Comparing strategies by least disagreement

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§0. Let (P, Σ) and (Q, Λ) be mouse pairs with scope H_{ω_2} such that $o(P) \leq \omega_1$ and $o(Q) \leq \omega_1$. We show that they can be compared in a process that lasts $\leq \omega_1$ steps, and in a natural sense iterates away least disagreements.

Sargsyan proved this in [1] in the case that P and Q have no initial segments satisfying ZFC + "There is a Woodin limit of Woodin cardinals". Our process is a variant of his. Like [1], we use various properties of mouse pairs, such as very strong hull condensation and full normalization, that hold for countable mouse pairs under AD^+ by [2]. The proofs in [2] use the iteration-to-a-background method of [3], so we aren't eliminating that method from the basic theory of mouse pairs. Sargsyan [4] used his process to prove AD^+ + "no lbr hod pairs with a Woodin limit of Woodins" implies HPC. Perhaps our process helps prove HPC higher up, but we don't see that. Nevertheless, ~~being able to~~ a process that compares mouse pairs with each other directly seems worthwhile. Theorem 1 states that our process is more effectively definable than the process of [3].

§1. Comparing pairs with scope H_{ω_1}

__Theorem 1__ Assume AD^+, and let (P,Σ) and (Q,Λ) be mouse pairs with scope HC of the same type. Let $Code(\Sigma) = p[T_\Sigma]$ and $Code(\Lambda) = p[T_\Lambda]$ be Suslin representations, and x_P, x_Q be reals coding P and Q respectively; then there are trees $\mathcal{T}$ and $\mathcal{U}$ on (P,Σ) and (Q,Λ) resp. with a common last pair (R,Ω) such that

(a) one of P-to-R and Q-to-R does not drop (so we have a successful comparison), and

(b) $\mathcal{T}, \mathcal{U}, R \in L[x_P, x_Q, T_\Sigma, T_\Lambda]$, and are countable in $L[x_P, x_Q, T_\Sigma, T_\Lambda]$.

__Remark__ [3] proves the weakening of Theorem 1 in which (b) is replaced by "$\mathcal{T}, \mathcal{U}, R$ are countable in N^*, where $(N^*, \ldots)$ is a Γ-Woodin tuple s.t. $x_P, x_Q \in HC^{N^*}$ and $Code(\Sigma)$, $Code(\Lambda)$ are in $\Gamma \cap \check{\Gamma}$". The process of [3] does not seem to be sufficiently effective to yield (b) of Theorem I.

Most of these notes are devoted to proving Theorem I. The main point is the comparison process itself. Later we shall generalize it to the case that $o(P) = \omega_1$ or $o(Q) = \omega_1$.

§2. General framework

The comparison process produces by induction pairs $(\mathcal{W}_\gamma, \mathcal{V}_\gamma)$ that are approximations to the desired final comparison. $\mathcal{W}_\gamma$ is a padded normal tree on (P, Σ), and $\mathcal{V}_\gamma$ is a padded normal tree on (Q, Δ). There will be weak tree embeddings

$$\Phi_{\alpha,\gamma} : \mathcal{W}_\alpha^* \longrightarrow \mathcal{W}_\gamma$$

and

$$\Psi_{\alpha,\gamma} : \mathcal{V}_\alpha^* \longrightarrow \mathcal{V}_\gamma$$

defined for certain $\alpha < \gamma$. Here $\mathcal{W}_\alpha^*$ is an extension of $\mathcal{W}_\alpha$ obtained by iterating away extender disagreements, and similarly for $\mathcal{V}_\alpha^*$. The step from $\mathcal{W}_\alpha^*$ to $\mathcal{W}_{\alpha+1}$ is generated by strategy disagreement.

~~to simplify the bookkeeping~~ We shall pad our trees in order to simplify bookkeeping.

<u>Def 2</u> A <u>padded normal tree</u> $\mathcal{T}$ is like a normal tree, but we allow $E_\alpha^{\mathcal{T}} = \emptyset$ (so $M_{\alpha+1}^{\mathcal{T}} = M_\alpha^{\mathcal{T}}$). If $E_\alpha^{\mathcal{T}} = \emptyset$, then $\mathcal{T}$ must specify $\lambda(E_\alpha^{\mathcal{T}})$ and $lh(E_\alpha^{\mathcal{T}})$ such that $\lambda(E_\beta^{\mathcal{T}}) < \lambda(E_\alpha^{\mathcal{T}})$ for all $\beta < \alpha$, and $\lambda(E_\alpha^{\mathcal{T}})$ is the largest cardinal of $M_\alpha \| lh(E_\alpha)$.

In the case $E_\alpha^{\mathcal{T}} = \emptyset$, the role of $\lambda(E_\alpha^{\mathcal{T}})$ is to determine whether for certain $\xi > \alpha$, $\mathcal{T}\text{-pred}(\xi+1) = \alpha$ or $\mathcal{T}\text{-pred}(\xi+1) = \alpha+1$. Of course, this only affects the bookkeeping.

<u>Def 3</u> Let P and Q be premice of the same type. A <u>slow comparison</u> of P with Q is a pair $(\mathcal{U}, \mathcal{T})$ of padded normal trees on P and Q of the same length such that for each $\alpha+1 < lh(\mathcal{U}) = lh(\mathcal{T})$, ~~either~~

~~(a) $E_\alpha^{\mathcal{T}} \neq \emptyset$ and $E_\alpha^{\mathcal{U}} \neq \emptyset$ and $E_\alpha^{\mathcal{T}} = E_\alpha^{\mathcal{U}}$~~

~~or (b) $E_\alpha^{\mathcal{T}} =$~~ ~~(a) $M_\alpha^{\mathcal{T}} \| lh E_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} \| lh E_\alpha^{\mathcal{U}}$, and~~

(b) (a) $M_\alpha^{\mathcal{T}} \| lh E_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} \| lh E_\alpha^{\mathcal{U}}$, and

(b) either $E_\alpha^{\mathcal{T}} \neq \emptyset$ or $E_\alpha^{\mathcal{U}} \neq \emptyset$, and

(c) If $E_\alpha^{\mathcal{T}} = \emptyset$ or $E_\alpha^{\mathcal{U}} = \emptyset$, then $\lambda(E_\alpha^{\mathcal{T}}) = \lambda(E_\alpha^{\mathcal{U}})$ and $lh(E_\alpha^{\mathcal{T}}) = lh(E_\alpha^{\mathcal{U}})$.

Remark By (a), $lh(E_\alpha^{\mathcal{T}}) = lh(E_\alpha^{\mathcal{U}})$, but we can have one (but not both) of them be ϕ.

Suppose $(\mathcal{T},\mathcal{U})$ is a slow comparison and let γ be least s.t. $M_\alpha^{\mathcal{T}} | \langle \gamma, 0\rangle \neq M_\alpha^{\mathcal{U}} | \langle \gamma, 0 \rangle$, and $\gamma > lh(E_\xi^{\mathcal{T}})$ for all $\xi < \alpha$. Then either $E_\alpha^{\mathcal{T}} = \dot{F}^{M_\alpha^{\mathcal{T}} | \gamma}$ and $E_\alpha^{\mathcal{U}} = \dot{F}^{M_\alpha^{\mathcal{U}} | \gamma}$, or there is a $\xi < \gamma$ s.t. $E_\xi^{M_\alpha^{\mathcal{T}}} \neq \phi$ and $E_\alpha^{\mathcal{T}} = E_\alpha^{\mathcal{U}} = E_\xi^{M_\alpha^{\mathcal{T}}} = E_\xi^{M_\alpha^{\mathcal{U}}}$. That is, either we hit an extender agreement, or we hit the least extender disagreement.

<u>Def. 3A</u> Let $(\mathcal{T},\mathcal{U})$ be a slow comparison.

(1) F is an <u>agreement extender</u> (A-extender) <u>of $(\mathcal{T},\mathcal{U})$</u> iff $\exists\alpha\,(F = E_\alpha^{\mathcal{T}} = E_\alpha^{\mathcal{U}})$. (So $F \neq \phi$.)

~~Otherwise~~ F is a <u>disagreement extender</u> (D-extender) iff $\exists \alpha < lh(\mathcal{T})\,(F = E_\alpha^{\mathcal{U}}$ or $F = E_\alpha^{\mathcal{T}})$ and F is not an A-extender of $(\mathcal{T},\mathcal{U})$.

(2) Let $\lambda < lh(\mathcal{T})$ be a limit ordinal; then $[0,\lambda)_T$ is an <u>A-branch</u> of $\mathcal{T}$ iff for all sufficiently large $\alpha < \lambda$

$$\alpha+1 \in [0,\lambda)_T \text{ iff } \alpha+1 \in [0,\lambda)_U$$

and

$$E_\alpha^{\mathcal{T}} = E_\alpha^{\mathcal{U}}.$$

Clearly $[0,\lambda)_T$ is an A-branch of $\mathcal{T}$ iff $[0,\lambda)_\mathcal{U}$ is an A-branch of $\mathcal{U}$. This can happen even if $[0,\lambda)_T \neq [0,\lambda)_\mathcal{U}$, because $<_T$ and $<_\mathcal{U}$ may be quite different, even though they agree on a tail of the two branches.

One can phrase Def. 3A in terms of the extender trees. The A-extenders of $(\mathcal{T},\mathcal{U})$ are just those $F \in Ext(\mathcal{T}) \cap Ext(\mathcal{U})$. If $s \in \mathcal{T}^{ext}$ and $t \in \mathcal{U}^{ext}$ are in the extender trees, put

$$s \equiv_{tail} t \text{ iff } \exists\alpha\,\exists\beta\,\big(\alpha\in dom(s) \text{ and } \beta\in dom(t) \text{ and } \forall k\,[(\alpha+k \in dom(s) \longleftrightarrow \beta+k\in dom(t)) \wedge s(\alpha+k)=t(\beta+k)]\big).$$

Then $[0,\lambda)_T$ is an A-branch of $\mathcal{T}$ iff $e^\mathcal{T}_\lambda \equiv_{tail} e^\mathcal{U}_\lambda$. (Here $e^\mathcal{T}_\lambda$ is the sequence of extenders used in $[0,\lambda)_T$. Cf. [3].)

Remark We allow $s \equiv_{tail} t$ when $lh(s)$ and $lh(t)$ are succ. ords. In this case, it just means they have the same last extender.

Def 3B Let $(\mathcal{T}, \mathcal{U})$ be a slow comparison, λ be a limit ordinal $< lh(\mathcal{T})$, $b = [0,\lambda)_T$ and $c = [0,\lambda)_U$. We say that (b, c) is a <u>branch divergence of $(\mathcal{T}, \mathcal{U})$</u> iff $e_b^{\mathcal{T}} \not\equiv_{\text{Tail}} e_c^{\mathcal{U}}$. We also say that $(\mathcal{T}, \mathcal{U})$ have a <u>branch divergence at λ</u> in this case.

If $e_b^{\mathcal{T}} \equiv_{\text{Tail}} e_c^{\mathcal{U}}$, that is, b and c are A-branches of ~~$\mathcal{E}$~~ $\mathcal{T}$ and $\mathcal{U}$, then we say that $(\mathcal{T}, \mathcal{U})$ has a <u>branch agreement at λ</u>.

Our comparison process produces systems $\hat{\mathcal{W}}$ and $\hat{\mathcal{V}}$. At stage α, we have $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$. We shall have

$$\alpha < \beta \Rightarrow \hat{\mathcal{W}}_\alpha \text{ "}\subseteq\text{" } \hat{\mathcal{W}}_\beta \text{ and } \hat{\mathcal{V}}_\alpha \text{ "}\subseteq\text{" } \hat{\mathcal{V}}_\beta$$

in a natural sense. One component of $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$ are sequences

$$\langle \mathcal{W}_\gamma \mid \gamma \le z(\alpha) \rangle, \langle \mathcal{V}_\gamma \mid \gamma \le z(\alpha) \rangle.$$

We set ~~$\mathcal{W}$~~ $\mathcal{W}_0 = \mathcal{V}_0 =$ empty tree, and

$$z(0) = 0.$$

We shall also have $\mathcal{W}^*_\gamma$ and $\mathcal{V}^*_\gamma$ for $\gamma \leq z(\alpha)$. Our induction hypotheses on $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$ include (7)

$(\dagger)_\alpha$

(a) For $0 \leq \gamma \leq z(\alpha)$, $(\mathcal{W}_\gamma, \mathcal{V}_\gamma)$ is a slow comparison, with $\mathcal{W}_\gamma$ by Σ and $\mathcal{V}_\gamma$ by Λ.

(b) For $0 < \xi \leq \alpha$, $lh(\mathcal{W}_{z(\xi)}) = \lambda + 1$ for some limit ordinal λ, and $(\mathcal{W}_{z(\xi)}, \mathcal{V}_{z(\xi)})$ has a branch divergence at λ.

(c) For $0 \leq \gamma \leq z(\alpha)$, $\mathcal{W}^*_\gamma$ is a normal extension of $\mathcal{W}_\gamma$ by Σ, $\mathcal{V}^*_\gamma$ is a normal extension of $\mathcal{V}_\gamma$ by Λ, and $(\mathcal{W}^*_\gamma, \mathcal{V}^*_\gamma)$ is a slow comparison.

(d) For $0 \leq \gamma \leq z(\alpha)$,

$$(\mathrm{Ext}(\mathcal{W}^*_\gamma) - \mathrm{Ext}(\mathcal{W}_\gamma)) \cap (\mathrm{Ext}(\mathcal{V}^*_\gamma) - \mathrm{Ext}(\mathcal{V}_\gamma)) = \emptyset$$

Moreover, if $\gamma \notin \mathrm{ran}(z)$, then $\mathcal{W}^*_\gamma = \mathcal{W}_\gamma$ and $\mathcal{V}^*_\gamma = \mathcal{V}_\gamma$

The step from $\mathcal{W}_\alpha^*$ to $\mathcal{W}_{\alpha+1}$ and $\mathcal{V}_\alpha^*$ to $\mathcal{V}_{\alpha+1}$ will be given by setting

$$\mathcal{W}_{\alpha+1} = \mathcal{W}_{\alpha+1}^* = X(\mathcal{W}_\beta^*, F_\alpha)$$

and $$\mathcal{V}_{\alpha+1} = \mathcal{V}_{\alpha+1}^* = X(\mathcal{V}_\beta^*, F_\alpha)$$

for some $F = F_\alpha$. It will be the same F and the same β on both sides. (So F is an A-extender of $(\mathcal{W}_{\alpha+1}, \mathcal{V}_{\alpha+1})$.) Here

$$\beta_{F_\alpha} = \beta = \text{least } \gamma \text{ s.t. } \forall \xi \in [\gamma, \alpha]$$
$$\operatorname{crit}(F_\alpha) < \lambda(F_\xi).$$

We then set
$$\beta = <^*\text{-pd}(\alpha+1).$$

We won't have that $lh(F_\alpha)$ increases with α, however.

F_α is the first extender used in one of $\mathcal{W}_\alpha, \mathcal{W}_{\alpha+1}$ but not the other. It is used in $\mathcal{W}_{\alpha+1}$ but not $\mathcal{W}_\alpha$. Similarly for $\mathcal{V}_\alpha$ and $\mathcal{V}_{\alpha+1}$.

Let γ be least s.t. F_α is on the sequence of $M_\gamma^{\mathcal{W}_\alpha}$ (again, it's the same γ on the $\mathcal{V}$-side).

Equivalently, γ is least such that $lh(F_\alpha) < lh(E_\gamma^{\mathcal{W}_\alpha})$ (7c)
or $\gamma = lh(\mathcal{W}_\alpha) - 1$. It is easy to see that

$$\mathrm{Ext}(\mathcal{W}_\alpha) \cap \mathrm{Ext}(\mathcal{W}_{\alpha+1}) = \{E_\xi^{\mathcal{W}_\alpha} \mid \xi < \gamma\}.$$

(Extenders G used in $\mathcal{W}_\alpha$ at γ or afterward are s.t. F_α is on the sequence before G. G determines the sequence up to G.) This agreement propagates:

$(\dagger)_\alpha$

(e) let γ be least s.t. F_α is on the sequence of $M_\gamma^{\mathcal{W}_\alpha}$ (equivalently, least s.t. F_α is on the sequence of $M_\gamma^{\mathcal{V}_\alpha}$), and let $\eta > \alpha$; then

$$\mathrm{Ext}(\mathcal{W}_\alpha) \cap \mathrm{Ext}(\mathcal{W}_\eta) \subseteq \{E_\xi^{\mathcal{W}_\alpha} \mid \xi < \gamma\}$$

and

$$\mathrm{Ext}(\mathcal{V}_\alpha) \cap \mathrm{Ext}(\mathcal{V}_\eta) \subseteq \{E_\xi^{\mathcal{V}_\alpha} \mid \xi < \gamma\}$$

In other notation, $F_\alpha = E_\gamma^{\mathcal{W}_{\alpha+1}} = E_\gamma^{\mathcal{V}_{\alpha+1}}$ in this situation. $\mathcal{W}_\alpha \upharpoonright \gamma+1 = \mathcal{V}_\alpha \upharpoonright \gamma+1$ and possibly they agree further. No $E_\xi^{\mathcal{W}_\alpha}$ for $\xi \geq \gamma$ is used in any later tree $\mathcal{W}_\eta$ ~~for $\eta \geq \alpha$~~ or $\mathcal{V}_\eta$, for $\eta > \gamma$. Similarly for $E_\xi^{\mathcal{V}_\alpha}$ with $\xi \geq \gamma$.

There will be further induction hypotheses (8) on $\hat{W}_\alpha$ and $\hat{\mathcal{V}}_\alpha$, involving other objects in them. These are mainly about the weak tree embeddings $\Phi_{\eta\gamma}$ and $\Psi_{\eta\gamma}$.

Set

$$P_\gamma = M_\infty^{W_\gamma}, \quad P_\gamma^* = M_\infty^{W_\gamma^*},$$

$$Q_\gamma = M_\infty^{\mathcal{V}_\gamma}, \quad Q_\gamma^* = M_\infty^{\mathcal{V}_\gamma^*}.$$

(All of W_γ, W_γ^*, $\mathcal{V}_\gamma$, and $\mathcal{V}_\gamma^*$ will have last models.)

§3. Definition of $\hat{\mathcal{W}}_1$ and $\hat{\mathcal{V}}_1$

We have already specified everything in $\hat{\mathcal{W}}_0$ and $\hat{\mathcal{V}}_0$. To get $\hat{\mathcal{W}}_1$ and $\hat{\mathcal{V}}_1$ we proceed in two steps.

<u>Step 1, D-phase</u> Let $(\mathcal{W}_0^*, \mathcal{V}_0^*)$ come from iterating away least extender disagreements using Σ and Λ until we reach $P_0^* = M_\infty^{\mathcal{W}_0^*}$ and $Q_0^* = M_\infty^{\mathcal{V}_0^*}$ such that for some

$$R \trianglelefteq P_0^* \text{ and } R \trianglelefteq Q_0^*$$

we have

$$\Sigma_R \neq \Lambda_R .$$

We let

$$R_0 = \text{first such } R$$

$$(\mathcal{W}_0^*, \mathcal{V}_0^*) = \text{shortest such trees}.$$

We pad so that $(\mathcal{W}_0^*, \mathcal{V}_0^*)$ is a slow comparison.

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We may assume that least-extender-disagreement does ~~the~~ reach such a $(\mathcal{W}_0^*, \mathcal{V}_0^*)$ and R, since otherwise the usual termination proof for mice (rather than mouse pairs) works. Also, since we are stopping as soon as the lined-up part of our iterates of P and Q supports a strategy-disagreement, the internal strategy predicates (if any) agree as far as the extenders, so

$$P_0^* \,\|\, \sup\{lh\, E_\alpha^{\mathcal{W}_0^*} \mid \alpha < lh(\mathcal{W}_0^*)\} = Q_0^* \,\|\, \sup\{lh\, E_\alpha^{\mathcal{V}_0^*} \mid \alpha < lh\, \mathcal{V}_0^*\}.$$

Step 1, A-phase

Let $\mathcal{Y}_0$ be a normal tree on R_0 that is by both Σ_{R_0} and Λ_{R_0}, and such that for

$$b_0 = \Sigma_{R_0}(\mathcal{Y}_0)$$

and

$$c_0 = \Lambda_{R_0}(\mathcal{Y}_0)$$

9c

we have

$$b_0 \neq c_0 .$$

We need to regard $\mathcal{Y}_0$ as being on $\mathcal{P}_0^*$ and $\mathcal{Q}_0^*$. For that, the following lemma is what we need.

<u>Lemma 3</u> (AD$^+$) Let (M, Ω) be a mouse pair and $N \trianglelefteq M$. Let $\mathcal{Y}$ be a normal tree on N by Ω_N; then there is a unique normal tree $\mathcal{Z}$ on M by Ω such that $lh(\mathcal{Z}) = lh(\mathcal{Y})$ and $E_\alpha^{\mathcal{Z}} = E_\alpha^{\mathcal{Y}}$ for all $\alpha < lh(\mathcal{Y})$.

This is a form of internal consistency for the strategy in a mouse pair that has not yet been proved, as far as we can tell. (It is not the same as internal lift consistency.) It should be provable by the strategy comparison technique of [2A]. For now we'll just assume it.

<u>Def 4</u> Let $\mathcal{Y}$ be a normal tree on N, and $N \trianglelefteq M$. Then $\mathcal{Z}(\mathcal{Y}, M)$ is the unique normal (so maximal) tree on M with the same tree order s.t. $E_\alpha^{\mathcal{Z}} = E_\alpha^{\mathcal{Y}}$ for all α.

We may also use the following notation from Sargsyan EIT. (10)

<u>Def. 5</u> If $\mathcal{T}$ and $\mathcal{U}$ are iteration trees, then $\mathcal{T} =^* \mathcal{U}$ iff $lh(\mathcal{T}) = lh(\mathcal{U})$, $<_T = <_U$, and $E_\alpha^{\mathcal{T}} = E_\alpha^{\mathcal{U}}$ for all $\alpha < lh(\mathcal{T})$.

In our particular context:

<u>Notation</u> If Y is a normal tree on R s.t. $R \trianglelefteq P_\alpha^*$, then $Y^w = Z(Y, P_\alpha^*)$. If Y is on some R s.t. $R \trianglelefteq Q_\alpha^*$, then $Y^v = Z(Y, Q_\alpha^*)$.

Returning to the definition of $\hat{\mathcal{V}}_1$ and $\hat{\mathcal{D}}_1$, we set

$$z(1) = lh(\Upsilon_0),$$

~~$\mathcal{K}_1^c = \mathcal{K}_1 =$ tree order of Υ_0~~

and for $\xi < lh(\Upsilon_0)$

$$\mathcal{W}_\xi = \mathcal{X}(\mathcal{W}_0^*, \Upsilon_0^{\mathcal{W}} \restriction \xi+1),$$

$$\mathcal{D}_\xi = \mathcal{X}(\mathcal{D}_0^*, \Upsilon_0^{\mathcal{D}} \restriction \xi+1).$$

<u>Remark</u> So $F_\eta = E_\eta^{\Upsilon_0}$ for all $\eta < lh(\Upsilon_0)$

For $\xi = z(1)$,

$$\mathcal{W}_{z(1)} = \mathcal{X}(\mathcal{W}_0^*, (\Upsilon_0{}^\frown b_0)^{\mathcal{W}}) \restriction \text{ extenders with length} < \delta(\Upsilon_0)$$

$$\mathcal{D}_{z(1)} = \mathcal{X}(\mathcal{D}_0^*, (\Upsilon_0{}^\frown c_0)^{\mathcal{W}}) \restriction \text{ extenders with length} < \delta(\Upsilon_0).$$

We include in $\mathcal{W}_{z(1)}$ and $\mathcal{D}_{z(1)}$ the branches generated by b_0 and c_0, so that they do have a last model.

~~In fact, it is clear that~~ Set for $0 < \xi < z(1)$

$$\mathcal{W}_\xi^* = \mathcal{W}_\xi \text{ and } \mathcal{D}_\xi^* = \mathcal{D}_\xi.$$

For $\xi <^{e}_{\Psi_0} \gamma$, ~~(illegible)~~

~~and~~ (so $\gamma < z(1)$), let

$$\Phi_{\xi,\gamma} : \mathcal{W}^*_\xi \longrightarrow \mathcal{W}^*_\gamma$$

and

$$\Psi_{\xi,\gamma} : \mathcal{V}^*_\xi \longrightarrow \mathcal{V}^*_\gamma$$

be the weak tree embeddings that come out of the full normalization construction. Note that $\xi <^{e}_{\Psi_0} \gamma \Rightarrow \gamma < z(1)$, so $\mathcal{W}_\gamma = \mathcal{W}^*_\gamma$. It is possible that $\xi = 0$ and $\mathcal{W}_\xi \neq \mathcal{W}^*_\xi$. It might be tempting to set $\xi <^{e}_{1} z(1)$ for $\xi \in b_0$, but $b_0 \neq c_0$, so we'd end up with $<^{e}_{1}$ ~~(illegible)~~ having different versions in $\hat{\mathcal{W}}$ and $\hat{\mathcal{V}}$.

In any case, for $\xi \in b_0$ there may be no weak tree embedding from $\mathcal{W}^*_\xi$ into $\mathcal{W}_{z(1)}$, because we have cut $X(\mathcal{W}^*_0, (\Upsilon_0{}^\frown b_0)^w)$ by removing all extenders with $lh \geq \delta(\Upsilon_0)$. We do keep $br(\mathcal{W}^*_0, \Upsilon_0, b_0)$, the branch generated by b_0.

We need to cut because otherwise $(\mathcal{W}_{z(1)}, \mathcal{V}_{z(1)})$ may not be a slow comparison.

Lemma 6

(a) For all $\xi \le z(1)$, $(\mathcal{W}_\xi, \mathcal{V}_\xi)$ is a slow comparison.

(b) $(\mathcal{W}_{z(1)}, \mathcal{V}_{z(1)})$ has a branch divergence at $lh(\mathcal{W}_{z(1)}) - 1$.

Proof (Sketch) $(\mathcal{W}_0^*, \mathcal{V}_0^*)$ is a slow comparison. For $\xi < z(1)$,

$$(\mathcal{W}_\xi^*, \mathcal{V}_\xi^*) = \left(X(\mathcal{W}_0^*, Y_0^w \restriction \xi+1), X(\mathcal{V}_0^*, Y_0^v \restriction \xi+1)\right).$$

But $Y_0^w =^* Y_0^v$, so the full normalization inflations of agreeing/disagreeing levels of the various $M_\gamma^{\mathcal{W}_0^*}$ and $M_\gamma^{\mathcal{V}_0^*}$ are by the same extenders and branch extenders. So agreement/disagreement is preserved, and $(\mathcal{W}_\xi, \mathcal{V}_\xi)$ is a slow comparison.

Because we have cut

$$\mathcal{W}_{z(1)} = \left(\bigcup_{\xi < lh(Y_0)} X(\mathcal{W}_0^*, Y_0^w \restriction \xi+1) \restriction \text{exts of } lh < lh(E_\xi^{Y_0})\right)^\frown br(\mathcal{W}_0^*, Y_0, b_0)$$

and

$$\mathcal{V}_{z(1)} = \left(\bigcup_{\xi < lh\, Y_0} X(\mathcal{V}_0^*, Y_0^v \restriction \xi+1) \restriction \text{exts of } lh < lh(E_\xi^{Y_0})\right)^\frown br(\mathcal{V}_0^*, Y_0, c_0).$$

So $(\mathcal{W}_{z(1)}, \mathcal{V}_{z(1)})$ is a slow comparison — adding the branches doesn't change that. The branches do not agree on a tail, so we get (b).

Remark This is why we cut $X(W_0^*, Z(Y_0, b_0))$ to extenders with length $< \delta(Y_0)$. The longer extenders ~~are~~ have been inflated by b_0. Since $b_0 \neq c_0$, including b_0-inflations and c_0-inflations could lead to $(W_{Z(1)}, V_{Z(1)})$ failing to be a slow comparison.

The idea of cutting this way is due to Sargsyan [1].

A more careful proof of Lemma 6 would proceed by induction on ξ, and use some simple properties of weak tree embeddings and the full normalization construction. For the reader's convenience, we summarize some of this background now. See [2] for further detail.

§4 Background on weak tree embeddings and full normalization.

Suppose $\mathcal{T}$ and $\mathcal{U}$ are normal trees, $\mathcal{T}\restriction\beta+1 = \mathcal{U}\restriction\beta+1$, F is an extender on the extended $M_\infty^{\mathcal{U}}$-sequence, and β is least s.t. $\kappa_F < \lambda(E_\beta^{\mathcal{U}})$. Suppose also $\mathrm{lh}(E_\beta^{\mathcal{T}}) \geq \mathrm{lh}(E_\beta^{\mathcal{U}})$ if they exists. Then

$$X(\mathcal{T},\mathcal{U},F) = \mathcal{U}\restriction\alpha+1 ^\frown \langle F\rangle ^\frown i_F'' \mathcal{T}^{\geq\beta}$$

where α is least s.t. F is on the extended $M_\alpha^{\mathcal{U}}$-sequence.

<u>Remark</u> "Extended" means $F = G^+$ for some G on the sequence is allowed. For now we are going to ignore the difference between G and G^+. In a more careful treatment, all ~~of~~ the $\mathcal{W}$'s, $\mathcal{V}$'s, and $\mathcal{Y}$'s would be λ-separated, near plus trees.

In this paper we shall only form $X(\mathcal{T},\mathcal{U},F)$ when $\mathcal{T}$ and $\mathcal{U}$ are by some strategy Ω of a mouse pair (M,Ω). In that case $X(\mathcal{T},\mathcal{U},F)$ is a normal tree that is also by Ω. There is a weak tree embedding

$$\Phi : \mathcal{T} \longrightarrow X(\mathcal{T},\mathcal{U},F)$$

given by

$$\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}\,\mathcal{T}\rangle, \langle t_\eta \mid \eta+1 < \mathrm{lh}(\mathcal{T})\rangle\rangle$$

where

$$u \restriction \beta = v \restriction \beta = \mathrm{id}$$

$$u(\xi) = \alpha+1+\xi \quad \text{for } \xi \geq \beta$$

$$v(\xi) = \begin{cases} \xi & \text{if } \xi \leq \beta \\ u(\xi) & \text{if } \xi > \beta \end{cases}$$

$X \restriction \alpha+1 = \mathcal{U} \restriction \alpha+1$, and the remaining models of X are $i_F'' \mathcal{T}^{\geq\beta}$, that is

$$M^X_{u(\xi)} = \begin{cases} \mathrm{Ult}(M^{*,X}_{\alpha+1}, F) & \text{if } \xi = \beta \\ \mathrm{Ult}(M^{\alpha}_{\xi}, F) & \text{if } \xi > \beta \end{cases}$$

For $\xi < \beta$, $t_\xi = \mathrm{id}$, and

$$t_\xi = \begin{cases} i_F^{M^{*,X}_{\alpha+1}} & \text{if } \xi = \beta \\ i_F^{M^{\alpha}_{\xi}} & \text{if } \xi > \beta \end{cases}$$

i = the canonical emb. into the ultrapower.

$s_\xi = \mathrm{id}$ for $\xi \le \beta$, and $s_\xi = t_\xi$ for $\xi > \beta$. In general

$$s_\xi : M^{\alpha}_{\xi} \to M^X_{v(\xi)},$$

$$t_\xi : M^{\alpha}_{\xi} \to M^X_{u(\xi)}.$$

s_ξ is total and elementary, but t_ξ may be partial. It is defined on $M^{\alpha}_{\xi} \mid lh(E^{\mathcal{T}}_{\xi})$ at least. It connects

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exit extenders, in that

$$E^X_{u(\xi)} = \hat{t}_\xi(E^{\mathcal{T}}_\xi)$$

where $\hat{t}_\xi$ is the canonical emb. associated to $\mathrm{Ult}(M^{\mathcal{T}}_\xi \| \mathrm{lh}(E^{\mathcal{T}}_\xi), F)$ — so that $\hat{t}_\xi$ factors into t_ξ. For $\mu = \mathrm{lh}(E^{\mathcal{T}}_\xi)$, we have the diagram

$$\begin{tikzcd}
M^{\mathcal{T}}_\xi \arrow[r, "t_\xi"] & M^X_{u(\xi)} = \mathrm{Ult}(M^{\mathcal{T}}_\xi, F) \\
\trianglerighteq & \trianglerighteq \\
M^{\mathcal{T}}_\xi | \mu \arrow[r, "t_\xi"] \arrow[dr, "\hat{t}_\xi"'] & t_\xi(M^{\mathcal{T}}_\xi | \mu) \\
& \hat{t}_\xi(M^{\mathcal{T}}_\xi | \mu) = \mathrm{Ult}(M^{\mathcal{T}}_\xi | \mu, F) \arrow[u, "rs_\xi"']
\end{tikzcd}$$

rs_ξ is the factor map between the two ultrapowers. Condensation tells us

$$\hat{t}_\xi(M^{\mathcal{T}}_\xi | \mu) \trianglelefteq M^X_{u(\xi)}.$$

Thus $E^X_{u(\xi)}$ is indeed on the (extended) $M^X_{u(\xi)}$-sequence.

The definition of $X(\mathcal{T}, \mathcal{U})$ proceeds by induction, defining

$$X_\xi = X(\mathcal{T}, \mathcal{U} \restriction \xi+1)$$

by

$$X_{\xi+1} = X(X_\beta, X_\xi, E^{\mathcal{U}}_\xi),$$

and for $\lambda < \mathrm{lh}(\mathcal{U})$ a limit ordinal

$$X_\lambda = \lim_{\xi <_U \lambda} X_\xi.$$

Here the direct limit is under the weak tree embeddings

$$\Phi_{\alpha,\beta} : X_\alpha \longrightarrow X_\beta$$

produced by the construction, for $\alpha <_U \beta$.

§4.2 Weak tree embeddings

In general, a <u>weak tree embedding</u>

$$\Phi : \mathcal{T} \to \mathcal{U}$$

is a system

$$\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\alpha \mid \alpha+1 < \mathrm{lh}(\mathcal{T})\rangle\rangle,$$

where $u : \{\alpha \mid \alpha+1 < \mathrm{lh}(\mathcal{T})\} \to \mathrm{lh}(\mathcal{U})$ and $v : \mathrm{lh}(\mathcal{T}) \to \mathrm{lh}(\mathcal{U})$ such that

$$s_\alpha : M_\alpha^{\mathcal{T}} \to M_{v(\alpha)}^{\mathcal{U}}$$

is total and elementary, and

$$t_\alpha = i^{\mathcal{U}}_{v(\alpha), u(\alpha)} \circ s_\alpha$$

is perhaps partial, but defined on $M_\alpha^{\mathcal{T}} \mid \mathrm{lh}\, E_\alpha^{\mathcal{T}}$. (In particular, $v(\alpha) \leq_U u(\alpha)$.)

t_α is used to connect exit extenders, in that

$$E^{\mathcal{U}}_{u(\alpha)} = \hat{t}_\alpha(E_\alpha^{\mathcal{T}}),$$

where

$$\hat{t}_\alpha = r s_\alpha^{-1} \circ t_\alpha$$

← **Rmk** This is the "type X" case in the defn.

is a map that "respects drops over $(M^{\mathcal{U}}_{u(\alpha)}, \eta, \lambda)$", for $\eta = \mathrm{lh}(E^{\mathcal{U}}_{u(\alpha)})$ and $\lambda = \mathrm{lh}(t_\alpha(E^{\mathcal{T}}_\alpha))$. The paradigm for $r s_\alpha^{-1}$ is the factor map above.

The diagram at a general successor step in the evolution of Φ is, with $\beta = T\text{-pred}(\alpha+1)$, ~~and~~ $v(\alpha+1) = u(\alpha)+1$ and

$$\begin{tikzcd}
M^{\mathcal{T}}_{\alpha+1} \arrow[r, "s_{\alpha+1}"] & M^{\mathcal{U}}_{v(\alpha+1)} & \\
 & M^{\mathcal{U}}_{\beta^*} \arrow[u, "E^{\mathcal{U}}_{u(\alpha)}"'] \arrow[r, dashed] & M^{\mathcal{U}}_{u(\beta)} \\
M^{\mathcal{T}}_{\beta} \arrow[uu, "E^{\mathcal{T}}_\alpha"] \arrow[ur, "\rho"] \arrow[r, "s_\beta"'] & M^{\mathcal{U}}_{v(\beta)} \arrow[u, "i^{\mathcal{U}}_{v(\beta),\beta^*}"'] &
\end{tikzcd}$$

with

$$s_{\alpha+1}([a,f]) = [\hat{t}_\alpha(a), \rho(f)]$$

The agreement between s, t, and $\hat{t}$ maps in a weak tree embedding is given by

(i) $s_{\alpha+1} = \hat{t}_\alpha$ on $lh(E_\alpha^{\tilde{\mathcal{T}}})$. (By the Shift Lemma.)

(ii) $t_{\alpha+1} = s_{\alpha+1}$ on $\lambda(E_\alpha^{\mathcal{T}})$. (Since $t_\alpha = i^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha$,

and $\mathrm{crit}(i^{\mathcal{U}}_{v(\alpha),u(\alpha)}) \geq s_\alpha$ " $lh(E_\alpha^{\mathcal{T}}) = s_{\alpha+1}$ " $lh(E_\alpha^{\mathcal{T}})$.)
by induction.

(iii) $t_{\alpha+1} = \hat{t}_{\alpha+1}$ on $lh(E_\alpha^{\mathcal{T}})$. (Since the projecta associated to drops being respected are $\geq lh(E_\alpha^{\mathcal{T}})$.)

(iv) $s_{\alpha+2} = \hat{t}_{\alpha+1}$ on $lh(E_{\alpha+1}^{\mathcal{T}})$. (Part (i) again.)

So

(v) $\hat{t}_\alpha$ and $s_{\alpha+1}$ agree on $\lambda(E_\alpha^{\mathcal{T}})$ with all s_η, t_η, and $\hat{t}_\eta$ for $\eta \geq \alpha+1$.

Of course $\hat{t}_\alpha$ and t_α can disagree on $\lambda(E_\alpha^{\mathcal{T}})$ if $r s_\alpha$ is nontrivial.

Direct limits along a sequence $\langle \Phi_{\alpha,\beta} \mid \alpha<\beta<\lambda \rangle$ of commuting weak tree embeddings are defined using the u-threads and the t-maps along them. Along any u-thread, the t maps and $\hat{t}$ maps are eventually the same, since you can't keep dropping to a proper initial segment by the wellfoundedness of the direct limit. (Here we assume everything is by an iteration strategy.) Letting $\mathcal{U} = \lim_{\alpha<\lambda} \mathcal{T}_\alpha$ be the limit tree, its extenders are given by

$$E \in Ext(\mathcal{U}) \text{ iff } \exists \alpha<\lambda\, \exists \xi\, [E = t_\xi^{\alpha,\lambda}(E_\xi^{\mathcal{T}_\alpha})$$
$$\text{and } \forall \beta \in (\alpha,\lambda)$$
$$\hat{t}_\xi^{\alpha,\beta}(E_\xi^{\mathcal{T}_\alpha}) = t_\xi^{\alpha,\beta}(E_\xi^{\mathcal{T}_\alpha})$$
$$= E^{\mathcal{T}_\beta}_{u_{\alpha,\beta}(\xi)}.]$$

$\text{Ext}(\mathcal{U})$ determines all of $\mathcal{U}$. It would not have worked to use v-threads to define the direct limit, since the v-maps are continuous at limit ~~oo~~ ordinals. This implies for example that if $\lambda = \omega$, and all $\tilde{\alpha}_\alpha$ are countable, then $\omega_1 \neq v_{\alpha,\lambda}(\xi)$ for all ξ. It is still possible that $\omega_1 < lh(\mathcal{U})$, i.e. $\omega_1 = u_{\alpha,\lambda}(\xi)$ for some ξ.

The whole of a weak tree embedding $\Phi : \mathcal{T} \to \mathcal{U}$ is determined by how the ~~ext~~ exit extenders of $\mathcal{T}$ are mapped. That is, setting

$$p^{\Phi}(E_\alpha^{\mathcal{T}}) = \hat{t}_\alpha(E_\alpha^{\mathcal{T}})$$
$$= E_{u(\alpha)}^{\mathcal{U}},$$

Φ is the unique weak tree embedding with p-map equal to p^{Φ}. It's sometimes useful to think of Φ as a nice map on exit extenders. The associated map on extender trees is also useful: for $\alpha < lh(\mathcal{T})$

$$e_\alpha^{\mathcal{T}} = \text{sequence of extenders used in } [0, \alpha)_T$$

and then $\mathcal{T}^{ext} = (\{e_\alpha^{\mathcal{T}} \mid \alpha < lh(\mathcal{T})\}, \subseteq)$

and $\hat{p}^{\Phi} : \mathcal{T}^{ext} \to \mathcal{U}^{ext}$ is given by

$$\hat{p}^{\Phi}(e_\alpha^{\mathcal{T}}) = e_{v(\alpha)}^{\mathcal{U}}.$$

$\hat{p}^{\Phi}$ preserves $\subseteq$ and $\perp$.

Suppose $\bar{\Phi}: \mathcal{T} \to \mathcal{U}$ is a (weak) tree embedding, that $\mathcal{T}$ and $\mathcal{U}$ have lengths $\alpha+1$ and $\beta+1$, and that $v^{\Phi}(\alpha) \leq_{\mathcal{U}} \beta$. Then we get an <u>extended (weak) tree embedding</u> $\dot{\Phi}$ by setting

$$u^{\dot{\Phi}}(\alpha) = \beta,$$

$$t_\alpha^{\dot{\Phi}} = i^{\mathcal{U}}_{v(\alpha),\beta} \circ s_\alpha^{\bar{\Phi}}.$$

$\dot{\Phi}$ can then be used to copy extensions $\mathcal{T}^\frown\langle E\rangle$. Basic lemmas like the Shift Lemma ([2], 3.20) apply to extended tree embeddings, not tree embeddings in general. In our construction, this is relevant, because of the step from W_ξ to its normal extension W_ξ^*. The identity map is a tree embedding from W_ξ to W_ξ^*, but it may not be extendible, since P_ξ^* may not be above P_ξ in W_ξ^*.

An extended tree embedding $\Phi: \mathcal{T} \to \mathcal{U}$ is completely determined by its last t-map $t_\infty^{\Phi}: M_\infty^{\mathcal{T}} \to M_\infty^{\mathcal{U}}$. So we can think of

(24b)

an extended tree embedding from $\mathcal{T}$ to $\mathcal{U}$ as a map $t : M_\infty^{\mathcal{T}} \to M_\infty^{\mathcal{U}}$ with a stronger elementarity property, in that it preserves facts about the way the two models were constructed.

§4.4 $X(\mathcal{T}, F)$ and $X(\mathcal{T}, s)$

Let $\mathcal{U}$ be a tree ~~on the last model of $\mathcal{T}$~~, and α be least such that F is on the (extended) sequence of $M_\alpha^{\mathcal{U}}$. We write

$$\alpha = \alpha(\mathcal{U}, F).$$

Let

$$\beta = \beta(\mathcal{U}, F) = \text{least } \gamma \text{ s.t. } \kappa_F < \lambda(E_\gamma^{\mathcal{U}})$$

as above. Suppose $\mathcal{T}\restriction\beta+1 = \mathcal{U}\restriction\beta+1$ and $lh(E_\beta^{\mathcal{T}}) \geq lh(E_\beta^{\mathcal{U}})$. We have defined

$$X(\mathcal{T}, \mathcal{U}, F) = \mathcal{U}\restriction\alpha+1 {}^\frown \langle F \rangle {}^\frown i_F{}'' \mathcal{T}^{\geq\beta}$$

above. But in the contexts to come, we shall almost always have that $\mathcal{T}, \mathcal{U}$ are trees on some mouse pair (P, Σ). In this case $\mathcal{U}\restriction\alpha+1$ is determined by F and (P, Σ), since

$$\mathcal{U}\restriction\alpha+1 = \text{unique normal } \mathcal{S} \text{ by } \Sigma \text{ s.t. } F \text{ is on the extended } M_\infty^{\mathcal{S}}\text{-sequence and } lh(E_\xi^{\mathcal{S}}) < lh(F) \text{ for all } \xi+1 < lh(\mathcal{S}).$$

So we can write

$$\alpha_F = \alpha(\mathcal{U}, F)$$

$$\beta_F = \beta(\mathcal{U}, F)$$

and

$$X(\mathcal{J}, F) = X(\mathcal{J}, \mathcal{U}, F),$$

if we are in a context where (P, Σ) is understood.

In such a context, for s a sequence of extenders we can define $X(\mathcal{J}, s)$ by

$$X_0 = X(\mathcal{J}, \emptyset) = \mathcal{J},$$

$$X_{i+1} = X(\mathcal{J}, s\restriction i+1) = X(X(\mathcal{J}, s\restriction i), s(i))$$

and

$$X_\lambda = X(\mathcal{J}, s\restriction\lambda) = \lim_{\eta<\lambda} X(\mathcal{J}, s\restriction\eta)$$

for λ a limit, where the limit is under the weak tree embeddings $\Phi_{\eta,\delta}: X_\eta \to X_\delta$ we get from full normalization. The tacit assumption here is that for all $i \in \mathrm{dom}(s)$, there is a unique minimal $\mathcal{U}_i$ by Σ that has $s(i)$ on $M_\infty^{\mathcal{U}_i}$, and we are setting

$$X(\mathcal{J}, s\restriction i+1) = X(X(\mathcal{J}, s\restriction i), \mathcal{U}_i, s(i)).$$

Proof of Lemma 6 again:

We show by induction ~~of~~ on $\xi < Z(1)$ that $(\mathcal{W}_\xi, \mathcal{V}_\xi)$ is a slow comparison.

Let $Z_0 = Z(\mathcal{W}_0^*, Y_0)$ and $Z_1 = Z(\mathcal{V}_0^*, Y_0)$ be the versions of Y_0 on P_0^* and Q_0^*.

Full normalization tells us

$$P_\xi = M_\xi^{Z_0}$$
$$Q_\xi = M_\xi^{Z_1},$$

and our assumed lemma 3 says $M_\xi^{Y_0} \trianglelefteq P_\xi, Q_\xi$.

Let $F = E_\xi^{Y_0}$. Let

$$\beta = \beta(\mathcal{W}_\xi, F) = \beta(\mathcal{V}_\xi, F),$$

noting that the fact that $(\mathcal{W}_\xi, \mathcal{V}_\xi)$ is a slow comparison plus our padding implies it's the same β. It is now easy to see that

$$\big(X(\mathcal{W}_\beta, F), X(\mathcal{V}_\beta, F)\big)$$ is a slow comparison,

because the various ultrapowers agree enough. The weak tree embs $\mathcal{W}_\beta \to \mathcal{W}_{\xi+1}$, $\mathcal{V}_\beta \to \mathcal{V}_{\xi+1}$ are essentially the same, so the induction gets past limit ordinals.

This completes the definition of $\hat{W}_1$ and $\hat{\mathcal{V}}_1$.

§5. Definition of $\hat{W}_2$ and $\hat{\mathcal{V}}_2$.

The step from $(\hat{W}_1, \hat{\mathcal{V}}_1)$ to $(\hat{W}_2, \hat{\mathcal{V}}_2)$ is pretty much the same as the general successor step, but we shall isolate it anyway. Let

$$W_{z(1)} \subseteq W^*_{z(1)}$$

and

$$\mathcal{V}_{z(1)} \subseteq \mathcal{V}^*_{z(1)}$$

be the normal extensions of $W_{z(1)}$ and $\mathcal{V}_{z(1)}$ you get by iterating away least extender disagreements until you expose a strategy disagreement, i.e. you have

$$R_1 \trianglelefteq P_1^*, \quad R_1 \trianglelefteq Q_1^*$$

and Y_1 on R_1 s.t.

$$\Sigma_{R_1}(Y_1) = b_1$$

$$\Lambda_{R_1}(Y_1) = c_1$$

and

$$b_1 \neq c_1 .$$

So $(W^*_{z(1)}, \nu^*_{z(1)})$ is a slow comparison.

If we never reach a strategy disagreement this way, then the standard argument for pure extender comparisons tells us our process terminates ~~[illegible]~~ in $< \omega_1$ steps, as desired.

~~We are going to set~~

$$\cancel{W_{z(2)} = X(W^*_{z(1)}, z(Y_1 \cap b_1, P_1^*)) \restriction \text{ext of } lh < \delta(Y_1)}$$

~~and~~

$$\cancel{W_{z(2)} = X(W^*_{z(1)}, z(Y_1 \cap c_1, Q_1^*)) \restriction \text{exts of } lh < \delta(Y_1)}$$

~~(with the cofinal branches given by b_1 and c_1 included),~~

We are going to set

$$\mathcal{W}_{z(2)} = X(\mathcal{W}^*_{z(1)}, (\Upsilon_1{}^\frown b_1)^{\mathcal{W}}) \upharpoonright \text{exts of } lh < \delta(\Upsilon_1)$$

and

$$\mathcal{V}_{z(2)} = X(\mathcal{V}^*_{z(1)}, (\Upsilon_1{}^\frown c_1)^{\mathcal{V}}) \upharpoonright \text{exts of } lh < \delta(\Upsilon_1),$$

(with the branches generated by b_1 and c_1 included).

We shall also include in $\hat{\mathcal{W}}_2$ and $\hat{\mathcal{V}}_2$ the trees that occur in the full normalization process applied to the stacks $\langle \mathcal{W}^*_{z(1)}, (\Upsilon_1{}^\frown b_1)^{\mathcal{W}} \rangle$ and $\langle \mathcal{V}^*_{z(1)}, (\Upsilon_1{}^\frown c_1)^{\mathcal{V}} \rangle$, along with the weak tree embeddings in the associated meta-tree. This was all we did in step I_e

This is all that is done in [1], but we must keep more structure going here. We can't afford to ignore the $\mathcal{W}^*_\xi$ and $\mathcal{V}^*_\xi$ for $\xi < z(1)$. Basically, we will be fully normalizing the meta-stacks $\langle \hat{\mathcal{W}}_1, (\Upsilon_1{}^\frown b_1)^{\mathcal{W}} \rangle$, and $\langle \hat{\mathcal{V}}_1, (\Upsilon_1{}^\frown c_1)^{\mathcal{V}} \rangle$, and looking at the weak tree embeddings that come out of those processes. ~~$<_n$ will basically be the tree order of the normal meta-tree these processes produce.~~

The standard meta-normalization process leading to $X(\hat{W}_1, (Y_1{}^\frown b_1)^{w})$ ~~called~~ would throw away the normal trees in your current meta-tree that are past the first tree that reaches your current exit extender. We won't do that here, because it seems to simplify bookkeeping. There's no harm in having trees in your sequence $\langle W_\xi \mid \xi \leq z(2)\rangle$ that will play no role in the rest of the construction.

We shall define $z(1,\xi)$ for $\xi \leq lh(Y_1)$.

For $\xi < lh(Y_1)$, we'll have

$$W_{z(1,\xi)} = W^*_{z(1,\xi)} = X(W_{z(1)}, Y_1^{W} \restriction \xi+1)$$

and

$$\mathcal{V}_{z(1,\xi)} = \mathcal{V}^*_{z(1,\xi)} = X(\mathcal{V}^*_{z(1)}, Y_1^{\mathcal{V}} \restriction \xi+1)$$

and

$$z(2) = \sup_{\xi < lh(Y_1)} z(1,\xi).$$

(So $F_{z(1,\xi)} = E^{Y_1}_\xi$.)

So this part is just the normalization of the stacks $\langle W^*_{z(1)}, Y_1^{W}\rangle$ and $\langle \mathcal{V}^*_{z(1)}, Y_1^{\mathcal{V}}\rangle$.

The remaining W_η and $\mathcal{V}_\eta$ for $\eta < z(2)$ come from letting Y_1 interact with earlier trees in $\hat{W}_1$ and $\hat{\mathcal{V}}_1$.

We define $z(1,\xi)$ by induction on ξ.

Set

$$z(1,0) = z(1).$$

Suppose we have defined $z(1,\xi)$ and $(W_i^*, \triangleright_i^*)$ for all $i \le z(1,\xi)$. We have set

$$F_{z(1,\xi)} = E_\xi^{\Upsilon_1}.$$

Let $F = F_{z(1,\xi)}$ and $\mu = \Upsilon_1\text{-pred}(\xi+1)$.

Put

$$\beta_F = \text{least } \gamma \text{ s.t. } \forall \theta \in [\gamma, z(1,\xi)]$$
$$\kappa_F < \lambda(F_\theta)$$
$$= \text{least } \gamma \text{ s.t. } \forall \theta \in [\gamma, z(1,\mu)]$$
$$\kappa_F < \lambda(F_\theta).$$

(Here and below $\kappa_F = \text{crit}(F)$.) (The 2nd equality holds because $\kappa_F < \lambda(F_{z(1,\mu)}) = \lambda(E_\mu^{\Upsilon_1}) \le \lambda(F_\eta)$ $\forall \eta \ge z(1,\mu)$ s.t. $\eta \le z(1,\xi)$.) Let

$$I_F = [\beta_F, z(1,\mu)]$$

and

$$z_F(i) = z(1,\xi) + 1 + (i - \beta_F)$$
$$J_F = z_F{}'' I_F,$$

so that

$$z_F : I_F \xrightarrow[\text{op}]{\text{onto}} J_F$$

Set

$$z(1,\xi+1) = \max(J_F)$$
$$= z_F(z(1,\beta))$$

and for $i \in I_F$

$$W_{z_F(i)} = W^*_{z_F(i)} = X(W^*_i, F),$$

$$\nu_{z_F(i)} = \nu^*_{z_F(i)} = X(\nu^*_i, F).$$

For $i \in I_F$ s.t. $i < z(1,\beta\mu)$, $F_{z_F(i)}$'s given by the full normalization construction,

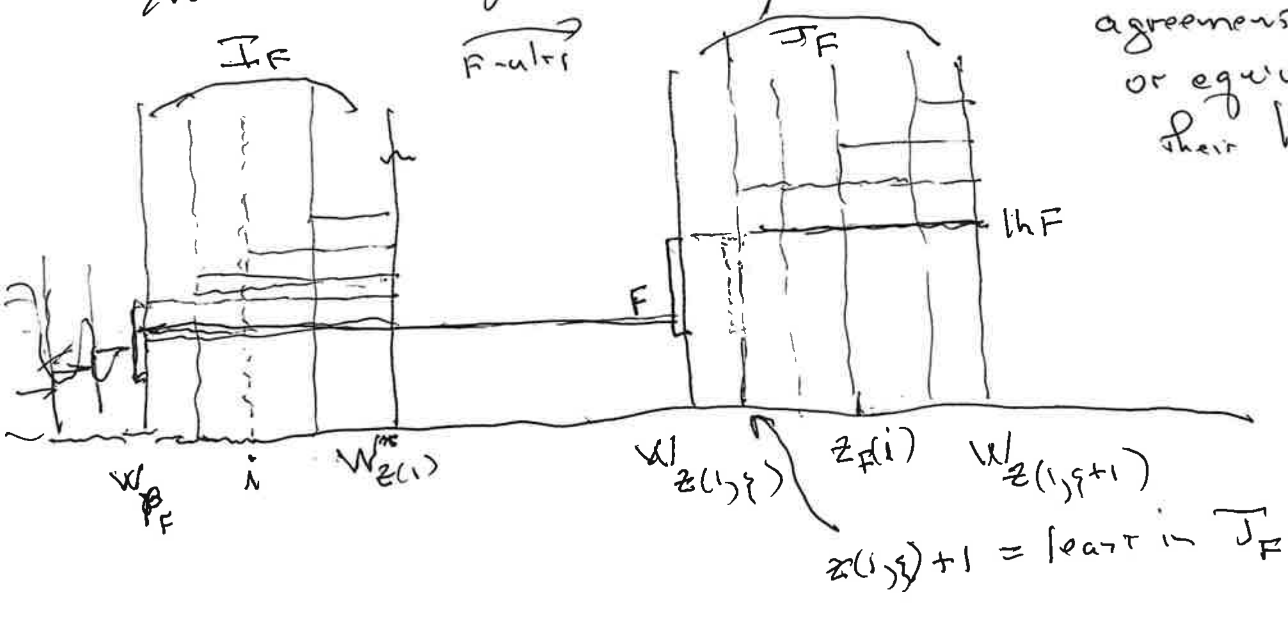
Insert 30a

so

$$F_{z_F(i)} = \text{first extender } G \text{ used in } W_{z_F(i)+1}$$
$$\text{s.t. } G \notin \mathrm{Ext}(W^*_{z_F(i)}).$$

Here is a diagram when $\beta = 0$:

Horizontal lines represent agreements of trees, or equivalently, of their last models.



Let us use the notation

$$\sigma_F = \max(I_F) = z(1, \mu)$$

$$\tau_F = \max(J_F) = z_F(\sigma_F)$$

$$\theta_F = \text{index of } F \text{ in } \hat{W}$$
$$= z(1, \xi).$$

So

$$I_F = [\beta_F, \sigma_F]$$

and

$$J_F = [\theta_F + 1, \tau_F].$$

It is possible that $\beta_F = \sigma_F$, in which case F is not inflating any of the F_i for $i < \sigma_F$. In general, F inflates the $\overline{F}_i$ for $\beta_F \le i < \sigma_F$. For such i, we have

$$F_{z_F(i)} = t_\infty^{\Phi_{i+1}, z_F(i+1)}(F_i)$$

Recall $F_i \in Ext(W^{\#}_{i+1})$ and $F_i \notin Ext(W^{\#}_i)$

$$= t_\infty^{\Psi_{i+1}, z_F(i+1)}(F_i).$$

Here $F_i \in Ext(W^{\#}_{i+1}) \cap Ext(\mathcal{V}_{i+1})$,

$P_i^* || lh(F_i) = Q_i^* || lh(F_i)$, and

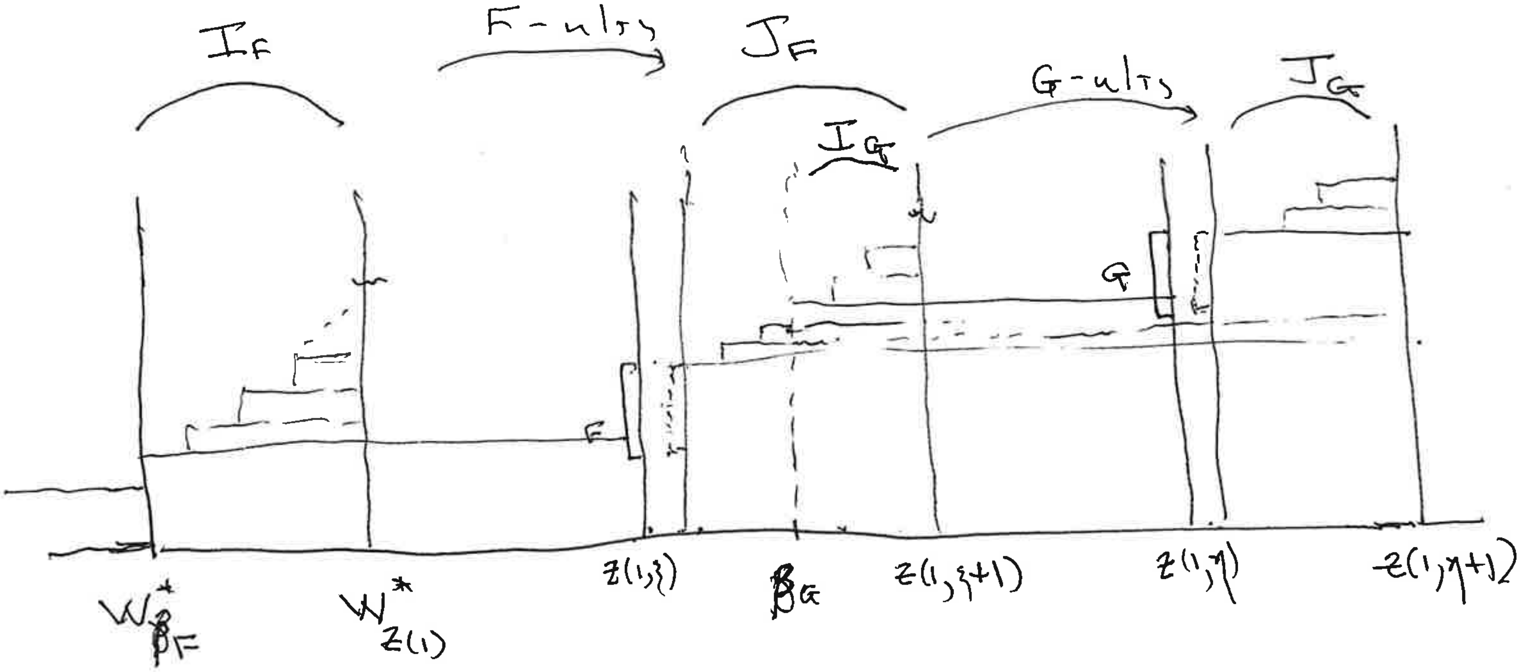
$F_{z_F(i)}$ is the image of F_i, i.e. the last extender, in $\mathrm{Ult}_0(P_i^* || lh(F_i), F)$.

For $i = \sigma_F = z(1, \mu)$,

$F_i = E_\mu^{Y_1}$. It is not inflated by F.

In general, extenders used in Y_1 do not inflate extenders used in Y_1. They inflate extenders in $\bigcup_{i \leq z(1)} \mathrm{Ext}(W_i^*)$, or the inflations earlier along the same Y_1 - branch of such extenders.

Here is a diagram for the case Y_1-pred$(\xi+1)=0$, Y_1-pred$(\eta+1)=\xi+1$, $F=E^{Y_1}_\xi$, $G=E^{Y_1}_\eta$.



The construction order $<^{\oplus 1}$ on $Z(1,\xi+1)+1$ is

$$<^{\oplus 1}\restriction Z(1,\xi+1)^{+1} = <^{\oplus 1}\restriction Z(1,\xi)^{+1} \cup \{\langle \alpha, Z_F(i)\rangle \mid i \in I_F \wedge \alpha \le^{c} i\}$$

where $F = E^{Y_1}_\xi$.

For $\alpha <^{\oplus 1} \gamma$,

$\Phi^1_{\alpha,\gamma}: W^*_\alpha \to W^*_\gamma$

$\Psi^1_{\alpha\gamma}: \mathcal{V}^*_\alpha \to \mathcal{V}_\gamma$

are the weak tree embeddings gotten by induction and composing with $W^*_i \to X(W^*_i, F)$, etc.

Letting $F = F_{z(1,\xi)}$, $\mu = Y_1\text{-pred}(\xi+1)$, and β_F, $\sigma_F = z(1,\mu)$, z_F, etc be as above, we let

$$\hat{\Phi}_F = \langle z_F, \langle \Phi_i \mid i \le \sigma_F \rangle\rangle$$

and

$$\hat{\Psi}_F = \langle z_F, \langle \Psi_i \mid i \le \sigma_F \rangle\rangle$$

where

$$\Phi_i : W_i^* \longrightarrow W_{z_F(i)} = X(W_i^*, F)$$

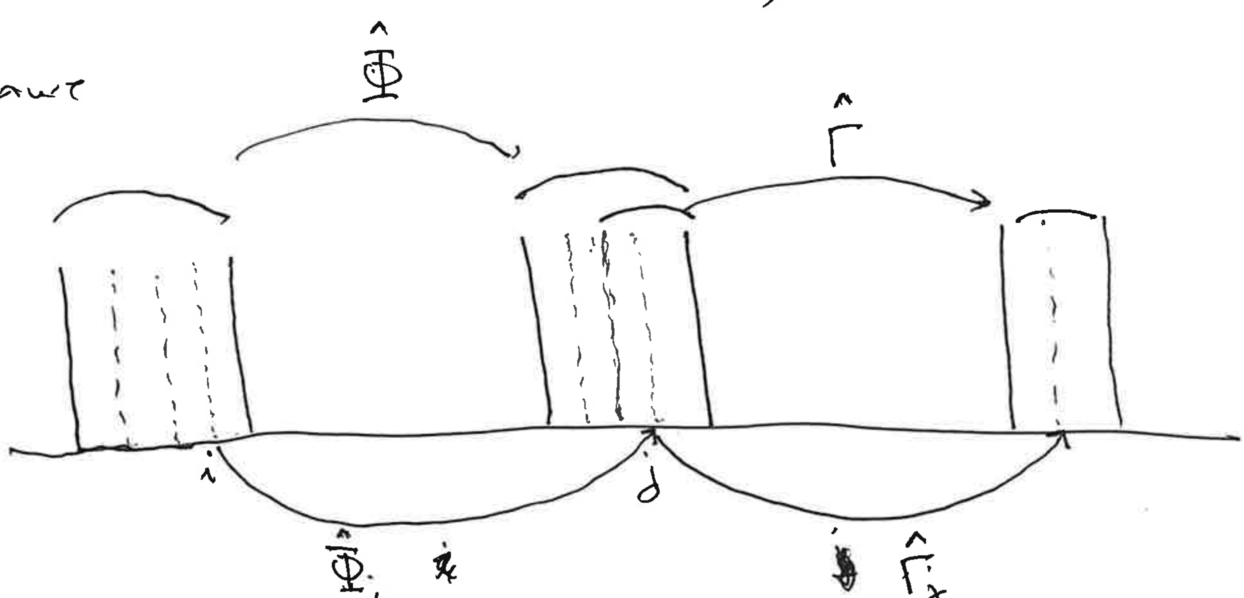
and

$$\Psi_i : \mathcal{V}_i^* \longrightarrow \mathcal{V}_{z_F(i)} = X(\mathcal{V}_i^*, F)$$

are the weak tree embedding from full normalization. (For $i < \beta_F$, set $z_F(i) = i$ and let $\Phi_i = \Psi_i =$ "id". So now $\beta_F = \text{crit}(z_F)$.) $\hat{\Phi}_F$ and $\hat{\Psi}_F$ are essentially the embeddings of meta-meta-trees that we might call $\hat{W}_{1,\mu}$ and $\hat{\mathcal{V}}_{1,\mu}$ into $\hat{W}_{1,\xi+1}$, $\hat{\mathcal{V}}_{1,\xi+1}$ that arise in the course of ~~come from~~ normalizing the meta-stacks $(\hat{W}_1, Y_1^W)$ and $(\hat{\mathcal{V}}_1, Y_1^{\mathcal{V}})$.

In the diagram on p. 31, we display all the individual trees that occur in any of the meta-trees in a sort of global agreement diagram, in which each column represents a tree W^*_η. The arrows on top taking one block of trees to another are the embeddings along branches of a "meta-meta-tree" whose nodes are the $\hat{W}_{1,\mu}$ and $\hat{W}_{1,\xi+1}$.

Picture



$\hat{\Phi}_i$ and $\hat{\Gamma}_j$ act on trees, and are part of a meta-tree structure. $\hat{\Phi}$ and $\hat{\Gamma}$ act on blocks of trees, and are part of a meta-meta-tree structure.

Before defining $\hat{W}_{1,\lambda}$ and $\hat{\mathcal{V}}_{1,\lambda}$ for λ a limit ordinal, let us introduce some notation. The notation reflects the fact that $\hat{W}_{1,\xi}$ is essentially $Y_1^W \restriction \{\xi+1\}$ regarded as a tree on $\hat{W}_1$. We define by induction on $i \le Z(2)$ such that $Z(1) \le i$

$$dw^1_i = e^{Y_1}_\tau, \text{ where } \tau \text{ is largest s.t. } z_0(1,\tau) \le i.$$

Here

$$z_0(1,\tau) = \begin{cases} z(1,\tau-1)+1 & \text{if } \tau \text{ is a succ. ord} \\ \sup_{\gamma<\tau} z(1,\gamma) & \text{if } \tau \text{ is a limit ord.} \end{cases}$$

If $F = E^{Y_1}_\tau$, then

$$J_F = [z_0(\tau+1), z(\tau+1)]$$

and for $i \in I_F$

$$dw^1_{z_F(i)} = dw^1_i \frown \langle F \rangle.$$

Similarly

$$dv^1_i = e^{Y_1}_\tau, \text{ where } \tau \text{ is largest s.t. } z_0(1,\tau) \le i.$$

So $dw'_i = dv'_i$ for $i < \mathcal{Z}(2)$. Set

$$dw'_{\mathcal{Z}(2)} = e^{\gamma_1}_{b_1},$$

$$dv'_{\mathcal{Z}(2)} = e^{\gamma_1}_{c_1}.$$

We shall have

$$W_i = \mathrm{Ult}\left(W^*_{\mathcal{Z}(1)}, dw^1_i\right)$$

Remark: for s a sequence of ascending $\mathrm{Ult}(W, s) =$ dir lim of $\mathrm{Ult}(W, s\restriction i)$'s.

and

$$\mathcal{V}_i = \mathrm{Ult}\left(\mathcal{V}^*_{\mathcal{Z}(1)}, dv^1_i\right)$$

for all $i \leq \mathcal{Z}(2)$. This is already consistent with our defn of $\hat{W}_{1,n}$ and $\hat{\mathcal{V}}_{1,n}$ for $n < \omega$.

$<^{\gamma_1}$ induces an order on intervals in $\mathcal{Z}(2)$. The intervals are equivalence classes $[\gamma]^{\sim}_1$, where

$\gamma \overset{1}{\sim} \delta$ iff $\gamma, \delta \leq \mathcal{Z}(1)$, or

$dw'_\gamma = dw'_\delta$

iff $\gamma, \delta \leq \mathcal{Z}(1)$, or

$dv'_\gamma = dv'_\delta$.

(Setting $dw'_\gamma = \emptyset$ for $\gamma \leq \mathcal{Z}(1)$, we can eliminate the first disjunct.) Notice that although $dw'_{\mathcal{Z}(2)} \neq dv'_{\mathcal{Z}(2)}$, we still have $[\mathcal{Z}(2)]^{\sim}_1 = \{\mathcal{Z}(2)\}$ is justified by both ~~$dw'_{\mathcal{Z}(2)}$~~ $dw'_{\mathcal{Z}(2)}$ and $dv'_{\mathcal{Z}(2)}$.

We then have the order

$$[i]^{*}_{\mathbb{I}} <^{1} [m]^{*}_{\mathbb{I}} \text{ iff } dw^{\mathbb{I}}_{i} \subsetneq dw^{\mathbb{I}}_{m}$$

$$\text{iff } dv^{\mathbb{I}}_{i} \subsetneq dv^{\mathbb{I}}_{m}$$

when $m < z(2)$. For $m = z(z)$, we shall leave "$[i]^{*}_{\mathbb{I}} <^{1} [m]^{*}_{\mathbb{I}}$" undefined.

Suppose $[i]_1 <^{1} [m]_1$, and let s be such that

$$dw^{1}_{m} = dw^{1}_{i} \frown s,$$

equivalently

$$dv^{1}_{m} = dv^{1}_{i} \frown s.$$

Let

$$\beta =_{df} \beta_s =_{df} \beta_{s,i,m}$$

$$= \text{least } \xi \text{ s.t. } \xi \frown i \text{ and } Ult(w^{*}_{\xi}, s) \text{ makes sense}$$

$$= \text{least } \xi \text{ s.t. } \xi \frown i \text{ and } Ult(v^{*}_{\xi}, s) \text{ makes sense}$$

Put

$$I_s = [\beta_s, \max([i]_1)]$$

If $\lambda < lh(\gamma_1)$ is a limit, then we ~~shall have~~ define

$$z(1,\lambda) = z_0(1,\lambda) + o.t.(I_s)$$

for all tails s of $dw^1_{z(\cdot,\lambda)} = e^{\gamma_1}_\lambda$.

Letting

$$z_s(k) = z_0(1,\lambda) + k.$$

for $k \in I_s$, we have a weak tree embedding

$$\Phi_{s,k} : W^*_k \longrightarrow W^*_{z_s(k)} = Ult(W^*_k, s)$$

We write

$$\hat{\Phi}_s = \langle z_s, \langle \Phi_{s,k} \mid k \in I_s \rangle \rangle$$

This defines the W^*_m and ν^*_m for $m \in [z_0(1,\lambda), z(1,\lambda)]$. ~~For $[i]_1$~~ We have

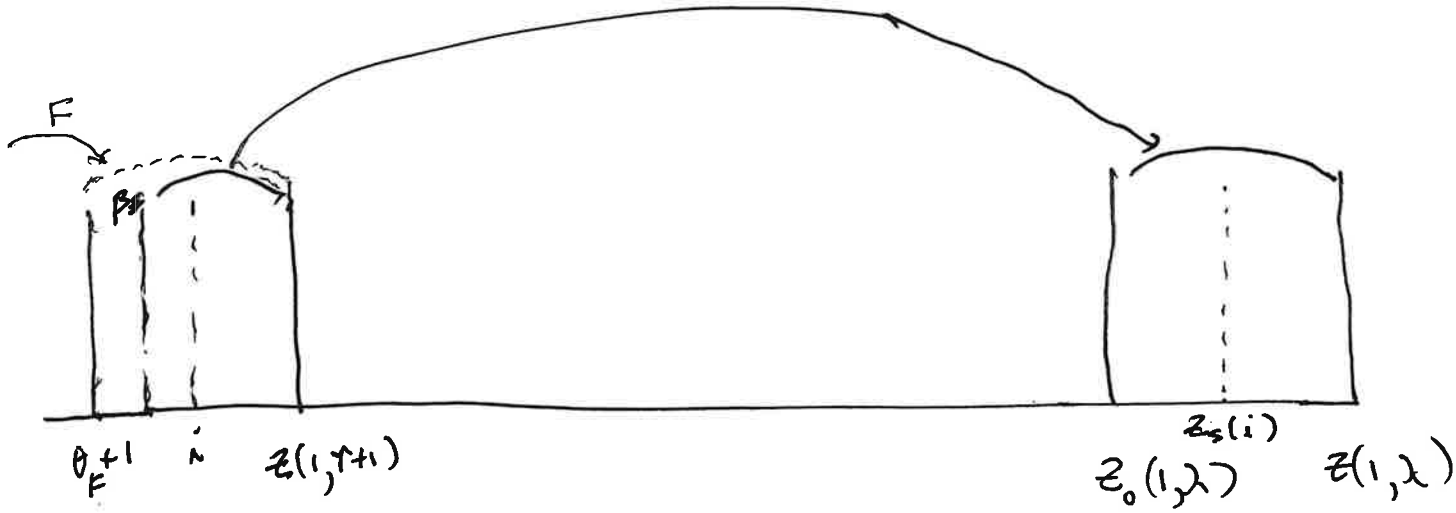
$$[m]_1 = [z_0(1,\lambda), z(1,\lambda)], \text{ for}$$

all m s.t. $z_0(1,\lambda) \le m \le z(m,\lambda)$.

For $[i]_1 < [m]_1$, we have

$$\hat{\Phi}_{[i]_1,[m]_1} = \hat{\Phi}_s, \text{ where } dw^1_i \frown s = dw^1_m$$

A picture is $\hat{\Phi}_s$



$F = E_\tau^{Y_1}$, $I_F = [\theta_F+1, z(1,\tau+1)]$

$d w_i^1 = r \wedge \langle F \rangle$

$d w^1_{z_s(i)} = d w_i^1 \wedge s = r \wedge \langle F \rangle \wedge s$, $I_s = [\beta_s, z(1,\tau+1)]$

$$\left(\hat{\Phi}_s\right)_i : W_i^* \longrightarrow \mathrm{Olt}(W_i^*, s) = W^*_{z_s(i)} .$$

This defines $\hat{\mathcal{W}}_2$ and $\hat{\mathcal{V}}_2$. Let

$$\mathcal{E}' = \{ [\alpha]_1 \mid \alpha \le z(2) \},$$

so that $<'$ is a tree order on $\mathcal{E}' - \{[z(2)]\}$.

For $a <' b$, we have $\hat{\Phi}_{a,b}$ and $\hat{\Psi}_{a,b}$. This all reflects the structure of $\hat{\mathcal{W}}_2$ and $\hat{\mathcal{V}}_2$ as meta-trees on $\hat{\mathcal{W}}_1$ and $\hat{\mathcal{V}}_1$ induced by the $X(\hat{\mathcal{W}}_1, \mathcal{Y}_1^{w})$ and $X(\hat{\mathcal{V}}_1, \mathcal{Y}_1^{v})$ constructions.

We also want to consider $\hat{\mathcal{W}}_2$ and $\hat{\mathcal{V}}_2$ as meta-trees on $\mathcal{W}_0^*$ and $\mathcal{V}_0^*$. The branch extenders here we shall call dw_γ^0 and dv_γ^0.

~~For~~ We set

$$dw_\gamma^0 = dv_\gamma^0 = e_\gamma^{\mathcal{Y}_0}, \quad \text{for } \gamma < z(1),$$

and

$$dw_{z(1)}^0 = e_{b_0}^{\mathcal{Y}_0}$$

$$dv_{z(1)}^0 = e_{c_0}^{\mathcal{Y}_0}.$$

Between $z(1)$ and $z(2)$ we proceed by induction. Suppose we have defined dw^0_γ and dv^0_γ for $\gamma \le z(1,\xi)$, and let

$$F = E^{Y_1}_\xi .$$

Let

$$\mu = Y_1\text{-pred}(\xi+1)$$

so that $\sigma_F = z(1,\mu)$. We shall define $dw^0_{z_F(i)}$ and $dv^0_{z_F(i)}$ for $i \le \sigma_F$. Recalling that $\beta_F =_{df} \text{crit}(z_F)$, we need only consider $i \in [\beta_F, \sigma_F]$. So fix such an i, and let

no hats

$$p_F = p\text{-map of } (\hat{\Phi}_F)_i$$

$$q_F = p\text{-map of } (\hat{\Psi}_F)_i .$$

These are F-ultrapower maps, and they agree on $\text{Ext}(W_i^*) \cap \text{Ext}(V_i^*)$. By induction, we have

$$\text{ran}(dw^0_i) \subseteq \text{Ext}(W_i) \cap \text{Ext}(V_i)$$

and

$$\text{ran}(dv^0_i) \subseteq \text{Ext}(W_i) \cap \text{Ext}(V_i).$$

<u>Remark</u> If e.g. $i = z(i)$, then dw_i^0 and dv_i^0 have distinct ranges. The ranges are not A-branches, but they do consist of A-extenders.

Let k_0 be least s.t. $\kappa_F < \lambda(dw_i^0(k_0))$, and $k_0 = \mathrm{dom}(dw_i^0)$ if no such k exists.

Then

$$dw^0_{z_F(i)} = \begin{cases} dw_i^0 \restriction k_0 {}^\frown \langle F \rangle {}^\frown \langle \dot{\rho}_F(dw_i^0(k)) \mid k_0 \le k \rangle & \text{if } \kappa_F \le \mathrm{crit}(dw_i^0(k_0)) \\ & \text{or } k_0 \notin \mathrm{dom}(dw_i^0) \\ dw_i \restriction k_0 {}^\frown \langle \dot{\rho}_F(dw_i^0(k)) \mid k_0 \le k \rangle & \text{if } \mathrm{crit}(dw_i^0(k_0)) < \kappa_F . \end{cases}$$

Similarly we define $dv^0_{z_F(i)}$. It's easy to see that

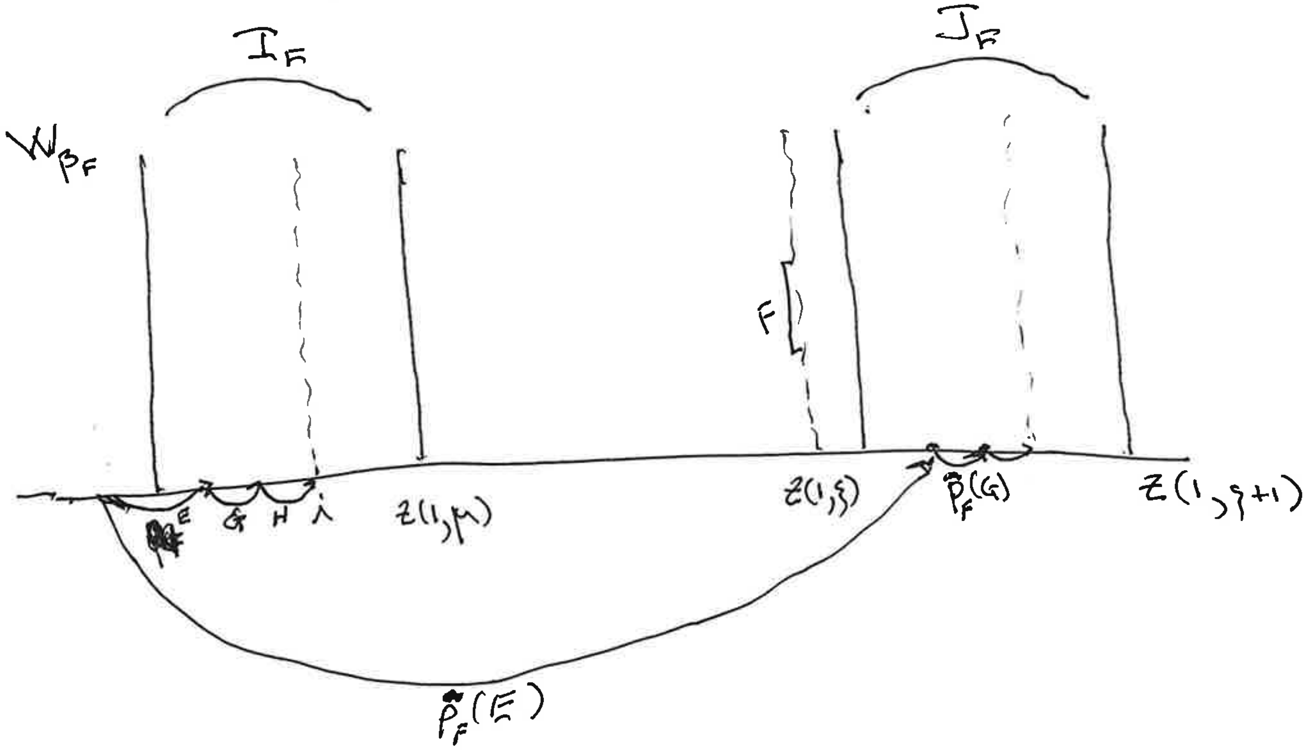
$$dw_i^0 = dw_j^0 \Longrightarrow i = j$$

and

$$dv_i^0 = dv_j^0 \Longrightarrow i = j .$$

So the analog of $\sim_1$ is trivial; i.e. $[i]_0 = \{i\}$.

Here is a possible picture:



Here $E = dw_i^\circ(k_0)$ and $K_E < K_F$.

$dw_i^\circ = s \frown \langle E, G, H\rangle$ for some s, and

$dw^\circ_{z_F(i)} = s \frown \langle \hat{p}_F(E), \hat{p}_F(G), \hat{p}_F(H)\rangle$.

For $\lambda < lh(\mathcal{T}_1)$ a limit ordinal

and

$i \in [z_0(1,\lambda), z(1,\lambda)]$

we define dw_i° and dv_i° by taking

a direct limit. Namely, let

$e = [0, \lambda)_{\gamma_1}$ and

$$i = z_e(k)$$

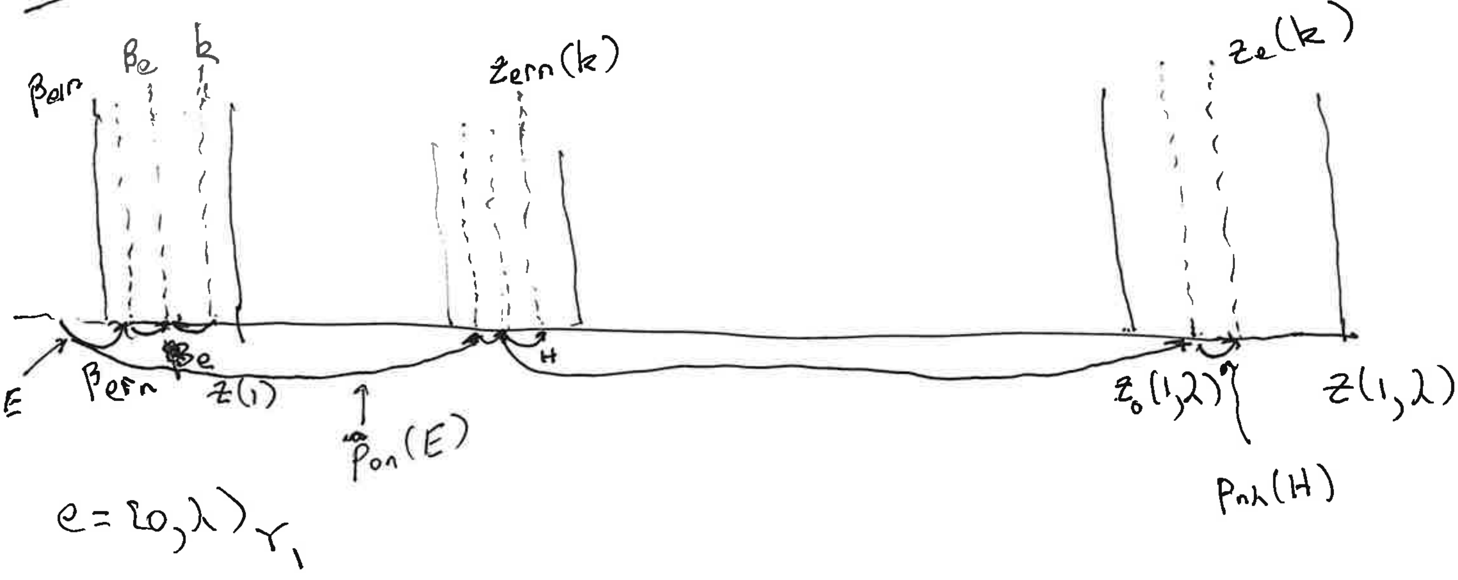
where $\beta_e \le k$. Then

$$dw_i^{\circ} = \lim_{n \le \mathrm{dom}(e)} dw^{\circ}_{z_{e\restriction n}(k)},$$

where the limit is under the $\hat{p}_{n,m}$-maps of the weak tree embeddings from $W^*_{z_{e\restriction n}(k)} \longrightarrow W_{z(e\restriction m)(k)}$ given by the $\hat{\vec{\Phi}}'$-system. Similarly

$$dv_i^{\circ} = \lim_{n \le \mathrm{dom}(e)} dv^{\circ}_{z_{e\restriction n}(k)}.$$

Messy picture:



Similarly, $dw^0_{z(2)}$ and $dv^0_{z(2)}$ are gotten by taking limits along b_1 and c_1:

$$dw^0_{z(2)} = \lim_{i \in b_1} dw^0_i \restriction \text{extenders of } lh < \delta(\gamma_1')$$

$$dv^0_{z(2)} = \lim_{i \in c_1} dv^0_i \restriction \text{extenders of } lh < \delta(\gamma_1).$$

It is easy to show

<u>Lemma 5.1</u>

(1) For all $\xi \le z(2)$,

$$\text{ran}(dw^0_\xi) \subseteq Ext(W_\xi) \cap Ext(\mathcal{V}_\xi)$$

and

$$\text{ran}(dv^0_\xi) \subseteq Ext(W_\xi) \cap Ext(\mathcal{V}_\xi).$$

(2) If $i < z(2)$ and $i \ne z(1,\xi)$ for all ξ, then $dw^0_i \equiv_{Tail} dv^0_i$.

"Pf" of (2): If $dw^0_i \not\equiv_{Tail} dv^0_i$, then we must have gotten dw^0_i as $(e^{\gamma_1}_\xi)(e^{\gamma_0}_{b_0})$, where $e^{\gamma_1}_\xi$ is shuffled cofinally into $e^{\gamma_0}_{b_0}$. So ξ is a limit and $i = z(1,\xi)$.

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The meta-tree whose branch extenders are the dw_i^0 can be defined by setting

$$np_i^{\hat{w}} = \text{closure of } \{\theta_F + 1 \mid F \in \operatorname{ran}(dw_i^0)\} \text{ in the order topology on ORD}$$

and

$$k <^0_{\hat{w}} i \text{ iff } k \in np_i^{\hat{w}} \wedge k \neq i .$$

If $k <^0_{\hat{w}} i$ then there is a unique non-empty s such that $dw_i^0 = dw_k^0 {}^\frown s$, and the natural branch embedding of our meta-tree from W_k^* to W_i is the map from W_k^* to $Ult(W_k^*, s) = W_i$.

Let us put for $k, i < z(2)$

$$k <^0 i \text{ iff } k <^0_{\hat{w}} i \text{ and } k <^0_{\hat{v}} i$$

$$\text{iff } \exists s\, (s \neq \emptyset \text{ and } dw_k^0 {}^\frown s = dv_k^0 {}^\frown s = dw_i^0 = dv_i^0)$$

__Remark__ The rhs of the second "iff" does not ~~[illegible]~~ imply $dw_k^0 = dv_k^0$, the (only) counterexamples being $k = z(1)$, and possibly $k = z(1, \xi)$ (if $e_\xi^{\gamma_1}$ has been inserted cofinally in ~~[illegible]~~ the images of dw_k^0 and dv_k^0)

Concerning 5.1 (2), it is not hard to show that $dv^{\circ}_{E(2)} \not\equiv_{\text{Tail}} dv^{\circ}_{E(2)}$. We prove something more general now.

<u>Def 5.2</u> Let E and F be extenders; then

(a) E fits into F iff $\kappa_F < \kappa_E < \lambda_E < \lambda_F$

(b) E is inconsistent with F iff

(i) ~~$lh(E) < lh(F)$ and $\kappa_E \leq \kappa_F$~~ $\kappa_E \leq \kappa_F < \lambda_E \leq \lambda_F$, or

(ii) ~~$lh(F) < lh(E)$ and~~ $\kappa_F \leq \kappa_E \leq \lambda_F \leq \lambda_E$.

Notice that if $lh(E) \neq lh(F)$, then E is inconsistent with F iff neither fits into the other and $|[\kappa_E, \lambda_E] \cap [\kappa_F, \lambda_F]| \geq 1$. Also, if E is inconsistent with F, then E and F cannot both be used along the same branch of a normal tree. The same is true if E fits into F. <u>Def</u> E and F overlap iff $|[\kappa_E, \lambda_E] \cap [\kappa_F, \lambda_F]| \geq 1$.

<u>Lemma 5.3</u> Let $\mathcal{U} = X(\mathcal{T}, F)$, and let $p_F : \mathrm{Ext}(\mathcal{T}) \to \mathrm{Ext}(\mathcal{U})$ be the p-map of the associated weak tree embedding. Let $G, H \in \mathrm{Ext}(\mathcal{T})$; then

(1) F is consistent with $p_F(G)$,

(2) F fits into $p_F(G)$ iff $\kappa_G < \kappa_F < \lambda_G$,

(3) G is consistent with H iff $p_F(G)$ is consistent with $p_F(H)$, and

(4) G fits into H iff $p_F(G)$ fits into $p_F(H)$.

<u>Proof</u> (1) and (2) are immediate. For (3) and (4), suppose $lh(G) \le lh(H)$. Let $i_1 : M_\infty^{\mathcal{T}} | lh(H) \to Ult_0(M_\infty^{\mathcal{T}} | lh(H), F)$ and $i_0 : M_\infty^{\mathcal{T}} | lh(G) \to Ult_0(M_\infty^{\mathcal{T}} | lh(G), F)$ be the two ultrapower maps, so that $p_F(H) = i_1(H)$ and $p_F(G) = i_0(G)$. Since $lh(G)$ is a regular cardinal in $M_\infty^{\mathcal{T}} | lh(H)$, $i_0 = i_1 \restriction M_\infty^{\mathcal{T}} \| lh(G)$. This easily implies (3) and (4). $\boxtimes$

<u>Def 5.4</u> Let s and t be sequences of extenders; then

(1) <u>s fits into t</u> iff $\forall i \in \mathrm{dom}(s)\, \exists j \in \mathrm{dom}(t)$ s.t. $s(i)$ fits into $t(j)$.

(2) For $k \le \mathrm{dom}(s)$, $s^{\ge k}(i) = s(k+i)$ is the tail of s starting at k.

(3) <u>A tail of s fits into t</u> iff $\exists k < \mathrm{dom}(s)$ ($s^{\ge k}$ fits into t)

(4) s fits cofinally into t iff s fits into t, and $\forall j \in \mathrm{dom}(t)\, \exists i \in \mathrm{dom}(s)\, \exists k \in \mathrm{dom}(t)$ $(j \le k \wedge s(i)$ fits into $t(j))$.

(5) A tail of s fits cofinally in t iff $\exists k \in \mathrm{dom}(s)$ ($s^{\geq k}$ fits cofinally into t).

Mostly we care about sequences of extenders that are the branch sequence of some iteration tree, meta-tree, or meta-meta tree. These sequences are (length) increasing and non-overlapping.

Def 5.5 Let s be a sequence of extenders; then

(i) s is increasing iff $i < j \in \mathrm{dom}(s) \Rightarrow \lambda_{s(i)} < \lambda_{s(j)}$,

(ii) s is non-overlapping iff $i < j \in \mathrm{dom}(s) \Rightarrow$
$$\lambda_{s(i)} \le \kappa_{s(j)}.$$

Lemma ~~Defn~~ 5.6 Let (P, Σ) be a mouse pair, $\mathcal{T}$ an iteration tree on (P, Σ), and s an increasing, non-overlapping sequence of extenders such that $X(\mathcal{T}, s)$ makes sense. (Cf. §4.4.) Let $\mathcal{U} = X(\mathcal{T}, s)$, and
$$\eta = \sup\{\xi + 1 \mid E^{\mathcal{U}}_{\xi} \in \mathrm{ran}(s)\};$$
then s fits cofinally into $e^{\mathcal{U}}_{\eta}$, and s fits into $e^{\mathcal{U}}_{\gamma}$ for all $\gamma \ge \eta$.

Proof Let $\mathcal{U}_i = X(\mathcal{T}, s \restriction i)$ and

$\eta_i = \sup\{\xi + 1 \mid E_\xi^{\mathcal{U}_i} = s(k)$ for some $k < i\}$.

We show by induction on i that $s \restriction i$ fits into $e_\gamma^{\mathcal{U}_i}$ for all $\gamma \geq \eta_i$, and fits cofinally into $e_{\eta_i}^{\mathcal{U}_i}$.

Let $i = 1$, and $F = s(0)$, so $\mathcal{U}_1 = X(\mathcal{T}, F)$.

$F = E_{\eta_1 - 1}^{\mathcal{U}_1}$, where $\alpha_F = \eta_1 - 1$. If $\gamma \geq \eta_1$, then $\gamma = u_F(\bar{\gamma})$ for some $\bar{\gamma} \geq \beta_F$ (where $\Phi_F : \mathcal{T} \to \mathcal{U}_1$ is the weak tree emb and u_F is its u-map).

So $e_\gamma^{\mathcal{U}_1} = e_{\bar{\gamma}}^{\mathcal{T}} \restriction k_0 \frown \langle F \rangle \frown \langle p_F(e_{\bar{\gamma}}^{\mathcal{T}}(k)) \mid k \geq k_0 \rangle$

(in which case F fits into F, so $s \restriction 1$ fits into $e_\gamma^{\mathcal{U}_1}$),

or $e_\gamma^{\mathcal{U}_1} = e_{\bar{\gamma}}^{\mathcal{T}} \restriction k_0 \frown \langle p_F(e_{\bar{\gamma}}^{\mathcal{T}}(k)) \mid k \geq k_0 \rangle$. In the latter case $\mathrm{crit}(e_{\bar{\gamma}}^{\mathcal{T}}(k_0)) < \kappa_F < \lambda(e_{\bar{\gamma}}^{\mathcal{T}}(k_0))$, so F fits into $p_F(e_{\bar{\gamma}}^{\mathcal{T}}(k_0))$, so again $s \restriction 1$ fits into $e_\gamma^{\mathcal{U}_1}$.

Moreover, F is the last extender in $e_{\eta_1}^{\mathcal{U}_1}$, so $s \restriction 1$ fits cofinally there.

Now let $i = \delta + 1$, $F = s(\delta)$, and

$$\mathcal{U}_\delta = X(\mathcal{I}, s \restriction \delta), \quad \mathcal{U}_{\delta+1} = X(\mathcal{U}_\delta, F).$$

So

$$\eta = \eta_{\delta+1} = \alpha_F + 1 = \text{index of } F \text{ in } \mathcal{U}_{\delta+1}.$$

Let

$$\beta = \beta_F = \beta(\mathcal{U}_\delta, F),$$

and $\Phi_F : \mathcal{U}_\delta \to \mathcal{U}_{\delta+1}$ be the weak tree embedding, with u-map u_F and p-map p_F. For $\gamma \geq \eta$, we have $\bar{\gamma} \geq \beta$ s.t.

$$\gamma = u_F(\bar{\gamma}).$$

Since s is increasing and non-overlapping, $\lambda_{s(i)} \leq \kappa_F$ for all $i < \delta$, so $\eta_\delta \leq \beta$. Thus $s \restriction \delta$ fits into $e_{\bar{\gamma}}^{\mathcal{U}_\delta}$, moreover p_F is the identity on ran$(s \restriction \delta)$. By 5.3(4), $s \restriction \delta$ fits into $e_\gamma^{\mathcal{U}_{\delta+1}}$. If there is $G \in \text{ran}(e_{\bar{\gamma}}^{\mathcal{U}_\delta})$ s.t. $\kappa_G < \kappa_F < \lambda_G$, then F fits into $p_F(G) \in \text{ran}(e_\gamma^{\mathcal{U}_{\delta+1}})$ by 5.3. Otherwise $F \in \text{ran}(e_\gamma^{\mathcal{U}_{\delta+1}})$. In both cases, $s \restriction \delta + 1$ fits into $e_\gamma^{\mathcal{U}_{\delta+1}}$. If $\gamma = \eta_{\delta+1}$, then $F = E_{\gamma-1}^{\mathcal{U}_{\delta+1}}$, so the fit is cofinal.

Now let λ be a limit ordinal and suppose 5.6 holds for $\mathcal{U}_\delta$, η_δ, and $S \upharpoonright \delta$ for all $\delta < \lambda$. We have

$$\mathcal{U}_\lambda = \lim_{\delta < \lambda} \mathcal{U}_\delta$$

where the limit is under the weak tree embeddings

$$\Phi_{\delta,\nu} : \mathcal{U}_\delta \to \mathcal{U}_\nu$$

with associated p-maps $p_{\delta,\nu}$ and u-maps $u_{\delta,\nu}$.
Let $\delta < \lambda$ and τ be s.t.

$$\eta_\lambda = u_{\delta,\lambda}(\tau).$$

We shall show that $S \upharpoonright \lambda$ fits cofinally into and non-overlapping, $e_{\eta_\lambda}^{\mathcal{U}_\lambda}$. Because S is increasing we have

$$\eta_\delta \leq \mathrm{crit}(u_{\delta,\lambda}) \leq \tau .$$

There are then two cases

<u>Case 1</u> For all γ s.t. $\delta < \gamma < \lambda$, $\mathrm{crit}(u_{\gamma,\lambda}) < u_{\delta\gamma}(\tau)$.

In this case $\hat{p}_{\delta,\lambda}(e_\tau^{\mathcal{U}_\delta})$ is gotten by successively fitting the extenders in ran $((S \upharpoonright \lambda)^{\geq \delta})$ into the $\hat{p}_{\delta,\nu}(e_\tau^{\mathcal{U}_\delta})$, either as themselves or inside some $p_{\delta,\nu}(E)$. The current $S(\nu)$ is over the last extender of $\hat{P}_{\delta,\nu+1}(e_\tau^{\mathcal{U}_\delta})$.

Case 2 $\exists \gamma > \delta$ s.t. $\mathcal{U}_{\delta\gamma}(\tau) = \operatorname{crit}(\mathcal{U}_{\delta,\lambda})$.

In this case, the extenders in $(S\upharpoonright\lambda)^{\geq\gamma}$ are inserted into images of $e_\gamma^{\mathcal{U}_\delta}$ at the end. That is, for $\nu \in (\gamma, \lambda]$,

$$e_{\mathcal{U}_{\delta,\nu}(\tau)}^{\mathcal{U}_\nu} = e_{\mathcal{U}_{\delta,\gamma}(\tau)}^{\mathcal{U}_\gamma} \frown (S\upharpoonright\nu)^{\geq\gamma}.$$

We leave the further checking here to the reader. See [2] and [3]. That $S\upharpoonright\lambda$ fits into $e_\gamma^{\mathcal{U}_\lambda}$ for all $\gamma \geq \eta_\lambda$ is similar, but case 2 ~~still~~ doesn't arise when $\gamma > \eta_\lambda$.

Lemma 5.6 $\boxtimes$

Remark In the construction of $X(\mathcal{I}, s)$ we do not allow a step from $X(\mathcal{I}, s\upharpoonright i)$ to a normal extension $X(\mathcal{I}, s\upharpoonright i)^*$, as we did in the construction of $\hat{W}_2$. If we allow such steps, 5.6 can fail. See remark 5.9.

We now use 5.6 to relate dw_i^0 to W_i^{ext} and dv_i^0 to $\mathcal{V}_i^{ext}$.

<u>Lemma 5.7</u> Let $i \le z(2)$, and $\eta = \sup\{\xi+1 \mid E_\xi^{W_i} \in \operatorname{ran}(dw_i^0)\}$; then

(1) a tail of dw_i^0 fits cofinally into $e_\eta^{W_i}$, and into $e_\gamma^{W_i}$ for all $\gamma \ge \eta$,

(2) $\eta = \sup\{\xi+1 \mid E_\xi^{\mathcal{V}_i} \in \operatorname{ran}(dv_i^0)\}$,

(3) a tail of dv_i^0 fits cofinally into $e_\eta^{\mathcal{V}_i}$, and into $e_\gamma^{\mathcal{V}_i}$ for all $\gamma \ge \eta$, and

(4) if $i = z(2)$, then $\eta+1 = lh(\mathcal{W}_{z(2)}) = lh(\mathcal{V}_{z(2)})$, so

(i) a tail of $dw_{z(2)}^0$ fits cofinally in $e_\eta^{W_{z(2)}}$

and (ii) a tail of $dv_{z(2)}^0$ fits cofinally in $e_\eta^{\mathcal{V}_{z(2)}}$.

<u>Remark 5.8</u> Let i, η be as in 5.7. We do not get that $dw_i^0 \equiv_{tail} e_\eta^{W_i}$. The reason

is that the extender in ran$(e_\eta^{W_i})$ that $dw_i^0(k)$ fits into may be a D-extender. In fact, ran$(e_\eta^{W_i})$ may consist entirely of D-extenders. Each $dw_i^0(k)$ is used in W_i, but the $dw_i^0(k)$'s could be used on different branches of W_i.

D-extenders are the only obstacle here. If $dw_i^0(k)$ fits into E, $E \in Ext(W_i)$, and E is an A-extender, then $dw_i^0(k) = E$.

<u>Remark 5.9</u> In 5.4, we don't necessarily get that all of dw_i^0 fits into $e_\eta^{W_i}$. For example, suppose $F = E_0^{Y_1}$ and $\beta_F = z(1)$. Let G be the first extender used in $W^*_{z(1)}$ after $W_{z(1)}$, i.e. $G = E_\gamma^{W^*_{z(1)}}$ where $\gamma = lh(W_{z(1)}) - 1$. Suppose $\kappa_G < \delta(Y_1) < lh(G) < \kappa_F$, and suppose $\kappa_F < \lambda_H$ for all H used in $W^*_{z(1)}$ after G.

Let
$$i = \theta_F + 1 = z(0,1) = z(1) + 1$$
$$= z_F(z(1)).$$

Then
$$dw_i^0 = dw_{z(1)}^0 {}^\frown \langle F \rangle.$$

Letting $F = E^{W_i}_{\eta-1}$ and $\mu = W^*_{z(1)}\text{-pred}(\gamma+1)$ be where we apply G in $W^*_{z(1)}$, we get

$$e^{W_i}_\eta = e^{W_{z(1)}}_\mu {}^\frown \langle G \rangle {}^\frown \langle F \rangle .$$

So dw_i^0 does not fit into $e^{W_i}_\eta$; again, the obstacle was a D-extender. The tail $\langle F \rangle$ of dw_i^0 does fit into $e^{W_i}_\eta$.

Proof of 5.7

Suppose first that $i \le z(1)$ and $i > 0$. Then we have for $i < z(1)$

$$W_i = X(W_0^*, e_i^{\gamma_0^W})$$

and

$$dw_i^0 = e_i^{\gamma_0^W}$$

so we can apply 5.6 with $s = dw_i^0$ and $\mathcal{T} = W_0^*$ and we get (1). The same proof gives (2) and (3), and (4) follows. For $i = z(1)$, the same proof works with $dw^0_{z(1)} = e^{\gamma_0^W}_{b_0}$ and $dv_{z(1)} = e^{\gamma_0^2}_{c_0}$.

Suppose next $z(1) < i < z(2)$. If $z(1) \not\leq^0 i$, i.e. $dw^0_{z(1)} \ntrianglelefteq dw_i^0$, then again

$$W_i = X(W_0^*, dw_i^\circ),$$

and we can apply 5.6 with $J = W_0^*$ and $s = dw_i^\circ$.

If $z(1) <^\circ i$, i.e. $dw_{z(1)}^\circ \lhd dw_i^\circ$, then 5.6 doesn't apply to W_0^* and dw_i°, because the path to W_i involved more than inserting dw_i°; we also took the step from $W_{z(1)}$ to $W_{z(1)}^*$. But letting

$$dw_{z(1)}^\circ = dw_i^\circ \restriction k$$

and

$$s = (dw_i^\circ)^{\geq k}$$

we have

$$W_i = X(W_{z(1)}^*, s).$$

Thus we can use 5.6 to show that s fits into W_i^{ext} the way we claim it does.

The same proof works on the v-side, and works on both sides when $i = z(2)$.

$\boxtimes$

So dw_i^0 fits into $e_\gamma^{w_i}$ for all suff large γ. In turn, we have

<u>Lemma 5.10</u> (1) For $\tau < lh(Y_1^-)$ and $i \in [z_0(1,\tau), z(1,\tau)]$, $e_\tau^{Y_1}$ fits into dw_i^0 and into dv_i^0.

(2) $e_{b_1}^{Y_1}$ fits into $dw_{z(2)}^0$.

(3) $e_{c_1}^{Y_1}$ fits into $dv_{z(2)}^0$.

<u>Proof</u> An induction on τ, like the proof of 5.6. ☒

Thus $(Y_1{}^\frown b_1)^{ext}$ is injected nicely into $\{dw_i^0 \mid i \leq z(2)\}$, and similarly on the v-side. In particular, the zipper pattern for $(e_{b_1}^{Y_1}, e_{c_1}^{Y_1})$ implies the zipper pattern for $(dw_{z(2)}^0, dv_{z(2)}^0)$, so $dw_{z(2)}^0$ and $dv_{z(2)}^0$ have disjoint tails.

§6. $\hat{W}_{\alpha+1}$ and $\hat{\mathcal{V}}_{\alpha+1}$ for α finite

The general successor step is similar to the step from $(\hat{W}_1, \hat{\mathcal{V}}_1)$ to $(\hat{W}_2, \hat{\mathcal{V}}_2)$. It involves keeping track of what happened at earlier limit steps, and we are deferring the limit step for now. So officially now we are assuming $(\hat{W}_\alpha, \hat{\mathcal{V}}_\alpha)$ is given, and $\alpha < \omega$. The objects associated to $(\hat{W}_\alpha, \hat{\mathcal{V}}_\alpha)$ include

(1) $\langle (W_\xi, \mathcal{V}_\xi) \mid \xi \le z(\alpha)\rangle$ and $\langle (W^*_\xi, \mathcal{V}^*_\xi) \mid \xi < z(\alpha)\rangle$ satisfying $(\dagger)_\alpha$.

(2) sequences of ~~extender~~ sequences $\langle dW_\xi^\gamma \mid \gamma \le \alpha \wedge \xi \le z(\alpha)\rangle$ and $\langle d\mathcal{V}_\xi^\gamma \mid \gamma \le \alpha \wedge \xi \le z(\alpha)\rangle$.

(3) meta-meta-tree embedding systems $\hat{\Phi}^\gamma_s$ defined for s a tail of some dW_ξ^γ, and $\hat{\Psi}^\gamma_s$ defined for s a tail of some $d\mathcal{V}_\xi^\gamma$.

We have already defined these objects for $\alpha \leq 2$. As in that case, dw_ξ^τ will be a branch extender (cut into its "missing" whole initial segments as usual) of a system we shall call $\hat{W}_\alpha^\tau$, which is just $\hat{W}_\alpha$ regarded as a normal meta-mite-tree on $\hat{W}_\tau$. The $\hat{\Phi}_s^\tau$ are just the branch embeddings of $\hat{W}_\alpha^\tau$. Similarly for dv_ξ^τ, $\hat{\mathcal{V}}_\alpha^\tau$, and the $\hat{\varphi}_s^\tau$. We shall have

$(\dagger)_\alpha$ $\quad \tau < \beta \leq \delta \leq \alpha \Rightarrow (\hat{W}_\delta^\tau$ extends $\hat{W}_\beta^\tau$ and $\hat{\mathcal{V}}_\delta^\tau$ extends $\hat{\mathcal{V}}_\beta^\tau)$.

So the dw_ξ^τ, $\hat{\Phi}_s^\tau$, dv_ξ^τ, and $\hat{\varphi}_s^\tau$ are indeed independent of α. Very heuristically, we might write

$$\hat{W}_\alpha = X(\hat{W}_\tau, \hat{W}_\alpha^\tau)$$

and

$$\hat{\mathcal{V}}_\alpha = X(\hat{\mathcal{V}}_\tau, \hat{\mathcal{V}}_\alpha^\tau),$$

for all $\tau < \alpha$.