§10. The general limit step

Let ϑ be a limit ordinal, and suppose that we have defined $\hat{W}_\alpha$ and $\hat{\vartheta}_\alpha$ for $\alpha < \vartheta$. At the same time the associated systems $\hat{W}^\alpha$, $\hat{\vartheta}^\alpha$ and their $\hat{\Phi}^\alpha_s$ and $\hat{\psi}^\alpha_s$ have been determined.

We shall define $\hat{W}_\vartheta$ and $\hat{\vartheta}_\vartheta$, a limit or "completion" of the $\langle(\hat{W}_\alpha, \hat{\vartheta}_\alpha) \mid \alpha < \vartheta\rangle$ system. These are defined in stages $\hat{W}_{\vartheta,\eta}$, $\hat{\vartheta}_{\vartheta,\eta}$ that we call "completion stages". We begin by setting

$$\hat{W}_{\vartheta,0} = \bigcup_{\alpha<\vartheta} \hat{W}_\alpha = \langle (W_i, W_i^*) \mid i < z_0(\vartheta)\rangle$$

and

$$\hat{\vartheta}_{\vartheta,0} = \bigcup_{\alpha<\vartheta} \hat{\vartheta}_\alpha = \langle (\vartheta_i, \vartheta_i^*) \mid i < z_0(\vartheta)\rangle,$$

where

$$z_0(\vartheta) = \sup_{\alpha<\vartheta} z(\alpha).$$

We discussed the steps ~~from $\hat{\vartheta}_{\vartheta,0}$ to $\hat{\vartheta}_{\vartheta,1}$ in the special case $\vartheta = \omega$ in §8.~~ from $(\hat{W}_{\vartheta,0}, \hat{\vartheta}_{\vartheta,0})$ to $(\hat{W}_{\vartheta,1}, \hat{\vartheta}_{\vartheta,1})$ and from $(\hat{W}_{\vartheta,1}, \hat{\vartheta}_{\vartheta,1})$ to $(\hat{W}_{\vartheta,2}, \hat{\vartheta}_{\vartheta,2})$ in the special case $\vartheta = \omega$ in §8. We are going to change those definitions slightly so that the step from $(\hat{W}_{\vartheta,\eta}, \hat{\vartheta}_{\vartheta,\eta})$ to $(\hat{W}_{\vartheta,\eta+1}, \hat{\vartheta}_{\vartheta,\eta+1})$

becomes uniform over all η. This will essentially amount to taking the step from $(\omega, 1)$ to $(\omega, 2)$ as our paradigm.

Let us describe first the ~~step for~~ first completion step, from $(\vartheta, 0)$ to $(\vartheta, 1)$.

§10.1 $\hat{W}_{\vartheta,1}$ and $\hat{\mathcal{V}}_{\vartheta,1}$

Let $W^-_{\xi_0(\vartheta)}$ and $\mathcal{V}^-_{\xi_0(\vartheta)}$ be the common part trees:

$$\mathcal{S} \trianglelefteq W^-_{\xi_0(\vartheta)} \text{ iff for all sufficiently large } \alpha < \vartheta, \ \mathcal{S} \trianglelefteq W_\alpha,$$

and

$$\mathcal{S} \trianglelefteq \mathcal{V}^-_{\xi_0(\vartheta)} \text{ iff for all sufficiently large } \alpha < \vartheta, \ \mathcal{S} \trianglelefteq \mathcal{V}_\alpha.$$

Lemma 10.1 $W^-_{\xi_0(\vartheta)}$ and $\mathcal{V}^-_{\xi_0(\vartheta)}$ have the same limit length.

Pf. See the proof of Lemma 8.2. Once $W_\alpha \restriction \xi+1$ stabilizes, $W_\alpha \restriction \xi+2$ can only change finitely often. $\boxtimes$

Let

$$b_0^\nu = \Sigma(\mathcal{W}^-_{z_0(\nu)})$$

and

$$c_0^\nu = \Lambda(\mathcal{V}^-_{z_0(\nu)}).$$

<u>Definition 10.2</u> (a) We say <u>b_0 justifies (γ, d)</u> iff $\gamma < \nu$, d is a cofinal branch of $(\hat{\mathcal{W}}^\gamma_{\nu,0})^{ext}$, all of whose proper initial segments are A-branches, and setting $\beta = \beta^\gamma_d$ (so that $u^\gamma_d(\beta) = z_0(\nu)$), there is

$$s \in (\mathcal{W}^*_\beta)^{ext}$$

such that

$$e_{b_0} = \hat{p}^\gamma_d(s) = s \circledast d.$$

(b) Similarly, <u>c_0 justifies (γ, d)</u> iff $\gamma < \nu$, d is a cofinal branch of $(\hat{\mathcal{V}}^\gamma_{\nu,0})^{ext}$, and setting $\beta = \beta^\gamma_d$ there is

$$s \in (\mathcal{V}^*_\beta)^{ext}$$

such that

$$e_{c_0} = \hat{q}^\gamma_d(s) = s \circledast d.$$

<u>Remarks</u>

(1) We ~~could~~ allow $\gamma = 0$. ~~probably~~ $\hat{\mathcal{W}}^0$ is a tree on $\mathcal{W}^*_0$, so we'd have $\beta^0_d = 0$ in this case.

(2) $s \in (\mathcal{W}^*_\beta)^{ext}$ implies that s is by Σ. So d is being justified on the Σ-side by inflating a branch according to Σ into a branch according to Σ.

(3) We allow $s \circledast d = (s \circledast d \restriction k)^\frown d^{\geq k}$, i.e. $s \circledast d \equiv_{tail} d$ is possible.

Lemma 10.3

(a) If b_0 justifies (γ, d), then d fits into e_{b_0}.

(b) For any γ, there is at most one d such that b_0 justifies (γ, d).

(c) Suppose b_0 justifies (γ, d); then for $\gamma < \delta < \nu$, there is a unique a such that b_0 justifies (δ, a); moreover, a fits into d. (d) Similarly on the ν-side.

Proof (a) follows from the fact that $e_{b_0} = s \circledast d$ for some s. Part (b) follows from the fact that at most one cofinal branch of $(\hat{W} \gamma)^{ext}$

can fit into e_{b_0}.

Remark The fit has to be cofinal in e_{b_0}, because for all $k \in \mathrm{dom}(e_{b_0})$, $e_{b_0} \upharpoonright k \in W_\alpha^{ext}$ for all $\alpha \geq \alpha_k$, some $\alpha_k < \nu$. So if $F \in \mathrm{ran}(d)$ and $F = F_\eta$ for $\eta > \alpha_k$, then $lh(F) > lh(e_{b_0}(n))$ for all $n < k$, so F must fit into some $e_{b_0}(j)$ for $j \geq k$.

Part (c) follows from the commutativity of our inflating maps, as described in 9.5 and 9.6. Let $\beta_d = \beta_d^\gamma$,

See also pp. 106–125

and

$$d = d_0{}^\frown d_1{}^\frown d_2$$

where the d_i's are $\neq \emptyset$, and let

$$i_0 = u^\gamma_{d_0}(\beta_d)$$

and

$$i_1 = u^\gamma_{d_1}(i_0).$$

Suppose d_0, d_1 are long enough that $[\beta_d]_\delta \neq [i_0]_\delta$ and $[i_0]_\delta \neq [i_1]_\delta$, so that

$$[\beta_d]_\delta <^\delta_A [i_0]_\delta <^\delta_A [i_1]_\delta .$$

~~By 9.6 we have $k_0 < Z(\delta)$ such that $\beta_d <^\gamma_A k_0$ and $k_0 <^\delta_A i_0$. Let~~

$$\cancel{a_0 = a(\delta, k_0, i_0),}$$

$$\cancel{l_0 = u^\delta_{a_0}(}$$

Let a_0 be the segment of $dw^\delta_{i_0}$ witnessing $[\beta_d]_\delta <^\delta_A [i_0]_\delta$, and a_1 the segment of $dw^\delta_{i_1}$ witnessing $[i_0]_\delta <^\delta_A [i_1]_\delta$. (So $a_0{}^\frown a_1$ witnesses $[\beta_d]_\delta <^\delta_A [i_1]_\delta$. Let

$$u^\delta_{a_0}(k_0) = i_0$$

and

$$u^\delta_{a_1}(l_1) = i_1 .$$

We have $k_0 <^{\delta}_{A} i_0$ and $a_0 = a(\delta, k_0, i_0)$,

and by 9.6

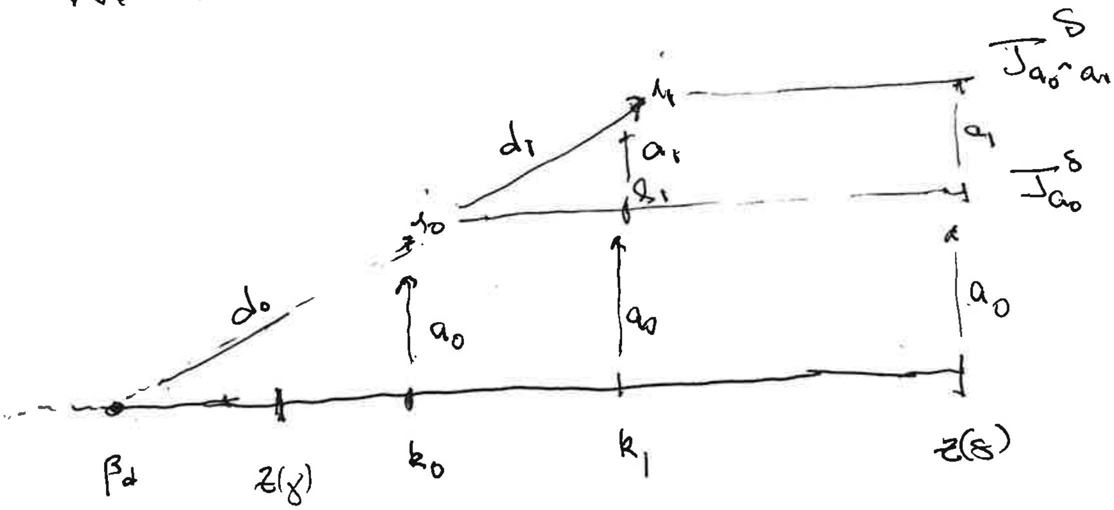
$$\beta_d <^{\delta}_{A} k_0$$

and

$$d_1 <^{\delta}_{A} i_1 .$$

Let $k_1 = (u^{\delta}_{a_0})^{-1}(d_1)$, so $k_1 < z(\delta)$ and $k_1 <^{\delta}_{A} i_1$.

We have the diagram



Clearly, $k_1 <^{\delta}_{A} i_1$ by transitivity, and $a(\delta, k_1, i_1) = a_0{}^\frown a_1$. We draw things as if $i_0 \neq d_1$, but $i_0 = d_1$ and $a_1 = d_1$ is possible.

<u>Claim</u> $k_0 \leq^{\gamma}_{A} k_1$

<u>Proof</u> It's enough to see $dw^{\gamma}_{k_0} \trianglelefteq dw^{\gamma}_{k_1}$, for then the u-maps commute in the way we want. If $t_0 \ntrianglelefteq t_1$, Let $t_0 = dw^{\gamma}_{k_0}$ and $t_1 = dw^{\gamma}_{k_1}$. then we have $H \in \operatorname{ran}(t_1)$ such that

(i) $\kappa_H < \lambda(t_0(n))$ for some n,

and (ii) $\lambda_H > \lambda(t_0(n))$ for all n.

(For (ii), notice $t_0 \circledast a_0 = d_0$ and $t_1 \circledast a_0{}^\frown a_1$ exists, so $\lambda_H < \lambda(t_0(n))$ for some n is impossible — ~~$X(W_{k_0}, d_0$~~ $X(W_{k_1}, a_0)$ would not be defined if W_{k_1} disagreed with W_{k_0} on some extender of $lh < \lambda(t_0(n))$ for some n.)

But then $p^\delta_{a_0}(H) \notin \operatorname{ran}(d_0)$, and $p^\delta_{a_1}(p^\delta_{a_0}(H)) \in \operatorname{ran}(d_1)$. Since $\operatorname{crit}(p^\delta_{a_0}(H)) < \lambda(d_0(n))$ for some n, ~~but $d_0 \triangleleft d_1$~~ and $\operatorname{crit}(d_1(0)) > \lambda(d_0(n))$ for all n, $\operatorname{crit}(p^\delta_{a_1}(p^\delta_{a_0}(H))) < \lambda(d_0(n))$ for some n. This contradicts $d_0 \triangleleft d_1$.

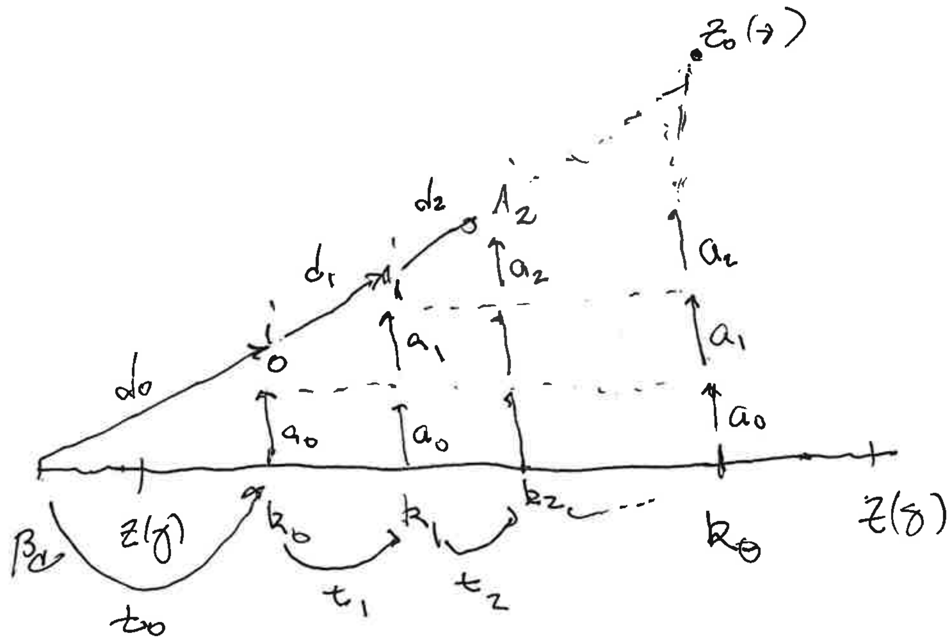
Claim. $\boxtimes$

Now let $\langle i_n \mid n < \theta \rangle$ be a strictly increasing sequence in $<_A^\gamma$ such that $\beta_\alpha <_A^\gamma i_0$, and $a(\gamma, \beta_\alpha, i_n) \triangleleft \alpha$ for all $n < \theta$, and $[i_n]_\gamma \neq [i_m]_\gamma$ if $m \neq n$.

Remark It is part of "b_0 justifies (γ, α)" that each proper initial segment of α is an A-branch of $\hat{W}^\gamma$.

Remark We can take $\theta = \omega$ unless we have reached $\nu = \omega_1$, in which case we can take $\theta = \omega_1$.

We have the diagram above, but extended:



Here

$$k_\Theta = \sup_{n<\Theta} k_n'$$

Let

$$a = \text{concatenation of } a_i\text{'s}$$
$$t = \text{concatenation of } t_i\text{'s}$$

So ~~since~~ by the way things fit together

$$\hat{p}_a(t) = d .$$

~~Otherwise~~ Let $s \in (W^{*}_{\beta_d})^{ext}$ be such that $\hat{p}_d(s) = e_{b_0}$. Let

$$\mathcal{S} = X(W^{*}_{\beta_d}, t) .$$

Notice that this makes sense, because all proper initial segments of t are A-branches, and thus no new D-extenders are introduced along t (beyond those with length $< lh(t(n))$, some n, that come from "side models/trees").

(In other words, thinking of t as a long extender, its measures concentrate on $P^{*}_{\beta_d}$.)

Let

$$v = p_t(s).$$

Note $\hat{p}_a$ gives a weak true embedding of $\mathcal{A}$ into $W^-_{z_0(\gamma)} \wedge b$, so that $\mathcal{A}$ is according to Σ. Moreover

$$\hat{p}_a(v) = \hat{p}_a(\hat{p}_t(s))$$

$$= \hat{p}_d(s)$$

$$= e_b,$$

So v justifies (γ, t).

<u>Claim 2</u> $t = dw^\gamma_{k_\theta}$ (= branch at $\hat{w}^\gamma$ chosen by $\hat{\Sigma}$ = meta-strategy induced by Σ)

"<u>Proof</u>" If $dw^\gamma_{k_\theta}$ is an A-branch, then this follows from the fact that W_{k_θ} was defined to be an inflation along $dw^\gamma_{k_\theta}$.

Hence $W_{k_\theta} = \mathcal{A}$

And so $dw^\gamma_{k_\theta}$ fits into v, so $= t$ by (b) of lemma.

It is possible that $dw^\gamma_{k_\theta}$ is not an A-branch, e.g. $k_\theta = z(\gamma)$ is possible. In that case, this seems to be an assumption on Σ: that it induces a strategy $\hat{\Sigma}$

such that ~~letting~~ defining $dw^\gamma_{k_\Theta}$ as $\hat{\Sigma}(\hat{W}^\gamma \restriction k_\Theta)$, and letting ~~W~~ W be any tree indexed in $\mathcal{J}^\gamma_{dw^\gamma_{k_\Theta}}$, W is by Σ. (In this case, W_{k_Θ} could be a proper initial segment of Δ, but it would still have $\hat{P}_t(s) \in W^{ext}_{k_\Theta}$.)

Lemma 10.3 $\boxtimes$

<u>Remark</u> We need to determine in much greater detail what is being assumed about Σ. The property we just described for $\hat{\Sigma}$ is like the way a meta-strategy Σ^* (for meta-trees on some $\mathcal{J}$ by Σ) is determined by Σ. See [2], [5].

<u>Remark</u> It might be better to drop the requirement that all proper initial segments of d are A-branches from "b_0 justifies (γ, d)". That way it becomes just a ~~requirement~~ property of the $\hat{W}$-system; $\hat{\Sigma}$ is not involved. "b_0 justifies (γ, d)" becomes equivalent to "$\hat{\Sigma}(\hat{W}^\gamma_{\omega,0}) = d$" for our $\hat{\Sigma}$ induced by Σ.

We would then add the requirement that all proper initial segments of d are A-branches to the case hypotheses of Cases 1 and 2 below.

We return to defining $\hat{W}_{\nu,1}$ and $\hat{\nu}_{\nu,1}$.

<u>Case 1</u> There is no ~~[illegible]~~ (γ, d) such that both b_0 and c_0 justify (γ, d).

In this case

$$z(\nu) = z_0(\nu)$$

$$W_{z(\nu)} = W^-_{z_0(\nu)} {}^\frown b_0$$

$$\nu_{z(\nu)} = \nu^-_{z_0(\nu)} {}^\frown c_0$$

$\hat{W}_\nu = \hat{W}_{\nu,1}$ is just $\hat{W}_{\nu,0}$ together with $W_{z(\nu)}$, and the cofinal branches of the $\hat{W}^\gamma_{\nu,0}$ for $\gamma < \nu$ that are chosen by $\hat{\Sigma}$. That is,

$$dw^\gamma_{z(\nu)} = \hat{\Sigma}(\hat{W}^\gamma_{\nu,0})$$

and

$$dv^\gamma_{z(\nu)} = \hat{\Lambda}(\hat{V}^\gamma_{\nu,0})$$

for $\gamma < \nu$. In the present case, none of these are A-branches.

<u>Case 2</u> There is a (γ, d) such that b_0 and c_0 both justify (γ, d).

Fix such a γ. For $\eta \geq \gamma$, let

d^η = unique d such that both b_0 and c_0 justify (η, d).

This is well-defined by 10.3.

<u>Lemma 10.4</u> $\beta_{d^{\gamma+1}} < z(\gamma+1)$.

<u>Proof</u> The same as that of Lemma 8.7.

(sketch: Otherwise $\beta_{d^{\gamma+1}} = z(\gamma+1)$. But then $\hat{P}_{d^{\gamma+1}}(e_{b_\gamma}) \trianglelefteq d^\gamma$ and $\hat{P}_{d^{\gamma+1}}(e_{c_\gamma}) \trianglelefteq d^\gamma$ by condensation for $\hat{\Sigma}$ and $\hat{\Lambda}$ .... or maybe its normalizing well. This contradicts $b_\gamma \neq c_\gamma$.)

$\boxtimes$

We now define $\hat{W}^\eta_{\nu,1}$ to be ~~the~~ $\hat{W}_{\nu,0}$ together with the union over $\eta < \nu$ of the inflation of the trees in the block $I^\eta_{d^\eta}$ by d^η. This gives us a block $J^\eta_{d^\eta}$ of trees, for each $\gamma \leq \eta < \nu$.

For $i \in I^{\eta}_{d^\eta}$, $k = u^{\eta}_{d^\eta}(i)$ is the index of $W_k = X(W_i^+, d^\eta)$ and of $\nu_k = X(\nu_i^+, d^\eta)$. $(\hat{\Phi}_{d^\eta})_i$ and $(\hat{\Psi}_{d^\eta})_i$ are the associated weak tree embeddings. The length of our ~~de~~ new sequence of slow $Z(\alpha, M)$ comparisons is

$$Z_1(\nu) = \sup_{\eta < \nu} (\max J^{\eta}_{d^\eta}).$$

It is a limit ordinal because

$$u^{\eta}_{d^\eta}(Z(\eta)) < u^{\xi}_{d^\xi}(Z(\xi))$$

for $\gamma < \eta < \xi$.

We need to define the $\hat{W}^{\xi}_{\nu,1}$ and $\hat{\nu}^{\xi}_{\nu,1}$ for $\xi < \nu$ as well. Fix ξ, and let $\eta > \gamma$ and $\eta > \xi$ and

$$u^{\eta}_{d^\eta}(i) = k,$$

then

$$dw^{\xi}_k = \hat{p}^{\eta}_{d^\eta}(dw^{\xi}_i)$$

and

$$dv^{\xi}_k = \hat{p}^{\eta}_{d^\eta}(dv^{\xi}_i).$$

dw_k^ξ and dv_k^ξ depend only on ξ and k, they are independent of η by the consistency of our inflation maps.

The dw_k^ξ and dv_k^ξ determine the most of $\hat{W}^\xi_{\nu,1}$ and $\hat{\mathcal{V}}^\xi_{\nu,1}$, in particular the orders $<^\xi_A$, and the maps $\Phi^\xi_{ik} : W_i^* \to W_k$ and $\Psi^\xi_{ik} : ~~W_i^*~~ \to \mathcal{V}_k$ for $i <^\xi_A k$, where $k < z_\eta(\nu)$.

This finishes the step from $(\hat{W}_{\nu,0}, \hat{\mathcal{V}}_{\nu,0})$ to $(\hat{W}_{\nu,1}, \hat{\mathcal{V}}_{\nu,1})$. The general step from ~~$(\hat{W}_{\eta,\nu}$~~ $(\hat{W}_{\nu,\eta}, \hat{\mathcal{V}}_{\nu,\eta})$ to $(\hat{W}_{\nu,\eta+1}, \hat{\mathcal{V}}_{\nu,\eta+1})$ is exactly the same. $\hat{W}_{\nu,\eta}$ contains a sequence of trees $\langle W_i, W_i^* \mid i < z_\eta(\nu)\rangle$ where $z_\eta(\nu)$ is a limit ordinal. (For $i \geq z_0(\nu)$, $W_i = W_i^*$.) $W^-_{z_\eta(\nu)}$ is the common part tree, and $b_\eta^\nu = \Sigma(W^-_{z_\eta(\nu)})$. Similarly we have

Remark "b_η^ν" is not good notation. We already used $b_\eta = \Sigma(Y_\eta)$ for something that's not really parallel.

$\bar{\mathcal{V}}_{\varepsilon_\eta(\nu)}$ and $c_\eta^\nu = \Delta(\bar{\mathcal{V}}_{\varepsilon_\eta(\nu)})$. We have $\hat{W}^\xi_{\nu,\eta}$ and $\hat{\mathcal{V}}^\xi_{\nu,\eta}$ as part of our data. If there is no (γ, d) such that $\gamma < \nu$ and d is a cofinal branch of both $\hat{W}^\gamma_{\nu,\eta}$ and $\hat{\mathcal{V}}^\gamma_{\nu,\eta}$ that is justified by b_η^ν and c_η^ν, then we are in case 1. We set $\hat{W}_{\nu,\eta+1} = \hat{W}_{\nu,\eta}$ plus $W_{\varepsilon_\eta(\nu)}$ $(= \bar{W}_{\varepsilon_\eta(\nu)} \frown b_\eta)$ and similarly for $\hat{\mathcal{V}}_{\nu,\eta+1}$. If there is such a (γ, d), then for all $\xi \geq \gamma$ we have a unique d^ξ such that b_η^ν and c_η^ν justify (ξ, d^ξ). We use the d^ξ's to inflate our system to $\hat{W}_{\nu,\eta+1}$ and $\hat{\mathcal{V}}_{\nu,\eta+1}$, as in Case 2.

Again, we get that $\beta_{d^{\gamma+1}} < \varepsilon(\gamma+1)$ if d^γ and $d^{\gamma+1}$ are defined, as in 8.7.

Remark In case 2 we are asking for (γ, δ) justified by both b^{ν}_{η} and c^{ν}_{η} with δ not just a branch tail in $\hat{W}^{\gamma}_{\nu,\eta}$) but an actual full branch.

So $\beta_{\delta\gamma} \leq z(\gamma)$).

At limit η, we let $\hat{W}_{\nu,\eta}$ be the "union" of all $\hat{W}_{\nu,\xi}$ for $\xi < \eta$, and similarly on the $\mathcal{V}$ side.

$$z_\eta(\nu) = \sup_{\xi<\eta} z_\xi(\nu))$$

if η is a limit.

The process goes on until we reach η such that Case I applies to $\hat{W}_{\nu,\eta}$ and $\hat{\mathcal{V}}_{\nu,\eta}$. When that happens, we set $z(\nu) = z_\eta(\nu)$, $\hat{W}_\nu = \hat{W}_{\nu,\eta+1}$, and $\hat{\mathcal{V}}_\nu = \hat{\mathcal{V}}_{\nu,\eta+1}$.

(So $W_{z(\nu)} = W^-_{z_\eta(\nu)} \frown b^\nu_\eta$ and $\mathcal{V}_{z(\nu)} = \mathcal{V}^-_{z_\eta(\nu)} \frown c^\nu_\eta$ in this case.) We show below that the process does stop.

<u>Lemma 10.5</u> Suppose $\eta \leq \gamma$ and $z_\gamma(\nu)$ is defined, and let $z_\eta(\nu) \leq i \leq z_\gamma(\nu)$; then $\left(W^-_{z_\eta(\nu)} \cap b^\nu_\eta\right) \trianglelefteq W_i$. Similarly on the η-side. Moreover, if $z_{\eta+1}(\nu) \leq i$, then $\left(W^-_{z_\eta(\nu)} \cap b^\nu_\eta\right) \trianglelefteq W_i$.

<u>Proof</u> If Case 1 applied at η, then $z(\nu) = z_\eta(\nu)$, so $i = z_\eta(\nu)$. Moreover, $W_i = W^-_{z_\eta(\nu)} \cap b^\nu_\eta$ by definition. So we may assume case 2 applied at η.

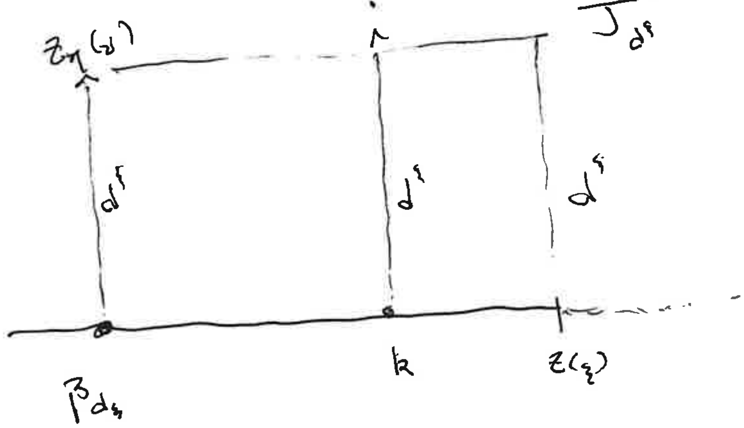
Suppose first that $z_\eta(\nu) \leq i < z_{\eta+1}(\nu)$.

Let d^ξ for $\xi \geq \gamma$ be such that b^ν_η and c^ν_η justified (ξ, d^ξ), (so i.e. $\hat{z}(\hat{W}^\xi_{\nu,\eta}) = d^\xi = \hat{\Lambda}(\hat{\nu}^\xi_{\nu,\eta})$.)

Let ξ be large enough that

$$u^\xi_{d^\xi}(k) = i$$

for some k. We have the diagram



letting $\beta = \beta^{\xi}_{d^{\xi}}$, and $\rho^{a}_{d^{\xi}}(s) = e_{b^{\nu}_{\eta}} = e_b$,
we have that d^{ξ} fits cofinally into e_b,

and
$$W_{z_\eta(\nu)} = X(W^{*}_{\beta}, d^{\xi})$$
and
$$W_i = X(W^{*}_{k}, d^{\xi}).$$

But since the fit is cofinal, and $X(W^{*}_{k}, d^{\xi})$ is defined, we must have also that all $s(n) \in \mathrm{Ext}(W^{*}_{k})$, and thus $s \in (W^{*}_{k})^{ext}$ since both W^{*}_{β} and W^{*}_{k} are by $\mathcal{E}$.

That is, we have
$$s = e^{W^{*}_{\beta}}_{\lambda} = e^{W^{*}_{k}}_{\lambda}$$
and $W^{*}_{\beta} \restriction \lambda+1 = W^{*}_{k} \restriction \lambda+1$, which gives
$$\begin{aligned} W^{-}_{z_\eta(\nu)} {}^\frown b^{\nu}_{\eta} &= X(W^{*}_{\beta} \restriction \lambda+1, d^{\xi}) \\ &= X(W^{*}_{k} \restriction \lambda+1, d^{\xi}) \\ &\trianglelefteq X(W^{*}_{k}, d^{\xi}) = W_i, \end{aligned}$$

as desired.

If $i = z_{\eta+1}(\nu)$, then W_i extends the common part tree at $\eta+1$, so $W^{-}_{z_\eta(\nu)} {}^\frown b^{\nu}_{\eta} \triangleleft W_i$.

For $z_{\eta+1}(\nu) < i \le z_{\eta+2}(\nu)$, we have $W^-_{z_{\eta+1}(\nu)} {}^\frown b^\nu_{\eta+1} \trianglelefteq W_i$ by what we just showed. Thus $W^-_{z_\eta(\nu)} {}^\frown b^\nu_\eta \triangleleft W_i$. Etc.

$\boxtimes$

There is a diagram of the argument between the W_i for $z_0(\nu) \le i \le z(\nu)$ on page 89.

The following lemma is important for showing that the process producing the $\hat{W}_\nu$ and $\hat{\nu}_\nu$ terminates at some countable ν.

<u>Lemma 10.6</u>

(1) Let $\nu < \omega_1$ be a limit ordinal and $\hat{W}_\alpha, \hat{\nu}_\alpha$ exist for all $\alpha < \nu$; then $z(\nu)$ exists, and $z(\nu) < \omega_1$.

(2) Suppose $\hat{W}_\nu$, $\hat{\nu}_\nu$ exist for all $\nu < \omega_1$; then $z(\omega_1)$ exists, and $z(\omega_1) < \omega_2$.

<u>Proof.</u> For (1): otherwise we have that $\hat{W}_{\nu,\omega_1}$ exists, and we have $\omega_1 = z_{\omega_1}(\nu)$, and the common part tree $W^-_{\omega_1}$.

Claim $lh(W^-_{\omega_1}) = \omega_1$.

Proof Otherwise $lh(W^-_{\omega_1}) = \lambda < \omega_1$.

For each $\xi < \lambda$ we get α_ξ s.t.

$W_i \upharpoonright \xi = W_{\alpha_\xi} \upharpoonright \xi$ for all $i \geq \alpha_\xi$.

Let $\gamma = \sup_{\xi<\lambda} \alpha_\xi$, so $\gamma < \omega_1$. Then

$W_i \upharpoonright \lambda = W^-_{\omega_1} \upharpoonright \lambda$ for all $i \geq \gamma$. But

then $E^{W_i}_\lambda$ can only change finitely often

at $i > \gamma$, since its length must go down

at each change point. The eventual

value is $E^{W^-_{\omega_1}}_\lambda$ by definition, so

$lh(W^-_{\omega_1}) > \lambda$, contradiction.

$\boxtimes$

Now let $b = \Sigma(W^-_{\omega_1})$. There are

ω_1 many distinct $E \in ran(e_b)$. By lemma

9.16, if E and F are distinct extenders

in $ran(e_b)$, then $ogn(E) \neq ogn(F)$. But

$ogn(E)$ is atomic, so it is an extender

used in $\hat{W}_{2,0}$. Thus there are only countably many possible ogn(E), contradiction.

The same proof yields (2). Note here that we need that $W_{\omega_2}^-$ has a cofinal branch b at this point. So we seem to need that Σ is an ω_2+1 strategy at this point.

$\boxtimes$