§6.1 Definitions of $\hat{\mathcal{W}}_{\alpha+1}$ and $\hat{\mathcal{V}}_{\alpha+1}$

Given that we have $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$ which include the objects above, let us define $\hat{\mathcal{W}}_{\alpha+1}$ and $\hat{\mathcal{V}}_{\alpha+1}$.

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Step I Iterate away least extender disagreements until you expose a strategy ~~Alt~~ disagreement.

This produces $\mathcal{W}^*_{z(\alpha)}$ and $\mathcal{V}^*_{z(\alpha)}$ normally extending $\mathcal{W}_{z(\alpha)}$ and $\mathcal{V}_{z(\alpha)}$, with $R_\alpha \trianglelefteq P^*_{z(\alpha)}, Q^*_{z(\alpha)}$, and Υ_α on R_α by Σ_{R_α} and Λ_{R_α}, with $b_\alpha \neq c_\alpha$ where $b_\alpha = \Sigma_{R_\alpha}(\Upsilon_\alpha)$ and $c_\alpha = \Lambda_{R_\alpha}(\Upsilon_\alpha)$. By Lemma 3, $(\Upsilon_\alpha{}^\frown b_\alpha)^{\mathcal{W}}$ on $P^*_{z(\alpha)}$ and $(\Upsilon_\alpha{}^\frown c_\alpha)^{\mathcal{V}}$ on $Q^*_{z(\alpha)}$ are by Σ and Λ.

Let $\hat{\mathcal{W}}^*_\alpha = \hat{\mathcal{W}}_\alpha$, but with $\mathcal{W}_{z(\alpha)}$ extended to $\mathcal{W}^*_{z(\alpha)}$. Similarly for $\hat{\mathcal{V}}^*_\alpha$.

Step 2 Form

$$\hat{W}_{\alpha+1} = X(\hat{W}^*_{\alpha}, (Y_\alpha{}^\frown b_\alpha)^{W} \restriction \text{exts of } lh < \delta(Y_\alpha^-))$$

and

$$\hat{\mathcal{V}}_{\alpha+1} = X(\hat{\mathcal{V}}^*_{\alpha}, (Y_\alpha{}^\frown c_\alpha)^{\mathcal{V}} \restriction \text{exts of } lh < \delta(Y_\alpha^-)).$$

To do this, we form by induction the partial normalizations

$$\hat{W}_{\alpha,\xi} = X(\hat{W}^*_{\alpha}, (Y_\alpha \restriction \xi+1)^{W})$$

and

$$\hat{\mathcal{V}}_{\alpha,\xi} = X(\hat{\mathcal{V}}^*_{\alpha}, (Y_\alpha \restriction \xi+1)^{\mathcal{V}})$$

for $\xi < lh(Y_\alpha)$. For $\xi <^{Y_\alpha} \eta$ we shall have embeddings

$$\dot{\Phi}^{\alpha}_{\xi,\eta} : \hat{W}_{\alpha,\xi} \longrightarrow \hat{W}_{\alpha,\eta}$$

and

$$\dot{\psi}^{\alpha}_{\xi,\eta} : \hat{\mathcal{V}}_{\alpha,\xi} \longrightarrow \hat{\mathcal{V}}_{\alpha,\eta}.$$

(We use the dot above Φ because $\hat{\Phi}$'s will use the global indexing, not that of Y_α. Otherwise they are the same.)

$\hat{W}_{\alpha,\xi}$ will have last tree $W_{z(\alpha,\xi)}$)

and $\hat{\mathcal{U}}_{\alpha,\xi}$ will have last tree $\mathcal{U}_{z(\alpha,\xi)}$.

The last models are

$$M_\infty^{W_{z(\alpha,\xi)}} = M_\xi^{Y_\alpha^W},$$

$$M_\infty^{\mathcal{U}_{z(\alpha,\xi)}} = M_\xi^{Y_\alpha^{\mathcal{U}}}.$$

We set $z(\alpha,0) = z(\alpha)$, and

$$F_{z(\alpha,\xi)} = E_\xi^{Y_\alpha}.$$

Letting $F = F_{z(\alpha,\xi)}$, set as before

$$\beta_F = \text{least } \beta \text{ s.t. } \kappa_F < \lambda(F_\eta) \text{ for all } \eta \in [\beta, z(\alpha,\xi)].$$

Letting $\mu = Y_\alpha\text{-pred}(\xi+1)$, it is easy to see that

$$\beta_F = \text{least } \beta \text{ s.t. } \kappa_F < \lambda(F_\eta) \text{ for all } \eta \in [\beta, z(\alpha,\mu)].$$

Put

$$\sigma_F = z(\alpha,\mu)$$

$$I_F = [\beta_F, \sigma_F]$$

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(no page 61)

$$\tau_F(i) = \begin{cases} i & \text{if } i < \beta_F \\ \theta_F + 1 + (i - \beta_F) & \text{for } i \in I_F, \end{cases}$$

where as we said, $\theta_F = \tau(\alpha, \xi)$.

Let

$$J_F = \tau_F{}'' I_F$$

and

$$\begin{aligned} \tau_F &= \max(J_F) \\ &= \tau_F(\sigma_F), \end{aligned}$$

so that $J_F = [\theta_F + 1, \tau_F]$ as before. Then

$$\begin{aligned} \dot{\Phi}^{\alpha}_{\mu, \xi+1} &=_{df} \hat{\Phi}_F \\ &=_{df} \langle \tau_F, \langle \Phi_i \mid i \leq \sigma_F \rangle \rangle, \end{aligned}$$

where

$$\Phi_i : W_i^* \longrightarrow X(W_i^*, F) =_{df} W_{\tau_F(i)}$$

is the natural weak tree embedding. We also let $W^*_{\tau_F(i)} = W_{\tau_F(i)}$.

If $\eta <^{\mathcal{T}_\alpha} \mu$, then we define $\dot{\Phi}_{\eta,\xi+1}$ by

$$\dot{\Phi}^\alpha_{\eta,\xi+1} = \dot{\Phi}^\alpha_{\mu,\xi+1} \circ \dot{\Phi}^\alpha_{\eta,\mu}$$

where the composition is understood in the natural way. Let us write

no dot

$$\dot{\Phi}^\alpha_{\gamma,\delta} = \langle u^\alpha_{\gamma,\delta}, \langle \Phi^\alpha_{i,u_{\gamma,\delta}(i)} \mid i \in \mathrm{dom}(u_{\gamma,\delta})\rangle\rangle$$

for the components. So $u^\alpha_{\mu,\xi+1} = Z_F$, and $u^\alpha_{\eta,\xi+1} = u^\alpha_{\mu,\xi+1} \circ u^\alpha_{\eta,\mu}$, and

$$\Phi^\alpha_{i,u_{\gamma\delta}(i)} : W^*_i \longrightarrow W_{u_{\gamma,\delta}(i)}$$

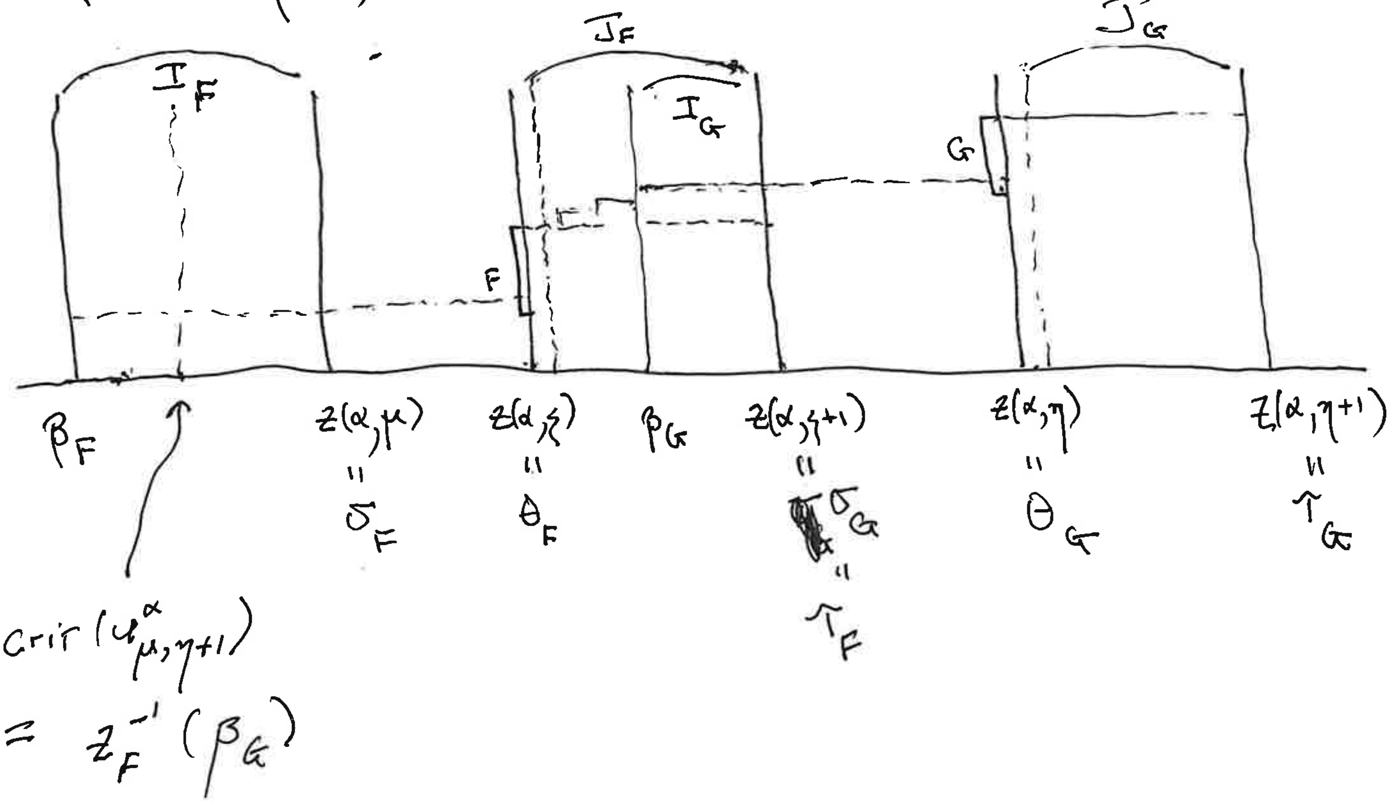
is the composed meta-tree component of $\dot{\Phi}^\alpha_{\gamma\delta}$.

As one goes up a branch of $\mathcal{T}_\alpha$, the critical points of the u^α-maps go up. That is,

$$u^\alpha_{\gamma\delta}(\mathrm{crit}(u^\alpha_{\gamma\delta})) \leq \mathrm{crit}(u_{\delta\rho})$$

for $\gamma <^{\mathcal{T}_\alpha} \delta <^{\mathcal{T}_\alpha} \rho$.

For example, if $\mu = \Upsilon_\alpha\text{-pred}(\xi+1)$ and $\xi+1 = \Upsilon_\alpha\text{-pred}(\eta+1)$, and $F = E_\xi^{\Upsilon_\alpha}$ and $G = E_\eta^{\Upsilon_\alpha}$, the picture is



If $\lambda < lh(\Upsilon_\alpha)$ is a limit ordinal,

then

$$\hat{W}_{\alpha,\lambda} = \lim_{\delta <_{\Upsilon_\alpha} \lambda} \hat{W}_{\alpha,\delta}$$

and

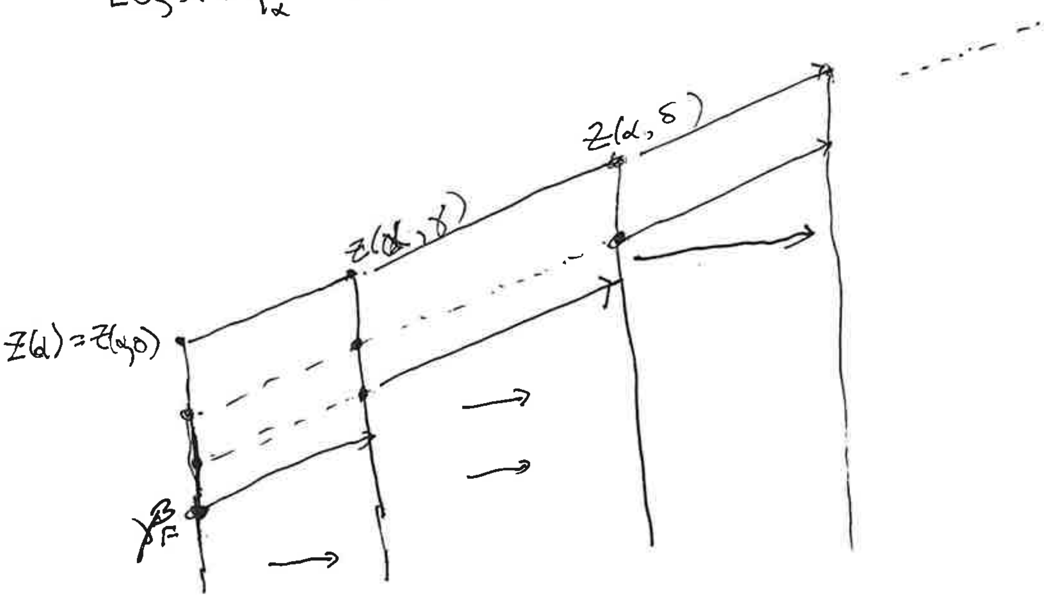
$$\hat{\mathcal{V}}_{\alpha,\lambda} = \lim_{\delta <_{\Upsilon_\alpha} \lambda} \hat{\mathcal{V}}_{\alpha,\delta}.$$

The limits are under the mouse-tree embeddings $\Phi^\alpha_{\gamma\delta}$ and $\Psi^\alpha_{\gamma\delta}$. So letting

$$z_0(\alpha,\lambda) = \sup_{\delta <_{\Upsilon_\alpha} \lambda} z(\alpha,\delta)$$

$$= \sup_{\gamma <_{\Upsilon_\alpha} \delta < \lambda} \text{crit}(u^\alpha_{\gamma\delta})$$

we list the direct limits along u^α-threads between $z_0(\alpha,\lambda)$ and $z(\alpha,\lambda)$. Our u-maps are order-preserving, and surjections except for the gap $[\text{crit}(u), u(\text{crit}(u)))$ in the range. So the u-maps along $[0,\lambda)_{\Upsilon_\alpha}$ look like



Letting F be the first extender used on $[0,\lambda)_{\Upsilon_\alpha}$, we have $\text{crit}(u_{0,\eta}) = \gamma^{\beta_F}$ for all $\eta \le_{\Upsilon_\alpha} \lambda$. If we let

$$\Upsilon = \text{least } \eta \text{ s.t. } \forall \delta <_{\Upsilon_\alpha} \lambda$$
$$\text{crit}(u_{\delta,\lambda}) \le u_{0,\delta}(\eta)$$

then

$$Z(\alpha,\lambda) = Z_0(\alpha,\lambda) + (Z(\alpha) - \tau).$$

The u-threads ~~whose listed~~ whose limits we list all start at $i \in [\tau, Z(\alpha)]$, with

$$W_{u_{0,\lambda}(i)} = W^*_{u_{0,\lambda}(i)} = \lim_{\delta <_{Y_\alpha} \lambda} W^*_{u_{0,\delta}(i)},$$

where the direct limit is under the tree embs

$$\Phi^\alpha_{u_{0,\delta}(i),\, u_{0,\gamma}(i)} : W_{u_{0,\delta}(i)} \longrightarrow W_{u_{0,\gamma}(i)}$$

that are components of $\Phi^\alpha_{\delta\gamma}$ for $\delta <_{Y_\alpha} \gamma$.

This finishes the definition of $\hat{W}_{\alpha,\beta}$ and $\hat{\nu}_{\alpha,\beta}$ for $\beta < lh(Y_\alpha)$. We set

See page 66a

$$\hat{W}^0_{\alpha+1} = \text{"}\bigcup_{\beta < lh(Y_\alpha)}\text{"}\ \hat{W}^*_{\alpha,\beta}$$

$$\hat{\nu}^0_{\alpha+1} = \text{"}\bigcup_{\beta < lh(Y_\alpha)}\text{"}\ \hat{\nu}_{\alpha,\beta}$$

where the unions are taken componentwise. Set

$$Z(\alpha+1) = \sup_{\beta < lh(Y_\alpha)} Z(\alpha,\beta)$$

and

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Remark It doesn't quite finish the definition because we need to define the F_i for i such that $i \neq z(\alpha, \xi)$ for all ξ. These are just the blow-ups of A-extenders F_k for $k < z(\alpha)$.

Namely, if $i \notin \{z(\alpha,\xi) \mid \xi < lh(\Upsilon_\alpha)\}$ and $z(\alpha+1) > i > z(\alpha)$, we can find δ s.t.

$$i = d^{\alpha}_{0,\delta}(k), \text{ where } k < z(\alpha),$$

and we set

$$F_i = \left(\mathring{\Phi}^{\alpha}_{0,\delta}\right)_{k+1}(F_k).$$

We have $F_k \in Ext(W_{k+1})$ and $(\mathring{p}_{0\delta})_{k+1} : Ext(W_{k+1}) \to Ext(W_{i+1})$. Then $F_i = (\mathring{p}_{0,\delta})_{k+1}(F_k)$

One can also just define the F_i by induction on i.

It's straightforward to show that $(\dagger)_{\alpha+1}$ still holds, e.g. $W_{i+1} = X(W_{\beta_{F_i}}, F_i)$ and $\mathcal{V}_{i+1} = X(W_{\beta_{F_i}}, F_i)$. There are more details in the next sub-section.

$$\mathcal{W}_{z(\alpha+1)} = X\left(\mathcal{W}^*_{z(\alpha)}, \left(\gamma_\alpha^\frown b_\alpha\right)^{\mathcal{W}}\right) \restriction \text{ext of lh} < \delta(\gamma_\alpha)$$

and

$$\mathcal{V}_{z(\alpha+1)} = X\left(\mathcal{V}^*_{z(\alpha)}, \left(\gamma'_\alpha{}^\frown c_\alpha\right)^{\mathcal{V}}\right) \restriction \text{exts of lh} < \delta(\gamma_\alpha),$$

where we include the branches generated by b_α and c_α. Another way to characterize them is as follows: let $\mathcal{S}^{\mathcal{W}}$ be the "common part tree" on the $\mathcal{W}$-side, i.e. $\mathcal{J} \trianglelefteq \mathcal{S}^{\mathcal{W}}$ iff for all suff. large $\xi < z(\alpha+1)$, $\mathcal{J} \trianglelefteq \mathcal{W}_\xi$, or equivalently

$$\mathcal{S}^{\mathcal{W}} = \bigcup_{\beta < \mathrm{lh}(\gamma_\alpha)} \mathcal{W}_{z(\alpha,\beta)} \restriction \text{exts of lh} < \mathrm{lh}(F_{z(\alpha,\beta)}).$$

Similarly for $\mathcal{S}^{\mathcal{V}}$. We have that $(\mathcal{S}^{\mathcal{W}}, \mathcal{S}^{\mathcal{V}})$ is a slow comparison, and

$$\mathcal{W}_{z(\alpha+1)} = \mathcal{S}^{\mathcal{W}} {}^\frown \Sigma(\mathcal{S}^{\mathcal{W}})$$

$$\mathcal{V}_{z(\alpha+1)} = \mathcal{S}^{\mathcal{V}} {}^\frown \Lambda(\mathcal{S}^{\mathcal{V}}),$$

so $(\mathcal{W}_{z(\alpha+1)}, \mathcal{V}_{z(\alpha+1)})$ is also a slow comparison. It is also a branch divergence, since

b_α fits into $\Sigma(\mathcal{A}^{w})$ and c_α fits into $\Lambda(\mathcal{A}^{v})$.

This finishes the definitions of $\hat{W}_{\alpha+1}$ and $\hat{v}_{\alpha+1}$; namely

$$\hat{W}_{\alpha+1} = \hat{W}^{0}_{\alpha+1} \text{``}{\frown}\text{''} \langle W_{z(\alpha+1)} \rangle$$

and

$$\hat{v}_{\alpha+1} = \hat{v}^{0}_{\alpha+1} \text{``}{\frown}\text{''} \langle v_{z(\alpha+1)} \rangle .$$

Notice that we did <u>not</u> take $\hat{w}_{\alpha+1} = \lim_{\beta \in b_\alpha} \hat{W}_{\alpha,\beta}$ and $\hat{v}_{\alpha+1} = \lim_{\beta \in c_\alpha} \hat{v}_{\alpha,\beta}$. The reason was explained in the step to $(\hat{W}_\lambda, \hat{v}_\lambda)$: if we did that, we would be inflating by branches that are not tail equivalent, so the pairs (W_i, v_i) with $i > \sup_{\beta < lh(\gamma_\lambda)} z(\alpha,\beta)$ might not be slow comparisons. The rule is that we only inflate by $\mathcal{A}$-branches.

We haven't yet defined the F_i for $i < z(\alpha+1)$ such that $i \neq z(\alpha,\xi)$ for all ξ. Of course, they are blow-ups of extenders of the form F_i for $i \le z(\alpha)$. That these blow-ups fit together so that we get $(\dagger)_{\alpha+1}$ we shall prove in the rest of this section.

§6.2 Definitions of $\hat{W}^{\tau}_{\alpha+1}$ and $\hat{\mathcal{V}}^{\tau}_{\alpha+1}$

We define $\hat{W}^{\tau}_{\alpha+1}$ for $\tau \le \alpha$, and its branch extenders dw^{τ}_{ξ} for $\xi \le z(\alpha+1)$. Similarly on the $\mathcal{V}$-side. We start with $\tau = \alpha$, where $\hat{W}^{\alpha}_{\alpha+1}$ is essentially just the normal meta-meta-tree on $\hat{W}_{\alpha}$ induced by $Y_{\alpha} \frown b_{\alpha}$. As in the case $\alpha = 1$, letting $i < z(\alpha+1)$, we set

$$dw^{\alpha}_{i} = \begin{cases} \emptyset & \text{if } i \le z(\alpha) \\ e^{Y_\alpha}_{\lambda} & \text{if } z_0(\alpha,\lambda) \le i \le z(\alpha,\lambda) \text{ for } \lambda \text{ a limit ord.} \\ e^{Y_\alpha}_{\tau+1} & \text{if } z(\alpha,\tau)+1 \le i \le z(\alpha,\tau+1) \\ & \text{otherwise} \end{cases}$$

Put

$$i \sim_{\alpha} j \text{ iff } dw^{\alpha}_{i} = dw^{\alpha}_{j}.$$

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and

$$[i]_\alpha = \{j \mid i \sim_\alpha j\}$$

and

$$i \leq^\alpha j \text{ iff } dw_i^\alpha \subseteq dw_j^\alpha .$$

$\leq^\alpha$ is a quasi-tree order, and becomes a tree order isomorphic to $<^{T_\alpha}$ if we mod out by $\sim_\alpha$. Then also

$$dw^\alpha_{z(\alpha+1)} = e^{\gamma_\alpha}_{b_\alpha}$$

and

$$dv_i^\alpha = \begin{cases} dw_i^\alpha & \text{if } i < z(\alpha+1), \\ e^{\gamma_\alpha}_{c_\alpha} & \text{if } i = z(\alpha+1). \end{cases}$$

So we get the same $\leq^\alpha$ and $[i]_\alpha$ using the dv_i^α's. Note $[z(\alpha+1)]_\alpha = \{z(\alpha+1)\}$.

The $\dot{\Phi}_{\gamma\delta}$ are the embeddings along $\leq^\alpha$-branches of $\hat{W}^\alpha_{\alpha+1}$. More precisely, for $i < z(\alpha+1)$ set

$$\xi_i = \text{unique } \tau \text{ such that } dw_i^\alpha = dv_i^\alpha = e^{\gamma_\alpha}_\tau .$$

So $[i]_\alpha = [z_0(\alpha, \xi_i), z(\alpha, \xi_i)]$. If

$i \leq^{\alpha} j$ and $j < z(\alpha+1)$, then

$$\hat{\Phi}^{\alpha}_{[i],[j]} = \dot{\Phi}^{\alpha}_{\xi_i, \xi_j} .$$

We may sometimes just write $\hat{\Phi}^{\alpha}_{i j}$ for $\hat{\Phi}^{\alpha}_{[i],[j]}$, which is not ambiguous because the ^ on top indicates that it acts on blocks of trees, not single ones.

Letting

$$dw_i^{\alpha} \frown s = dw_j^{\alpha}$$

we have

$$\hat{\Phi}^{\alpha}_{[i],[j]} : \hat{W}^{\alpha}_{\alpha+1} \upharpoonright [i] \longrightarrow \hat{W}^{\alpha}_{\alpha+1} \upharpoonright [j]$$

Insert 70b

comes from taking an s-ultrapower in some sense. Note that s determines α, since $Ext(Y_\alpha) \cap Ext(Y_\gamma) = \phi$ for $\alpha \neq \gamma$. (We assume $s \neq \phi$.) But then it determines dw_j^{α}, as its downward closure in Y_α^{ext}, and hence dw_i^{α}, and hence $[i]_\alpha$ and $[j]_\alpha$. So we can define

$$\hat{\Phi}_s = \hat{\Phi}^{\alpha}_s = \hat{\Phi}^{\alpha}_{[i],[j]} = \dot{\Phi}^{\alpha}_{\xi_i, \xi_j} .$$

(70b)

More precisely, for any $\delta \in [i]_\alpha$,

$$W_{\mathcal{U}^\alpha_{[i],[j]}}(\delta) = X(W^*_\delta, \delta).$$

Let us also write

$$u_s = u_s^\alpha = u^\alpha_{\xi_i, \xi_j},$$

$$\left(\hat{\Phi}_s\right)_k = \left(\mathring{\Phi}^\alpha_{\xi_i, \xi_j}\right)_k \quad \text{for } k \in \mathrm{dom}(u_s)$$

so that $\left(\hat{\Phi}_s\right)_k : W^*_k \longrightarrow W_{u_s(k)}$ is a weak tree embedding. Let

$$\theta_s = \begin{cases} \theta_F + 1 & \text{if } F = s(\mathrm{dom}(s)-1) \\ \sup\{\theta_{s(m)} \mid m \in \mathrm{dom}(s)\} & \text{if } \mathrm{dom}(s) \text{ is a limit ordinal} \end{cases}$$

($\theta_s = z_0(\alpha, \xi_j)$) and

$$\beta_s = u_s^{-1}(\theta_s),$$

$$\sigma_s = \sigma_{s(0)} \quad (= z(\alpha, \xi_i)),$$

$$\tau_s = u_s(\sigma_s) \quad (= z(\alpha, \xi_j)),$$

$$I_s = [\beta_s, \sigma_s],$$

and

$$J_s = [\theta_s, \tau_s].$$

So $u_s : I_s \xrightarrow[\text{op}]{1\text{-}1} J_s$, and for $i \in I_s$,

$\left(\hat{\Phi}_s\right)_i : W^*_i \longrightarrow W_{u_s(i)}$ is the weak tree embedding corresponding to the fact that "$W_{u_s(i)} = \mathrm{Ult}(W_i^*, s)$".

Our definitions easily imply

<u>Prop. 6.2.1</u> For any $i \in (z(\alpha), z(\alpha+1))$ there is a unique $s \in Y_\alpha^{ext}$ such that $i \in J_s$; moreover, $[i]_\alpha = J_s$ in this case, and $s = dw_i^\alpha$.

Now let us define dw_i^τ and dv_i^τ for $\tau < \alpha$ and $z(\alpha) < i \leq z(\alpha+1)$. Suppose first that $i < z(\alpha+1)$, and let

$$[i]_\alpha = J_s$$

where $s \in Y_\alpha^{ext}$. Suppose first

$$s = e_{q+1}^{Y_\alpha}$$

and let

$$F = E_q^{Y_\alpha}.$$

We then have a unique $k \in I_F$ such that

$$i = z_F(k).$$

We have dw_k^τ and dv_k^τ by induction, moreover $\mathrm{ran}(dw_k^\tau) \cup \mathrm{ran}(dv_k^\tau) \subseteq \mathrm{Ext}(W_k^*) \cap \mathrm{Ext}(\mathcal{V}_k^*)$ (by induction).

let (no hat. $\hat{p}_F$ acts on $(W_k^*)^{ext}$.)

$$p_F : \mathrm{Ext}(W_k^*) \longrightarrow \mathrm{Ext}(W_i)$$

and

$$q_F : \mathrm{Ext}(W_k^*) \longrightarrow \mathrm{Ext}(V_i^*)$$

be the maps of $\hat{\Phi}_F$ and $\hat{\Psi}_F$. Let m be least s.t. $\kappa_F < \lambda(dw_k^\tau(m))$ if such exists, and $m = \mathrm{dom}(dw_k^\tau)$ if none exists. Then

$$dw_i^\tau = \begin{cases} dw_k^\tau \frown \langle \hat{p}_F(dw_k^\tau(n)) \mid n \geq m \rangle & \text{if } m < \mathrm{dom}(dw_k^\tau) \text{ and } \kappa_F < \lambda(dw_k^\tau(m)) \\ dw_k^\tau \frown \langle F \rangle \frown \langle \hat{p}_F(dw_k^\tau(n)) \mid n \geq m \rangle & \text{otherwise.} \end{cases}$$

dv_i^τ is defined from dv_k^τ and F in parallel fashion, using q_F.

~~Def 6.2.2 $i \sim_\tau j$ iff dw~~

~~Prop 6.2.2 $dw_i^\tau = dw_j^\tau$ iff $dv_i^\tau = dv_j^\tau$~~ If $s = e_\lambda^{\tau_\alpha}$ for $\lambda < lh(\tau_\alpha)$ a limit, then we define dw_i^τ by taking the corresponding limit.

A more efficient way to ~~define~~ describe dw_i^{τ} is:

let $s \in Y_\alpha^{ext}$ and let k_0 be such that $u_s(k_0) = i$, and $i \in J_s$. Let p be the p-map of $(\hat{\Phi}_s)_{k_0}$, so that $p: Ext(W_{k_0}^{*}) \longrightarrow Ext(W_i)$. We have

$$ran(dw_{k_0}^{\tau}) \subseteq Ext(W_{k_0})$$

by induction. Then

$$p''(ran\, dw_{k_0}^{\tau}) \subseteq ran(dw_i^{\tau})$$

and the remaining extenders in dw_i^{τ} are extenders in $ran(s)$ that have been inserted "as themselves".

Finally, let $dw_{z(\alpha+1)}^{\tau}$ be the image of $dw_{z(\alpha)}^{\tau}$ under the $\hat{p}$-map of $\hat{\Phi}_{e_{b\alpha}^{Y_\alpha}}$, with appropriate extenders from $e_{b\alpha}^{Y_\alpha}$ inserted as themselves. Similarly for $dv_{z(\alpha+1)}^{\tau}$.

One can show

Lemma 6.2.2

(a) $dw_i^\Upsilon \equiv_{Tail} dw_j^\Upsilon \Longrightarrow dw_i^\Upsilon = dw_j^\Upsilon$, and similarly $dv_i^\Upsilon \equiv_{Tail} dv_j^\Upsilon \Longrightarrow dv_i^\Upsilon = dv_j^\Upsilon$

In particular, $dw_i^\Upsilon(k)$ determines $dw_i^\Upsilon \restriction k$.

(b) $\forall i \forall \gamma \exists j \left(dw_i^\Upsilon \restriction \gamma = dw_j^\Upsilon \right)$, and similarly for the $\mathcal{V}$-side.

(c) If $n \in dom(dw_i^\Upsilon) \cap dom(dw_j^\Upsilon)$ and $dw_i^\Upsilon(n) \neq dw_j^\Upsilon(n)$, then for all $m, m' \geq n$ such that $m \in dom(dw_i^\Upsilon)$ and $m' \in dom(dw_j^\Upsilon)$, $dw_i^\Upsilon(m) \perp dw_j^\Upsilon(m')$. Similarly on $\mathcal{V}$-side.

(d) Either $dw_i^\Upsilon \equiv_{Tail} dv_i^\Upsilon$, or $\{j \mid dw_j^\Upsilon = dw_i^\Upsilon\} = \{i\} = \{j \mid dv_j^\Upsilon = dv_i^\Upsilon\}$.

In (c), we use the notation $E \perp F$ for extenders E and F to mean neither is an initial segment of the other.

We omit the proof of 6.2.2 for now. Concerning (d), recall that for $\alpha = 1$ and $\tau = 0$, we have $dw^0_{Z(1)} = e^{\Upsilon_0}_{b_0}$ and $dv^0_{Z(1)} = e^{\Upsilon_0}_{c_0}$, so the two are not tail-equivalent. The tail-inequivalence can be inflated by agreeing branches $e^{\Upsilon_1}_\xi$ for $\xi < lh(\Upsilon_1)$, leading to

$~~d_{Z(1,\xi)}~~ \, dw^0_{Z(1,\xi)} \not\equiv_{Tail} dv^0_{Z(1,\xi)}$ for $0 < \xi < lh(\Upsilon_1)$.

(ξ must be a limit ordinal here. Cofinal inflation of a branch divergence in $(W_{Z(\tau)}, V_{Z(\tau)})$ is the only way to have $|[i]_\tau| = I$.)

<u>Def 6.2.3</u> $i \frown_\tau j$ iff $dw^\tau_i = dw^\tau_j$.

$[i]_\tau = \{j \mid i \frown_\tau j\}$.

By 6.2.2,

$i \frown_\tau j$ iff $dw^\tau_i \equiv_{Tail} dw^\tau_j$

iff $dv^\tau_i \equiv_{Tail} dv^\tau_j$

iff $dv^\tau_i = dv^\tau_j$.

So 6.2.3 is justified in omitting "$\mathcal{W}$" on the left-hand side. We can also omit reference to α, because $\hat{W}^{\tau}_{\alpha}$ will be extended by $\hat{W}^{\tau}_{\beta}$ when $\alpha < \beta$.

Remark It is possible that both $dw^{\tau}_{i} \equiv_{Tail} dw^{\tau}_{i}$ and $[i]_{\tau} = \{i\}$. For example, let $\tau = 0$ and $\alpha = 1$ and $i = z(1, \omega)$, where $\mathrm{crit}(e^{Y_1}_{\omega}) > \delta(Y_0)$ and $\omega < \mathrm{lh}(Y_0)$.

Def 6.2.4 $\mathrm{Ext}(\hat{W}^{\tau}_{\gamma}) = \bigcup_{i \le z(\gamma)} \mathrm{ran}(dw^{\tau}_{i})$, and $\mathrm{Ext}(\hat{\mathcal{V}}^{\tau}_{\gamma}) = \bigcup_{i \le z(\gamma)} \mathrm{ran}(dv^{\tau}_{i})$,

Def 6.2.5 $\left(\hat{W}^{\tau}_{\gamma}\right)^{ext} = \left(\{ dw^{\tau}_{i} \mid i \leq z(\gamma)\}, \subseteq\right)$

and $\left(\hat{\mathcal{V}}^{\tau}_{\gamma}\right)^{ext} = \left(\{ dv^{\tau}_{i} \mid i \leq z(\gamma)\}, \subseteq\right)$.

By 6.2.2, these are trees. Also, if $s \in \hat{W}^{\tau, ext}$ then $s(k)$ determines $s \restriction k$.

For $s \in \left(\hat{W}^{\tau}_{\gamma}\right)^{ext}$, let

$$J^{\tau}_{s} = \{ i \mid dw^{\tau}_{i} = s \}$$

We can omit "γ" in specifying J^{τ}_{s}, but not "τ". What we called "J_s" before was defined in the special case that $s \in \psi(Y_\alpha)^{ext}$, and so is J^{α}_{s}.

We have

$$J^{\tau}_{s} = [i]_{\tau},$$

for all i s.t. $s = dw^{\tau}_{i}$.

~~Insert page 78a~~

Example 6.2.6 Let $F \in Ext(Y_1)$ and $\beta_F = 0$. Then for $i = \theta_F + 1$, $dw^{0}_{i} = dw^{1}_{i} = \langle F \rangle$. But $J^{0}_{\langle F \rangle} = \{\theta_F + 1\}$, while $J^{1}_{\langle F \rangle} = [\theta_F + 1, z_F(z(1))]$.

So $[\theta_F + 1]_0 \subsetneq [\theta_F + 1]_1$.

Concerning the relationship between the dw_i^τ for varying τ, we have

<u>Lemma 6.2.7</u> Let $\tau < \gamma \leq \alpha$ and $i \leq z(\alpha+1)$; then

(1) $[i]_\tau \subseteq [i]_\gamma$,

and

(2) dw_i^γ fits into dw_i^τ.

<u>Proof</u> By induction on α, with a subinduction on i. If we have it for $\alpha' < \alpha$, the new cases to consider involve i s.t. $z(\alpha) < i \leq z(\alpha+1)$. Fix such an i, and let $dw_i^\alpha = e_{\xi+1}^{Y_\alpha}$. (We leave the case that the dw_i^α is of limit length to the reader; the relationships in (1) and (2) are preserved as we move i out in the relevant direct limits.) Let $F = E_\xi^{Y_\alpha}$, and

$$z_F(k) = i.$$

Then $[k]_\tau \subseteq [k]_\gamma$ by induction, moreover

$$[i]_\tau = z_F{}''([k]_\tau - \beta_F)$$

and

$$[i]_\gamma = z_F{}''([k]_\gamma - \beta_F).$$

Thus $[i]_\tau \subseteq [i]_\gamma$, as desired.

For (2), we have that dw_k^γ fits into dw_k^τ by induction. Suppose first that we have $G \in \operatorname{ran}(dw_k^\gamma)$ such that

$$\kappa_G < \kappa_F < \lambda_G .$$

In this case $\operatorname{ran}(dw_i^\gamma) = p_F{}'' \operatorname{ran}(dw_k^\gamma)$, and since p_F preserves "fitting into" by 5.3, dw_i^γ fits into dw_i^τ. Suppose next there is no such G; then $\operatorname{ran}(dw_i^\gamma) = p_F{}'' \operatorname{ran}(dw_k^\gamma) \cup \{F\}$. Since p_F preserves fitting into, it is enough to see that F fits into some $H \in \operatorname{ran}(dw_i^\tau)$. But if there is a $G \in \operatorname{ran}(dw_k^\tau)$ s.t. $\kappa_G < \kappa_F < \lambda_G$, then F fits into $p_F(G) \in \operatorname{ran}(dw_i^\tau)$, and otherwise $F \in \operatorname{ran}(dw_i^\tau)$, so in either case we're done.

$\boxtimes$

Associated to the dw_i^τ there are embeddings acting on blocks of trees, just as we had when $\tau = \alpha$. Namely, if

$$dw_k^\tau \frown s = dw_i^\tau,$$

i.e. $[k]_\tau <^* [i]_\tau$ with s being the witness, then we get

$$\mathring{\Phi}^\tau_{[k],[i]} = \hat{\Phi}^\tau_s$$

where

$$\hat{\Phi}^\tau_s = \langle u_s^\tau, \langle \Phi_i^\tau \mid i \in I_s^\tau \rangle \rangle$$

with

$$u_s^\tau : I_s^\tau \xrightarrow[\text{op}]{1-1} J_s^\tau$$

Rmk See §4.4 for the defn of Φ_i^τ. See also 5.6.

and

$$\Phi_i^\tau : W_i^* \longrightarrow W_{u_s^\tau(i)}$$

the weak tree embedding we get by composing all the $J(n)$-ultrapower maps for $n \in \mathrm{dom}(s)$.

Remark 6.2.8 Let us ignore the D-extenders for a moment, by assuming that there are none. (That could be the case.) Then we can regard $\langle \mathcal{Y}^\frown_\gamma b_\gamma \mid \tau \le \gamma \le \alpha \rangle$ as a stack of normal trees on $\hat{W}_\tau$, and $\hat{W}^\tau_{\alpha+1}$ is the normalization of $\langle \mathcal{Y}^\frown_\gamma b_\gamma \mid \tau \le \gamma \le \alpha \rangle$. The $\vec{dw}^\tau_i$ are the branch extender sequences of $\hat{W}^\tau_{\alpha+1}$.

Because we have padded the way we have we don't need to write $\vec{dw}^{\tau,\alpha}_i$; the i tells you what the relevant α is. Similarly, since $\alpha < \beta \Rightarrow \hat{W}^\tau_\alpha$ is an "initial segment" of $\hat{W}^\tau_\beta$, we can just write $\hat{W}^\tau$ for the "post τ structure of $\hat{W}$". Using this notation, we have the heuristic formula

$$\hat{W}^\tau = X(Y_\tau{}^\frown b_\tau, \hat{W}^{\tau+1}).$$

The formula explains why ~~the~~ $dw_i^{\tau+1}$ fits into dw_i^τ.

Remark Of course we also have $\hat{\varphi}_s^\tau$ and "$\hat{V}^\tau = X(Y_\tau{}^\frown c_\tau, \hat{V}^{\tau+1})$".

The generalization of 5.7 is

Lemma 6.2.9 Let $\eta = \sup\{\xi+1 \mid E_\xi^{W_i} \in \mathrm{ran}(dw_i^0)\}$; then

(i) a tail of dw_i^0 fits cofinally into $e_\eta^{W_i}$, and

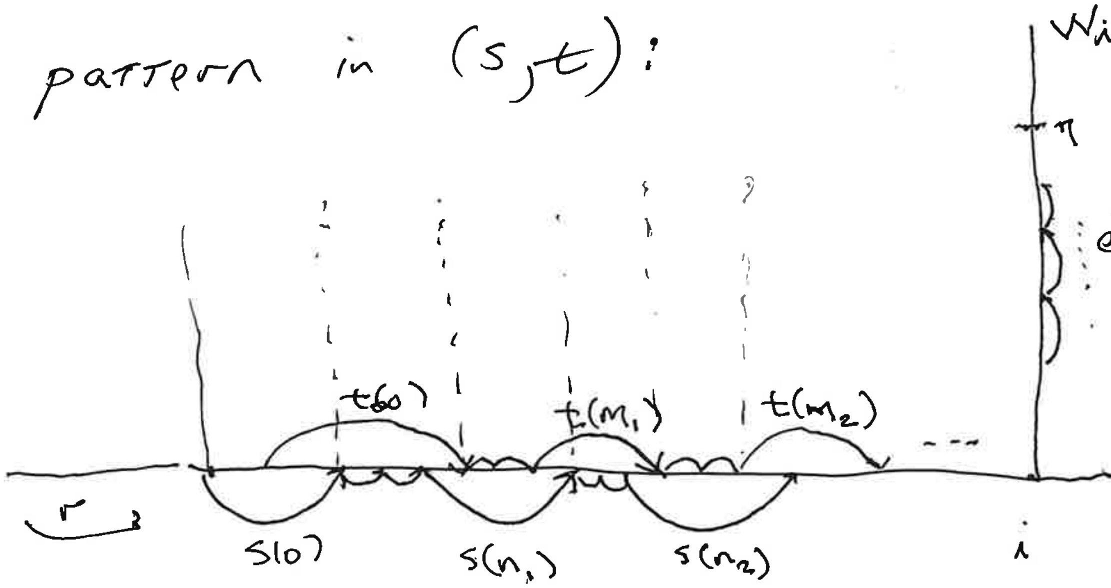
(ii) if s is a branch of $(\hat{W}^0)^{ext}$ such that $\eta = \sup\{\xi+1 \mid E_\xi^{W_i} \in \mathrm{ran}(s)\}$, and η is a limit ordinal, ~~then $s = dw_i^0$. Moreover, the same is true on the V-side.~~

and a tail of s fits cofinally into $e_\eta^{W_i}$, then $s = dw_i^0$.

Moreover, the counterparts ~~of~~ (i) and (ii) hold for dv_i^0.

Proof For (i), see the proof of 5.7.

For the uniqueness clause (ii), by passing to tails we may assume we have a tail t of dw_i^0 that fits into $e_\eta^{W_i}$, and a branch $r \frown s$ of $(\hat{W}^0)^{ext}$ s.t. ~~that r~~ s fits into $e_\eta^{W_i}$, and (toward contradiction) $s(0)$ is inconsistent with $t(0)$. We then get the zipper pattern in (s,t):



We have $G \in \mathrm{ran}(e_\eta^{W_i})$ s.t. $s(0)$ fits into G, and $H \in \mathrm{ran}(e_\eta^{W_i})$ s.t. $t(0)$ fits into H. Since $s(0)$ and $t(0)$ are inconsistent (see 5.2), $G = H$. But then $s(n_1)$ fits into some $K \in \mathrm{ran}(e_\eta^{W_i})$, and since it is inconsistent with $t(0)$, $K = G$.

(83b)

Proceeding by induction, all $t(m_i)$ and $s(m_i)$ fit into the fixed G. But this implies $\lambda_G > \lambda_{t(m)}$ for all m, so $G \notin \operatorname{ran}(e_\eta^{W_i})$, contradiction.

$\boxtimes$

We can sharpen 6.2.9 in the case that $e_\eta^{W_i}$ has a tail consisting of A-extenders.

<u>Lemma 6.2.10</u> Let i be a limit ordinal, $\eta = \sup\{\xi+1 \mid E_\xi^{W_i} \in \operatorname{ran}(d_{W_i}^0)\}$, and suppose that s is a tail of $e_\eta^{W_i}$ such that every $F \in \operatorname{ran}(s)$ is an A-extender; then s is a tail of $d_{W_i}^0$. Similarly on the $\mathcal{V}$-side.

Proof Let $\lambda = \operatorname{dom}(s)$, and for $1 \leq \nu \leq \lambda$, let

$$\eta_\nu = \sup\{\xi+1 \mid E_\xi^{W_i} \in \operatorname{ran}(s\restriction\nu)\}$$

and

$$\theta_\nu = \sup\{\theta_F+1 \mid F \in \operatorname{ran}(s\restriction\nu)\}.$$

So $\eta_\lambda = \eta$ and $\theta_\lambda = i$. Notice that

$$W_{\theta_\nu} \restriction \eta_\nu + 1 = W_i \restriction \eta_\nu + 1$$

for all $\nu < \lambda$, because the extenders in $\operatorname{ran}(s \restriction \nu)$ are cofinal in the extenders used on both sides. (This holds whether ν is a successor or a limit ordinal.) Let

$$d = dw^0_{\beta_{s(0)}}.$$

<u>Claim 6.2.11</u> For $1 \le \nu < \lambda$,

(a) $d^\frown s \restriction \nu = dw^0_{\theta_\nu}$, and

(b) $\theta_\nu = \beta_{s(\nu)}$.

<u>Proof</u> By induction on ν. Take first $\nu = 1$. ~~We~~ Letting $F = s(0)$, we have $\theta_1 = \theta_F + 1$ and $d = dw^0_{\beta_F}$. But $dw^0_{\theta_F+1} = dw^0_{\beta_F}{}^\frown \langle F \rangle$, so we have (a). For (b), we must show that $\theta_F + 1 = \beta_G$, where $G = s(1)$.

The key is that F and G are used consecutively along a branch of the normal tree $\mathcal{W}_i$. This implies $\lambda_F \leq \kappa_G$, and hence $\theta_F + 1 \leq \beta_G$. Note that F is used in all $\mathcal{W}_k$ for $k \in [\theta_F + 1, i]$, and G is used in all $\mathcal{W}_k$ for $k \in [\theta_G + 1, i]$.

Let

$$F = E_\gamma^{\mathcal{W}_{\theta_F+1}},$$

then

$$(*) \qquad \mathcal{W}_\eta \restriction \gamma+1 = \mathcal{W}_{\eta'} \restriction \gamma+1 \quad \forall \eta, \eta' \in [\theta_F + 1, i),$$

so

$$(**) \qquad \eta < \eta' \Rightarrow \lambda(H_{\eta'}) \leq \lambda(H_\eta),$$

$$\text{where} \quad H_\xi = E_{\gamma+1}^{\mathcal{W}_\xi} \quad \forall \xi \in [\theta_F + 1, i).$$

By $(*)$

$$\lambda(H_{\eta+1}) \leq \lambda(F_\eta) \qquad \forall \eta \in [\theta_F + 1, i),$$

because $F_\eta \in \mathrm{Ext}(\mathcal{W}_{\eta+1})$ and $F_\eta \neq E_\xi^{\mathcal{W}_{\eta+1}}$ for $\xi < \gamma$.

Remark The "η" in the lines above is variable, it's not the η in the statement of 6.2.10, which has become η_λ.

But also

$$\kappa_G < \lambda(H_{\theta_G+1}),$$

because G is the next extender used after F on a branch of W_{θ_G+1}. Thus

$$\kappa_G < \lambda(F_\eta)$$

for all $\eta \in [\theta_F+1, \theta_G]$, which implies $\beta_G \leq \theta_F + 1$, as desired.

This proves 6.2.11 when $\nu = 1$. The general successor step is similar, so we omit it. Suppose now ν is a limit ordinal. We have $e_{\eta_\nu}^{W_{\theta_\nu}} = e_{\eta_\nu}^{W_1}$ since $W_{\theta_\nu} \restriction \eta_\nu + 1 = W_1 \restriction \eta_\nu + 1$, and $s \restriction \nu$ is a tail of ~~$\hat{W}_1$~~ $e_{\eta_\nu}^{W_1}$. Moreover $d \frown s \restriction \nu$ is a branch of $\hat{W}^{0,\mathrm{ext}}$ by (a) at $\nu' < \nu$. 83f

From 6.2.9 (ii) we get that $d^\frown s \restriction \nu = d_{W^0_{\theta_\nu}}$, so we have (a) of Subclaim 6.2.11. For part (b), let $G = s(\nu)$. Using that $s \restriction \nu^\frown \langle G \rangle$ is a ~~tail~~ segment of a branch of W_i, we get that $\beta_G = \sup\{\theta_F + 1 \mid F \in \operatorname{ran}(s \restriction \nu)\}$ just as in the successor step. This yields (b).

Subclaim 6.2.11 $\boxtimes$

From the subclaim we get that $d^\frown s$ is a branch of $\hat{W}^{0,\mathrm{ext}}$. By 6.2.9 (ii), we have then that $d^\frown s = e^{W_i}_{\eta_\lambda}$, as desired.

Lemma 6.2.10 $\boxtimes$

§7. The limit step – outline

Because we cut to extenders of length $\delta(Y_\alpha)$, the successor step did not produce an embedding from $\hat{W}_\alpha$ to $\hat{W}_{\alpha+1}$. When we get to limit λ, we'd like to set $\hat{W}_\lambda = \lim_{\alpha<\lambda} \hat{W}_\alpha$, but it's not clear what sort of limit to take. We outline the solution to this problem now.

Set

$$\hat{W}^0_\lambda = \text{"}\bigcup_{\alpha<\lambda}\text{"}\, \hat{W}_\alpha$$

and

$$\hat{\mathcal{V}}^0_\lambda = \text{"}\bigcup_{\alpha<\lambda}\text{"}\, \hat{\mathcal{V}}_\alpha,$$

where the "union" comes from the fact that $\alpha<\beta \Rightarrow \hat{W}_\alpha$ is extended by $\hat{W}_\beta$, and similarly for the $\hat{\mathcal{V}}$'s. Let

$$z_0(\lambda) = \sup_{\alpha<\lambda} z(\alpha),$$

and let $\mathcal{S}^{w}$ and $\mathcal{S}^{v}$ be the common part trees, i.e.

$$\mathcal{T} \trianglelefteq \mathcal{S}^{w} \text{ iff for all sufficiently large } \xi < z_0(\lambda),\ \mathcal{T} \trianglelefteq \mathcal{W}_\xi$$

and

$$\mathcal{T} \trianglelefteq \mathcal{S}^{v} \text{ iff for all sufficiently large } \xi < z_0(\lambda),\ \mathcal{T} \trianglelefteq \mathcal{V}_\xi.$$

Then $(\mathcal{S}^{w}, \mathcal{S}^{v})$ is a slow comparison. Set

$$b = \Sigma(\mathcal{S}^{w}),$$

$$c = \Delta(\mathcal{S}^{v}),$$

and

$$W^0_{z_0(\lambda)} = \mathcal{S}^{w} \frown b,$$

$$V^0_{z_0(\lambda)} = \mathcal{S}^{v} \frown c.$$

If $b \neq_{\text{Tail}} c$, then we can stop here, and set $\hat{W}_\lambda = \hat{W}^0_\lambda {}^\frown \langle W^0_{z_0(\lambda)} \rangle$ and $\hat{\mathcal{V}}_\lambda = \hat{\mathcal{V}}^0_\lambda {}^\frown \langle V^0_{z_0(\lambda)} \rangle$. We have that ~~both~~ $(W^0_{z_0(\lambda)}, \mathcal{V}^0_{z_0(\lambda)})$ is both a slow comparison and a branch divergence. Moreover, the usual reflection argument shows that this can't be what's happening at stationary many λ. [It it happened at ω_1, letting $\pi : H \to V$ with $\mathrm{crit}(\pi) = \lambda$, we'd have $b \equiv_{\mathrm{Tail}} c$ for b and c the branches of the common part trees chosen by at ω_1 Σ and Λ. This is because their extender sequences $e_b^{\mathcal{S}\omega_1}$ and $e_c^{\mathcal{S}\omega_1}$ would be consistent with π. But then $b \cap \lambda \equiv_{\mathrm{Tail}} c \cap \lambda$ by elementarity.]

So let us assume $b \equiv_{Tail} c$.

Insert 87a

By 6.2.9 (ii), we have branches S and T of $\hat{W}^0_\lambda$ and $\hat{\nu}^0_\lambda$ determine by $e_b^{W^0_{z_0(\lambda)}}$ and $e_c^{\nu^0_{z_0(\lambda)}}$. The $\hat{W}^0$ and $\hat{\nu}^0$ systems now choose these as $dw^0_{z_0(\lambda)}$ and $dv^0_{z_0(\lambda)}$. By 6.2.7 (2), these in turn determine branches $dw^\tau_{z_0(\lambda)}$ and $dv^\tau_{z_0(\lambda)}$ for all $\tau < \lambda$. The tail agreements propagate, so

$$dw^\tau_{z_0(\lambda)} \equiv_{Tail} dv^\tau_{z_0(\lambda)}$$

for all $\tau < \lambda$. This means that for each $\tau < \lambda$ we can lift $\hat{W}_\tau$ along the agreeing tail of $dw^\tau_{z_0(\lambda)}$ and $dv^\tau_{z_0(\lambda)}$.

Insert from p. 87

Lemma 6.2.9 just applied to the finite stages $\hat{W}_n$, $\hat{\mathcal{U}}_n$ for $n<\omega$ so far. Moreover, it was just the uniqueness part, ~~of the~~ showing that at most one branch of $(\hat{W}^0)^{ext}$ fits into $\mathcal{C}_b^{\delta^W}$ on a tail. We also need to prove existence.

Lemma 6.2.9 (1) proves existence by using the way a cofinal branch b of Y generates a branch of $X(\mathcal{T}, Y^\frown b)$. At the general limit step we need to go in the other direction, from a branch of X to a branch of Y.

The result is systems we call $\hat{W}^1_\lambda$ and $\hat{V}^1_\lambda$ that extend $\hat{W}^0_{z_1(\lambda)}$ and $\hat{V}^0_{z_1(\lambda)}$, and look like them. All W_i for $i \in [z_0(\lambda), z_1(\lambda))$ extend $\Delta_0^w{}^\frown b_0$, and all V_i for $i \in [z_0(\lambda), z_1(\lambda))$ extend $\Delta_0^v{}^\frown c_0$. Let Δ_1^w, Δ_1^v be the common part, i.e.

$$\sigma \trianglelefteq \Delta_1^w \text{ iff } \forall \text{ suff large } \gamma < z_1(\lambda),\ \sigma \trianglelefteq W_\gamma$$

~~iff $\forall$ suff large $\gamma < z_1(\lambda)$ $\sigma \trianglelefteq V_\gamma$.~~

and similarly for Δ_1^v.

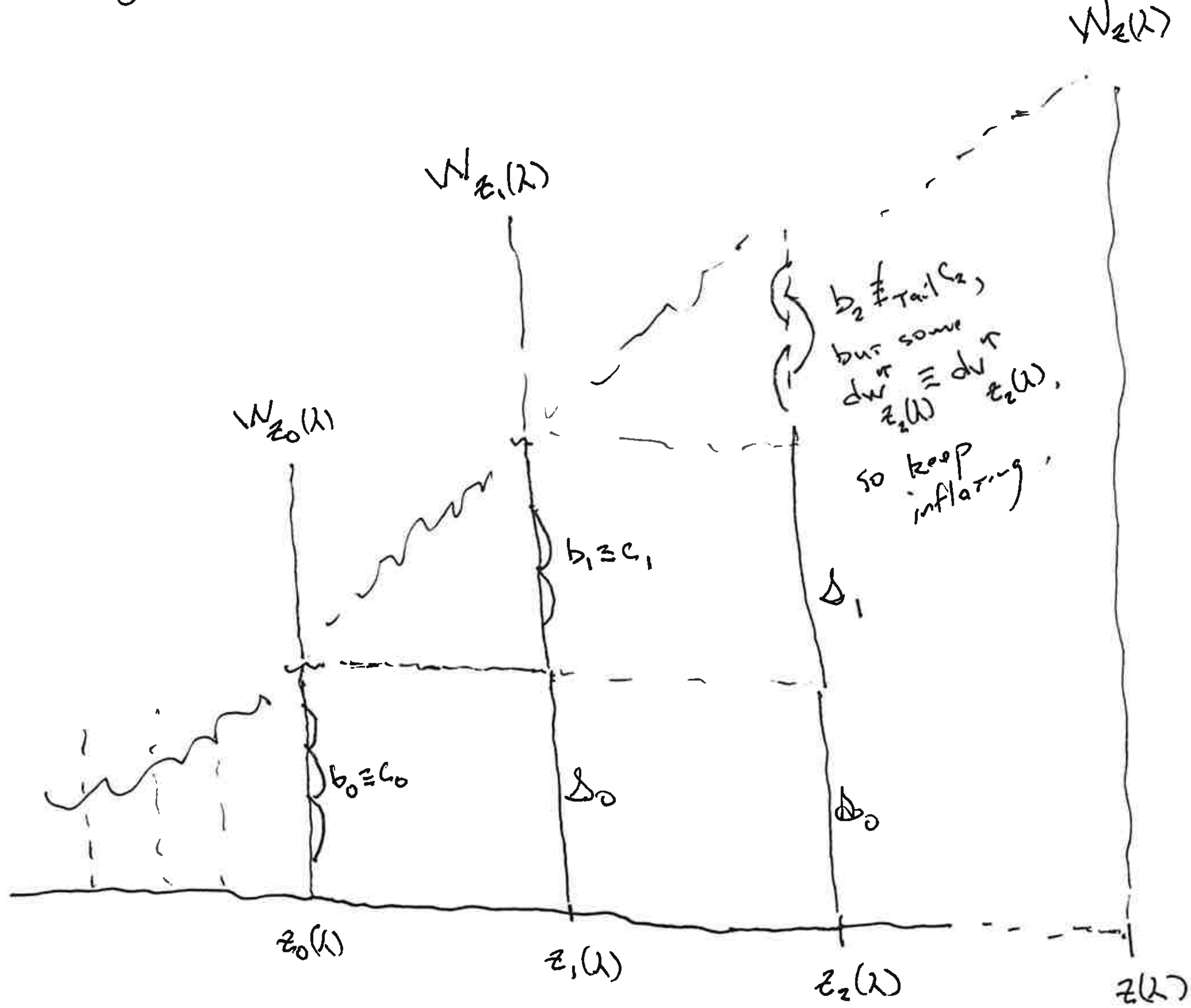
Let $b_1 = \Sigma(\Delta_1^w)$ and $c_1 = \Lambda(\Delta_1^v)$, and now repeat the previous round. ~~Etc.~~ The one wrinkle is that we may have

$$e_{b_1}^{\Delta_1^w} \not\equiv_{tail} e_{c_1}^{\Delta_1^v}$$

but some τ s.t. (using $e_{b_1}^{\Delta_1^w}$ and $e_{c_1}^{\Delta_1^v}$ to determine $dw^\tau_{z_1(\lambda)}$ and $dv^\tau_{z_1(\lambda)}$),

$$dw^\tau_{z_1(\lambda)} \equiv_{tail} dv^\tau_{z_1(\lambda)}.$$

In that case, we lift by all the agreeing tails corresponding to $\tau' \geq \tau$. A diagram is



This "completion process" can go into the transfinite, but it must halt in countably many stages. (Since no new D-extenders are being added from outside, we are getting canonical maps of $z_0(\lambda)$ onto all the $z_\alpha(\lambda)$.) We let $(W_{z(\lambda)}, \vec{\mathcal{T}}_{z(\lambda)})$ be the last slow comparison we produce this way. We've stopped because there is ~~not~~ no $\gamma < \lambda$ such that $dw^\tau_{z(\lambda)} \equiv_{Tail} d\gamma^\tau_{z(\lambda)}$.

Note The next section, § 8, comes from an earlier version. It needs to be re-written, but it is basically correct.

§8

~~§~~ Definition of $\hat{W}_\omega$ and $\hat{\mathcal{V}}_\omega$.

We have defined the $\hat{W}_n$ and $\hat{\mathcal{V}}_n$ for n finite. The definition of $\hat{W}_\omega$ and $\hat{\mathcal{V}}_\omega$ is much like the general limit case, but we isolate it for now. Let

$$\hat{W}^0_\omega = \text{“}\bigcup_{n<\omega}\text{”}\ \hat{W}_n, \quad z_0(\omega) = \sup_n z(n),$$

and

$$\hat{\mathcal{V}}^0_\omega = \text{“}\bigcup_{n<\omega}\text{”}\ \hat{\mathcal{V}}_n.$$

Let $\mathcal{S}^W$ be the common part tree;

$$\mathcal{T} \trianglelefteq \mathcal{S}^W \text{ iff for all sufficiently large } n,$$
$$\mathcal{T} \trianglelefteq W_{z(n)}.$$

Similarly for $\mathcal{S}^\mathcal{V}$. $(\mathcal{S}^W, \mathcal{S}^\mathcal{V})$ is almost a stow comparison, but the last branch is missing. It is easy to see that $\mathcal{S}^W$ has limit length. Let

$$W^0_{z_0(\omega)} = \mathcal{S}^W \frown \Sigma(\mathcal{S}^W)$$

$$\mathcal{V}^0_{z_0(\omega)} = \mathcal{S}^\mathcal{V} \frown \Lambda(\mathcal{S}^\mathcal{V}).$$

If $\Sigma(\mathcal{S}^W) \not\equiv_{\text{tail}} \Lambda(\mathcal{S}^\mathcal{V})$ (i.e. they two branches don't use the same extenders on a tail), then we are done. We set

$$z_0(\omega) = z(\omega),$$

$$W_{z(\omega)} = W^0_{z_0(\omega)},$$

$$\nu_{z(\omega)} = \nu^0_{z_0(\omega)},$$

$$\hat{W}_\omega = \hat{W}^0_\omega {}^\frown \langle W_{z(\omega)} \rangle,$$

and

$$\hat{\nu}_\omega = \hat{\nu}^0_\omega {}^\frown \langle \nu_{z(\omega)} \rangle .$$

$(W_{z(\omega)}, \nu_{z(\omega)})$ is a slow comparison and a branch divergence, and our other inductive hypotheses are met.

So suppose $Z(\mathcal{S}^W) \equiv_{\text{tail}} \Lambda(\mathcal{S}^\nu)$. Let $\lambda = lh(\mathcal{S})$, and $\eta <^{W^0_\omega} \lambda$, $\eta <^{\nu^0_\omega} \lambda$ be s.t. η begins an agreeing tail, i.e. $[\eta, \lambda)_{W^0_\omega} = [\eta, \lambda)_{\nu^0_\omega}$. Let s enumerate the extenders used in this tail in incr. order. Our goal now is to show that s determines a common cofinal-in-$z_0(\omega)$ ~~branch~~ path in $\hat{W}^0_\omega$ and $\hat{\nu}^0_\omega$. ~~in that it is the sequence of~~

We shall use the proof of Lemma 6.2.10 for this. Let $\lambda = \mathrm{dom}(s)$, and for $1 \le \nu < \lambda$

$$\eta_\nu = \sup\{\xi + 1 \mid E_\xi^{\delta^w} \in \mathrm{ran}(s \restriction \nu)\}$$

and

$$\theta_\nu = \sup\{\theta_F + 1 \mid F \in \mathrm{ran}(s \restriction \nu)\}.$$

Set $\eta_\lambda = lh(W^0_{z_0(\omega)})$ and $\theta_\lambda = z_0(\omega)$. Let

$$d = dw^0_{\beta_{s(0)}})$$

$$c = dv^0_{\beta_{s(0)}}.$$

<u>Lemma 8.1.1</u> For $1 \le \nu < \lambda$,

(a) $d^\frown s \restriction \nu = dw^0_{\theta_\nu}$,

(b) $\theta_\nu = \beta_{s(\nu)}$, and

(c) $c^\frown s \restriction \nu = dv^0_{\theta_\nu}$.

<u>Proof</u> This was proved in 6.2.11.

$\boxtimes$

We defined $\theta_\lambda = z_0(\omega)$; this is justified by

Lemma 8.2 $\{\theta_F + 1 \mid F \in \operatorname{ran}(s)\}$ is cofinal in $z_0(\omega)$.

Proof Let $\gamma = \sup\{\theta_F + 1 \mid F \in \operatorname{ran}(s)\}$, and suppose toward contradiction that $\gamma < z_0(\omega)$. Then all $F \in \operatorname{ran}(s)$ are used in W_γ, So since W_γ and W_ω^0 are both by Σ and use the same extenders (those in $\operatorname{ran}(s)$) cofinally-in-$\cancel{\ }$ $\operatorname{lh}(\Delta)$ often, $W_\omega^0 \trianglelefteq W_\gamma$. But then $W_\omega^0 \trianglelefteq W_\eta$ for $\gamma \le \eta < z_0(\omega)$. However, let

$$\xi + 1 = \operatorname{lh}(W_\omega^0),$$

and consider the exit extender $E_\xi^{W_\eta}$ for $\eta \ge \gamma$. We have

$$\gamma \le \eta \le \delta < z_0(\omega) \Rightarrow \operatorname{lh}(E_\xi^{W_\eta}) \ge \operatorname{lh}(E_\xi^{W_\delta}).$$

So $E_\xi^{W_\eta}$ can only change finitely often at $\eta \ge \gamma$. Its eventual value is then in the common part $\operatorname{Ext}(\Delta)$, contrary to $\operatorname{Ext}(\Delta) = \operatorname{Ext}(W_\omega^0)$.

$\boxtimes$

By 8.1.1 and 8.2, $d \frown s$ is a cofinal branch of $(\hat{W}^{0,ext}, \subseteq)$ and $c \frown s$ is a cofinal branch of $(\hat{V}^{0,ext}, \subseteq)$.

We set

$$dw^0_{z_0(\omega)} = d \frown s,$$

and

$$dv^0_{z_0(\omega)} = c \frown s.$$

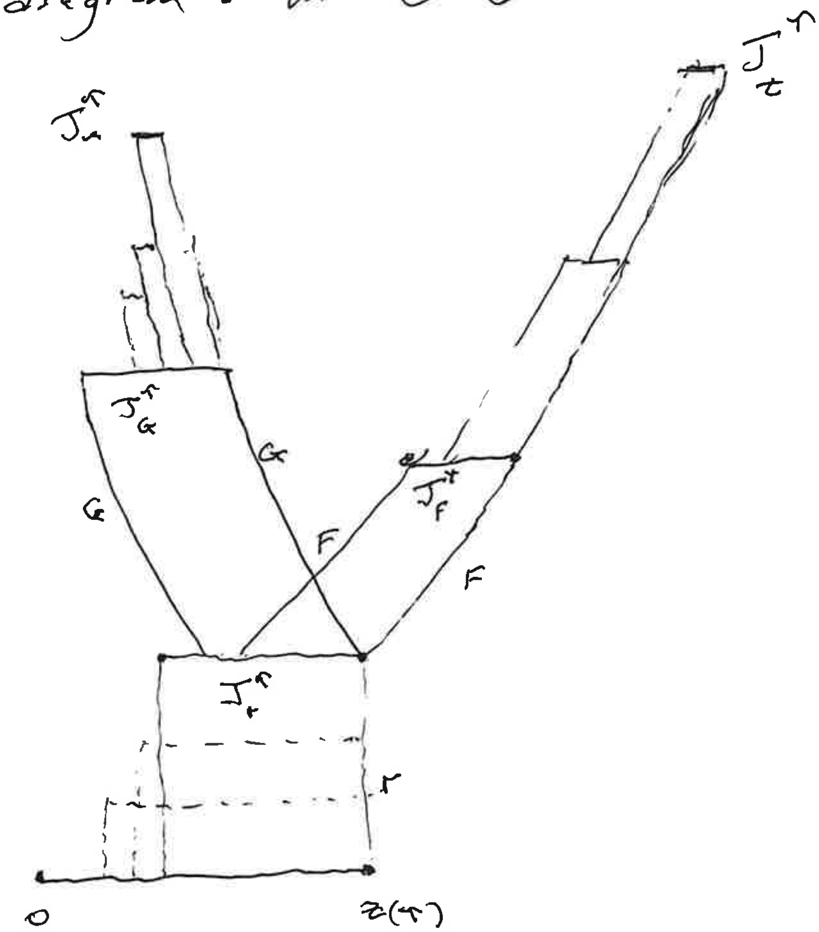
Those are then the unique cofinal branches that fit into the branches of $W^0_{z_0(\omega)}$ and $V^0_{z_0(\omega)}$ that are chosen by Σ and Λ.

We show now that for $\gamma < \omega$ there are unique cofinal branches of $\hat{W}^{\gamma,ext}_\omega$ and $\hat{V}^{\gamma,ext}_\omega$ that fit into $dw^0_{z_0(\omega)}$ and $dv^0_{z_0(\omega)}$.

<u>Lemma 8.3</u> Let $t, u \in \hat{W}^{\gamma,ext}_\omega$ where $\gamma < \omega$, and suppose $\exists n \in dom(t) \cap dom(u)\ (t(n) \neq u(n))$; then $J^\gamma_t \cap J^\gamma_u = \emptyset$.

Let
$$t = r^\frown\langle F\rangle^\frown t_0$$
and
$$u = r^\frown\langle G\rangle^\frown u_0$$
where $F \neq G$. Say $F = F_\eta$ and $G = F_\gamma$ where $\eta < \gamma$.

Here is a diagram. ~~The horizontal~~



The horizontal lines represent inflations $\overline{J}^r_v$ of $\hat{W}_r$ along initial segments v of t or u. $\overline{J}^r_G \cap \overline{J}^r_F = \emptyset$ because G was used strictly after F in the inflation process, so $\overline{J}^r_F \subseteq \Theta_G$ and $\min(\overline{J}^r_G) = \Theta_G + 1$.

We now show inductively that the disjointness propagates. For the limit step, note that if λ, λ' are limit ordinals, then $\sup\{\Theta_H \mid H \in \operatorname{ran}(t_0 \restriction \lambda)\} \neq \sup\{\Theta_H \mid H \in \operatorname{ran}(u_0 \restriction \lambda')\}$. (Otherwise, letting $i = \sup\{\Theta_H \mid H \in \operatorname{ran}(t_0\restriction\lambda)\}$,

$dw_i^\tau \trianglelefteq t$ and $dw_i^\tau \trianglelefteq u$ by the proof of 6.2.2 (b).

Remark $J_t^\tau \cap J_u^\tau = \emptyset$ implies that if $i+1 \in J_t^\tau$ and $k+1 \in J_u^\tau$, then $F_i \neq F_k$. So A-extenders used in $\hat{W}_\tau$ are inflated differently along incompatible branches of $\hat{W}^\tau$.

Lemma 8.3 fails if we allow $t \in \hat{W}_\omega^{\tau, ext}$ and $u \in \hat{\mathcal{V}}_\omega^{\tau, ext}$. {Take $\tau = 0$, $t = dw^0_{z(1)}$ and $u = dv^0_{z(1)}$; then $J_t^0 = \{z(1)\} = J_u^0$. Lemma 8.4 says this is the only sort of counterexample.

<u>Lemma 8.4</u> Let $t \in \hat{W}_\omega^{\tau, ext}$ and $u \in \hat{\mathcal{V}}_\omega^{\tau, ext}$, where $\tau < \omega$, and suppose $\exists n \in \mathrm{dom}(t) \cap \mathrm{dom}(u)$ $(t(n) \neq u(n))$; then either

(a) $J_t^\tau \cap J_u^\tau = \emptyset$, or

(b) $J_t^\tau = J_u^\tau = \{i\}$, for some limit ordinal i.

Proof Let $t = dw_i^\tau$, so that $J_t^\tau = [i]_\tau$. We are assuming $J_t^\tau \cap J_u^\tau \neq \emptyset$, as otherwise (a) holds. But then $J_u^\tau = [i]_\tau$, as we saw after 6.2.3. So

$$dw_i^\tau = J_t^\tau = [i]_\tau = J_u^\tau = dv_i^\tau.$$

By 6.2.2.(d), either $[i]_\tau = \{i\}$ and we're done, or $dw_i^\tau \equiv_{Tail} dv_i^\tau$. But $t \not\equiv_{Tail} u$, so $dw_i^\tau \not\equiv_{Tail} dv_i^\tau$, so we're done.

$\boxtimes$

Remark We need to fill in the proof of 6.2.2 (d).

<u>Lemma 8.5</u> Let $\tau < \omega$; then there is exactly one cofinal branch of $\hat{\mathcal{W}}^{\tau,ext}_{\omega}$ that fits into $dw^0_{z_0(\omega)}$, and exactly one cofinal branch of $\hat{\mathcal{V}}^{\tau,ext}_{\omega}$ that fits into $dv^0_{z_0(\omega)}$.

<u>Proof</u> Let

$$F_{i-1}, F_{k-1} \in ran(s)$$

with $i < k$. We shall show that $dw^{\tau}_{i} \trianglelefteq dw^{\tau}_{k}$, and similarly on the $\mathcal{V}$-side. The desired branch on the $\mathcal{W}$-side is then

$$d = \bigcup_{F_{i-1} \in ran(s)} dw^{\tau}_{i}.$$

By 6.2.7 (2), d fits into $\bigcup_{F_{i-1} \in ran(s)} dw^0_i =$

$dw^0_{z_0(\lambda)}$.

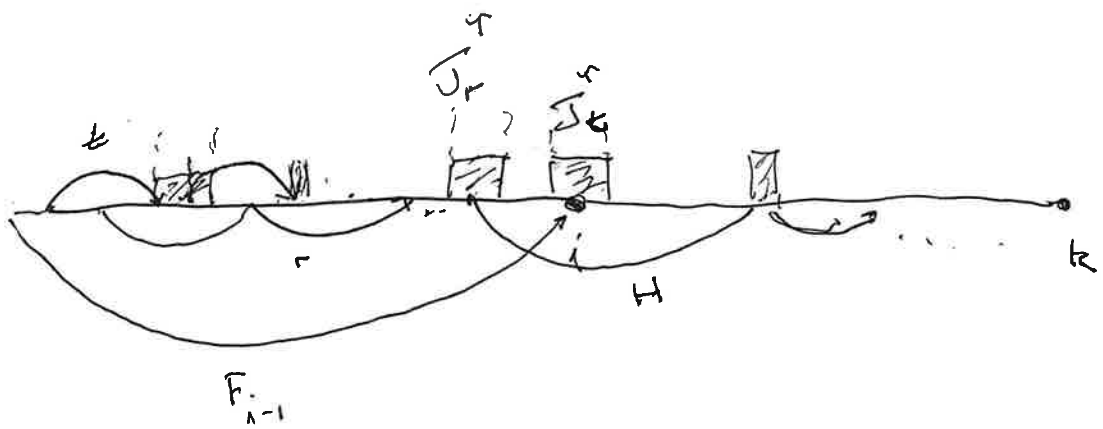
Let $t = dw_i^\tau$ and $u = dw_k^\tau$, and suppose that $\exists n \in \mathrm{dom}(t) \cap \mathrm{dom}(u)\ (t(n) \neq u(n))$.

Let

$$r = \text{longest } v \trianglelefteq t \text{ s.t. } J_v^\tau \subseteq i.$$

We have $i \in J_t^\tau$, and $J_v^\tau \cap J_r^\tau$ by 8.4.

The picture is (collapsing the vertical lines for W_γ's in our usual diagram)



In the picture, $H = u(\mathrm{dom}(r))$. It has to cross F_{i-1} because it is inflating a tail of J_r^τ and $J_r^\tau \subseteq i$. That is, H does not fit into F_{i-1}, and therefore does not fit into any $F \in \mathrm{ran}(dw^0_{\xi_0}(\omega))$.

Contradiction.

$\boxtimes$

In the same vein

<u>Lemma 8.4.1</u> If $s, t \in W^{\gamma,\mathrm{ext}}$ and $s(n) = t(m)$, then $n = m$ and $s \restriction n = t \restriction m$.

Proof. Like 8.4. $\boxtimes$

So we can define

<u>Def 8.4.2</u> Let $F \in \mathrm{Ext}(\hat{W}^{\tau})$; then

$$\hat{\Phi}_F^{\tau} = \langle z_F, \langle \Phi_i^{\tau} \mid i \in I_F^{\tau} \rangle \rangle,$$

where those are as defined on p. 81, with $dw_k^{\tau} \wedge \langle F \rangle = dw_i^{\tau}$. (There are unique such k and i by 8.4.1.) Let

$$J_F^{\tau} = z_F{}'' I_F^{\tau}.$$

(So $J_F^{\tau} = J_{dw_i^{\tau}}^{\tau}$ in the sense defined before.)

<u>Def 8.4.3</u> If F is an A-extender, then $I_F = \bigcup_\gamma I_F^\gamma$ and $J_F = \bigcup_\gamma J_F^\gamma$.

It's easy to see that if $\gamma < \gamma'$, then I_F^γ is an initial segment of $I_F^{\gamma'}$ and J_F^γ is an initial segment of $J_F^{\gamma'}$, but the inclusions can be proper.

Example: Let $F \in \mathrm{Ext}(Y_{\gamma+1})$ and $\beta_F < z(\gamma)$; then $J_F^\gamma = [\theta_F + 1, z_F(z(\gamma))]$ and $J_F^{\gamma+1} = [\theta_F + 1, z_F(z(\gamma+1))]$.

If F is an atomic A-extender, then I_F and J_F are just what we defined in §6.1 (page 62). We have now extended the defn to arbitrary A-extenders.

Definition 8.5.1 For $\tau < \omega$

$$dw^{\tau}_{z_0(\omega)} = \text{unique cofinal branch of } \hat{W}^{\tau,\text{ext}}_{\omega} \text{ that fits into } dw^{0}_{z_0(\omega)},$$

and

$$dv^{\tau}_{z_0(\omega)} = \text{unique cofinal branch of } \hat{\mathcal{V}}^{\tau,\text{ext}}_{\omega} \text{ that fits into } dv^{0}_{z_0(\omega)}.$$

Lemma 8.5.2

The proof of 8.~~5~~ shows that $dw^{\tau}_{z_0(\omega)} \equiv_{\text{Tail}} dv^{\tau}_{z_0(\omega)}$. To repeat it: Suppose not, and let t and u be tails of $dw^{\tau}_{z_0(\omega)}$ and $dv^{\tau}_{z_0(\omega)}$ that fit into s, where s is a tail of $dw^{0}_{z_0(\omega)}$ and of $dv^{0}_{z_0(\omega)}$, and such that $\text{ran}(t) \cap \text{ran}(u) = \emptyset$. Let $F_{i-1} \in \text{ran}(s)$ be such that $t(0)$ fits into F_{i-1}, and let k be largest s.t. all $G \in \text{ran}(t \restriction k)$ fit into F_{i-1}. Thus

$$[i]_{\tau} = \mathcal{J}^{\tau}_{t \restriction k}.$$

Let l be largest s.t. for all $G \in \text{ran}(u \restriction l)$, ~~Gets~~ $\theta_G < i-1$.

We apply Lemma 8.4 now: since $\exists n\ (t\restriction k)(n) \neq (u\restriction l)(n)$, and (b) of the conclusion fails because $i \in J^{\tau}_{t\restriction k}$ and i is a successor ordinal, we get that $J^{\tau}_{t\restriction k} \cap J^{\tau}_{u\restriction l} = \emptyset$. But this implies $J^{\tau}_{u\restriction l} \subseteq i-1$, and $H = u(l)$ must then be such that $\beta_H < i-1 < \theta_H$, so $\kappa_H < \lambda_{F_{i-1}} < \lambda_H$, so H doesn't fit into any $G \in \operatorname{ran}(s)$, contradiction.

Lemma 8.5.2 ☒

From this we get

<u>Lemma 8.6</u> There is a $\gamma < \omega$ such that for all $\tau \in I_{\gamma,\omega})$, $dw^{\tau}_{z_0(\lambda)} = dv^{\tau}_{z_0(\lambda)}$, and $dw^{\tau}_{z_0(\lambda)}$ fits into s.

Proof

Let $dw^{0}_{z_0(\omega)} = t \frown s$ and $dv^{0}_{z_0(\omega)} = u \frown s$. Let $\gamma < \omega$ be large enough that for all $G \in ran(t) \cup ran(u)$, $\theta_G < z(\gamma)$. (This is true if $\theta_{s(0)} < z(\gamma)$.)

~~Since dw~~ Let $\tau \geq \gamma$. Since $dw^{\tau}_{z_0(\omega)}$ fits into $dw^{0}_{z_0(\lambda)}$, every $G \in ran(dw^{\tau}_{z_0(\omega)})$ fits into some $F \in ran(t) \cup ran(s)$. But $\tau \geq \gamma$, so $\theta_G \geq z(\gamma)$, so we must have $F \in ran(s)$. So $dw^{\tau}_{z_0(\omega)}$ fits into s, and similarly $dv^{\tau}_{z_0(\omega)}$ fits into s. By the proof of 8.5.2, we get $dw^{\tau}_{z_0(\omega)} = dv^{\tau}_{z_0(\omega)}$.

$\boxtimes$

Recall the "meta-tree embeddings"

$\hat{\Phi}_a^{\mathcal{T}}$, defined when there are i, k

such that $dw_k^{\mathcal{T}} \cap a = dw_i^{\mathcal{T}}$:

$$\hat{\Phi}_a^{\mathcal{T}} = \langle u_a^{\mathcal{T}}, \langle \Phi_i^{\mathcal{T}} \mid i \in I_a^{\mathcal{T}} \rangle\rangle$$

with

$$u_a^{\mathcal{T}} : I_a^{\mathcal{T}} \xrightarrow[\text{o.p}]{\text{onto}} J_a^{\mathcal{T}}$$

and

$$\Phi_i^{\mathcal{T}} : W_i^{*} \longrightarrow W_{u_a^{\mathcal{T}}(i)}$$

a weak tree embedding. Our notation was

$$\begin{aligned} \theta_a^{\mathcal{T}} &= \sup\{\theta_G + 1 \mid G \in \operatorname{ran}(a)\} \\ &= \min(J_a^{\mathcal{T}}) \end{aligned}$$

and

$$\begin{aligned} \beta_a^{\mathcal{T}} &= (u_a^{\mathcal{T}})^{-1}(\theta_a^{\mathcal{T}}) \\ &= \min(I_a^{\mathcal{T}}). \end{aligned}$$

These were defined for $i < z_0(\omega)$, but now they also make sense for $i = z_0(\omega)$. In parallel fashion we have $\hat{\psi}_a^\tau$ when $dv_m^\tau \frown a = dv_n^\tau$, defined when $n \le z_0(\omega)$ now:

$$\hat{\psi}_a^\tau = \langle u_a^\tau, \langle \psi_i^\tau \mid i \in I_\alpha^\tau \rangle\rangle.$$

When both $dv_m^\tau \frown a = dv_n^\tau$ and $dw_i^\tau \frown a = dw_k^\tau$, then $i = m$, $k = n$, and $\hat{\Phi}_a^\tau$ and $\hat{\psi}_a^\tau$ have the same I_a^τ, J_a^τ, u_a^τ associated, which is why we can specify them using only a and τ.

The proof of the following lemma is a key element.

<u>Lemma 8.7</u> Let $\gamma < \omega$ be such that $dw^{\tau}_{z_0(\omega)} = dv^{\tau}_{z_0(\omega)}$ for all $\tau \geq \gamma$, and let $d = dw^{\gamma+1}_{z_0(\omega)}$; then $\beta_d < z(\gamma+1)$.

<u>Remark</u> So $I^{\gamma+1}_d = [\beta_d, z(\gamma+1)]$ gets mapped by $\cancel{\hat{\Phi}^{\gamma+1}}$ $\hat{u}^{\gamma+1}_d$ to $[z_0(\omega), \rho] = J^{\gamma+1}_d$, where $z_0(\omega) < \rho$. Thus $\hat{\Phi}^{\gamma+1}_d$ and $\hat{\psi}^{\gamma+1}_d$ inflate tails of $\hat{W}_{\gamma+1}$ and $\hat{\mathcal{V}}_{\gamma+1}$ to proper extensions of $\hat{W}^0_\omega$ and $\hat{\mathcal{V}}^0_\omega$.

<u>Proof</u> Let

$$d^\gamma = dw^\gamma_{z_0(\omega)} = dv^\gamma_{z_0(\omega)},$$

and

$$d^{\gamma+1} = dw^{\gamma+1}_{z_0(\omega)} = dv^{\gamma+1}_{z_0(\omega)}.$$

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Let

$$p : \mathrm{Ext}(\mathcal{Y}_\gamma) \longrightarrow \mathrm{Ext}(\hat{W}_\omega^\gamma)$$

be the inflationary map corresponding to the fact that

$$\hat{W}_\omega^\gamma \frown d^\gamma = X(\mathcal{Y}_\gamma \frown b_\gamma, \hat{W}_\omega^{\gamma+1} \frown d^{\gamma+1}).$$

(This is heuristic; we haven't defined the X-operation formally, and we are ignoring D-extenders when we write this formula.)

Let

$$a = \text{longest } a \in (\mathcal{Y}_\gamma \frown b_\gamma)^{\mathrm{ext}} \text{ such that } \hat{p}(a) \trianglelefteq d^\gamma.$$

Here $\hat{p}(a)$ is the downward closure of $p``\mathrm{ran}(a)$ in the extender tree $(\hat{W}_\omega^\gamma \frown d^\gamma)^{\mathrm{ext}}$.

Claim 8.7.1 $\sup\{lh(E) \mid E \in ran(a)\}$ is bounded in $\delta(Y_\gamma)$.

Proof Otherwise a is cofinal in Y_γ^{ext}. We show that this implies $a = e_{b_\gamma}^{Y_\gamma}$, by appealing to condensation (very strong hull) for Σ. But the whole situation is symmetric now, so the same proof shows $a = e_{c_\gamma}^{Y_\gamma}$ by vshc for Λ. This is a contradiction.

Let $e_b = e_{b_\gamma}^{Y_\gamma}$, and let us show $a = e_b$.

Let
$$i = \theta_{\hat{p}(a)} = \theta_{d^\gamma \upharpoonright k}$$
~~so that~~ where $\hat{p}(a) = d^\gamma \upharpoonright k$. Then
$$dw_i^\gamma = \hat{p}(a).$$

On the other hand, let

$$\eta = \sup\{\xi+1 \mid E_\xi^{W_{z(\gamma+1)}} \in \mathrm{ran}(e_b)\};$$

then e_b ~~fits is a ... of~~ $= d^{\gamma}_{z(\gamma+1)}$ and e_b fits into $e_\eta^{W_{z(\gamma+1)}}$. $\hat{\rho}$ is the $\hat{\rho}$-map of $\hat{\Phi}^{\gamma+1}_r$, where $r \trianglelefteq d^{\gamma+1}$ is the minimal segment of $d^{\gamma+1}$ consisting of extenders inserted into images of extenders in $Ext(\mathcal{T}_\gamma)$.

In particular, we have

$$\left(\hat{\Phi}^{\gamma+1}_r\right)_{z(\gamma+1)} : W^*_{z(\gamma+1)} \longrightarrow W_i$$

a weak tree emb. Let $\hat{g}$ be the associated map on extender trees; then

$$\hat{g}(e_b) = e^{W_i}_{v(\eta)}$$

where v is the v-map of $\left(\hat{\Phi}^{\gamma+1}_r\right)_{z(\gamma+1)}$.

But since e_b fits into $e_\eta^{W_{z(\gamma+1)}}$, and $\hat{p}$ agrees with $\hat{q}$ on ~~ws~~ $\mathrm{Ext}(W_{z(\gamma+1)})$ because they both come from taking the r-ultrapower, we get that $\hat{p}(e_b)$ fits into $e_{v(\eta)}^{W_i}$. Thus

$$\hat{p}(e_b) = d_{W_i}^{\gamma}$$
$$= \hat{p}(a).$$

It follows that $e_b = a$. That proves Claim 8.7.1.

$\boxtimes$

<u>Remark</u> In [2] it is shown that if ~~strategy~~ (P, Σ) is a mouse pair, and $\mathcal{T}$ is a tree on (P, Σ) with last model, then Σ induces a meta-strategy for $\mathcal{T}$ via

$$\Sigma^*(\mathcal{U}) = \text{unique cofinal } b \text{ s.t. } \mathcal{T}_b \text{ is by } \Sigma.$$

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Our $\hat{\mathcal{W}}^{\gamma}$ is not really a meta-tree on $\hat{\mathcal{W}}_{\gamma}$, but the situation is similar. (The result of [2] has a heavy component due to Schlutzenberg. [5].)

We can now finish the proof of Lemma 8.7. Let

$$a = e_{\xi}^{\Upsilon_{\gamma}}$$

where $\xi < lh(\Upsilon_{\gamma})$. (Note here that a is by Σ, by the proof of the claim 8.7.1.) Let $E_{\xi}^{\Upsilon_{\gamma}} = F_{\beta}$, so $\beta = z(\gamma, \xi)$. (it's an A-extender). We claim that

$\beta_d \leq \beta$, hence $\beta_d < z(\gamma+1)$. For letting $r \trianglelefteq d^{\gamma+1}$ be the initial segment that produces $\hat{p}(a)$ as above, and

$$d^{\gamma+1} = r^\frown t$$

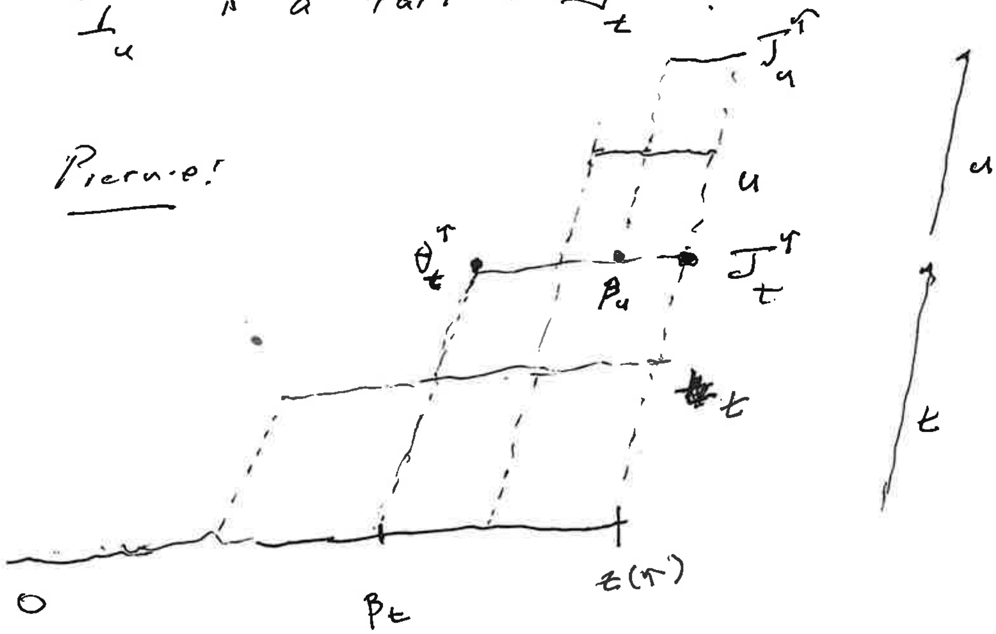
we have

$$d^{\gamma} = \hat{p}(a)^\frown t.$$

The following is easy to prove

Lemma 8.7.2 For any τ and any $t^\frown u \in \hat{W}^{\tau,\text{ext}}$, $\beta_u^\tau \in J_t^\tau$. Thus I_u^τ is a tail of J_t^τ.

Picture!



Proof $\theta_t^\tau \leq \beta_u^\tau$ because otherwise $u(0)$ would overlap some $E \in \text{ran}(t)$.

$\beta_u^\tau \leq \max J_t^\tau$ because otherwise $dw_{\beta_u^\tau}^\tau$ is incompatible with t, contrary to $dw_{\beta_u^\tau}^\tau {}^\frown u \in \hat{W}^{\tau,\text{ext}}$. $\boxtimes$

Returning to 8.7, note that with $F = F_\beta$

$$\beta = z(\gamma,\xi) = \max J^\gamma_a$$

$$\beta+1 = z(\gamma,\xi)+1 = \min(J^\gamma_F)$$

and

$$J^\gamma_a \cap J^\gamma_F = \emptyset.$$

This propagates via r to yield $J^\gamma_{\hat{p}(a)} \cap J^\gamma_{p(F)} = \emptyset$.

But $\hat{p}(a)^\frown t \in \hat{W}^{\gamma,ext}$, so by 8.7.2,

$\beta^\gamma_t \in J^\gamma_{\hat{p}(a)}$. But

$$\beta^\gamma_t = u^\gamma_{\hat{p}(a)}(\beta^\gamma_{d^\gamma})$$

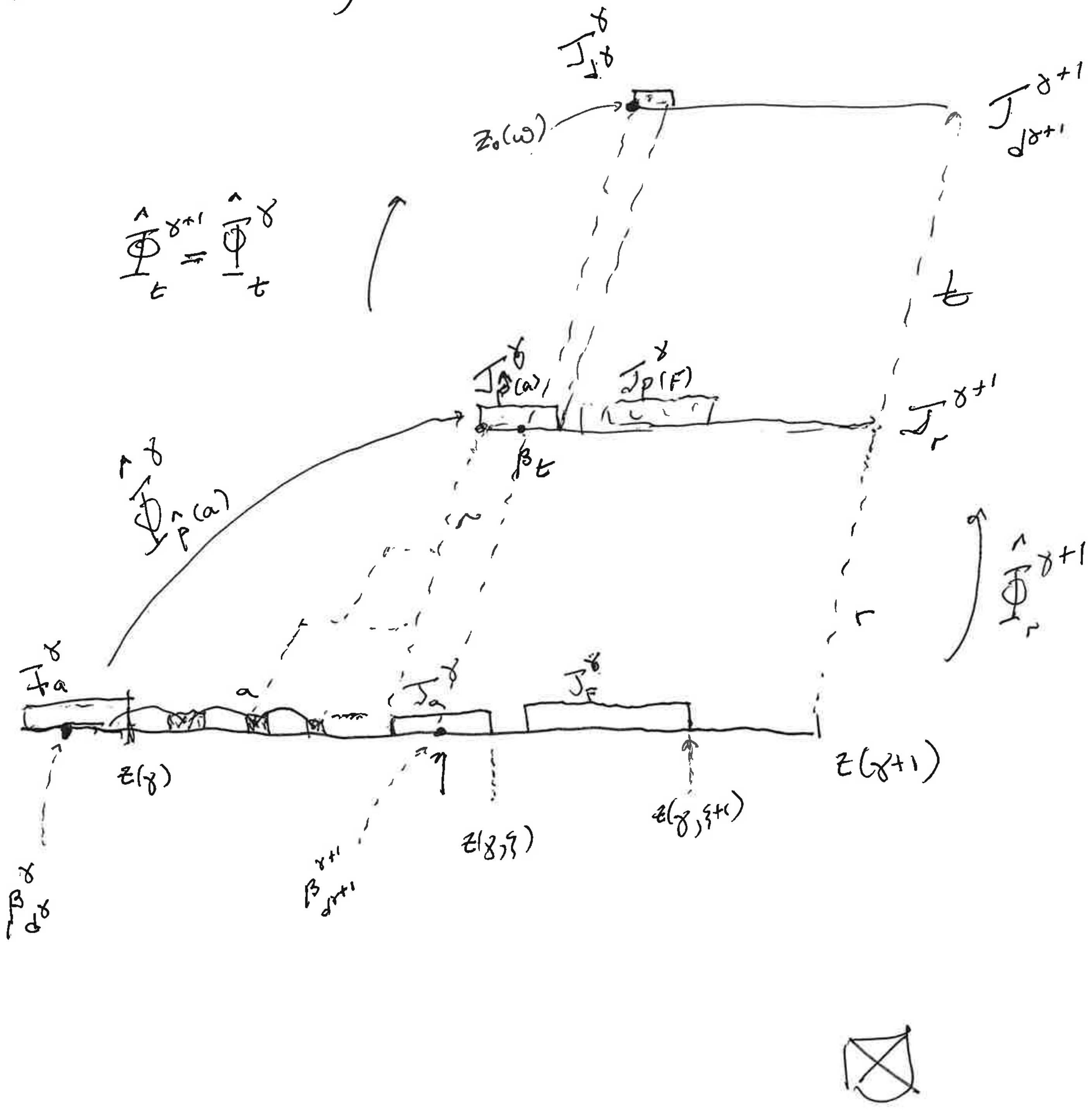
$$= u^{\gamma+1}_r \circ u^\gamma_a(\beta^\gamma_{d^\gamma}).$$

This means that letting $\eta = u^\gamma_a(\beta^\gamma_{d^\gamma})$, we have $\eta \in J^\gamma_a$, so $\eta < \beta$, and $\cancel{u_r(\eta) = \beta_t}$, so $\eta \cancel{\in u^{\gamma+1}_r d_t}$ $u^{\gamma+1}_r(\eta) = \beta^\gamma_t = \beta^{\gamma+1}_t$ (since t is actually a tail of $d^{\gamma+1}$). This implies

$$\eta = \beta^{\gamma+1}_{r^\frown t},$$

and since $\eta < \beta$ and $r^\frown t = d^{\gamma+1}$, we're done.

Here is a diagram



By lemma 8.7, we can properly extend $\hat{W}^0_\omega$ and $\hat{\mathcal{V}}^0_\omega$ by inflating along

$$d^\nu = dw^\nu_{z_0(\omega)} = dv^\nu_{z_0(\omega)}$$

for $\nu < \lambda$ such that the two branch extender sequences are indeed equal. This is true for arbitrary large $\nu < \omega$, and it yields proper extensions by 8.7. Let

$$\cancel{\hat{W}^0_{\omega,\nu} = \lim}$$

$$\mathcal{J}^\nu_{d^\nu} = \{ u^\nu_{d_\nu}(i) \mid i \in \vec{I}^\nu_{d^\nu} \}$$

and for $i \in I^\nu_{d^\nu}$, and $k = u^\nu_{d^\nu}(i)$,

$$W_k = \lim_{s \triangleleft d^\nu} W^*_{u^\nu_s(i)}$$

and

$$\mathcal{V}_k = \lim_{s \triangleleft d^\nu} \mathcal{V}^*_{u^\nu_s(i)}$$

where the limits are under the $\hat{\Phi}$ and $\hat{\Psi}$ maps. In the $\hat{\Phi}$ case, for example, ~~and~~ we have whenever

$$s^\frown t \trianglelefteq d^\nu$$

a map

$$\left(\hat{\Phi}^\nu_t\right)_{u^\nu_s(i)} : W^*_{u^\nu_s(i)} \longrightarrow W_{u^\nu_{s^\frown t}(i)}$$

that is a weak tree embedding, and $W_{u^\nu_{d^\nu}(i)}$ is the direct limit under these maps.

~~and~~ This gives us maps when $s^\frown t = d^\nu$, namely

$$\left(\hat{\Phi}^\nu_t\right)_k : W^*_k \longrightarrow W_{u^\nu_t(k)}$$

defined when $k \in \vec{I}^\nu_t$. (Our notation sets $\vec{J}^\nu_{d^\nu} = \vec{J}^\nu_t$ when t is a tail of d^ν.)

For $s \in \hat{W}^{\nu, ext}$, let

$$\hat{W}^*_{s,\nu} = \langle (W_\alpha, W^*_\alpha) \mid \alpha \leq \max(J^\nu_s) \rangle,$$

together with the additional structure (the F'_β's, the Z-functions, and the dw^η_i for $i \leq \max(J^\nu_s)$ and $\eta < \omega$, for example) we have put on it.

This also makes sense for $s = d^\nu$. We set

$$\hat{W}^0_{\omega,\nu} = \hat{W}_{d^\nu,\nu}.$$

For $s \wedge t = d^\nu$, we have

$$\hat{\Phi}^\nu_t : \hat{W}_{s,\nu} \longrightarrow \hat{W}_{\omega,\nu}.$$

Similarly on the ν-side, where we end up with

$$\hat{\Psi}^\nu_t : \hat{\nu}_{s,t} \longrightarrow \hat{\nu}_{\omega,\nu}$$

when $s \wedge t = d^\nu$. Since we are inflating by the same d^ν on both sides, $\mathrm{lh}(\hat{W}_{\omega,\nu}) = \mathrm{lh}(\hat{\nu}_{\omega,\nu})$, and (W_α, ν_α) is a slow comparison for each $\alpha < \mathrm{lh}(\hat{W}_{\omega,\nu})$. Moreover, for $\alpha > z_0(\omega)$, W_α extends $W_{z_0(\omega)}{}^\frown b$ and ν_α extends $\nu_{z_0(\omega)}{}^\frown c$.

Suppose that $\eta < \gamma < \omega$ and $dw^{\eta}_{z_0(\omega)} = dv^{\eta}_{z_0(\omega)}$, so that $dw^{\tau}_{z_0(\omega)} = dv^{\tau}_{z_0(\omega)}$ for all $\tau \geq \eta$. (See the proof of 8.6.)

We need to show that $\hat{W}_{\omega,\eta}$ is extended by $\hat{W}_{\omega,\gamma}$, and that the inflationary maps $\hat{\Phi}^{\eta}_{t}$ for $t \lhd d^{\eta}$ interact properly with the inflationary maps $\hat{\Phi}^{\gamma}_{s}$ for $s \lhd d^{\gamma}$. Similarly on the $\mathcal{V}$-side.

The diagram on p. 102m already shows the consistency-of-inflations we need to prove. (There it is between $\hat{\Phi}^{\gamma}_{d^{\gamma}}$ and $\hat{\Phi}^{\gamma+1}_{d^{\gamma+1}}$.)

The consistency of inflations comes down to the following very familiar commutative diagrams. Let $W_{z_F(i)} = X(W_i^*, F)$ and $W_{z_G(i)} = X(W_i^*, G)$, and suppose $\kappa_G < \lambda_F$, then

(a) If $\kappa_G \leq \kappa_F$, then

$$\begin{tikzcd}
W_{z_G(i)} \arrow[r, "(\hat{\Phi}_{G(F)})_{z_G(i)}"] & W_l \\
W_i^* \arrow[u, "(\hat{\Phi}_G)_i"] \arrow[r, "(\hat{\Phi}_F)_i"'] & W_{z_F(i)} \arrow[u, "(\hat{\Phi}_G)_{z_F(i)}"']
\end{tikzcd}$$

commutes. Here $l = z_{G(F)} \circ z_G(i) = z_G \circ z_F(i)$.

(b) If $\kappa_F < \kappa_G$, then

$$\begin{tikzcd}
 & W_l \\
W_i^* \arrow[ur, "(\hat{\Phi}_{G(F)})_i"] \arrow[r, "(\hat{\Phi}_F)_i"'] & W_{z_F(i)} \arrow[u, "(\hat{\Phi}_G)_{z_F(i)}"']
\end{tikzcd}$$

commutes. Here $l = z_{G(F)}(i) = z_G \circ z_F(i)$.

Here is a diagram that puts the diagrams above together in a representative case. Suppose $\eta < \gamma < \omega$ and $z(\eta) < k < z(\gamma)$. Let

$$dw^{\eta}_{k} = r^\frown \langle F, G, H \rangle$$

and suppose

$$u^{\eta}_{r^\frown\langle F,G,H\rangle}(n) = k.$$

Suppose

$$dw^{\gamma}_{z_0(\omega)} = \langle K, L\rangle^\frown s$$

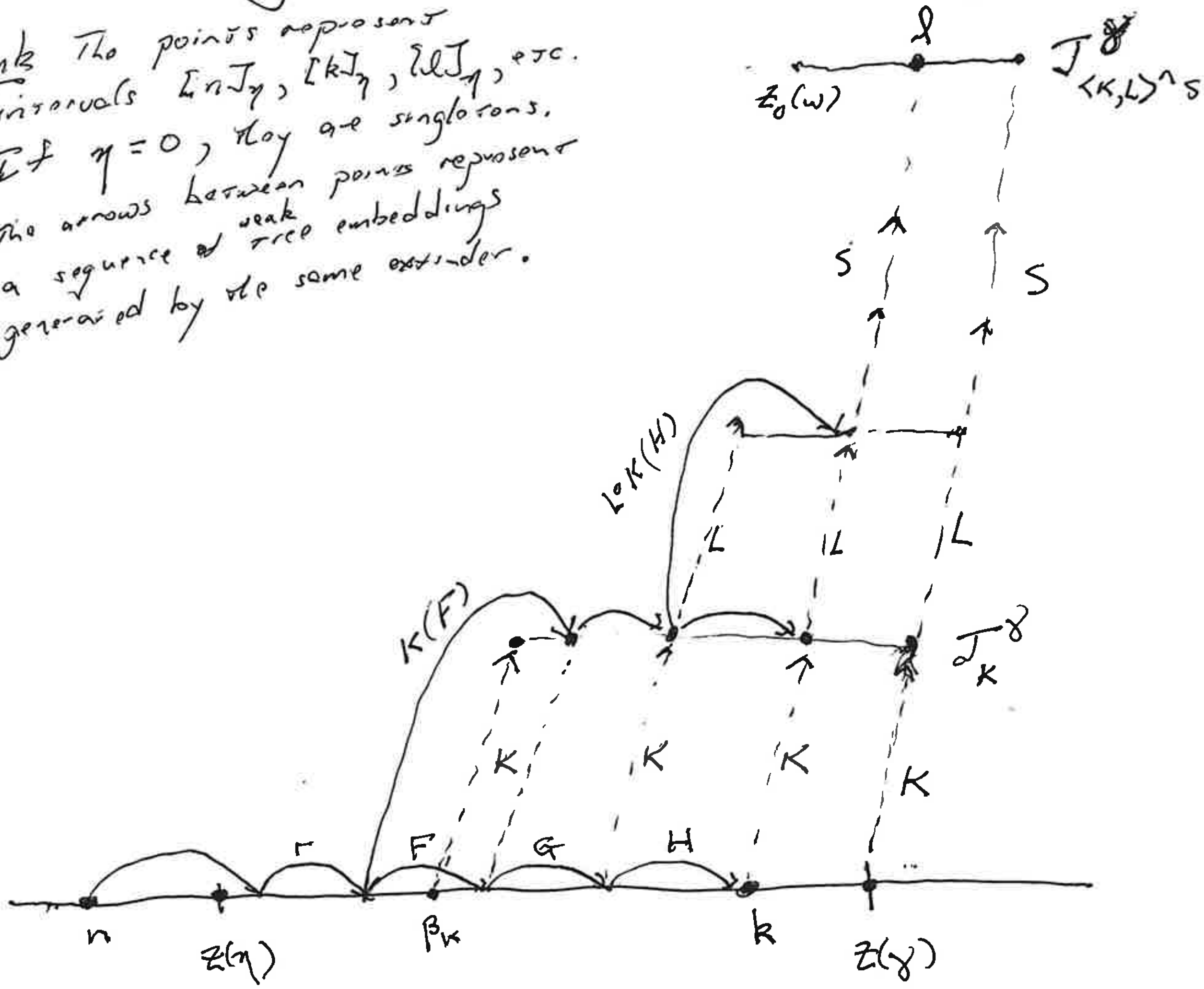
and

$$u^{\gamma}_{\langle K,L\rangle^\frown s}(k) = \ell.$$

Let us draw horizontal lines to represent intervals in $[0, z_0(\omega)]$, with the points i corresponding to intervals $[i,j]_\eta$ of normal trees. We put the intervals on different levels, so that the arrows representing weak tree embeddings show up better.

We might have to pattern

Rmk The points represent intervals $[n]_\eta$, $[k]_\eta$, $[l]_\eta$, etc. If $\eta = 0$, they are singletons. The arrows between points represent a sequence of weak tree embeddings generated by the same extender.



In this situation, dw^η_k has been inflated by $\langle K, L\rangle$, then left behind by S, so

$$dw^\eta_l = r ^\frown \langle K(F), K(G), L \circ K(H)\rangle ^\frown S$$

and

$$u^\eta_{0,\omega}(n) = u^\gamma_{0,\omega}(k) = l$$

and

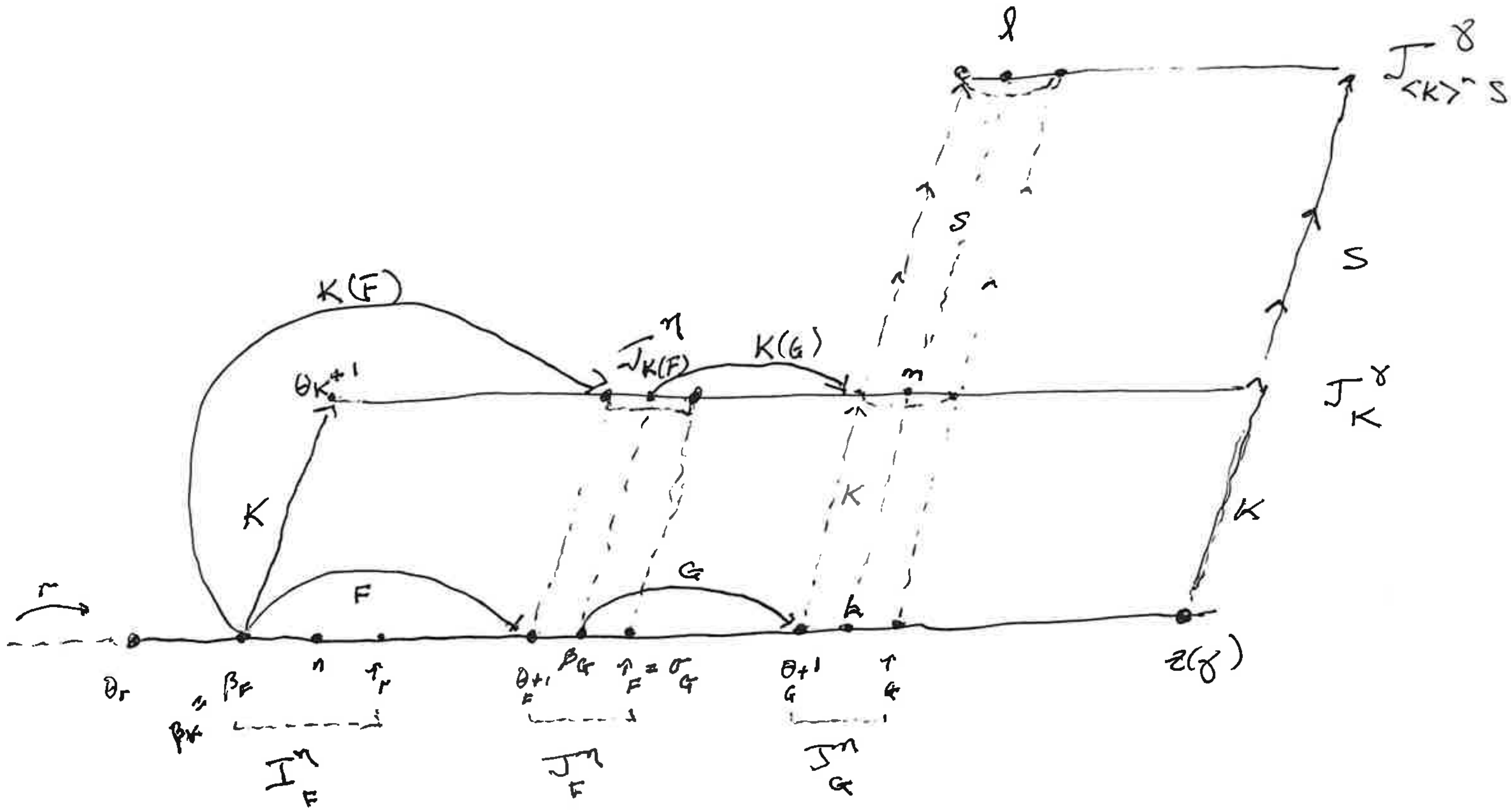
$$\left(\hat{\Phi}^\eta_{0,\omega}\right)_n = \left(\hat{\Phi}^\eta_{dw^\eta_l}\right)_n = \left(\hat{\Phi}^\gamma_{dw^\gamma_l}\right)_k \circ \left(\hat{\Phi}^\eta_{dw^\eta_k}\right)_n$$

Here is a version of the diagram above that shows the block structure – the fact that F, G, etc. act on blocks of trees. We drop H and L to simplify the picture. So $dw^\eta_k = r^\frown\langle F, G\rangle$ and $dw^\gamma_\ell = \langle K\rangle^\frown s$ and $dw^\eta_\ell = r^\frown\langle K(F), K(G)\rangle^\frown s$ in the situation we draw. The drawing also assumes that I^η_F is a proper subset of J^η_r and I^η_G is a proper subset of $J^\eta_F = J^\eta_{r^\frown\langle F\rangle}$.

In the diagram, the points are connected by arrows representing weak tree embeddings. J-intervals are connected by a sequence of weak tree embeddings which are all generated by ultrapowers by the same sequence of extenders.

The embeddings of $\hat{W}^\eta$ commute with those of $\hat{W}^\gamma$ (i.e. the $(\hat{\Phi}^\eta_t)_k$ and the $(\hat{\Phi}^\gamma_\nu)_m$) whenever it makes sense.

Here the points represent single normal trees W_i, not blocks $\langle W_i \mid i \underset{\eta}{\sim} j \rangle$.



In the diagram, $J^{\eta}_{r^\frown K(F)} = J^{\eta}_{K(F)} = z_K^{\alpha} J^{\eta}_F$,

and $J^{\eta}_{K(G)} = z_K^{\alpha} J^{\eta}_G$. At the top line

$$[l]_\eta = J^{\eta}_{r^\frown\langle K(F), K(G)\rangle^\frown S} = u^{\gamma}_{\langle K\rangle^\frown S}{}^{\alpha} J^{\eta}_{r^\frown\langle F, G\rangle}.$$

(The diagram assumes that $\beta_S = \theta_{K(G)}+1$.) Thus $[l]_\eta$ is a proper initial segment of $[l]_\gamma = J^{\gamma}_{\langle K\rangle^\frown S}$.

Commutativity yields $z_K(k) = m = u^{\eta}_{r^\frown\langle K(F), K(G)\rangle}(n)$.

More generally, let $\eta < \gamma < \omega$ and let $a \in \hat{W}^{\eta,\mathrm{ext}}$ be such that $\theta_a < z(\gamma)$. Let $r \in \hat{W}^{\gamma,\mathrm{ext}}$. Let

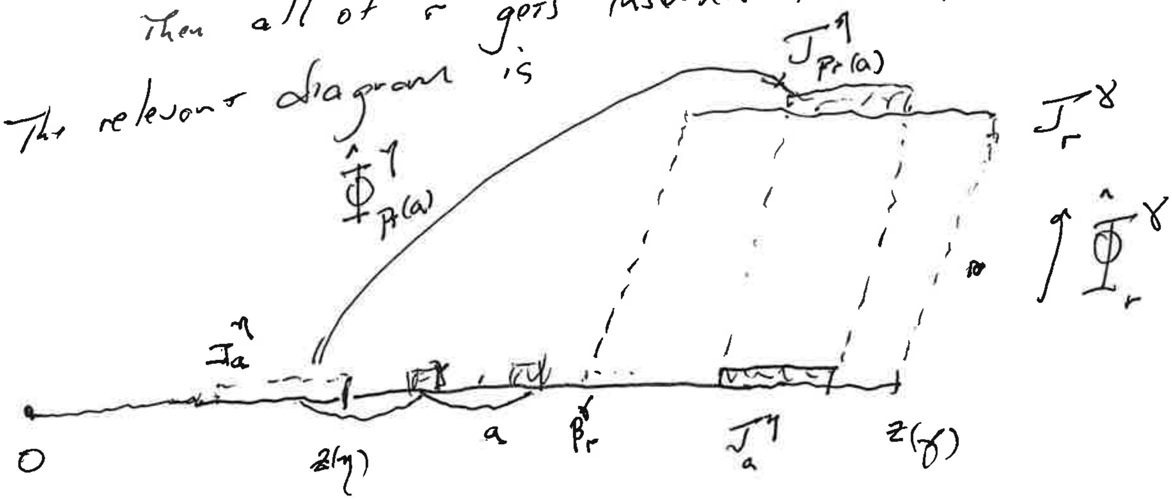
$$p_r^\gamma = p\text{-map of } \hat{\bar{\Phi}}_r^\gamma .$$

(Recall that each $(\hat{\bar{\Phi}}_r^\gamma)_i$ has its own p-map from $\mathrm{Ext}(W_i^*)$ into $\mathrm{Ext}(W_{\alpha_r^\gamma(i)})$, and these maps agree at places in their common domain. p_r^γ is the union of them. $\mathrm{Ext}(W_{z(\gamma)}) \subseteq \mathrm{dom}(p_r)$, if we allow $(\bar{\Phi}_r^\gamma)_i$ to be defined when $i \leq \beta_r$ in the natural way.

Case (a) $\beta_r^\gamma < \theta_a$.

Then all of r gets inserted into a to form $p_r^\gamma(a) \in \hat{W}^{\eta,\mathrm{ext}}$.

The relevant diagram is



We have $J^\eta_{p_r(a)} = u_r^\gamma{}'' J_a^\eta$ and

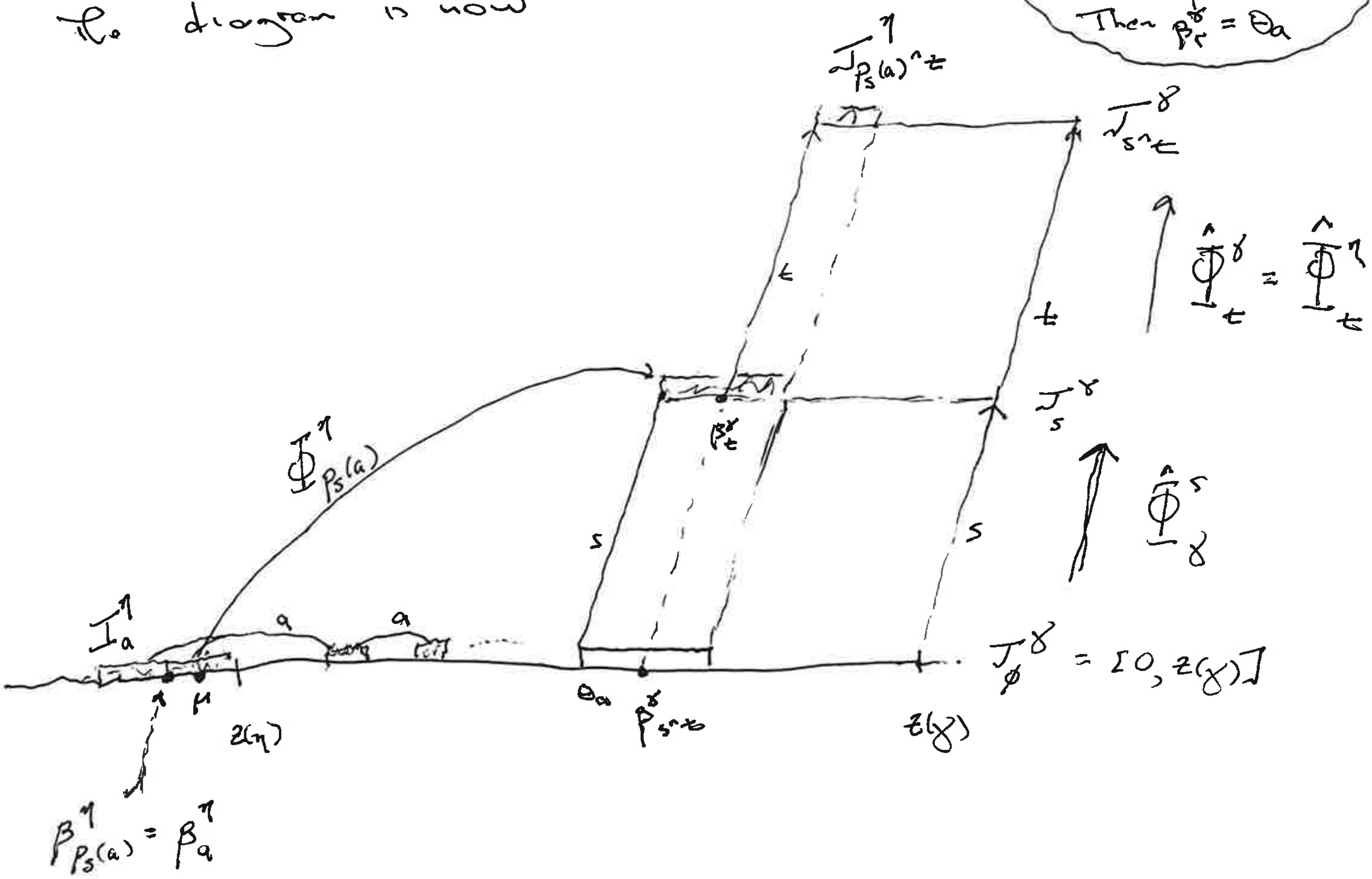
$$\hat{\bar{\Phi}}^\eta_{p_r^\gamma(a)} = \hat{\bar{\Phi}}_r^\gamma \circ \Phi_a^\eta$$

in the component-wise sense.

Case (b) $\quad \theta_a \le \beta^\gamma_r \le \max(J^\eta_a)$

Let $s \trianglelefteq r$ be shortest s.t. $\theta_a \le \beta^\gamma_s$.
So all of s is inserted below the image of a
into $p^\gamma_s(a) = p^\gamma_r(a)$, and $r = s^\frown t$ for some t.
The diagram is now

Rmk: $t = \phi$ is possible. Then $\beta^\gamma_r = \theta_a$



We have $J^\eta_{p_s(a)^\frown t} = \text{“} u^\gamma_r \text{''} [\beta^\gamma_r, \max J^\eta_a]$

and $I^\eta_{p_s(a)^\frown t} = [\mu, z(\eta)]$ where $\mu = (u^\eta_a)^{-1}(\beta^\gamma_r)$.

$\hat{\Phi}^\eta_{p_s(a)^\frown t} = \hat{\Phi}^\gamma_r \circ \hat{\Phi}^\eta_a$.

The assertions in cases ⓐ and ⓑ above can be proved by induction on $\mathrm{dom}(r)$, with the atomic step coming from the diagrams on p. 107. We skip detailed verification for now.

We can now see how the inflations by $d^\eta = dw^\eta_{\tau_0(\omega)} = dv^\eta_{\tau_0(\omega)}$ fit together in the tail of $\eta < \omega$ such that $dw^\eta_{\tau_0(\omega)} = dv^\eta_{\tau_0(\omega)}$. Fix $\eta < \gamma$ in this tail. We have $\beta^\gamma_{d^\gamma} < \tau(\gamma)$ by Lemma 8.7. Let $a \in \hat{W}^{\eta,\mathrm{ext}}$ be such that

$$\beta^\gamma_{d^\gamma} \in J^\eta_a .$$

This identifies a uniquely, moreover $\max J^\eta_a < \tau(\gamma)$ by our construction. Let

$$\mu = (\varphi^\eta_a)^{-1}(\beta^\gamma_{d^\gamma}).$$

We can write

$$d^\gamma = r \frown t$$

where r is the maximal initial segment of d^γ consisting of extenders inserted into or between images of extenders in $\mathrm{ran}(a)$; that is,

$\theta_{\hat{p}_r(a)} \leq \beta(t(0))$, and r is the shortest such initial segment of d^γ. (It is possible that $r = d^\gamma$ and $t = \phi$.) We have that

$\cancel{\hat{p}_r(a)^\frown t}$

d^γ fits into $\hat{p}(r)^\frown t$

and $\hat{p}_r(a)^\frown t$ is a ~~cofinal~~ branch of $W^{\eta,ext}$ because $\beta^\gamma_{d^\gamma} \in J^\eta_a$, so $\beta^\gamma_t =$

$u^\gamma_r(\beta^\gamma_{d^\gamma}) \in J^\eta_{\hat{p}_r(a)}$, but $u^\gamma_r(\beta^\gamma_{d^\gamma}) = u^\eta_{\hat{p}_r(a)}(\mu)$,

so

$$\theta^\eta_{\hat{p}_r(a)} \leq \beta(t(0)) \leq \beta^\eta_t = \beta^\gamma_t \leq \max J^\eta_{\hat{p}_r(a)} .$$

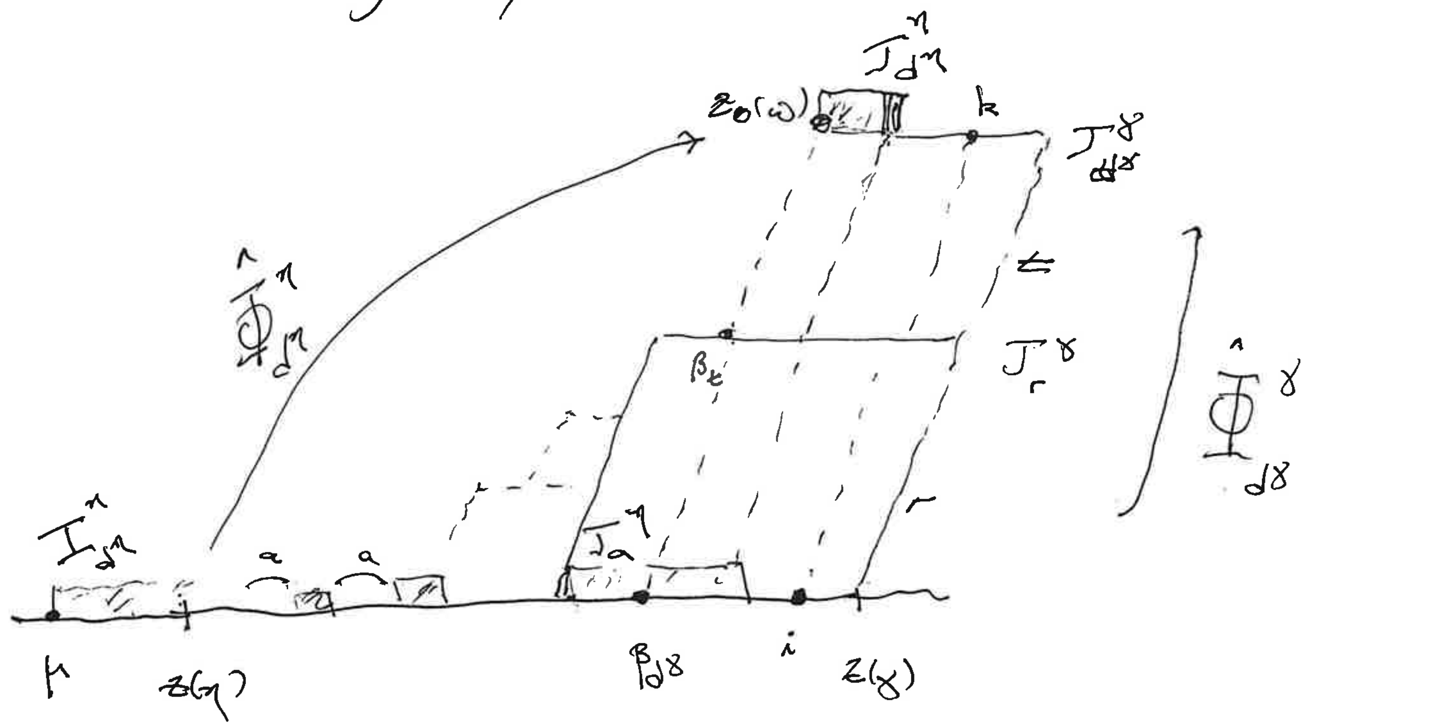
This implies that $\hat{p}_r(a)^\frown t$ is indeed a branch of $\hat{W}^{\eta,ext}$, and clearly its cofinal. Since d^γ fits into it

$$\hat{p}(r)^\frown t = d^\eta .$$

Our commutative diagrams imply $u^\eta_{\hat{p}(r)^\frown t}(\mu) = u^\gamma_{d^\gamma}(\beta^\gamma_{d^\gamma}) = \bar{\delta}_0(a)$, i.e.

$$\mu = \beta^\eta_{d^\eta} .$$

The earlier diagram specializes now to



We have $\hat{\Phi}^\eta_{d\eta} = \hat{\Phi}^\gamma_{d\gamma} \circ \hat{\Phi}^\eta_a$. Clearly $J^\eta_{d\eta}$ is a proper initial segment of $J^\gamma_{d\gamma}$.

There is more to be said about the consistency of inflations by the d^γ's. Given $i < z(\gamma)$ and letting $k = u^\gamma_{d\gamma}(i)$, $\hat{\Phi}^\gamma_{d\gamma}$ (and $\hat{\Psi}^\gamma_{d\gamma}$) inflates dw^ξ_i to dw^ξ_k, for all $\xi < \gamma$. (All of them, not just ξ in the agreeing tail where $dw^\xi_{z_0(\omega)} = dv^\xi_{z_0(\omega)}$.) Similarly, $\hat{\varphi}^\gamma_{d\gamma}$ (and $\hat{\psi}^\gamma_{d\gamma}$) inflates dv^ξ_i to dv^ξ_k. If $i < z(\eta)$

and d^η is defined, then $\hat{\Phi}^\eta_{d^\eta}$ and

$\hat{\Phi}^\gamma_{d^\gamma}$ inflate dw_i^ξ and ~~dw_i^ξ~~ to the same

thing, and similarly for dv_i^ξ. So

letting

$$z_1(\omega) = \sup_{\gamma<\omega} \max J^\gamma_{d^\gamma}$$

we have for $k < z_1(\omega)$ and $\xi < \omega$

$$dw_k^\xi = \overset{\text{common}}{\text{inflation}} \text{ of } dw^\xi_{(u^\gamma_{d^\gamma})^{-1}(k)}$$

by $\hat{\Phi}^\gamma_{d^\gamma}$, for all γ s.t.

$(u^\gamma_{d^\gamma})^{-1}(k)$ exists.

$$dv_k^\xi = \text{common inflation of } dv^\xi_{(u^\gamma_{d^\gamma})^{-1}(k)}$$

by $\hat{\Phi}^\gamma_{d^\gamma}$, for all γ s.t.

$(u^\gamma_{d^\gamma})^{-1}(k)$ exists

We can write $\hat{\Psi}^\gamma_{d^\gamma}$ on the rhs too —

it's a case of A-branches inflating A-extenders,

so the two sides act the same way. Another notation would be

$$dw_k^\xi = \hat{P}^\gamma_{d^\gamma}\left(dw_i^\xi\right), \text{ for all suff large } \gamma\text{'s}$$

and call dw_k^ξ the d^γ-inflation of dw_i^ξ.

We let (Rmk The objects below were already defined on p. 105.)

$$\hat{W}^0_{\omega,\gamma} = \left\langle W_i \mid k \leq \max\left(J^\gamma_{d^\gamma}\right)\right\rangle,$$

with the structure induced by the $F_{i^\gamma_{d^\gamma}(i)} = P^\gamma_{d^\gamma}(F_i)$ and

$$dw_k^\xi \text{ for } \xi < \gamma.$$

We have

$$\hat{\Phi}^\gamma_{d^\gamma} : \hat{W}_\gamma \longrightarrow \hat{W}^0_{\omega,\gamma}$$

is structure-preserving. Similarly, we have $\hat{V}^0_{\omega,\gamma}$ and

$$\hat{\Psi}^\gamma_{d^\gamma} : \hat{V}_\gamma \longrightarrow \hat{V}^0_{\omega,\gamma}.$$

The new (W_i, V_i) are slow comparisons, and the F_i's are the A-extenders, and the same ones in both $\hat{W}^0_{\omega,\gamma}$ and $\hat{V}^0_{\omega,\gamma}$.

Finally, we complete our first round by setting

$$z_0(\omega,\gamma) = \max J^{\gamma}_{d^{\gamma}}$$

and

$$z_1(\omega) = \sup_{\gamma<\omega} z_0(\omega,\gamma),$$

and

$$\hat{W}^1_\omega = \text{"}\bigcup_{\gamma<\omega}\text{"}\ \hat{W}^0_{\omega,\gamma}$$

$$\hat{\nu}^1_\omega = \text{"}\bigcup_{\gamma<\omega}\text{"}\ \hat{\nu}^0_{\omega,\gamma}.$$

Note

~~Prop. 8.7 For $\eta<\gamma$ s.t. d^η (hence d^γ)~~

defined,

$$u^\eta_{d^\eta}(\beta^\eta z(\eta)) \le u^\eta_{d^\eta}(z(\eta))$$

Lemma 8.8 For $\eta<\gamma$, $z_0(\omega,\eta) < z_0(\omega,\gamma)$.

So $z_1(\omega)$ is a limit ordinal.

Pf. See the diagrams above. $\boxtimes$

We are now ready for round 2.

Let Δ_1^w be the common part tree, i.e.

$$\mathcal{T} \trianglelefteq \Delta_1^w \text{ iff for all sufficiently large } i < z(1),\ \mathcal{T} \trianglelefteq W_i^*.$$

(Rmk: $W_i = W_i^*$ for $i \in [z_0(w), z_1(w))$. The completion process doesn't involve normal extension by D-extenders, it just inflates earlier work.)

Also

$$\mathcal{T} \trianglelefteq \Delta_1^v \text{ iff for all sufficiently large } i < z(1),\ \mathcal{T} \trianglelefteq V_i^*.$$

<u>Prop 8.9</u> (a) $W^0_{z_0(w)} \trianglelefteq \Delta_1^w$ and $V^0_{z_0(w)} \trianglelefteq \Delta_1^v$

(b) $lh(\Delta_1^w) = lh(\Delta_1^v)$ is a limit ordinal, and ~~(Δ_1^w, Δ_1^v) is a slow comparison~~

$$\sup\{lh\, F_\eta^{\Delta_1} \mid \eta < lh(\Delta_1)\} = \liminf_{i \to z(1)} lh(F_i)$$

(c) Letting $b = \Sigma(\Delta_1^w)$ and $c = \Lambda(\Delta_1^v)$, $(\Delta_1^w{}^\frown b, \Delta_1^v{}^\frown c)$ is a slow comparison.

Proof Routine.

Let $b = \Sigma(\Delta_1^{w})$ and $c = \Lambda(\Delta_1^{\nu})$.

In round 1, we asked whether $b \equiv_{Tail} c$, and ended the completion process with $z(\omega) = z_0(\omega)$ and $W_{z(\omega)} = W_\omega^0 \wedge b$ and $\nu_{z(\omega)} = \nu^0_{z_0(\omega)} \wedge c$ if $b \not\equiv_{Tail} c$. This was ok because the termination argument at the end will show ~~that~~ $b \equiv_{Tail} c$ if ω is replaced by ω_1. It doesn't seem that we can be so lazy about completion in rounds $\alpha > 1$.

Namely, we might have $b \not\equiv_{Tail} c$, but have some $\gamma < \omega$ and cofinal ~~branches r of $\hat{W}_\omega^{1,ext}$ and s of $\hat{\nu}^{1,ext}$ such that $r \equiv_{Tail} s$. If $b \not\equiv_{Tail} c$, then $\gamma > 0$.~~

branches r of $\hat{W}^{1,\tau,ext}_\omega$ and s of $\boxed{122}$

$\hat{\mathcal{V}}^{1,\tau,ext}_\omega$ (ie each $r_0 \triangleleft r$ is of the form

dw_i^τ for some $i < \mathbb{Z}(1)$, and those i's are

cofinal in $\mathbb{Z}(1)$, and similarly for s)

such that

(1) some tail of r fits into e^Δ_b,

(2) some tail of s fits into e^Δ_c,

and (3) $r \equiv_{tail} s$.

If $e^\Delta_b \not\equiv_{tail} e^\Delta_c$, then $\tau > 0$, but it may nevertheless exist. If τ has the property, then all $\gamma \in [\tau, \omega)$ have the property. (Since e.g. ~~[illegible]~~ exactly one cofinal branch of $\hat{W}^{1,\gamma,ext}$ can have a tail that fits into r.)

By (1), r is a good branch of $\hat{W}^{1,\tau,ext}$ from the point of view of Σ (it's by a meta-strategy that Σ determines), and by (2), s is good according to Λ. The same is true of the unique cofinal branches $\bar{r}_\gamma, s_\gamma$ of $\hat{W}^{1,\gamma,ext}$ and $\hat{\mathcal{V}}^{1,\gamma,ext}$ that fit into them. Passing to tails, we can pick for all $\gamma \geq \tau$ ~~tails~~ common tails ~~diff~~ d^γ s.t.

d^γ is a tail of r_γ and s_γ, for all $\gamma \geq \tau$.

We then use the d^γ for $\gamma \geq \tau$ to inflate, producing $\hat{W}^1_{\omega,\gamma}$ and $\hat{\mathcal{V}}^1_{\omega,\gamma}$ for $\gamma \in [\tau, \omega)$, of length $z_1(\omega,\gamma)$, and set

$$\hat{W}^2_\omega = \text{“}\bigcup_{\gamma \in [\tau,\omega)}\text{”}\ \hat{W}^1_{\omega,\gamma}$$

$$\hat{\mathcal{V}}^2 = \text{“}\bigcup_{\gamma \in [\tau,\omega)}\text{”}\ \hat{\mathcal{V}}^1_{\omega,\gamma}.$$

These systems have length
$$z(2) = \sup_{\gamma < \omega} z(1,\gamma).$$

$z(2)$ is a limit ordinal, so we can go on to form $\mathcal{S}_2^w$, $\mathcal{S}_2^v$, and so ~~and~~ on.

At limit ~~steps~~ Rounds λ in the completion process we have $\hat{W}_\omega^\lambda = $ "$\bigcup_{\alpha<\lambda} \hat{W}_\omega^\alpha$" and $\hat{\mathcal{V}}_\omega^\lambda = $ "$\bigcup_{\alpha<\lambda} \hat{\mathcal{V}}_\omega^\alpha$" of length $z_\lambda(\omega)$. We complete them ~~to~~ to $\hat{W}_\omega^{\lambda+1}$ and $\hat{\mathcal{V}}_\omega^{\lambda+1}$ of length $z_{\lambda+1}(\omega)$ in the same way. ~~then~~

This process has to stop at some countable stage ξ because it is producing effectively countings of the $z_\xi(\omega)$. (At some stage $< \omega_1^{L[T_z, T_\lambda, P, Q]}$ in fact.)

That is the sketch again, but let's give some more details on how round ~~one~~ two goes.

Remark We could have been more aggressive about completion in round one, proceeding in the same way we do in rounds $\alpha > 1$.

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