§9 Extender origins.

Recall from 6.2.5a the ordering $<_A^\tau$ on $\mathcal{B}_\tau =_{df} \{[i]_\tau \mid (W_i, \nu_i) \text{ exists}\}$, namely $[i]_\tau <_A^\tau [j]_\tau$ iff $\exists a\,(a \neq \emptyset \wedge dw_j^\tau = dw_i^\tau \cap a \wedge dv_j^\tau = dv_i^\tau \cap a)$. We refine this ordering to a partial order on the τ-blocks on $\{i \mid (W_i, \nu_i) \text{ exists}\}$ as follows:

<u>Def 9.1</u> $i <_A^\tau j$ iff $[i]_\tau <_A [j]_\tau$, and for $a = a(\tau, i, j)$, $u_a^\tau(i) = j$.

So for $x, y \in \mathcal{B}_\tau$, $x <_A^\tau y$ iff $\exists i \in x\, \exists j \in y\, (i <_A^\tau j)$. The order on $\mathcal{B}_\tau$ and the order on $\bigcup \mathcal{B}_\tau$ are easy to distinguish from context, so we shall use $<_A^\tau$ for both.

If $i <_A^\tau j$ for $i, j \in \bigcup \mathcal{B}_\tau$, then letting $a = a(\tau, i, j)$, we have

$$\Phi_{ij}^* : W_i^* \longrightarrow W_j$$

the canonical emb. associated to $\mathrm{Ult}(W_i^*, a) = W_j$, given by

$$\Phi^{\tau}_{ij} = \left(\hat{\Phi}^{\tau}_{a}\right)_{i}.$$

Similarly, $\Psi^{\tau}_{ij} : V^{*}_{i} \longrightarrow V_{j}$ is given by

$$\Psi^{\tau}_{ij} = \left(\hat{\Psi}^{\tau}_{a}\right)_{i}.$$

<u>Lemma 9.2</u> As an order on $\bigcup \mathcal{B}_{\tau}$

(i) $<^{\tau}_{A}$ is transitive,

(ii) For any $j \in \bigcup \mathcal{B}_{\tau}$, $\{i \mid i <^{\tau}_{A} j\}$ is wellordered by $<^{\tau}_{A}$,

(iii) If $i <^{\tau}_{A} j <^{\tau}_{A} k$, then

$$\Phi^{\tau}_{ik} = \Phi^{\tau}_{jk} \circ \Phi^{\tau}_{ij} \text{ and } \Psi^{\tau}_{ik} = \Psi^{\tau}_{jk} \circ \Psi^{\tau}_{ij}.$$

<u>Proof</u> (i) and (iii) are clear. Let us check through (ii).

Proof of ~~9.2~~ 9.2 (ii) Suppose wlog $i < k$.

Let $i <^\tau_A j$ and $k <^\tau_A j$, and τ ,

and
$$s = a(\tau, i, j)$$
$$t = a(\tau, k, j).$$

s and t are tails of dw^τ_j, so they are compatible. Since $i < k$, it must be that t is a tail of s,

that is
$$s = \mu ^\frown t$$
for some $\mu \neq \emptyset$. But then μ is an A-path,

so
$$i <^\tau_A u^\tau_\mu(i)$$
is witnessed by μ. But $u^\tau_t(u_\mu(i)) = u^\tau_s(i) = j$,

so $u^\tau_\mu(i) = u^{\tau\,-1}_t(j) = k$, and
$$i <^\tau_A k$$
is witnessed by μ.

Lemma 9.2 ☒

Some further observations:

Prop 9.2.1 Equivalent are:

(a) $dw_j^\sigma \equiv_{Tail} dv_j^\sigma$,

(b) $\exists i \,(i <^\sigma_A j)$.

Proof. Immediate. ☒

~~Then~~

Definition 9.2.2 If $i <^\sigma_A j$, then we set $np^\sigma_{ij} = \{ k \mid k = i \text{ or } k = j \text{ or } i <^\sigma_A j \}$. We call np^σ_{ij} the normal σ-path from i to j.

np^σ_{ij} is wellordered by $<^\sigma_A$. There are weak tree embeddings $\Phi^\sigma_{k,l}$ and $\Psi^\sigma_{k,l}$ along it, given by segments of $a(\sigma, i, j)$. There are no "z-levels" $z(\gamma)$ in np^σ_{ij}, except perhaps $z(\gamma) = i$, by the following proposition.

~~Sublemma~~ Prop 9.~~2~~.3 Equivalent are

(i) $\exists\sigma\,\exists i\ (i <^{\sigma}_{A} j)$,

(ii) for all γ, $j \neq z(\gamma)$

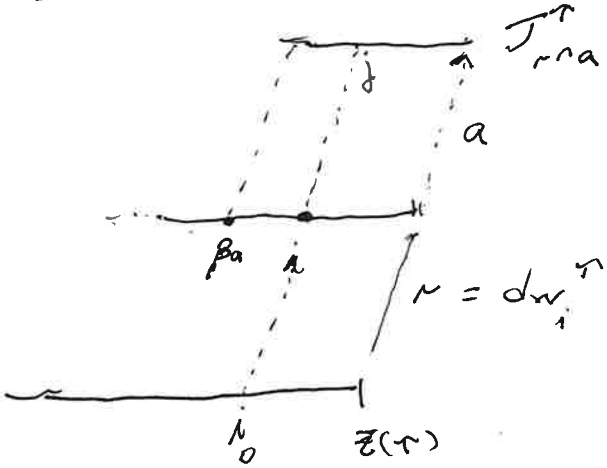
Proof Inspecting the construction, we see that $(W_j, \mathcal{V}_j)$ is obtained via inflation iff $j \neq z(\gamma)$ for all γ. So (ii) → (i). But if $j = z(\gamma)$, it is a root in $<^{\sigma}_{A}$ for all σ by our definitions. So (i) → (ii). ☒

Remark 9.~~2~~.3 will be true at all stages, not just for $j < z(\omega)$. At levels $z(\gamma)$ we shall have $(W^{\sigma}_{z(\gamma)}, \mathcal{V}^{\sigma}_{z(\gamma)})$ a common part tree, with $W_{z(\gamma)} = W^{\sigma}_{z(\gamma)} \cap \Sigma(W^{\sigma}_{z(\gamma)})$ and $\mathcal{V}_{z(\gamma)} = \mathcal{V}^{\sigma}_{z(\gamma)} \cap \Lambda(\mathcal{V}^{\sigma}_{z(\gamma)})$. We will have stopped the completion process because the two sides could not agree on how to inflate any $(W^{*}_{i}, \mathcal{V}^{*}_{i})$ to an extension of $(W_{z(\gamma)}, \mathcal{V}_{z(\gamma)})$. So (i) will fail. On the other hand, if it is possible to inflate along a common A-path, we will, and the completion continues.

We show now that Φ_{ij}^{τ} and Ψ_{ij}^{τ} are independent of τ.

<u>Lemma 9.3</u> Suppose $i <_A^{\sigma} j$ and $i <_A^{\tau} j$; then $\Phi_{ij}^{\sigma} = \Phi_{ij}^{\tau}$ and $\Psi_{ij}^{\sigma} = \Psi_{ij}^{\tau}$.

<u>Proof</u> Assume wlog $\sigma < \tau$. Let $a = a(\tau, i, j)$. Since $i <_A^{\tau} j$, $[j]_\tau \not\subseteq [0]_\tau$, so $j > z(\tau)$. We have the picture from before



Our definition of the inflation process along a leads to

$$dw_j^{\sigma} = dw_i^{\sigma} \circledast a .$$

But $i <_A^{\sigma} j$, so $dw_i^{\sigma} \subseteq dw_j^{\sigma}$. It

Remark $\tau \circledast a = \hat{p}_a(\tau)$ in our previous notation. It's the result of shuffling a into τ. It's Schlutzenberg's notation.

follows that $\lambda(dw_i^\sigma(n)) \leq \text{crit}(a(0))$ for all $n \in \text{dom}(dw_i^\sigma(n))$. (If n is the least counterexample, $dw_j^\sigma(n)$ is a proper reflection of $dw_i^\sigma(n)$.) Thus

$$dw_j^\sigma = dw_i^\sigma \wedge a.$$

But this means $a = a(\sigma, i, j)$ as well, so $\Phi_{ij}^\sigma = \Phi_{ij}^\tau$ and $\psi_{ij}^\sigma = \psi_{ij}^\tau$.

$\boxtimes$

<u>Remark 9.3.1</u> The proof used: if $t \triangleleft t \circledast a$, then $t \circledast a = t \wedge a$.

<u>Lemma 9.4</u> Let $\sigma < \tau$ and suppose $[i]_\sigma <_A^\sigma [j]_\sigma$; then either $[i]_\tau = [j]_\tau$, or $[i]_\tau <_A^\tau [j]_\tau$.

Proof of 9.4

(a) If $\{i\}_T \not<^{T}_{A} \{j\}_T$, then dw_j^T must skip over $\{i\}_T$, in that for some H,

$$dw_j^T = s^\frown\langle H\rangle^\frown t$$

where

$$\beta_H < \min(\{i\}_T) \leq \max(\{i\}_T) < \theta_H + 1.$$

[Here s is the longest initial segment of dw_j^T such that $\theta_s \leq \min(\{i\}_T)$. We have $\max(J_s^T) < \min(\{i\}_T)$ since otherwise $J_s^T = \{i\}_T$ and $\{i\}_T <^{T}_{A} \{j\}_T$. For the same reason, $\theta_H + 1 > \max(\{i\}_T)$: otherwise $J^T_{s^\frown\langle H\rangle} = \{i\}_T$.] But dw_j^T fits into dw_j^σ, so H fits into some extender in dw_j^σ. Since

$$\beta_H < \min(\Sigma i J_\tau) \le \min(\Sigma \lambda J_\sigma),$$

we have on n s.t.

$$\text{crit}(H) < \lambda_{dw_i^\sigma(n)}.$$

But $dw_i^\sigma \subseteq dw_j^\sigma$ and H fits into the latter, so $\lambda_H < \lambda_{dw_i^\sigma(n)}$. Let $K = dw_i^\sigma(n)$.

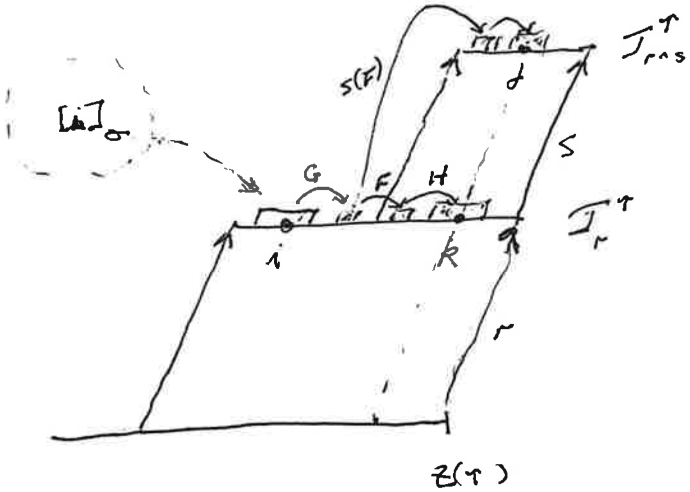
We have

$$\max(\Sigma i J_\sigma) \le \max(\Sigma i J_\tau) \le \theta_H,$$

and H is on the sequence of $M_\infty^{W_{\theta_H}}$, which agrees with the sequence of $M_\infty^{W|\theta_K}$ strictly below $lh(K)$. Thus H is on the sequence of $M_\infty^{W_{\theta_K}}|||lh(K)$. But K is in dw_j^σ, so H is on the sequence of $M_\infty^{W_j}$. This is impossible because $H = F_{\theta_H}$, so it is not on the sequence of any $M_\infty^{W_\ell}$ for $\ell > \theta_H$.

9.4 ☒

Remark: Another way of putting $Q_{\sigma\tau}(a)$ is: if $\sigma < \tau$ and $[i]_\sigma <^\sigma_A [j]_\sigma$, then either $[i]_\tau = [j]_\tau$ or $[i]_\tau <^\tau_A [j]_\tau$.) Here is a diagram:



$s = a(r, k, j)$

$r = dw^\tau_k = dw^\tau_i$

Here $\langle G, F, H\rangle = a(\sigma, i, k)$ and

$\langle G, s(F), s(H)\rangle = \langle G, F, H\rangle * S = a(\sigma, i, j)$. The diagram shows that $[k]_\tau <^\tau_A [j]_\tau$ but $[k]_\sigma \not<^\sigma_A [j]_\sigma$ is possible. In fact, $k <^\tau_A j$ but $[k]_\sigma \not<^\sigma_A [j]_\sigma$ in the situation shown.

Here are two simple lemmas about the relationship of $<^\sigma_A$ to $<^\tau_A$ when $\sigma < \tau$.

Lemma 9.5 Let $\sigma < \tau$, $i <^{\sigma}_{A} k$, and $k <^{\tau}_{A} j$; then equivalent are:

(a) $i <^{\sigma}_{A} j$

(b) letting $a = a(\tau, k, j)$,
$\lambda_E \leq \mathrm{crit}(a(0))$ for all $E \in \mathrm{ran}(dw_i^{\sigma})$,

(c) letting $a = a(\tau, k, j)$ and $dw_k^{\sigma} = dw_i^{\sigma} \frown b$,

$$dw_j^{\sigma} = dw_i^{\sigma} \frown (b \circledast a).$$

Moreover, if one of (a)–(c) holds, then

$$\Phi^{\sigma}_{i,j} = \Phi^{\tau}_{k,j} \circ \Phi^{\sigma}_{i,k} \quad \text{and} \quad \psi^{\sigma}_{i,j} = \psi^{\tau}_{k,j} \circ \psi^{\sigma}_{i,k}$$

Proof By construction, $dw_j^{\sigma} = dw_k^{\sigma} \circledast a$, where $a = a(\tau, k, j)$. Let $b = a(\sigma, i, k)$, so $dw_k^{\sigma} = dw_i^{\sigma} \frown b$. Then

$$dw_j^{\sigma} = (dw_i^{\sigma} \frown b) \circledast a.$$

From this we see that $dw_i^{\sigma} \subseteq dw_j^{\sigma}$ iff (b) iff (c). See remark 9.3.1.

The commutativity if (a) - (c) holds is the consistency of inflations we discussed already: if (a)-(c), then

$$\hat{\Phi}^{\sigma}_{b \circledast a} = \hat{\Phi}^{\tau}_{a} \circ \hat{\Phi}^{\sigma}_{b},$$ ← Need to explain this formula somewhere.

and specializing using that $u_b^{\sigma}(i') = k$ and $u_a(k) = j$,

$$\left(\hat{\Phi}^{\sigma}_{b * a}\right)_i = \left(\hat{\Phi}^{\tau}_{a}\right)_k \circ \left(\hat{\Phi}^{\sigma}_{b}\right)_i,$$

which is just our formula in 9.5 in other notation.

Remark (b) and (c) hold if $dw_i^{\sigma} = \phi$, that is, $i \leq z(\sigma)$. ⊠

<u>Lemma 9.6</u> Let $i \prec_A^{\sigma} j$, and $\sigma < \tau$; then there is exactly one $k \in [i]_{\tau}$ such that $i \precsim_A^{\sigma} k$ and either $k = j$ or $k \prec_A^{\tau} j$.

Proof If $\{j\}_\tau = \{i\}_\tau$, then we can take $k = j$ (and must). So suppose not; then $\{i\}_\tau <^\tau_A \{j\}_\tau$ by 9.4. Let

$$dw_j^\tau = dw_i^\tau \frown a .$$

The u-map u_a^τ of $\hat{\Phi}_a^\tau$ and $\hat{\psi}_a^\tau$ has range $\{j\}_\tau$, so we can ~~thus~~ let $k \in \{i\}_\tau$ with $u_a^\tau(k) = j$. Thus

$$dw_j^\sigma = dw_k^\sigma \circledast a ,$$

so

$$dw_i^\sigma \trianglelefteq dw_k^\sigma \circledast a .$$

This implies $dw_i^\sigma \trianglelefteq dw_k^\sigma$. [Proof: ~~[illegible]~~ If $dw_j^\sigma = dw_k^\sigma \frown a$ and $dw_i^\sigma \ntrianglelefteq dw_k^\sigma$, then $a(0) \in \text{ran}(dw_i^\sigma)$, contrary $\{i\}_\sigma \subseteq \{k\}_\tau$ and $a = a(\tau, k, j)$. So let n be least s.t. $\text{crit}(a(0)) < \lambda(dw_k^\sigma(n))$. Thus

$dw_j^\sigma = dw_k^\sigma \restriction n \frown ((dw_k^\sigma)^{\geq n} \circledast a)$.

None of the extenders in $(dw_k^\sigma)^{\geq n} \circledast a$ occur in our system until $a(0)$ is introduced, which is after $[\dot{i}]_\tau$ has been produced, so after all extenders in dw_i^σ show up. Thus $dw_i^\sigma \trianglelefteq dw_k^\sigma \restriction n$.]

So $[i]_\sigma <_A^\sigma [k]_\sigma$. We~~assume~~ let $dw_k^\sigma = dw_i^\sigma \frown s$ and let $k_0 = u_s^\sigma(i)$, and $\dot{j}_0 = u_a^\tau(k_0)$. It is enough to see that $\dot{j}_0 = \dot{j}$, for then $k_0 = k$ and $i <_A^\sigma k$ as desired. But $i <_A^\sigma k_0$ and $k_0 <_A^\tau \dot{j}_0$, so since $\operatorname{crit}(a(0)) \geq \lambda_E$ for all $E \in \operatorname{ran}(dw_i^\sigma)$, 9.5 tells us that $i <_A^\sigma \dot{j}_0$. But $k_0 \in [k]_\sigma$, so $\dot{j}_0 \in [\dot{j}]_\sigma$, and $i <_A^\sigma \dot{j}$, so $\dot{j} = \dot{j}_0$ ⊠

Let us now look at how extenders in $\bigcup_i Ext(W_i^*) \cup Ext(V_i^*)$ originate.

<u>Definition 9.7</u> For $E \in \bigcup_i Ext(W_i^*) \cup Ext(V_i^*)$ we let

$$\theta_E^* = \text{least } i \text{ such that } E \in Ext(W_i^*) \cup Ext(V_i^*)$$

If E is an A-extender, then $E = F_{\theta_E}$, so

$$\theta_E^* = \theta_E + 1.$$

~~If E is a D-extender, then θ_E^* is a limit ordinal.~~ But θ_E^* is also defined for D-extenders.

<u>Def. 9.8</u> E is <u>atomic</u> iff either

$$E \in \bigcup_i \left(Ext(W_i^*) - Ext(W_i)\right) \cup \left(Ext(V_i^*) - Ext(V_i)\right),$$

or $E \in \bigcup_\gamma Ext(Y_\gamma)$.

The atomic extenders are the ones we insert, at their moment of insertion.

Prop 9.9.

(a) E is an atomic $\mathcal{D}$-extender iff $\theta^*_E = Z(\gamma) = Z(\gamma, 0)$ for some γ.

(b) E is an atomic $\mathcal{A}$-extender iff $\theta^*_E = Z(\gamma, \xi)^{+1}$ for some γ and ξ.

Proof. Just the definitions. ☒

Every E traces back to a unique atomic extender:

Def 9.10 If E is an atomic $\mathcal{D}$-extender, then $at(E)$ is the unique γ s.t. $\theta^*_E = Z(\gamma)$.
If E is an atomic $\mathcal{A}$-extender, then $at(E) = \gamma + 1$, where γ is such that $\theta^*_E = Z(\gamma, \xi)^{+1}$ for some ξ.

Lemma 9.11 Let $E \in \bigcup_i Ext(W_i^*) \cup Ext(\mathcal{V}_i^*)$, and suppose E is not atomic; then there is a unique atomic extender G such that letting $\gamma = at(G)$, we have

$$\Theta_G^* <_A^\gamma \Theta_E^*$$

and

(i) If G is a D-extender and $G \in Ext(W_{Z(\gamma)}^*)$, then for p the p-map of $\Phi^\gamma_{\Theta_G^*, \Theta_E^*}$, we have $p(G) = E$,

(ii) If G is a D-extender and $G \in Ext(\mathcal{V}_{Z(\gamma)}^*)$, then for q the p-map of $\Psi^\gamma_{\Theta_G^*, \Theta_E^*}$, we have $q(G) = E$, and

(iii) If G is an A-extender, then for p and q the p-maps of $\Phi^\gamma_{\Theta_G^*, \Theta_E^*}$ and $\Psi^\gamma_{\Theta_G^*, \Theta_E^*}$, we have $p(G) = q(G) = E$.

Remark In cases (i) and (ii), $\Theta_E^* = Z(\gamma)$. In case (iii) $G \in Ext(\mathcal{Y}_{\gamma-1})$. In both cases, γ is least s.t. $\hat{W}^\gamma$ and $\hat{\mathcal{V}}^\gamma$ can inflate G.

The proof will use

Sublemma 9.12 Let $[i]_\sigma \neq [0]_\sigma$, and suppose there is no k such that $k <_A^\sigma i$; then either

(i) $i = z(\eta)$ for some $\eta \geq \sigma+1$, or

(ii) for some $\eta \geq \sigma+1$, ~~and some $\eta \geq \sigma+1$~~,
we have $z(\eta) <_A^\eta i$, and for $a = a(\eta, z(\eta), i)$ (so that $dw_i^\sigma = dw_{z(\eta)}^\sigma \circledast a$), we have that a fits cofinally into both dw_i^σ and dv_i^σ.

Remarks The hypothesis is equivalent to $dw_i^\sigma \not\equiv_{Tail} dv_i^\sigma$. Clearly, this implies $i \geq z(\sigma+1)$, since $\hat{w}^\sigma$ and $\hat{v}^\sigma$ are essentially Y_σ below $z(\sigma+1)$.

The converse of the lemma is true, in that if either (i) or (ii) holds, then $dw_i^\sigma \not\equiv_{Tail} dv_i^\sigma$. For if $i = z(\eta)$ where $\eta \geq \sigma+1$, then letting $\eta = \beta+1$, $e_{b_\beta}^{Y_\beta}$ fits into $dw_{z(\eta)}^\sigma$ cofinally and $e_{c_\beta}^{Y_\beta}$ fits into $dv_{z(\eta)}^\sigma$ cofinally, so $dw_{z(\eta)}^\sigma \not\equiv_{Tail} dv_{z(\eta)}^\sigma$. If (ii) holds, then the tail inequivalence of $dw_{z(\eta)}^\sigma$ and $dv_{z(\eta)}^\sigma$ is propagated when we insert a cofinally into both of them. Rmk Need to discuss η limit case.

<u>Proof of 9.12</u> Suppose that $dw_i^\sigma \not\equiv_{\text{Tail}} dv_i^\sigma$; then $i \geq z(\sigma+1)$. Suppose that $i \neq z(\eta)$ for all $\eta \geq \sigma+1$. Then dw_i^σ and dv_i^σ must have been obtained by inflation, so we have $\eta > \sigma$ and k such that $k <_A^\eta i$, and for $a = a(\eta, k, i)$,

$$dw_i^\sigma = dw_k^\sigma \circledast a,$$

and

$$dv_i^\sigma = dv_k^\sigma \circledast a$$

If $dw_k^\sigma \equiv_{\text{tail}} dv_k^\sigma$, then we get $dw_i^\sigma \equiv_{\text{tail}} dv_i^\sigma$, so $dw_k^\sigma \not\equiv_{\text{tail}} dv_k^\sigma$. By induction, either (i) or (ii) holds for k.

Suppose (i) holds, i.e. $k = z(\eta)$ for $\eta \geq \sigma+1$. So $\sup\{\lambda(E) \mid E \in \text{ran}(dw_k^\sigma)\} = \sup\{\lambda(E) \mid E \in \text{ran}(dv_k^\sigma)\}$. It follows that a fits $\left(\begin{smallmatrix}\text{Here } a = \\ a(\eta, z(\eta), k)\end{smallmatrix}\right)$ cofinally into dw_i^σ and dv_i^σ, for otherwise $dw_i^\sigma = (dw_k^\sigma \circledast a \restriction n)^\frown a^{\geq n}$ and $dv_i^\sigma = (dv_k^\sigma \circledast a \restriction n)^\frown a^{\geq n}$ for some $n \in \text{dom}(a)$, so that $a^{\geq n}$ is an agreeing tail for them. But then (ii) holds at i, as desired.

Suppose that (i) holds at k, and let

$$dw_k^\sigma = d^\sigma_{z(\eta)} \circledast d,$$

$$dv_k^\sigma = d^\sigma_{z(\eta)} \circledast d,$$

and $d = a(\eta, z(\eta), k) = dw_k^\eta = dv_k^\eta$

where d fits cofinally into dw_k^σ and dv_k^σ. Once again this implies $\sup\{\lambda_E \mid E \in \mathrm{ran}(dw_k^\sigma)\} = \sup\{\lambda_E \mid E \in \mathrm{ran}(dv_k^\sigma)\}$, so a fits cofinally into dw_i^σ and dv_i^σ by the argument above.

But then $\eta > \eta$ (since a inflated an η-inflation), and

$$d \circledast a = dw^\eta_{z(\eta),i} = dv^\eta_{z(\eta),i} = a(\eta, z(\eta), i)$$

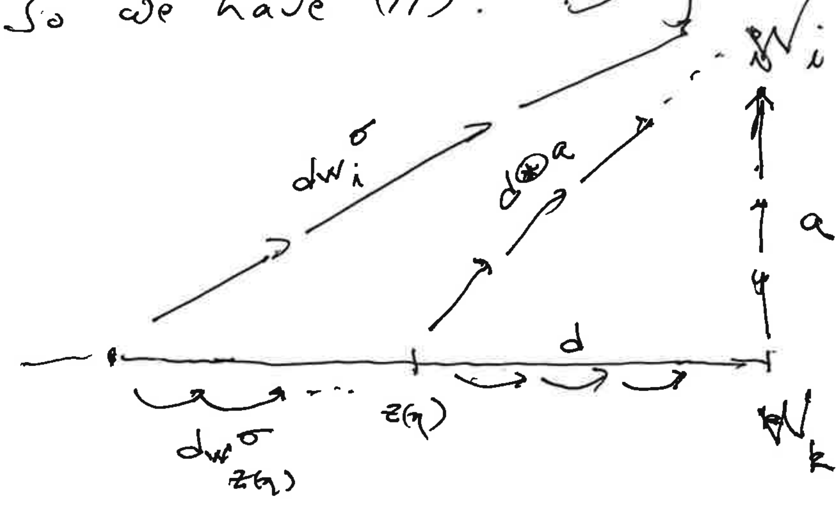
Also

$$dw_i^\sigma = \left(dw^\sigma_{z(\eta)} \circledast d\right) \circledast a = dw^\sigma_{z(\eta)} \circledast (d \circledast a)$$

and

$$dv_i^\sigma = \left(dv^\sigma_{z(\eta)} \circledast d\right) \circledast a = dv_{z(\eta)} \circledast (d \circledast a).$$

So we have (ii). Diagram:



This is rough. The block structure is ignored.

9.12 ☒

Proof of 9.11. Suppose first that E is a D-extender and $E \in Ext(W^*_{\theta^*_E})$. Since E is not atomic, $\theta^*_E \neq \varepsilon(\gamma)$ for all γ by 9.9. By 9.2.3, $\exists \sigma\, \exists k\;\; k <^\sigma_A \theta^*_E$. Let σ be the least such, and fix $k <^\sigma_A \theta^*_E$. Let

$$a = a(\sigma, k, \theta^*_E).$$

Notice that for all $n \in dom(a)$,

$$\lambda(a(n)) < \lambda_E,$$

Remark: Since E is a D-ext., $E \neq a(n)$ for all n.

since otherwise E would be in $Ext(W_i)$ for some $i < \theta^*_E$. But $W_{\theta^*_E} = Ult(W^*_k, a)$, so the full normalization process leads to

$$E = p_{k,\theta^*_E}(F)$$

where p_{k,θ^*_E} is the p-map of $\bar{\Phi}^\sigma_{k,\theta^*_E}$. But then

$$k = \theta^*_F,$$

since if $\ell = \theta^*_F < k$, then $(\hat{\Phi}_a)_\ell$ exists (i.e. $\beta_a \leq \ell$) because $\forall n \in dom(a)$ $crit(a \upharpoonright n) \leq \lambda(i_{a \upharpoonright n}(F))$. But then

$(\hat{\Phi}_a)_\ell (F) = E$, so $\theta^*_E = u^\sigma_a(\ell) < u^\sigma_a(k) = \theta^*_E$.

So $k = \theta^*_F$. If $k = z(\sigma)$, then F is atomic, and (i) of 9.11 holds with $G = F$ and $\gamma = \sigma$. (If we had assumed $E \in \mathrm{Ext}(\mathcal{V}^*_{\theta^*_E})$, we would have gotten (ii) of 9.11 at this point.)

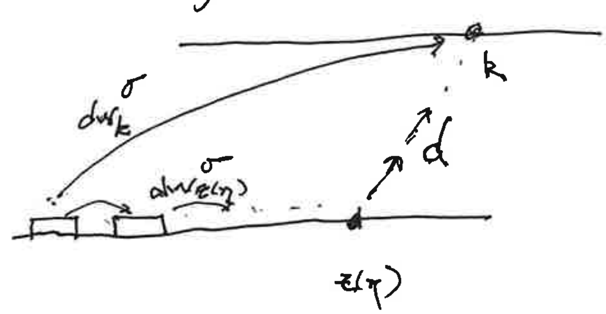
If $k > z(\sigma)$, then 9.12 and the minimality of k in $<^\sigma_A$ tell us that either $k = z(\eta)$ for some $\eta \geq \sigma + 1$, or dw^σ_k is the cofinal inflation of some $dw^\sigma_{z(\eta)}$ for $\eta \geq \sigma + 1$. But if $k = z(\eta)$, and then since F is a D-extender, $a\tau(F) = \eta$, and we have (i) with $G = F$ and $\gamma = \eta$.

($k <^\eta_A \theta^*_E$ in this case, with $a(\sigma, k, \theta^*_E) = a = a(\eta, k, \theta^*_E)$.)

Finally, suppose dw_k^σ is the cofinal inflation of $dw_{z(\eta)}^\sigma$, where $\eta = \sigma + 1$. So $z(\eta) <^\eta k$; let

$$d = a(\eta, z(\eta), k).$$

We have the diagram



$\mathcal{J}_k^\sigma = \{k\}$

$\mathcal{J}_{z(\eta)}^\sigma = \{z(\eta)\}$

$$(\hat{\Phi}_d)_{z(\eta)} = \Phi^\eta_{z(\eta),k} : W^*_{z(\eta)} \longrightarrow \mathrm{Ult}(W^*_{z(\eta)}, d) = W_k.$$

Since d is inserted cofinally in $dw^\sigma_{z(\eta)} \circledast d = dw^\sigma_k$, we must have $\forall n \in \mathrm{dom}(d)\,(\mathrm{lh}(d(n)) < \mathrm{lh}(F))$, for otherwise $F \in \mathrm{Ext}(W_i^*)$ for some $i < k$.

But this implies

$$F = \Phi^\eta_{z(\eta),k}(G)$$

for some $G \in \mathrm{Ext}(W^*_{z(\eta)})$. Since d was inserted cofinally into $dw^\sigma_{z(\eta)}$, $G \notin \mathrm{Ext}(W_{z(\eta)})$. Thus G is atomic, with $\mathrm{st}(G) = \eta$. So we get (i) again, with $\gamma = \eta$. (Note $k <^\eta_A \theta^*_E$, since d is cofinally inserted in dw^σ_k.)

The case that $E \in Ext(\mathcal{D}^*_{\theta^*_E})$ and E is a D-extender is parallel, as we remarked above.

Suppose now that E is an A-extender, but not atomic. Since it is not atomic, it must have been obtained by inflation. So we get η, k such that

$$k <^{\eta}_{A} \theta^*_E$$

and for

$$a = a(\eta, k, \theta^*_E)$$

we have for

$$p = p\text{-map of } \Phi^{\eta}_{k, \theta^*_E},$$

~~that~~ an F such that

$$p(F) = E.$$

~~Let η be least s.t.~~ ~~All~~ Note that

$lh(a(n)) < lh(E)$ for all $n \in dom(a)$.

(Because $E \in ran(p)$ and $E \notin Ext(W^*_i)$ for $i < \theta^*_E$. The mere fact that $k <^{\eta}_{A} \theta^*_E$ would allow that $E = a(n)$ for $n+1 = dom(a)$. But then a would not be witnessing that E is obtained by inflation.) (I.e., it's that $a = a(\eta, k-1, \theta_E)$ holds too.)

(Note that if $\theta_E = z(\gamma)$, then $E = E_0^{Y_\gamma}$, so E is atomic.)

Let η be least s.t. there are such k and $a = a(\eta, k, \theta^*_E)$ with $p(F) = E$ for some (unique) F. Let k be least for η, and now fix F and p and a as above.

Claim 1. $\theta^*_F = k$.

Proof We have $\theta^*_E = \theta_E + 1$ because E is an A-extender. Also $u^\eta_a(k) = \theta^*_E$. Since a is inflating F, $\beta^\eta_a \leq k-1$, and one can check then that $\theta_F = k-1$ and F is an A-extender. ⊠

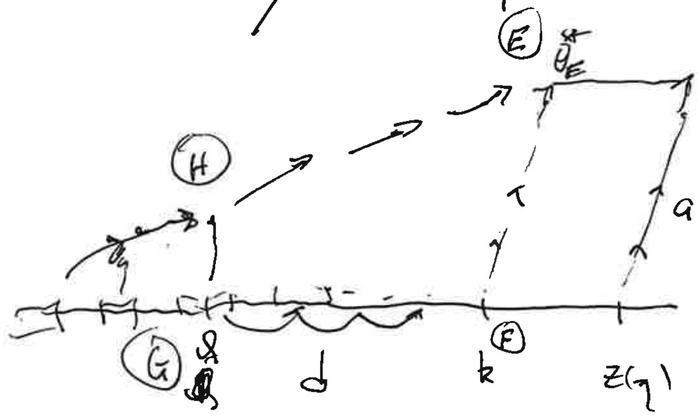
Claim 2 $k < z(\eta)$.

Proof If $k \geq z(\eta)$, then k is either a z-level, or a cofinal inflation of one. I.e. either $k = z(\eta)$, or $[k]_\eta = \{k\}$. In either case, k is a limit ordinal, contrary to $k = \theta_F + 1$.

⊠

<u>Claim 3</u> F is atomic

<u>Proof</u> Otherwise the argument above gives us $l <_A^\gamma k$ and $d = d(\gamma, l, k)$ and G s.t. for g = p-map of $\Phi^\gamma_{l,k}$, $g(G) = F$. ~~Then~~ Since $k < z(\eta)$, we must have $\gamma < \eta$. But then $dw^\gamma_{\theta^*_E} = dw^\gamma_k \circledast a$, and this yields some $d^\eta_a(l) = j <_A^\gamma \theta^*_E$ and H inflated by $a(\gamma, j, \theta^*_E)$ to E, contrary to the minimality of η. Diagram:



Since F is atomic, and η is minimal, we get $\eta = at(F)$. This gives (iii) of 9.11.

∎

Finally, we must show the uniqueness part of 9.11. Suppose E is not atomic, and F and G are distinct atomic origins of E. So we have σ, τ s.t.

$$\theta^*_F <^\sigma_A \theta^*_E$$

$$\theta^*_G <^\tau_A \theta^*_E$$

and F, G atomic with $F \neq G$ and

$$p\,\Phi^\sigma_{\theta^*_F, \theta^*_E}(F) = E$$

$$p\,\Phi^\tau_{\theta^*_G, \theta^*_E}(G) = E.$$

(We assume wlog that F, G are on the N-side.)

If $\sigma = \tau$, then θ^*_F and θ^*_G are $\leq^\sigma_A$ comparable (since $<^\sigma_A \theta^*_E$) so $\theta^*_F = \theta^*_G$ and $F = G$ by atomicity. Suppose then wlog $\sigma < \tau$. By 9.6 we get $\theta^*_F <^\sigma_A \theta^*_G$, and by 9.5 we get $\Phi^\sigma_{\theta^*_F, \theta^*_E} = \Phi^\tau_{\theta^*_G, \theta^*_E} \circ \Phi^\sigma_{\theta^*_F, \theta^*_G}$. So $\Phi^\sigma_{\theta^*_F, \theta^*_G}(F) = G$, contrary to G being atomic.

∎

Def. 9.13 Let $E \in \bigcup_i \mathrm{Ext}(\mathcal{W}_i^*) \cup \mathrm{Ext}(\mathcal{V}_i^*)$; then $\mathrm{ogn}(E) = E$ if E is atomic, and if E is not atomic, $\mathrm{ogn}(E)$ is the unique atomic G such that $\theta_G^* < \theta_E^*$, and for $\gamma = \mathrm{at}(G)$, and ρ the ρ-map of $\Phi^\gamma_{\theta_G^*, \theta_E^*}$ or $\Psi^\gamma_{\theta_G^*, \theta_E^*}$) we have $\rho(G) = E$.

Here "ogn" stands for "origin".

Def 9.14 For any E, we let $\mathrm{at}(E) = \mathrm{at}(\mathrm{ogn}(E))$.

We now show that if E and F are used on the some branch of some $\mathcal{W}_i^*$ or $\mathcal{V}_i^*$, then they have distinct origins.

~~**Lemma 9.15** Suppose E and F are used on the same branch~~

Remark Suppose $E = E_\gamma^{\mathcal{W}_k^*}$ and $F = E_\delta^{\mathcal{W}_k^*}$ where $\gamma < \delta$; then $\theta_E^* \le \theta_F^*$. Similarly on the $\mathcal{V}$-side.

Remark If E is a D-extender and $\alpha\tau(E)=\gamma$, then $ogn(E) \in \left(Ext(W^*_{Z(\gamma)}) - Ext(W_{Z(\gamma)})\right) \cup \left(Ext(\gamma^{\alpha}_{Z(\gamma)}) - Ext(\gamma)_{Z(\gamma)}\right)$.

If E is an A-extender and $\alpha\tau(E)=\gamma$, then $ogn(E) \in Ext(Y_{\gamma-1})$.

Definition 9.15 Suppose $i <^{\sigma}_{A} j$; then

$$p^{\sigma}_{ij} = p^{\Phi^{\sigma}_{ij}}$$

and

$$q^{\sigma}_{ij} = p^{\Psi^{\sigma}_{ij}}$$

are the p-maps of Φ^{σ}_{ij} and Ψ^{σ}_{ij}.

Lemma 9.16 Suppose E and F are used on the same branch of W^*_{φ}, with E used strictly before F; then for $\tau = \alpha\tau(F)$,

(a) $[\Theta^*_E]_{\tau} \leq^{\tau}_{A} [\Theta^*_F]_{\tau}$, and

(b) $ogn(E) \neq ogn(F)$.

Similarly on the D-side.

Proof Since E is used before F in W^*_{φ}, $\Theta^*_E \leq \Theta^*_F \leq \varphi$, and E is used before F on the same branch of $W^*_{\Theta^*_F}$. So we may assume $\varphi = \Theta^*_F$.

Let us consider first the case that F is atomic. So $F = ogn(F)$ and

$\theta^*_E \leq \theta^*_F \leq z(\tau)$, so $[\theta^*_F]_\tau = [0]_\tau = [\theta^*_E]_\tau$, and we have (a). If $\mathrm{ogn}(E) = \mathrm{ogn}(F)$, then $\mathrm{at}(E) = \tau$ too, so $E = \mathrm{ogn}(E) = \mathrm{ogn}(F) = F$. But $E \neq F$, contradiction. This proves (b).

So we assume F is not atomic, i.e., $[\theta^*_F]_\tau \neq [0]_\tau$, i.e. $\theta^*_F > z(\tau)$. Let us prove (a). Suppose toward contradiction that $[\theta^*_E]_\tau \not\leq^\tau_A [\theta^*_F]_\tau$. Thus $[\theta^*_E]_\tau \neq [0]_\tau$, and the branch of $(B_\tau, \leq^\tau_A)$ to $[\theta^*_F]_\tau$ must "skip over" $[\theta^*_E]_\tau$. This gives us k and i such that

$$k < \theta^*_E < i$$

and

$$k <^\tau_A i \leq^\tau_A \theta^*_F$$

and $[k]_\tau \neq [\theta^*_E]_\tau$ and $[i]_\tau \neq [\theta^*_E]_\tau$, and i is an $<^\tau_A$ immediate successor of k, so that

$$a(\tau, k, i) = \langle G \rangle$$

is a single extender. But then $\theta_G + 1 = \min([i]_\tau)$, so $\theta_G \geq \max([\theta^*_E]_\tau)$, so E is used in W_{θ_G} whereas G is on the sequence of $M_\infty^{W_{\theta_G}^*}$, so $G \neq E$. Also, E is used in W_{θ_G+1}, so $lh(G) > lh(E)$.

Now let
$$p^{\mathcal{T}}_{k,\ell}(F_0) = F,$$
and let s_0 be such that $s_0{}^\frown\langle F_0\rangle \in (W^*_k)^{ext}$.

Then
$$\hat{p}^{\mathcal{T}}_{k,\ell}(s_0{}^\frown\langle F_0\rangle) = s^\frown\langle F\rangle \in W^{ext}_\ell,$$
so
$$E \in \operatorname{ran}(\hat{p}_{k,\ell}(s_0{}^\frown\langle F_0\rangle)).$$
But all extenders in $Q(\mathcal{T}, i, \ell)$ have critical point $\geq \lambda_G > lh(E)$, so
$$E \in \operatorname{ran}(\hat{p}_G(s_0{}^\frown\langle F_0\rangle)).$$
Since $E \neq G$, this means
$$E = p_G(H)$$
for some $H \in \operatorname{ran}(s_0{}^\frown\langle F_0\rangle)$. Note $H \neq E$, because $\theta^*_E > k$, so $E \notin Ext(W^*_k)$. But this means G fits into E, contrary to $lh(G) > lh(E)$.

That proves (a).

Let us prove (b). Let $n <^{\mathcal{T}}_A \theta^*_F$, and $n \leq z(\mathcal{T})$, so that
$$F_0 =_{df} \operatorname{ogn}(F) \in Ext(W^*_n)$$
and
$$p^{\mathcal{T}}_{n,\theta^*_F}(F_0) = F.$$

Let $E_0 = \text{lgh}(E)$, and assume toward contradiction that $E_0 = F_0$. So $\text{at}(E) = \tau$, and

$$P^{\tau}_{\eta, \theta^*_E}(F_0) = E.$$

(Note $\eta = \theta^*_{E_0} = \theta^*_{F_0}$.) It will be enough to show that $\theta^*_E = \theta^*_F$, since then $E = F$, contrary to F being used strictly after E in $W^*_{\theta_F}$. If not, then letting

$$a = a(\tau, \theta^*_E, \theta^*_F)$$

we have $a \neq \emptyset$. ($\theta^*_E <^{\eta}_{A} \theta^*_F$ by part (a) and our assumption that $\text{lgh}(E) = \text{lgh}(F)$.)

Let

$$G = a(0),$$

and

$$s^\frown\langle E\rangle \in W^{ext}_{\theta^*_E};$$

then

$$\hat{P}^{\eta}_{\theta^*_E, \theta^*_F}(s^\frown\langle E\rangle) = t^\frown\langle F\rangle \in W^{ext}_{\theta^*_F}$$

so $E \in \text{ran}(t)$, so

$$E \in \text{ran}(\hat{p}_G(s^\frown\langle E\rangle)).$$

But $\hat{p}_a(E) = F$, and $F \neq E$, so $p_G(E) \neq E$. Moreover, $G \neq E$ since G is an A-extender with $\Theta_{G+1} > \theta^*_E$, and $lh(G) > lh(E)$ since

E is used in W_{θ_G+1}. These facts imply $E \notin \operatorname{ran}(\hat{p}_G(s^\frown\langle E\rangle))$, contradiction.

9.16 ☒

Clearly the proofs of (a) and (b) work on the $\mathcal{D}$-side too.

9.16 ☒

<u>Remark 9.16a</u> Assume the hypotheses of 9.16, and let $n \le z(\tau)$ be such that $n <_A^\tau \theta_F^*$.

Let

$$a_0 = dw^\tau_{\theta_E^*}$$

and

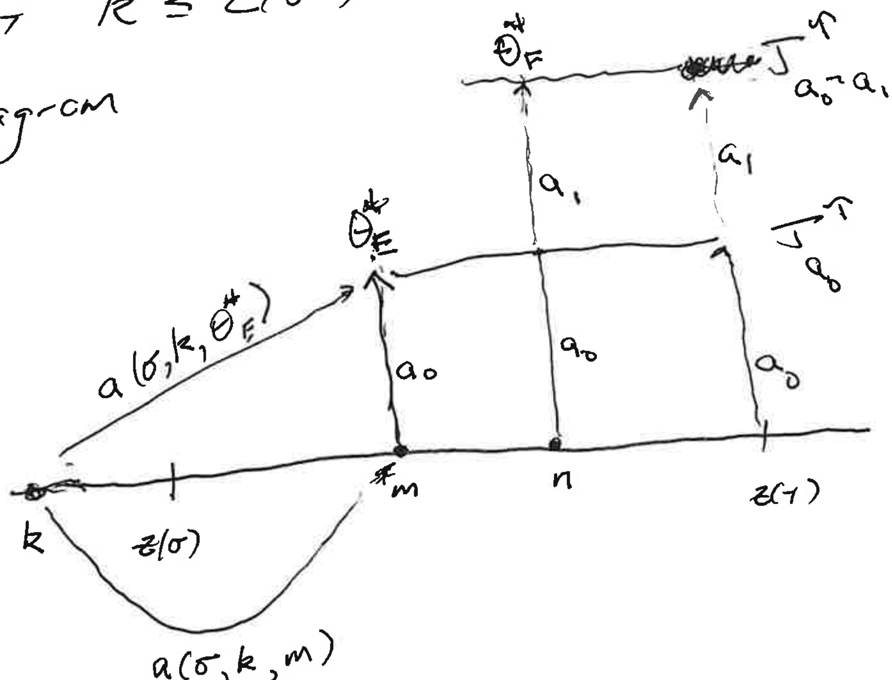
$$a_0{}^\frown a_1 = a = dw^\tau_{\theta_F^*}.$$

(So $a = a(\tau, n, \theta_F^*)$.) Let $u^\tau_{a_0}(n) = i$, so $[i]_\tau = [\theta_E^*]_\tau$ and $i <_A^\tau \theta_F^*$. We get that $\theta_E^* \le i$. Let $F_0 = \operatorname{ogn}(F)$ and $F_1 = p^\tau_{a_0}(F) = p^\tau_{ni}(F)$. Since $a_1(0)$ is on the last model of some W_j^* with $j \ge \theta_E^*$ (since $\theta_F^* \sim_\tau i$), and E is used in W_j and in $W_{\theta_{a_1(0)}+1}$, so $lh(a_1(0)) > lh(E)$. Thus since

W_i and $W_{\theta_F^*}$ are the same extenders of length $< lh(a_1(0))$, $E \in Ext(W_i)$, so $\theta_E^* \leq i$.]

Thus we have $m \leq n$ such that $d_{a_0}^{\tau}(m) = \theta_E^*$.

Now suppose $at(E) = \sigma$, where $\sigma \leq \tau$. Let $k \leq z(\sigma)$ and $k <_A^{\sigma} \theta_E^*$. We get the diagram



We shall need to analyze this situation further in the termination proof.

It's also possible that $at(E) > \tau$ in the situation above. In that case, one can show that $E = a_0(\text{dom}(a_0)-1)$, since $ogn(E)$ was inserted after n. (In particular, E must be an A-extender if $\alpha_T(E) > \alpha_T(F)$ but E is used before F on the same branch of some W_ℓ^*.) The diagram here is (160)