from ~~4~~.0.

$\boxtimes$

Corollary 4.1e $e_b \equiv_{tail} e_c$; in fact, there is an e such that

$$e_b = e_\gamma^{\mathcal{W}^-} {}^\frown e$$

and

$$e_c = e_\gamma^{\mathcal{V}^-} {}^\frown e.$$

Proof e is the sequence of missing initial segments of factors of $i^{\mathcal{W}^-}_{\gamma, b} \restriction N = i^{\mathcal{V}^-}_{\gamma, c} \restriction N$. $\boxtimes$

Let us fix e as in the corollary.

<u>Subclaim 4.1f</u> $d^\frown e$ is a branch of $(\hat{\mathcal{W}}^{\delta})^{ext}$, and a branch of $(\hat{\mathcal{V}}^{\nu})^{ext}$.

<u>Proof</u> The elements of $\mathrm{ran}(e)$ are A-extenders. Note that d fits (non-cofinally in this case) into $e_\gamma^{\mathcal{W}^-}$ and $e_\gamma^{\mathcal{V}^-}$. Since $\mathrm{crit}(e(0)) \geq \varepsilon_\gamma^{\mathcal{W}^-} = e_\gamma^{\mathcal{V}^-}$, we get

$$\lambda(d(i)) \leq \mathrm{crit}(e(0))$$

for all $i \in \mathrm{dom}(d)$. ~~Recall our notation $\theta_s = \sup\{\alpha+1 \mid F_\alpha \in \mathrm{ran}(s)\}$ for s a sequence of A-extenders. Then~~

We claim that $\beta_{e(0)} = \delta$. To see $\beta_{e(0)} \geq \delta$, note that $\kappa_{e(0)} \geq \varepsilon_\gamma$, $e(0)$ applies to $M_\gamma^{\mathcal{W}_\delta} = M_\gamma^{\mathcal{V}^-}$, and no $\mathcal{W}_\xi$ for $\xi < \delta$ agrees with $\mathcal{W}_\delta$ up to γ. Thus $\delta \leq \beta_{e(0)}$. To see $\beta_{e(0)} \leq \delta$, note $\kappa_{e(0)} < \lambda(E_\gamma^{\mathcal{V}^-})$, and $\lambda(E_\gamma^{\mathcal{W}_\xi}) \geq \lambda(E_\gamma^{\mathcal{V}^-})$ for all $\xi \in [\delta, z_{\pi(\gamma)}(\omega_1)]$ because once $\mathcal{W}_\xi \restriction \gamma$ has stabilized, the

$\lambda(E_\gamma^{W_\xi})$ can only go down as ξ increases.

This implies $\kappa_{e(0)} < \lambda_{F_\xi}$ for all $\xi \in [\delta, \tau_{\pi(\eta)}(\omega_1)]$,

so $\beta_{e(0)} \leq \delta$. Thus $\beta_{e(0)} = \delta$.

Now $\delta \in J_d^{\vec{w}}$.

This implies $d^\frown\langle e(0)\rangle \in (\hat{\mathcal{W}}^{\vec{w}})^{ext}$, and similarly we get $d^\frown\langle e(0)\rangle \in (\hat{\mathcal{V}}^{\vec{w}})^{ext}$. We now show by induction on $i \in dom(e)$ that

$$d^\frown e\restriction i \in (\hat{\mathcal{W}}^{\vec{w}})^{ext} \cap (\hat{\mathcal{V}}^{\vec{w}})^{ext}.$$

and $\beta_{d^\frown e\restriction i} = \delta$.

We just did the case $i = 1$. The general successor step is similar: suppose $d^\frown e\restriction i \in (\hat{\mathcal{W}}^{\vec{w}})^{ext} \wedge (\hat{\mathcal{V}}^{\vec{w}})^{ext}$; we must show it for $i+1$. For that, we must show that $\beta_{e(i)} \in J^{\vec{w}}_{d^\frown e\restriction i}$, and in fact

$$\beta_{e(i)} = u^{\hat{\vec{\Phi}}_{e\restriction i}}(\delta).$$

But letting $\delta_1 = u^{\hat{\vec{\Phi}}_{e\restriction i}}(\delta)$

we have $$W_{\delta_1} = X(W_\delta, e\restriction i),$$

and letting $\mathcal{U} : \mathrm{lh}\, W_\delta \to \mathrm{lh}\, W_{\delta_1}$ be $\mathcal{U}^{\vec{\Phi}_{\delta,\delta_1}}$ (not the same as $\mathcal{U}^{\hat{\vec{\Phi}}_{e(i)}}$, which acts on $\vec{\mathcal{T}}_{d \wedge e(i)}$, a block of trees; $\vec{\Phi}_{\delta,\delta_1}$ is one of the weak tree embeddings sitting inside $\hat{\vec{\Phi}}_{e(i)}$), set

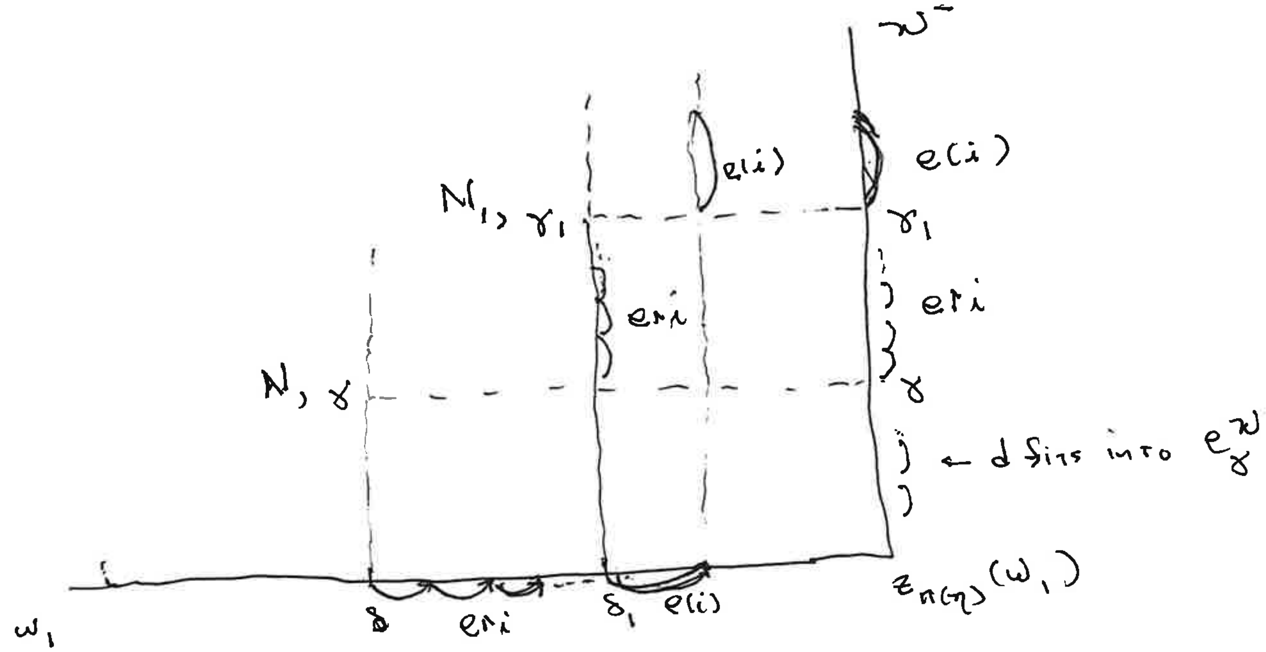
$$\gamma_1 = \mathcal{U}^{\vec{\Phi}_{\delta,\delta_1}}(\gamma).$$

Then

$$W_{\delta_1} \upharpoonright \gamma_1 + 1 = W^- \upharpoonright \gamma_1 + 1,$$

moreover $e(i)$ is applied to $M_{\gamma_1}^{W^-}$ in W^-. As in the case $i = 0$, this implies $\beta_{e(i)} = \delta_1$, which in turn implies $\beta_{e(i+1)} = \delta$.

Here is a diagram and a bit of elaboration:



Let $N_1 = \mathrm{Ult}(\bar{R}, E_\pi \restriction \varepsilon_{\gamma_1}^{w_{\delta_1}})$; then

$N_1 = i_{\gamma,\gamma_1}^{\mathcal{W}^-}(N) = i_{\gamma_1\gamma_1}^{\mathcal{V}^-}(N)$, and

~~$\varepsilon_{\gamma_1} \trianglelefteq K_{e(i)}$~~ $N_1 \trianglelefteq M_{\gamma_1}^{\mathcal{W}_\xi}$ for all $\xi \in [\delta_1, z_{\pi(\eta)}(\omega_1)]$.

Similarly $N_1 \triangleleft M_{\gamma_1}^{\mathcal{V}_\xi}$ for all such ξ.

$\kappa_{e(i)} < o(N_1)$, so $\kappa_{e(i)} < \lambda_{F_\xi}$ for

all $\xi \in [\delta_1, z_{\pi(\eta)}(\omega_1)]$. Thus $\beta_{e(i)} \leq \delta_1$.

But $\varepsilon_{\gamma_1} \leq \kappa_{e(i)}$ and no W_η for $\eta < \delta_1$

agrees with W_{δ_1} to $\varepsilon_{\gamma_1} = \sup\{\lambda_{e(j)} \mid j < i\}$.

So $\beta_{e(i)} = \delta_1$.

For $\lambda \in \mathrm{dom}(e)$ a limit, we just need to see that $d \frown e \restriction \lambda \in (\hat{\mathcal{W}}^\nu)^{ext} \cap (\hat{\mathcal{V}}^\nu)^{ext}$, i.e. it is chosen by the strategies producing $\hat{\mathcal{W}}^\nu$ and $\hat{\mathcal{V}}^\nu$. But letting

$$\gamma_1 = \sup\{\xi \mid E_\xi^{\mathcal{W}^-} \in \mathrm{ran}(e \restriction \lambda)\}$$

we get that

$$e_{\gamma_1}^{\mathcal{W}^-} = \hat{P}_{d \frown e \restriction \lambda}(e_{\bar{b}}^{\bar{\mathcal{W}}})$$

and

$$e_{\gamma_1}^{\mathcal{V}^-} = \hat{P}_{d \frown e \restriction \lambda}(e_{\bar{c}}^{\bar{\mathcal{V}}}),$$

so that $d \frown e \restriction \lambda$ is justified on the $\mathcal{W}$-side by moving $e_{\bar{b}}^{\bar{\mathcal{W}}}$ into $e_{\gamma_1}^{\mathcal{W}^-}$, both branches being by Σ, and $d \frown e \restriction \lambda$ is justified on the $\mathcal{V}$-side by moving $e_{\bar{c}}^{\bar{\mathcal{V}}}$ into $e_{\gamma_1}^{\mathcal{V}^-}$, both branches being by Λ.

4.1f $\boxtimes$

<u>Subclaim 4.1g</u> $(\nu, d^\frown e)$ is justified by both b and c.

<u>Proof</u> $\hat{p}_d(e_b^-) = e_\gamma^{w^-}$, so $\hat{p}_{d^\frown e}(e_b^-) = e_\gamma^{w^-} \frown e$ (since $\mathcal{E}_\gamma \leq \kappa_{e(0)}$), i.e. $\hat{p}_{d^\frown e}(e_b^-) = e_b$. This shows $(\nu, d^\frown e)$ is justified by b. Similarly, it is justified by e.

$\boxtimes$

The subclaims imply $z_{\pi(\eta)}(\omega_1) < z(\omega_1)$. Pulling back by π, we get $z_\eta(\nu) < z(\nu)$. By claim 2, $z_{\pi(\eta)} + \omega_1 \leq z(\omega_1)$, so $z_\eta(\nu) + \nu \leq z(\nu)$. But ~~$z_{\eta+1}(\nu) = z_\eta(\nu) + \nu$~~ then $z_{\pi(\eta)}(\omega_1) + \omega_1 = z_{\pi(\eta)+1}(\omega_1) \leq \omega_1$. Pulling back $z_{\eta+1}(\nu) = z_\eta(\nu) + \nu \leq z(\nu)$.

For the rest of $(\dagger)_{\eta+1}$:

<u>Subclaim 4.1d</u> For $\alpha \in [z_\eta(\nu), z_\eta(\nu)+\nu)$, $\alpha <_A^{\vec{\nu}} \pi(\alpha)$ and $\Phi^{\vec{\nu}}_{\alpha,\pi(\alpha)} = \Gamma_{\pi, w_\alpha}$ and $\Psi^{\vec{\nu}}_{\alpha,\pi(\alpha)} = \Gamma_{\pi, \nu_\alpha}$.

<u>Proof</u> We just showed that $z_\eta(\nu) <_A^{\vec{\nu}} z_{\pi(\eta)}(\omega_1)$ is witnessed by $d^\frown e$, i.e. in fact

$$a(\nu, z_\eta(\nu), z_{\pi(\eta)}(\omega_1)) = d^\frown e.$$

(Part of this is that $u^{\hat{\Phi}^{\vec{\nu}}_{d^\frown e}}(z_\eta(\nu)) = z_{\pi(\eta)}(\omega_1)$. We showed that $u^{\hat{\Phi}^{\vec{\nu}}_{d^\frown e\restriction i}}(z_\eta(\nu)) = u^{\hat{\Phi}_{e\restriction i}} \circ u^{\vec{\nu}}_{d}(z_\eta(\nu)) = u^{\hat{\Phi}_{e\restriction i}}(\delta)$ has limit $z_{\pi(\eta)}(\omega_1)$ as $i \to \mathrm{dom}(e)$, which implies $u^{\hat{\Phi}^{\vec{\nu}}_{d^\frown e}}(z_\eta(\nu)) = z_{\pi(\eta)}(\omega_1)$.)

Our definition of the $\hat{W}$ system is then such that for all $\alpha \in [z_\eta(\nu), z_\eta(\nu)+\nu)$,

$$\alpha <_A^{\nu} \pi(\alpha)$$

and

$$a(\nu, \alpha, \pi(\alpha)) = d^\frown e.$$

(Let $\alpha = z_\eta(\nu) + \beta$; then
$u^{\vec{\Phi}_{d^\frown e}}(\alpha) = u^{\Phi^\nu_{d^\frown e}}(z_\eta(\nu)) + \beta = \pi(\alpha)$, and
$d^\frown e$ witnesses that $\alpha <^\nu_A \pi(\alpha)$.)

We must show that for $\alpha \in [z_\eta(\nu), z_\eta(\nu)+\nu)$,
$\Phi^\nu_{\alpha,\pi(\alpha)} = \Gamma_{\pi, W_\alpha}$ and $\Psi^\nu_{\alpha,\pi(\alpha)} = \Gamma_{\pi, \nu_\alpha}$.
The proof is by induction on α.

We begin with $\alpha = z_\eta(\nu)$. Let

$$\Phi = \Phi^\nu_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)}$$

and

$$\Gamma = \Gamma_{\pi, W_{z_\eta(\nu)}} .$$

Both Φ and Γ are extended weak tree embeddings from $W_{z_\eta(\nu)}$ to $W_{z_{\pi(\eta)}(\omega_1)}$, so they are equal if they have the same p-map. Letting p^Φ be the p-map of Φ, then, we must show that $p^\Phi = \pi \restriction \mathrm{Ext}(W_{z_\eta(\nu)})$. Note first that

$$p^\Phi \restriction \mathrm{Ext}(\bar{W}^-) = \pi \restriction \mathrm{Ext}(\bar{W}^-),$$

since if $E \in \mathrm{Ext}(\bar{W}^-)$, then $E \in \mathrm{Ext}(W_\xi)$ for all sufficiently large $\xi < z_\eta(\nu)$, and $p^\Phi(E) = p^{\Phi^\nu_{\xi,\pi(\xi)}}(E) = \pi(E)$ for all suff. lg. $\xi < z_\eta(\nu)$ by $(\dagger)_\eta$.

Now let $E \in \mathrm{Ext}(W_{z_\eta(\nu)}) - \mathrm{Ext}(\bar{W}^-)$.

By the elementarity of π,

$$H \models \exists \gamma <^u \nu\ \exists a\ (\ \bar{b} \text{ and } \bar{c} \text{ justify } \gamma, a)$$
$$\text{and } \exists \xi < \nu (\ W_{z_\eta(\nu)} = X(W_\xi, a) \text{ and}$$
$$\nu_{z_\eta(\nu)} = X(\nu_\xi, a)\ ).$$

Fix γ, a, and ξ as on the rhs. So $\xi <^\gamma_A z_\eta(\nu)$ and $a = a(\gamma, \xi, z_\eta(\nu))$. Let

$$p^\gamma_{\xi, z_\eta(\nu)} = p^{\Phi^\gamma_{\xi, z_\eta(\nu)}} = (p^\gamma_a)_\xi$$

be the p-map from $\mathrm{Ext}(W_\xi)$ to $\mathrm{Ext}(W_{z_\eta(\nu)})$.

Since a fits into $e_{\bar{b}}$ and $lh(E) > lh(G)$ for all $G \in \mathrm{ran}(e_{\bar{b}})$, we have

$$E = p^\gamma_{\xi, z_\eta(\nu)}(G)$$

for some $G \in \mathrm{Ext}(W_\xi)$. But then

$$\pi(E) = p^\gamma_{\xi, z_{\pi(\eta)}(\omega_1)}(G)$$
$$= p^\nu_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)}(p^\gamma_{\xi, z_\eta(\nu)}(G))$$
$$= p^\Phi(E),$$

as desired.

To elaborate the last calculation:

$\pi(\xi) = \xi$, $\pi(\gamma) = \gamma$, and $\pi(G) = G$. It suffices then to see line 2, that is,

$$P^{\gamma}_{\xi, z_{\pi(\eta)}(\omega_1)} = P^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)} \circ P^{\gamma}_{\xi, z_\eta(\nu)}.$$

But this follows from Lemma 9.5. To check its hypotheses: we have $\gamma < \nu$,

$\xi <^{\gamma}_{A} z_\eta(\nu)$, $\xi <^{\nu}_{A} z_{\pi(\eta)}(\omega_1)$ (by applying π to that)

and $z_\eta(\nu) <^{\nu}_{A} z_{\pi(\eta)}(\omega_1)$ (as witnessed by $d^\frown e$).

By Lemma 9.5, $\Phi^{\gamma}_{\xi, z_{\pi(\eta)}(\omega_1)} = \Phi^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)} \circ \Phi^{\gamma}_{\xi, z_\eta(\nu)}$,

as desired.

Now let $\alpha \geq z_\eta(\nu)$, and

~~Now~~ suppose we have $\Gamma_{\pi, W_\beta} = \Phi^{\nu}_{\beta, \pi(\beta)}$ for all $\beta \leq \alpha$. (When $\beta < \nu$, both are the identity.) We wish to show $\Gamma_{\pi, W_{\alpha+1}} = \Phi^{\nu}_{\alpha+1, \pi(\alpha)+1}$.

Let

$$F = F_\alpha$$

and

$$\beta = \beta_F,$$

so that $W_{\alpha+1} = X(W^*_\beta, F)$. (If $\beta \geq \nu$, $W^*_\beta = W_\beta$. $\beta < \nu$ is possible.) We shall apply the Shift Lemma for ~~weak~~ tree embeddings. We have

$$W_\alpha \xrightarrow{\Phi^{\nu}_{\alpha, \pi(\alpha)} = \Gamma_{\pi, W_\alpha}} W_{\pi(\alpha)}$$

Note that this is a full tree embedding, not just a weak one. Let

$$G = \pi(F) = t_\infty^{\Phi^{\nu}_{\alpha, \pi(\alpha)}}(F).$$

The Shift lemma for tree embeddings (whose hypotheses should be verified at this point!) gives us $\Phi^{\nu}_{\alpha+1, \pi(\alpha)+1}$ making the diagram

$$\begin{tikzcd} W_{\alpha+1} \approx X(W_\beta^*, F) \arrow[r, "\vec{\Phi}_{\alpha+1,\pi(\alpha)+1}"] & X(W_{\pi(\beta)}, G) = W_{\pi(\alpha)+1} \\ W_\beta^* \arrow[u, "\Phi_F"] \arrow[r, "\vec{\Phi}_{\beta,\pi(\beta)}"] & W_{\pi(\beta)} \arrow[u, "\Phi_G"'] \end{tikzcd}$$

Let $p = p^{\vec{\Phi}_{\alpha+1,\pi(\alpha)+1}}$ be the p-map.

Let ξ be least such that F is on the $M_{\xi+1}^{W_\alpha}$-sequence, so heuristically

$$W_{\alpha+1} = W_\alpha \restriction \xi+1 \frown \langle F \rangle \frown i_F{}''(W_\beta^*)^{\geq\theta}$$

for a certain θ. In terms of extenders used

$$\mathrm{Ext}(W_{\alpha+1}) = \mathrm{Ext}(W_\alpha \restriction \xi+1) \cup \{F\} \cup p_F{}''\,\mathrm{Ext}(W_\beta^*)$$

where $p_F = p^{\Phi_F}$. To show that $\vec{\Phi}_{\alpha,\pi(\alpha)+1} = \Gamma_{\pi, W_{\alpha+1}}$ it is enough to see that p and π agree on $\mathrm{Ext}(W_{\alpha+1})$. But

$$p \restriction \mathrm{Ext}(W_\alpha \restriction \xi+1) = p^{\vec{\Phi}_{\alpha,\pi(\alpha)}} \restriction \mathrm{Ext}(W_\alpha \restriction \xi+1)$$
$$= \pi \restriction \mathrm{Ext}(W_\alpha \restriction \xi+1)$$

by the Shift Lemma and our induction hypo.

$$p(F) = G = \pi(F)$$

by Shift Lemma defn of p. Finally,

for $E \in \mathrm{Ext}(\mathcal{W}_\beta^*)$

$$p(\rho_F(E)) = \rho_G(p^{\Phi^\nu_{\beta,\pi(\beta)}}(E))$$

$$= \rho_{\pi(F)}(\pi(E))$$

$$= \pi(\rho_F(E)),$$

by the Shift Lemma and induction.

We leave the case α is a limit ordinal to the reader.

4.1h $\boxtimes$

Altogether, the subclaims yield $(\dagger)_{\eta+1}$.

SubClaim 4.1 $\boxtimes$

<u>Subclaim 4.2</u> If η is a limit ordinal and $cf^H(\eta) = \omega$, and $(\dagger)_\eta$ holds, then $(\dagger)_{\eta+1}$ holds.

<u>Proof</u> Again $\bar{\mathcal{W}}^- = \mathcal{W}^-_{z_\eta(\nu)}$ and $\bar{\mathcal{V}}^- = \mathcal{V}^-_{z_\eta(\nu)}$ are the common part trees at ν, η, and $\mathcal{W}^- = \pi(\bar{\mathcal{W}}^-)$ and $\mathcal{V}^- = \pi(\bar{\mathcal{V}}^-)$ are also common part trees at $\omega_1, \pi(\eta)$. Let

$$b = \Sigma(\mathcal{W}^-),$$

$$c = \Lambda(\mathcal{V}^-),$$

$$\langle \bar{b}, \bar{c} \rangle = \pi^{-1}(\langle b, c \rangle),$$

$$\theta = lh(\mathcal{W}^-) = lh(\mathcal{V}^-),$$

$$\bar{\theta} = \pi^{-1}(\theta).$$

It is easy to see that $cf^H(\bar{\theta}) = \omega$. Thus $cf(\pi(\eta)) = cf(\theta) = \omega$, and

$$\theta = \sup \pi''\bar{\theta},$$

and $\pi''\bar{b}$ is cofinal in b, and $\pi''\bar{c}$ is cofinal in c.

We shall find a cofinal branch a of $\hat{\mathcal{W}}^{v,\text{ext}}_{\omega_1,\pi(\eta)}$ such that (v,a) is justified by e_b, because $\hat{p}_a(e_{\bar{b}}) = e_b$.

The reason for that will be that

$$\left(\hat{\Phi}^v_a\right)_{z_\eta(v)} = \Gamma_{\pi, W_{z_\eta(v)}}.$$

Similarly we get (v,d) justified by e_c with $\left(\hat{\Phi}^v_d\right)_{z_\eta(v)} = \Gamma_{\pi, v_{z_\eta(v)}}$. But then the agreement between $\Gamma_{\pi, W_{z_\eta(v)}}$ and $\Gamma_{\pi, v_{z_\eta(v)}}$ implies that $a = d$, which then implies $z_{\pi(\eta)}(\omega_1) < z(\omega_1)$, and leads to the rest of $(T)_{\eta+1}$.

We focus now on obtaining (v,a) justified by b.

<u>Subclaim 4.2a</u> There is a cofinal branch a of $\hat{\mathcal{W}}^{\vec{\nu}, \text{ext}}_{\omega_1, \pi(\eta)}$ such that $(\vec{\nu}, a)$ is justified by b, and $\cancel{(\Phi^{\vec{\nu}}_{a})_{\mathcal{E}_\eta(\gamma)} = \Gamma_{\vec{\nu}, W_{\eta(\gamma)}}}$. $p_a \restriction \text{Ext}(\bar{\mathcal{W}}^-) = \pi \restriction \text{Ext}(\bar{\mathcal{W}}^-)$ and $\hat{p}_a(e_{\bar{b}}) = e_b$.

<u>Proof</u> Let $\langle E_n \mid n < \omega \rangle$ be such that

(i) $E_n \in \text{ran}(\pi) \cap \text{ran}(e_b^{\mathcal{W}^-})$, and

(ii) E_{n+1} is used strictly after E_n in b,

(iii) $\{E_n \mid n < \omega\}$ is cofinal in $\text{ran}(e_b)$, and

(iv) $\theta^*_{E_0} > \omega_1$.

Let
$$k_n = \theta^*_{E_n},$$
$$\bar{E}_n = \pi^{-1}(E_n), \text{ and}$$
$$i_n = \pi^{-1}(k_n) = \theta^*_{\bar{E}_n}.$$

So $i_n <^{\vec{\nu}}_A k_n$ by $(\dagger)_\eta$. Let
$$a_n = a(\vec{\nu}, i_n, k_n).$$

We shall take $a = \bigcup_{n<\omega} a_n$. The main point is

<u>Subclaim 4.2a.1</u> If $n \le m$, then $a_n \subseteq a_m$.

<u>Proof</u> Suppose ~~a_n~~ $n \le m$ and $a_n \not\subseteq a_m$, or equivalently, $[k_n]_\nu \not\le^\nu_A [k_m]_\nu$. Then the branch of $(\mathcal{B}_\nu, <^\nu_A)$ to $[k_m]_\nu$ skips over $[k_n]_\nu$. This gives us j, l s.t.

$$j <^\nu_A l <^\nu_A k_m,$$

$$j < i_n < l, \; j \not\prec_\nu i_n, \; l \not\prec_\nu i_n,$$

and

$$a(\nu, j, l) = \langle H \rangle$$

for some single A-extender H. Let $E = E_n$ and $F = E_m$. We argue as in 9.16 (and several times before). We have

$$\theta^*_H = \theta_H + 1 \in [l]_\nu,$$

so

$$\theta^*_E = i_n < \theta^*_H.$$

So $E \neq H$. Also, E is used in both W_{i_n} and W_j, so ~~lh(E)~~ $lh(E) < lh(H)$. Now let

$$a(i, j, k_m) = \langle H \rangle ^\frown d,$$

and since $F \in \operatorname{ran}(p^{\nu}_{\langle H \rangle ^\frown d})$, let $p^{\nu}_{\langle H \rangle ^\frown d}(F_0) = F$. This gives us s such that

$$s ^\frown \langle F_0 \rangle \in W_j^{ext}$$

and

$$\hat{p}^{\nu}_{\langle H \rangle ^\frown d}(s ^\frown \langle F_0 \rangle) = t ^\frown \langle F \rangle \in W_{k_m}^{ext}.$$

Since $E \in \operatorname{ran}(t)$ and $lh(E) < lh(H)$, this gives

$$E \in \operatorname{ran} \hat{p}^{\nu}_H(s).$$

But $E \neq H$, and $lh(E) < lh(H)$, so the only way this can happen is that $\hat{p}^{\nu}_H(E) = E$. That implies $\theta^*_E \leq j$. But $j < i_n = \theta^*_E$, contradiction.

4.2a.1 $\boxtimes$

So $k_n <^*_A k_m$ when $n < m$.

<u>Subclaim 4.2a.2</u> Let $d = a(\vec{v}, k_n, k_m)$ where $n < m$; then $\mathrm{crit}(d(0)) \geq \lambda_{E_n}$.

<u>Proof</u> Let $a_n = a(\vec{v}, i_n, k_n)$; then

$$p^{\vec{v}}_{a_m} = p^{\vec{v}}_d \circ p^{\vec{v}}_{a_n}.$$

By $(\dagger)_\eta$, the $p^{\vec{v}}_{a_i}$'s agree with π, so

$$\cancel{p^{\vec{v}}_{a_m}(\bar{E}_n)}$$

$$\begin{aligned} E_n &= \pi(\bar{E}_n) \\ &= p^{\vec{v}}_{a_m}(\bar{E}_n) \\ &= p^{\vec{v}}_d(p^{\vec{v}}_{a_n}(\bar{E}_n)) = p^{\vec{v}}_d(\pi(\bar{E}_n)) \\ &= p^{\vec{v}}_d(E_n). \end{aligned}$$

This implies $\mathrm{crit}(d(0)) \geq \lambda_{E_n}$.

$\boxtimes$

Now set

$$a = \bigcup_{n<\omega} a_n.$$

The subclaims imply

$$p_a^{\nu}(\bar{E}_n) = p_{a_n}^{\nu}(\bar{E}_n)$$

$$= \pi(E_n)$$

for all n. Letting $E_n = e_b(\gamma_n)$ and $\bar{\gamma}_n = \pi^{-1}(\gamma_n)$, this implies

$$\hat{p}_a^{\nu}(e_{\bar{b}} \upharpoonright \bar{\gamma}_{n+1}) = p_{a_n}^{\nu}(e_{\bar{b}} \upharpoonright \bar{\gamma}_n+1)$$

$$= e_b \upharpoonright \gamma_n+1,$$

and hence

$$\hat{p}_a^{\nu}(e_{\bar{b}}) = e_b.$$

<u>Subclaim 4.2a.3</u> $W^- \frown b = Ult(\bar{W}^- \frown \bar{b}, a)$, and ~~$p_a^{\nu}$~~ $\pi \upharpoonright Ext(\bar{W}^-)$ is the p-map of the canonical tree embedding. Similarly on the ν-side.

<u>Proof</u> Let $E_n = E_{\xi_n}^{W^-}$, and $\bar{\xi}_n = \pi^{-1}(\xi_n)$. Then

$$\mathcal{W}^- \restriction \xi_n+1 = \mathcal{W}_{k_n} \restriction \xi_n+1$$

$$= \mathrm{Ult}(\mathcal{W}_{i_n} \restriction \bar{\xi}_n+1, a_n)$$

~~subsection~~ $$= \mathrm{Ult}(\bar{\mathcal{W}}^- \restriction \bar{\xi}_n+1, a_n).$$

Moreover $\bigcup_{n<\omega} \mathrm{Ult}(\bar{\mathcal{W}}^- \restriction \bar{\xi}_n+1, a_n) = \mathrm{Ult}(\bar{\mathcal{W}}^-, a)$.

$\boxtimes$

We have shown that e_b justifies (ν, a), and $\hat{p}_a \restriction \mathrm{Ext}(\mathcal{W}^-_{z_\eta(\nu)}) = \pi \restriction \mathrm{Ext}(\mathcal{W}^-_{z_\eta(\nu)})$

4.2a $\boxtimes$

<u>Subclaim 4.2b</u> There is a cofinal branch d of $\hat{\mathcal{V}}^{\nu,\mathrm{ext}}_{\omega_1, \pi(\eta)}$ such that (ν, d) is justified by c, and $\hat{p}_d \restriction \mathrm{Ext}(\bar{\mathcal{V}}^-) = \pi \restriction \mathrm{Ext}(\bar{\mathcal{V}}^-)$.

<u>Proof.</u> The same.

$\boxtimes$

<u>Subclaim 4.2C</u> $a = d$.

<u>Proof</u> Let $\varepsilon = \sup\{\lambda_E \mid E \in \mathrm{ran}(\rho_b)\} = \sup\{\lambda_E \mid E \in \mathrm{ran}(\rho_c)\}$, and $\bar{\varepsilon} = \pi^{-1}(\varepsilon)$.

So $cf(\varepsilon) = cf(\bar{\varepsilon}) = \omega$, and $\varepsilon = \sup \pi``\bar{\varepsilon}$. We have

$$M_{\bar{b}}^{\bar{w}^-} \restriction \bar{\varepsilon} = M_{\bar{c}}^{\bar{w}^-} \restriction \bar{\varepsilon}$$

$$\begin{aligned} M_b^{w^-} \restriction \varepsilon &= Ult(M_{\bar{b}}^{\bar{w}^-} \restriction \bar{\varepsilon}, \check{E}_\pi \restriction \varepsilon) \\ &= Ult(M_{\bar{c}}^{\bar{w}^-} \restriction \bar{\varepsilon}, \check{E}_\pi \restriction \varepsilon) \\ &= M_c^{w^-} \restriction \varepsilon . \end{aligned}$$

Here we are regarding $\check{E}_\pi$ as E_π, but restricted to measuring sets in $M_{\bar{b}}^{\bar{w}^-} \restriction \bar{\varepsilon}$. ($\pi$ is continuous at $\bar{\varepsilon}$, so only those sets need be measured, unlike the Subclaim 4.1 case, when unbounded subsets of $\bar{\varepsilon}$ were relevant.)

But then both a and d are the sequence of missing-from-$M_b^{w^-} \restriction \varepsilon$ initial segments of tail factors of E_π. So $a = d$. $\boxtimes$

This finishes the proof of 4.2a

Subclaim 4.2a $\boxtimes$

By 4.2a, $z_{\pi(\eta)}(\omega_1) < z(\omega_1)$, as witnessed by (ν, a) satisfying 4.2a. Let us verify the rest of $(\dagger)_{\eta+1}$.

Subclaim 4.2d $\left(\hat{\Phi}^{\nu}_{a}\right)_{z_\eta(\nu)} = \Gamma_{\pi, W_{z_\eta(\nu)}}$ and $\left(\hat{\Psi}^{\nu}_{a}\right)_{z_\eta(\nu)} = \Gamma_{\pi, \nu_{z_\eta(\nu)}}$.

Proof $z_\eta(\nu) < z(\nu)$ by elementarity of π, moreover ~~[illegible]~~ η is a limit ordinal. Thus there is (γ, a_0) justified by $\bar{b}$ and $\bar{c}$ such that $\gamma < \nu$, and

$$W_{z_\eta(\nu)} = X(W_\xi, a_0)$$

and

$$\nu_{z_\eta(\nu)} = X(\nu_\xi, a_0)$$

where $\xi < \nu$. Here $\xi <^{\gamma}_{A} z_\eta(\nu)$, as witnessed by a_0, and $\xi < z(\gamma) < \nu$. The two weak tree embeddings are $\bar{\Phi}^{\gamma}_{\xi, z_\eta(\nu)} = \left(\hat{\bar{\Phi}}^{\gamma}_{a_0}\right)_\xi$ and $\psi^{\gamma}_{\xi, z_\eta(\nu)} = \left(\hat{\psi}^{\gamma}_{a_0}\right)_\xi$.

Their associated p-maps are $p^{\gamma}_{\xi, z_\eta(\nu)}$ and $q^{\gamma}_{\xi, z_\eta(\nu)}$.

Let $p^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)}$ be the p-map of $\left(\hat{\bar{\Phi}}^{\nu}_{a}\right)_{z_\eta(\nu)}$. We have already shown that $p^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)}$ agrees with π on $Ext(\bar{W}^-)$. But if $E \in Ext(W_{z_\eta(\nu)}) - Ext(\bar{W}^-)$, then $E \in ran(p^{\gamma}_{\xi, z_\eta(\nu)})$.

~~[We are using here that we are in the $cof(z_\eta(\nu)) = \omega$ case, so that a fits cofinally into e_b, and a_0 fits cofinally into e_f.]~~

Letting $E = p^{\gamma}_{\xi, z_\eta(\nu)}(G)$, we get

$$\pi(E) = p^{\gamma}_{\xi, z_{\pi(\eta)}(\omega_1)}(G)$$

$$= p^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)} \circ p^{\gamma}_{\xi, z_\eta(\nu)}(G)$$

Using Lemma 9.5

$$= p^{\nu}_{z_\eta(\nu), z_{\pi(\eta)}(\omega_1)}(E).$$

So π agrees with the p-map of $(\hat{\Phi}^{\nu}_a)_{z_\eta(\nu)}$ on all of $\mathrm{Ext}(W_{z_\eta(\nu)})$, and thus

$$(\hat{\Phi}^{\nu}_a)_{z_\eta(\nu)} = \Gamma_{\pi, W_{z_\eta(\nu)}}.$$

Parallel calculations show $(\hat{\Psi}^{\nu}_a)_{z_\eta(\nu)} = \Gamma_{\pi, \nu_{z_\eta(\nu)}}$.

4.2d $\boxtimes$

<u>Subclaim 4.2e</u> For all $\alpha < \nu$,

$$(\hat{\Phi}^{\nu}_a)_{z_\eta(\nu)+\alpha} = \Gamma_{\pi, W_{z_\eta(\nu)}} \text{ and } (\hat{\Psi}^{\nu}_a)_{z_\eta(\nu)} = \Gamma_{\pi, \nu_{z_\eta(\nu)}}.$$

<u>Proof</u> The argument given at the end of the proof of 4.1h works here. $\boxtimes$

This finishes the proof of Subclaim 4.2, the general cof $= \omega$ case.

Subclaim 4.2 $\boxtimes$

Finally, we deal with the rest of the cof $= \omega_1$ case.

<u>Subclaim 4.3</u> Suppose η is a limit ordinal such that $(\dagger)_\eta$ holds and $\mathrm{cof}^H(\eta) = \nu$; then $(\dagger)_{\eta+1}$ holds.

<u>Proof</u> The proof is very close to the proof of 4.1, which was the $\eta = \gamma+1$ case. In the present case $\mathrm{cf}^H(z_\eta(\nu)) = \nu$ and $\mathrm{cf}(z_{\pi(\eta)}(\omega_1)) = \omega_1$. ~~Because the map $\xi \to$ stabilizing point~~ Let $\mathcal{W}^- = \mathcal{W}^-_{z_{\pi(\eta)}(\omega_1)}$ and $\mathcal{V}^- = \mathcal{V}^-_{z_{\pi(\eta)}(\omega_1)}$ be the common part trees, ; and $\theta = lh(\mathcal{W}^-) = lh(\mathcal{V}^-)$. $\mathrm{cf}(\theta) = \omega_1$ because the map $\xi \to$ point where $\mathcal{W}^- \restriction \xi$ is stabilized can't have range bounded in ω_1. Let again

$$b = \Sigma(w^-)$$
$$c = \Lambda(v^-)$$
$$\varepsilon = \sup\{\lambda_E \mid E \in \operatorname{ran}(e_b^{w^-})\}$$
$$= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_c^{v^-})\}$$

and

$$\langle \bar{w}^-, \bar{v}^-, \bar{b}, \bar{c}, \bar{\theta}, \bar{\varepsilon} \rangle = \pi^{-1}(\langle w^-, v^-, b, c, \theta, \varepsilon \rangle).$$

Sticking further with the notation in #1, set

$$\gamma = \sup \pi'' \bar{\theta}$$
$$\varepsilon_\gamma = \sup \pi'' \bar{\varepsilon}$$
$$= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_{b\cap\gamma}^{w^-})\}.$$
$$= \sup\{\lambda_E \mid E \in \operatorname{ran}(e_\gamma^{v^-})\}.$$

Notice here that $\gamma \in b \cap c$, so $e_\gamma^{w^-} = e_{b\cap\gamma}$ and $e_\gamma^{v^-} = e_{c\cap\gamma}$. ~~They are the downward closures~~ Equivalently, $e_\gamma^{w^-} = \bigcup_{i \in \operatorname{dom} e_{\bar{b}}^{\bar{w}^-}} \pi(e_{\bar{b}} \restriction i)$, and similarly for $e_\gamma^{v^-}$.

Let

$$\delta = \sup \pi'' \bar{\varepsilon}_\gamma(v).$$

<u>Subclaim 4.3a</u> $z_\eta(\nu) <^\nu_A \delta$.

<u>Proof</u> In the proof of 4.1, it was that $z_\eta(\nu)+\nu <^\nu_A \delta = z_{\pi(\tau)}(\omega_1)+\nu$. In the current case the witnesses $a(\nu, \alpha, \pi(\alpha))$ are not necessarily constant as $\alpha \longrightarrow z_\eta(\nu)$. But we can piece together a ~~[illegible]~~ union of them. The way to do this was described in the proof of 4.2, the $d = a(\nu, z_\eta(\nu), z_{\pi(\tau)}(\omega_1))$ that witnessed the parallel fact, cof $= \omega$ case.

<u>Subclaim 4.3a.1</u> There is a (unique) branch a^0 of $\hat{K}^{\nu,ext}$ such that ~~$z_\eta(\nu) \in$ dom~~ $u^\nu_{a^0}(z_\eta(\nu)) = \delta$ and $p^\nu_{a^0} \upharpoonright Ext(W^-_{z_\eta(\nu)}) = \pi \upharpoonright Ext(W^-_{z_\eta(\nu)})$.

<u>Proof</u> Let $\langle \bar{E}_n \mid n<\omega \rangle$ be an increasing sequence of extenders in $ran(e^{\bar{w}^-}_{\bar{b}})$ that are cofinal, and let $E_n = \pi(\bar{E}_n)$.

Let

$$i_n = \Theta^*_{\bar{E}_n}$$

and

$$k_n = \Theta^*_{E_n}.$$

So $\pi(k_n) = i_n$, so

$$i_n <_A^{\nu} k_n$$

for all n. Let

$$a_n^{\circ} = a(\nu, i_n, k_n).$$

Then E_n's a cofinal in $ran(e_{b \restriction \chi})$, so the k_n's are cofinal in δ and the i_n's are cofinal in $\tau_\eta(\nu)$. The proof of 4.29.1 shows

$$n \le m \Rightarrow a_n^{\circ} \subseteq a_m^{\circ}.$$

Let

$$a^{\circ} = \bigcup_n a_n^{\circ}.$$

We claim that a° works. For given G

For $\vec{p}_a^{\,\nu} \restriction \{G \in Ext(\bar{W}^-) \mid lh(G) < lh(\bar{E}_n)\} = p_{a_n}^{\nu} \restriction \{G \in Ext(W|_{i_n}) \mid lh(G) < lh(\bar{E}_n)\} \subseteq \pi$, for all $n < \omega$. So $p_a^{\nu} \restriction Ext(\bar{W}^-) \subseteq \pi$.

$\boxtimes$

<u>Subclaim 4.3a.2</u> There is a (unique) branch a^* of $\hat{\mathcal{T}}_{\nu,\mathrm{EXT}}$ such that $i^{\nu}_{a'}(z_\eta(\nu)) = \delta$ and $P^{\nu}_{a'} \restriction \mathrm{EXT}(\bar{\nu}_{z_\eta(\nu)}) = \pi \restriction \mathrm{EXT}(\bar{\nu}_{z_\eta(\nu)})$.

<u>Proof</u> Parallel. $\boxtimes$

<u>Subclaim 4.3a.3</u> $a^0 = a'$

<u>Proof</u> a^0 induces a tree embedding from $\bar{W}^- \cap \bar{\delta}$ into $W_\delta \restriction \gamma+1 = W^- \restriction \gamma+1$. Namely, for $n \in \omega$ let γ_n be s.t.

$$E_n = E_{\gamma_n}$$

and let $\bar{\gamma}_n = \pi^{-1}(\gamma_n)$. Then the γ_n's are cofinal in γ, and

$$k_n = \text{least } \xi \text{ s.t. } W_\xi \restriction \gamma_n+1 = W^- \restriction \gamma_n+1$$

$$i_n = \text{least } \xi \text{ s.t. } W_\xi \restriction \bar{\gamma}_n+1 = \bar{W}^- \restriction \bar{\gamma}_n+1.$$

We then have that

$$\Phi^{\nu}_{a^0_n} \restriction \bar{\gamma}_n+1 = \Gamma_{\pi, W_{i_n}} \restriction \bar{\gamma}_n+1$$

$$= \Phi^{\nu}_{a^0_m} \restriction \overline{\gamma_n+1} \quad \text{for all } m \geq n$$

Remark If $\Phi : \mathcal{T} \to \mathcal{U}$ is a tree embedding, then $\Phi \restriction \xi$ is the part of Φ that acts on $\mathcal{T} \restriction \xi$.

Set $\bar{\Phi}^-_{a^0} = \bigcup_n \bar{\Phi}^{\nu}_{a^0_n} \restriction \bar{\gamma}_n + 1$. Then $\bar{\Phi}^-_{a_0}$ is a tree embedding from $\bar{\mathcal{W}}^-$ to $\mathcal{W}^- \restriction \gamma$ whose ρ-map agrees with π, and whose t and s maps do as well, since $\bar{\Phi}^-_{a_0} = \bigcup_n \Gamma_{\pi, W_{i_n}} \restriction \bar{\gamma}_n + 1$. Also, since $\pi(\bar{b}) = b$, we can extend $\bar{\Phi}^-_{a_0}$ to $\bar{\Phi}_{a_0} : \bar{\mathcal{W}}^- {}^\frown \bar{b} \longrightarrow \mathcal{W}^- \restriction \gamma + 1$ by setting $u^{\bar{\Phi}}(\bar{\theta}) = \gamma$ and

$$t^{\bar{\Phi}_{a_0}}_{\bar{\theta}}\left(i^{\bar{W}^-}_{\xi, \bar{b}}(x)\right) = i^{W^-}_{\xi, b\cap\gamma}\left(t^{\bar{\Phi}^-_{a^0}}_{\xi}(x)\right).$$

We then have

$$\begin{aligned} t^{\bar{\Phi}_{a_0}}_{\bar{\theta}} &= \text{canonical embedding of } M^{\bar{W}^-}_{\bar{\theta}} \\ &\qquad \text{into } Ult(M^{W^-}_{\bar{\theta}}, a^0) \\ &= \text{canonical embedding of } M^{\bar{W}^-}_{\bar{\theta}} \text{ into} \\ &\qquad Ult(M^{\bar{W}^-}_{\bar{\theta}}, E_\pi \restriction \mathcal{E}^{W^-}_{\gamma}) \end{aligned}$$

It follows that a^0 enumerates the missing whole initial segments of tail factors of the ~~$E_\pi M_{\bar\theta}$~~ E_π ultrapower of $M_{\bar\theta}^{\bar w^-}$ that are of lh $< \varepsilon_\gamma^{w^-}$. But this much of the E_π ultrapower's factor sequence is determined by $M_{\bar\theta}^{\bar w^-} \restriction \bar\varepsilon$, and $M_{\bar\theta}^{\bar w^-} \restriction \bar\varepsilon = M_{\bar\theta}^{\bar\nu^-} \restriction \bar\varepsilon$. ~~So what we~~

Since what we just said about a^0 holds for a' vis-a-vis $\bar\nu^-$, a' is the factor sequence associated to $Ult(M_{\bar\theta}^{\bar\nu^-}, E_\pi)$, restricted to lh $< \varepsilon_\gamma$. So ~~a A~~

$a^0 = a'$ $\boxtimes$

This proves Subclaim 4.3a, that $z_\eta(\nu) <_A^\nu \delta$. $\boxtimes$

Notation Set $\alpha = a^0 = a'$. $= a(\cancel{\gamma}, z_\eta(\nu), \delta)$

~~Remark~~ The E_π-ultrapower, or the a^0 ultrapower, could be discontinuous at $\bar\varepsilon$. We have $\varepsilon_\gamma = \sup i_{a^0}{}'' \bar\varepsilon = \sup \pi'' \bar\varepsilon$ and only have $M_\gamma^{w^-} | \varepsilon_\gamma = M_\gamma^{\nu^-} | \varepsilon_\gamma$ at this point. But now we argue as before.

<u>Subclaim 4.3b</u> $M_b^{\mathcal{W}^-} \restriction \mathcal{E}^{+, M_b^{\mathcal{W}^-}} = M_c^{\mathcal{V}^-} \restriction \mathcal{E}^{+, M_c^{\mathcal{V}^-}}$.

<u>Proof</u> Same as the proof of Subclaim 4.1b. ⊠

The rest of the proof of 4.3 is the same as the proof of 4.1. In outline:

Let

$$R = M_b^{\mathcal{W}^-} \restriction \mathcal{E}^{+, M_b^{\mathcal{W}^-}}$$

$$\bar{R} = \pi^{-1}(R)$$

$$t = s_{\bar{\theta}}^{\ulcorner \pi, \bar{\mathcal{W}}^- \frown \bar{b}} \restriction \bar{R}$$

$$= s_{\bar{\theta}}^{\ulcorner \pi, \bar{\mathcal{V}}^- \frown \bar{c}} \restriction \bar{R}$$

$t = t_\infty^{\mathcal{R}_0} \restriction \bar{R}$, where $\mathcal{R}_0 : \bar{\mathcal{W}} \frown \bar{b} \to \mathcal{W}^- \restriction \gamma+1$ is associated to the α-ultrapower (i.e. $\mathcal{W}^- \restriction \gamma+1 = \mathcal{X}(\bar{\mathcal{W}}^- \frown \bar{b}, \alpha)$)

and $t = t_\infty^{\Omega_1}$ where $\Omega_1 : \bar{\mathcal{V}}^- \restriction \bar{c} \longrightarrow$
$\mathcal{V}^- \restriction_{\gamma+1} = X(\bar{\mathcal{V}}^- \restriction \bar{c}, \alpha)$. Let

$$N = M_\gamma^{W_s} / t(\bar{\varepsilon})^{+, M_\gamma^{W_s}}$$
$$= \mathrm{Ult}(\bar{R}, \alpha).$$

Subclaim 4.3$\overset{c}{\cancel{\text{d}}}$

(1) $N = \mathrm{Ult}(\bar{R}, E_\pi \restriction \varepsilon_\gamma)$, and t is the canonical embedding.

(2) Let $k : N \longrightarrow \pi(R)$ be the factor map; then

$$k = i_{\gamma, b}^{W^-} \restriction N$$
$$= i_{\gamma, c}^{\mathcal{V}^-} \restriction N.$$

Proof See 4.1e.

$\boxtimes$

Corollary $e_{b-\gamma}^{\mathcal{W}^-} = e_{c-\gamma}^{\mathcal{V}^-}$.

Let $e = e^{\nu^-}_{b-\gamma} = e^{\nu^-}_{c-\gamma}$.

Subclaim ~~4.3a.iv~~ 4.3d $\quad d^\frown e$ is a branch of $\hat{\mathcal{K}}^{\nu,\mathrm{ext}}$ and of $\hat{\mathcal{J}}^{\nu,\mathrm{ext}}$.

Pf See 4.1f. $\boxtimes$

Subclaim ~~4.3a.v~~ 4.3e $\quad (\nu, d^\frown e)$ is justified by b and c.

Proof See 4.1g $\boxtimes$

~~Subclaims 4.3a.1 – 4.3a.7 show that~~ $\cancel{z_{\pi(\eta)}(\omega_1) < z(\omega_1)}$

Subclaim 4.3f $\quad z_\eta(\nu) < z(\nu)$.

Proof $z_{\pi(\eta)}(\omega_1) < z(\omega_1)$ by 4.3a – 4.3e. $\boxtimes$

<u>Subclaim 4.3g</u> For $\alpha \in \{z_\eta(\nu), z_\eta(\nu)+\nu\}$,

$\alpha <_A^{\vec{\nu}} \pi(\alpha)$ and $\Phi^{\vec{\nu}}_{\alpha,\pi(\alpha)} = \Gamma_{\pi,\nu_\alpha}$ and

$\psi^{\vec{\nu}}_{\alpha,\pi(\alpha)} = \Gamma_{\pi,\nu_\alpha}$.

<u>Proof</u> Same as the proof of 4.1h.

$\boxtimes$

Together 4.3a – 4.3g prove $(\dagger)_{\eta+1}$.

So we have proved 4.3.

Subclaim 4.3 $\boxtimes$

Subclaims 4.1 – 4.3 prove Claim 4,

that $(\dagger)_\eta \longrightarrow (\dagger)_{\eta+1}$ for all η.

Claim 4 $\boxtimes$

Claims 1–4 show $z_\eta(\nu)$ is defined for all $\eta < \omega_1$,
contrary to 10.6. Thus $z(\omega_1)$ is undefined,
i.e. our process terminates at some countable stage.

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