The JSZ anomaly in strategy mouse comparison

J. Steel

Sept. 2025

The proofs of solidity, universality, and condensation for strategy mice must deal with an anomalous case first identified by Jensen in the pure extender mouse case. Schindler and Zeman found the arguments that work for this "JSZ anomaly" in the pure extender mouse case. [CPMP] simply asserts that the Schindler-Zeman argument works in the strategy mouse case. (See the "Simplifying assumption" on p. 423, and the discussion afterward.) In fact, ~~the~~ there is some additional work to do; the adaptation is not completely straightforward. We describe it here.

In outline, the JSZ anomalous case is the following. We are comparing a phalanx (M, H, α) with M. We have $\pi : H \to M$ with $\mathrm{crit}(\pi) \geq \alpha$. In the JSZ case we have

(i) $\mathrm{crit}(H) = \alpha$,

(ii) $E^M_\alpha \neq \emptyset$.

Letting $\mathcal{S}$ on (M, H, α) and $\mathcal{U}$ on M be the coiteration trees, the problem arises when we get $E^{\mathcal{S}}_\eta = L$ where $\mathrm{crit}(L) = \lambda_E$. (This might happen with $M^{\mathcal{S}}_\eta = H$, for example.) The phalanx rules require

$$M^{\mathcal{S}}_{\eta+1} = \mathrm{Ult}(M|\alpha, L),$$

but then $M^{\mathcal{S}}_{\eta+1}$ is not a premouse, since $E^M_\alpha = E$ is a proper initial segment of $\dot{F}^{M^{\mathcal{S}}_{\eta+1}} = i_L(E) =$ extender of $L \circ E$, and $E \notin M^{\mathcal{S}}_{\eta+1}$.

The SZ solution to this problem is to just continue comparing anyway. We'll get $E^{\mathcal{S}}_{\eta+1} = i_L(E)$ $= L \circ E$ applied to M to form $M^{\mathcal{S}}_{\eta+2}$. The failure of ISC for it doesn't bother us, because if $L \circ E$ is ~~used~~ an initial segment of the main branch on the $\mathcal{U}$ side, where ISC holds everywhere, then the main branch of $\mathcal{U}$ uses first E and then, at its next extender, L. Thus

L is used in both $\mathcal{T}$ and $\mathcal{U}$. That cannot happen in a comparison by least extender disagreement — no extender is used on both sides.

This last step is what causes trouble in the strategy mouse case. We compare by iterating both sides into $(M_{\nu,l}, \Sigma_{\nu,l})^{\mathbb{C}}$, for some background construction $\mathbb{C}$ and level $\langle \nu, l \rangle$ of $\mathbb{C}$. There will be many stages at which the current last models have the same extender L as their least disagreement with $M_{\nu,l}$.

Our solution is simply to avoid the SZ move in this case. That is, adopting the notation above, with $L = E^{\mathcal{T}}_{\eta}$ and $\mathrm{crit}(L) = \lambda_E$, if L is used in $\mathcal{U}$, then we shall set $M^{\mathcal{T}}_{\eta+1} = \mathrm{Ult}(H, L)$, and if L is not used in $\mathcal{U}$, we shall follow SZ and set $M^{\mathcal{T}}_{\eta+1} = \mathrm{Ult}(M|\alpha, L)$ and $M^{\mathcal{T}}_{\eta+2} = \mathrm{Ult}(M, i_L(E))$.

More generally, we might have reached $(M_\xi^\Delta, M_{\xi+1}^\Delta, \alpha_\xi)$ with ξ unstable and have $\xi+1 \leq_\Delta \eta$, with $s = e_{\xi+1,\eta}^\Delta$ = sequence of extenders used in $[\xi+1,\eta)_\Delta$ being such that $Ult(M_\xi^\Delta | \alpha_\xi, s)$ makes sense. If s occurs as a branch extender in $\mathcal{U}$, then we simply continue from $M_\eta^\Delta = Ult(M_{\xi+1}^\Delta, s)$. If it does not occur, we make the SZ move, and set E_η^Δ = last extender of $Ult(M_\xi^\Delta | \alpha_\xi, s) = i_s(E_{\alpha_\xi}^{M_\xi^\Delta})$. This guarantees that E_η^Δ cannot be an initial segment of a branch extender in $\mathcal{U}$, which is what we want in this case. [Let $E = E_{\alpha_\xi}^{M_\xi^\Delta}$. If $i_s(E)$ is the extender of $[\tau, \gamma)_\mathcal{U}$, then $e_{\tau,\gamma}^\mathcal{U} = \langle E \rangle ^\frown s ^\frown t$ because E and s are identified by least missing whole initial segments. (ISC holds in $\mathcal{U}$!). So in fact s did occur as some $e_{\sigma,\theta}^\mathcal{U}$.]

This solution seems to work in all the JSZ-anomalous cases. We illustrate the details in one of them now.

Acknowledgement The author noticed this problem while working on Dodd solidity with Nam Trang. Thanks to Trang for his comments on an earlier draft of this note.

§1. Parameter solidity

We fill in the JSZ case in the proof of Lemma 9.6.2 of [CPMP]. Let us adopt the hypotheses of that lemma: AD^+, and (M, Ψ) is a strongly stable lbr hod pair with scope HC. Let us assume $k(M) = 0$ for definiteness. (This implies strong stability.)

Let

$$\rho = \rho_1(M)$$

$$r = p_1(M).$$

~~Our Apparent~~ q = longest solid initial segment of r,

and

$$r = s \cup q$$

where $s \neq \emptyset$. (So we are trying to obtain a contradiction.) We have $Th_1^M(\max(s) \cup q) \notin M$.

Let

$$\alpha_0 = \text{least } \beta \text{ s.t. } Th_1^M(\beta \cup q) \notin M,$$

so that $\alpha_0 \leq \max(s)$. Let

$$H = cHull_1^M(\alpha_0 \cup q)$$

and

$$\pi : H \to M$$

be the anticollapse. By Claim 0 in the proof of [CPMP, 9.6.2] we have

$$H \models \alpha_0 \text{ is a cardinal}$$

and

$$\rho < \alpha_0 \leq \mathrm{crit}(\pi) \leq \max(s).$$

Our assumption that we are in the JSE case amounts to

$$E^M_{\alpha_0} \neq \emptyset,$$

so that α_0 is not a cardinal of M, and

$$\mathrm{crit}(\pi) = \alpha_0.$$

Letting $\lambda_0 = \lambda(E^M_{\alpha_0})$, we also have

<u>Claim 0</u> $M \models \lambda_0$ is a limit cardinal.

<u>Proof</u> $M \| \alpha_0 \models \lambda_0$ is a limit cardinal, so $H \models \lambda_0$ is a limit cardinal, because α_0 is a cardinal of H. So $M \models \pi(\lambda_0)$ is a limit cardinal. But $\pi(\lambda_0) = \lambda_0$. $\otimes$

If λ_0 is not measurable in H, then the JSE anomalous case cannot actually occur in the proof of 9.6.2 given in [CPMP].

Since we are assuming toward contradiction that solidity has failed for $p_1(M)$, we get

<u>Claim 1</u> $H \models \lambda_0$ is measurable by the sequence

<u>Proof</u> Otherwise [CPMP] shows $Th_1^M(\alpha_0 \cup \gamma) \in M$, contradiction. ☒

<u>Claim 2</u> Let μ be the order 0 measure on λ_0 in the $Ult(M, E_\gamma^M)$ sequence; then μ is the order 0 measure of H on λ_0.

<u>Proof</u> Let $\nu = E_\gamma^M$ be the order 0 measure of M on λ_0. We want to see that $\pi^{-1}(\nu) = \mu$. (Note here that $\gamma \in ran(\pi)$.) Letting $\pi(\bar{\gamma}) = \gamma$, and $\sigma = \pi \restriction H|\bar{\gamma}+1$, we can apply condensation to $\sigma : H|\bar{\gamma}+1 \longrightarrow M|\gamma+1$. (Note

$\gamma < o(M)$, so this is ok by induction on mouse-rank.) Condensation implies $H | \bar{\gamma}+1 \trianglelefteq \mathrm{Ult}(M|\gamma+1, E^M_{\alpha_0})$, which easily implies our claim.

$\boxtimes$

<u>Remark 1</u> Claim 2 is useful because our trees $\mathcal{U}$ on M will be λ-separated, so if E is part of a least disagreement with $M^{\mathbb{C}}_{\nu\ell}$, $\mathcal{U}$ will use E^+ and not E.

Now let $((N^*, \in, w, \mathcal{F}, \Phi), \Phi^*)$ be a coarse strategy pair that captures $\mathrm{Code}(\Psi)$, and let $\mathbb{C}$ be its maximal $(w, \mathcal{F}, \Phi)$-construction. We work mainly in N^*. Let

$$m^{\mathbb{C}}_{\nu,\ell} = (M^{\mathbb{C}}_{\nu,\ell}, \Omega^{\mathbb{C}}_{\nu,\ell})$$

be the $\langle \nu, \ell \rangle$-th level of $\mathbb{C}$. By 9.5.8

of [CPMP], we can fix ν_0 such that (M, Ψ) iterates to $M^{\mathfrak{c}}_{\nu_0,0}$, and iterates strictly past $M^{\mathfrak{c}}_{\nu,l}$ for all $\nu < \nu_0$. For $\langle \nu, l\rangle \leq_{lex} \langle \nu_0, 0\rangle$, let $\mathcal{U}_{\nu,l}$ be the λ-separated tree on (M, Ψ) witnessing this.

For $\langle \nu, l\rangle \leq_{lex} \langle \nu_0, 0\rangle$ we define $\mathcal{S}_{\nu,l}$ on (M, H, α_0), by induction on $\langle \nu, l\rangle$. The induction stops when we reach $\langle \nu, l\rangle$ such that the last model of $\mathcal{S}_{\nu,l}$ is an initial segment of $M^{\mathfrak{c}}_{\nu,l}$, and $\mathcal{S}_{\nu,l}$ does not drop on its branch to the last model. The definition of $\mathcal{S}_{\nu,l}$ is the same as that in the proof of 9.6.2 of [CPMP], except for the additional clauses related to the JSZ anomaly. We shall use the notation of that proof here, and just focus on the additional clauses.

Fix $\langle \nu, \ell \rangle \leq_{lex} \langle \nu_0, 0 \rangle$ such that $S_{\eta, j}$ has been defined for all $\langle \eta, j \rangle <_{lex} \langle \nu, \ell \rangle$, and the induction has not stopped at $\langle \nu, \ell - 1 \rangle$. Let $\mathcal{U} = \mathcal{U}_{\nu, \ell}$. We define $S = S_{\nu, \ell}$ by induction, adding one or two models at a time. Its models are

$$P_0 = M_0^{\mathcal{S}} = M$$
$$P_1 = M_1^{\mathcal{S}} = H$$

and in general

$$P_\theta = M_\theta^{\mathcal{S}}.$$

Let

$$Q_\theta = M_\theta^{\mathcal{U}}$$

We also lift $\mathcal{S}$ to $\mathcal{T}$ on M via $(\mathrm{id}, \pi) = (\pi_0, \pi_1)$. So letting

$$P_\theta^* = M_\theta^{\mathcal{T}}$$

we have

$$\pi_\theta : P_\theta \longrightarrow P_\theta^*.$$

(So $P_0^* = P_1^* = M$.) The branch embs are $i_{\alpha\beta} = i_{\alpha\beta}^{\mathcal{S}}$, $i_{\alpha\beta}^* = i_{\alpha\beta}^{\mathcal{T}}$, and $j_{\alpha\beta} = i_{\alpha\beta}^{\mathcal{U}}$.

The strategies attached are

$$\Sigma_\theta^* = \Psi_{\Sigma \restriction \theta+1},$$

$$\Sigma_\theta = (\Sigma_\theta^*)^{\pi_\theta},$$

$$\Lambda_\theta = \Psi_{\mathcal{U} \restriction \theta+1}.$$

Nodes of $\mathcal{S}$ are either stable or unstable. The current last model is always stable. The unstable θ satisfy the induction hypotheses on p. 421 of [CPMP].

Because (M, H, α) has the JSE form, we add that for θ unstable, and $i_{0,\theta} : P_0 \to P_\theta$,

$$\alpha_\theta = i_{0\theta}(\alpha_0) = \sup i_{0\theta}'' \alpha_0$$

and define

$$\lambda_\theta = i_{0\theta}(\lambda_0) = \lambda(E_{\alpha_\theta}^{P_\theta})$$

$$g_\theta = i_{0\theta}(g)$$

We shall have that $i_{0\theta}$ is given by an extender on $P_0 \mid \lambda_0$, that is

$$\eta \in [0, \theta)_{\mathcal{S}} \Rightarrow \mathrm{crit}(i_{\eta,\theta}) < \lambda_\eta$$

for θ (and hence η) being unstable. In particular, we aren't going to lift a phalanx $(P_\eta, P_{\eta+1}, \alpha_\eta)$ at an unstable η by an extender L s.t. $\mathrm{crit}(L) = \lambda_\eta$.

If θ is unstable, we have

$$P_{\theta+1} = \mathrm{cHull}_1^{P_\theta}(\alpha_\theta \cup q_\theta)$$

and

$$\sigma_{\theta+1} : P_{\theta+1} \longrightarrow P_\theta$$

the anticollapse. We have also

$$\pi_{\theta+1} = \pi_\theta \circ \sigma_{\theta+1}.$$

0 is unstable and 1 is stable. The roots of $\mathcal{S}$ are 0, 1, and the $\theta+1$ for θ unstable. All roots except 0 are thus stable. For γ stable,

$$\mathrm{st}(\gamma) = \text{least } \xi \leq_{\mathcal{S}} \gamma \text{ such that } \xi \text{ is stable.}$$

~~$\mathrm{st}(\gamma)$ is the stable~~

$$\mathrm{st}^*(\gamma) = \begin{cases} \mathrm{st}(\gamma) & \text{if } \mathrm{st}(\gamma) = \theta+1 \text{ for some unstable } \theta \\ \text{undefined} & \text{otherwise.} \end{cases}$$

<u>Def 1</u> If $\mathcal{W}$ is an iteration tree or pseudo-iteration tree and $\alpha \leq_{W} \beta$, then $e^{\mathcal{W}}_{\alpha\beta}$ is the sequence of extenders used in $[\alpha, \beta]_{W}$. (I.e., all $E^{\mathcal{W}}_{\mu}$ for $\alpha <_{W} \mu+1 \leq_{W} \beta$.)

We call $e^{\mathcal{W}}_{\alpha\beta}$ the <u>branch extender sequence</u> of $[\alpha, \beta]_{W}$.

<u>Def. 2</u> Let $\gamma < lh(\mathcal{S})$; then γ is <u>$\mathcal{U}$-accessible</u> iff

(1) γ is stable and $st^{*}(\gamma)$ exists

(2) letting $\theta+1 = st^{*}(\gamma)$ and $e^{\mathcal{U}}_{\tau} = e^{\mathcal{S}}_{\theta}$, we have

$$e^{\mathcal{S}}_{\theta+1,\gamma} = \langle D \rangle^\frown s = \langle E^{\mathcal{S}}_{\theta+1} \rangle^\frown s,$$

where D is the order 0 measure on λ_{θ} of $P_{\theta+1}$, and

$$E^{\mathcal{U}}_{\tau} = E^{P_{\theta}}_{\alpha_{\theta}}\text{-then-}D$$

and (i) $\mathrm{crit}(i_{\eta,\gamma}) \leq i_{\theta+1,\eta}(\lambda_{\theta})$ for all $\eta \in [\theta+1, \gamma)_{S}$,

and (ii) $s = e^{\mathcal{U}}_{\tau+1,\beta}$ for some β, or $s = \emptyset$.

13a

Remark In clause 2, the fact that $e^{\delta}_{\theta+1,\gamma} = \langle D \rangle ^\frown S$ ~~implies~~ and $D = E^{\delta}_{\theta+1}$ implies that

$$P_{\theta+2} = \mathrm{Ult}(P_{\theta+1}, D),$$

so

$$\theta+2 \leq_{\delta} \gamma ;$$

and

$$S = e^{\delta}_{\theta+2,\gamma} = e^{u}_{\gamma+1,8} .$$

Condition (2)(i) is equivalent to:

$\mathrm{Ult}(P_\theta \mid \alpha_\theta, \langle\theta\rangle^\frown s)$ makes sense. Thus

$i_{\theta+1,\gamma}(E^{P_\theta}_{\alpha_\theta})$ makes sense.

The motivation for Def. 2 is: suppose θ is unstable, so we have τ such that $e^{\mathcal{T}}_\theta = e^{\mathcal{U}}_\tau$, and hence $P_\theta = Q_\tau$. We have $P_{\theta+1} = \mathrm{cHull}^{P_\theta}_1(\alpha_\theta \cup g_\theta)$, and $\alpha_\theta = (\lambda_\theta^+)^{P_{\theta+1}}$. Suppose

The case that $\lambda_\theta < \mathrm{lh}(E^{\mathcal{T}}_{\theta+1}) < \alpha_\theta$ is discussed below.

$$P_\theta \| \alpha_\theta = Q_\tau \| \alpha_\theta = M_{\nu,e} \| \alpha_\theta.$$

(If $\mathrm{lh}(E_{\theta+1}) < \alpha_\theta$, then $\theta+1$ becomes a "dead node".)

The next extender $E^{\mathcal{T}}_{\theta+1}$ has to come from $P_{\theta+1}$, where α_θ is a cardinal, so $\mathrm{lh}(E^{\mathcal{T}}_{\theta+1}) > \alpha_\theta$. We are proving that $M^{\mathcal{U}}_{\nu,e}$ does not move, so

$$M^{\mathcal{U}}_{\nu,e} \models E_{\alpha_\theta} = \emptyset.$$

It follows that Q_τ has its least extender disagreement with $M^{\mathcal{U}}_{\nu,e}$ at α_θ, so

$$E^{\mathcal{U}}_\tau = (E^{P_\theta}_{\alpha_\theta})^+ = (E^{P_\theta}_{\alpha_\theta}\text{-then-}D)$$

where D is the order 0 measure of $Ult(Q_\tau, E_{\alpha_\theta}^{P_\theta})$ on λ_θ. Our Claim 2 above shows that D is the order 0 measure of $P_{\theta+1}$ on λ_θ. It's not on the $M_{\nu,\ell}^{\mathcal{C}}$-sequence because $\mathcal{U}$ used it (in effect). So we shall use it immediately on the $\mathcal{S}$ side, setting

$$P_{\theta+2} = Ult(P_{\theta+1}, D).$$

<u>Remark 2</u> D may not be the least extender in Jensen indexing on $P_{\theta+1}$ but not $M_{\nu,\ell}^{\mathcal{C}}$, but it's ok to hit it next. We let the dummy generators of D that are $> \lambda_\theta$ be moved along $[\theta+2, \gamma)_S$. This corresponds to what is happening in $\mathcal{U}$, where the generators of $E_{\alpha_\theta}^{+}$ are thought of as being $\leq \lambda_\theta + 1$, and the ~~generators~~ coordinates of D above λ_θ can be moved by branches $[\tau+1, \eta)_\mathcal{U}$.

The following claim shows how we shall use $\mathcal{U}$-accessibility.

<u>Claim 3</u> Suppose $\gamma < lh(\mathcal{S})$ is $\mathcal{U}$-accessible. Let $\theta+1 = st^*(\gamma)$ and $e^{\mathcal{U}}_\gamma = e^{\mathcal{S}}_\theta$, and let D be the order 0 measure of $P_{\theta+1}$ on λ_θ. Let $s = e^{\mathcal{S}}_{\theta+2,\gamma}$, so that

$$P_\gamma = Ult(P_{\theta+1}, \langle D\rangle^\frown s),$$

and let δ be such that

$$e^{\mathcal{U}}_{\gamma+1,\delta} = s,$$

then ~~[illegible]~~

$$P_\gamma \ntrianglelefteq Q_\delta .$$

<u>Proof</u> Suppose $P_\gamma \trianglelefteq Q_\delta$ toward contradiction. Let $\alpha = \alpha_\theta$ and $t = \sigma_\theta^{-1}(q_\theta)$. Since $(P_\theta, P_{\theta+1}, \alpha_\theta)$ is a lift of (P_0, P_1, α_0), it is still "problematic", in that

$$Th_1^{P_{\theta+1}}(\alpha \cup t) \notin P_\theta .$$

By $\mathcal{U}$-accessibility,

$$E^{\mathcal{U}}_\gamma = (E^{P_\theta}_\alpha)^+ .$$

Let

$$\beta = (\alpha^+)^{P_{\theta+1}}.$$

We have, for $E_\alpha = E_\alpha^{P_\theta}$,

$$\begin{aligned} P_{\theta+1}|\beta &= \mathrm{Ult}(P_\theta, E_\alpha)|\beta \\ &= \mathrm{Ult}(Q_\gamma, E_\alpha)|\beta \\ &= \mathrm{Ult}(Q_{pd_{\mathcal{U}}(\gamma+1)}, E_\alpha)|\beta. \end{aligned}$$

The first line holds by condensation, and the third line because $E_\alpha = (E_\gamma^{\mathcal{U}})^-$ and Q_γ agrees with $Q_{pd_{\mathcal{U}}(\gamma+1)}$ past $\mathrm{dom}(E_\alpha)$. Letting

$$R_{\gamma+1} = \mathrm{Ult}(Q_{pd_{\mathcal{U}}(\gamma+1)}, E_\alpha),$$

we have

$$Q_{\gamma+1} = \mathrm{Ult}(R_{\gamma+1}, D),$$

and

$$\begin{aligned} Q_\delta &= \mathrm{Ult}(Q_{\gamma+1}, S) \\ &= \mathrm{Ult}(R_{\gamma+1}, \langle D\rangle^\frown S). \end{aligned}$$

As we noted, $R_{\gamma+1}|\beta = P_{\theta+1}|\beta$.

β may not be a cardinal in $\mathrm{Ult}(P_\theta, E_\alpha)$,
so it may not be a cardinal in $R_{\gamma+1}$.
$\alpha = \lambda_\theta^+$ holds in both $P_{\theta+1}$ and $R_{\gamma+1}$.
We have that $i^{\alpha}_{\theta+1,\gamma}$ is given by the $\langle D\rangle^\frown s$ ultrapower of $P_{\theta+1}$, and $j_{\gamma+1,\delta} \circ i_D$ is given by the $\langle D\rangle^\frown s$ ultrapower of $R_{\gamma+1}$, so

$$i_{\theta+1,\gamma} \restriction \beta = j_{\gamma+1,\delta} \circ i_D \restriction \beta.$$

<u>Subclaim 3A</u> $R_{\gamma+1} \models \beta$ is not a cardinal.

<u>Proof</u> Otherwise, $\beta = (\alpha^+)^{R_{\gamma+1}}$. But then

$j_{\gamma+1,\delta} \circ i_D(\beta) = \sup j_{\gamma+1,\delta} \circ i_D{}''\beta = \sup i_{\theta+1,\gamma}{}''\beta$

$= i_{\theta+1,\gamma}(\beta)$, so $Q_\delta \models i_{\theta+1,\gamma}(\beta) = i_{\theta+1,\gamma}(\alpha)^+$.

But $P_\gamma \triangleleft Q_\delta$, and P_γ is the collapsing structure for $i_{\theta+1,\gamma}(\beta)$, since

$P_\gamma = \mathrm{Hull}_1^{P_\gamma}(i_{\theta+1,\gamma}(\alpha) \cup i_{\theta+1,\gamma}(t))$. So

$Q_\delta \models i_{\theta+1,\gamma}(\beta)$ is not a cardinal, contradiction.

$\boxtimes$

Now let $N \triangleleft R_{\gamma+1}$ be the collapsing level for β.

<u>Subclaim 3B</u> $N = P_{\theta+1}$.

<u>Proof</u> $\mathrm{Ult}(N, \langle D\rangle^\frown s) \trianglelefteq Q_\delta$ because we have a factor map $\sigma : \mathrm{Ult}(N, \langle D\rangle^\frown s) \to j_{\gamma+1,\delta} \circ i_D(N)$, and we can apply condensation to σ. (If $\sigma \neq id$, then $\mathrm{crit}(\sigma) = \sup j_{\gamma+1,\delta} \circ i_D{}''\beta = i_{\theta+1,\gamma}(\beta)$ and $R_{\gamma+1} \models \beta$ is singular.) Clearly, $\mathrm{Ult}(N, \langle D\rangle^\frown s)$ is the collapsing level for $i_{\theta+1,\gamma}(\beta)$ in Q_δ, so

$$\mathrm{Ult}(N, \langle D\rangle^\frown s) = P_\gamma$$
$$= \mathrm{Ult}(P_{\theta+1}, \langle D\rangle^\frown s).$$

We claim now that $N = P_{\theta+1}$. For notice that α is a cardinal of $R_{\tau+1}$, so $j_{\tau+1} \circ i_D(\alpha) = \alpha^*$ is a cardinal of Q_δ. Since $\mathrm{Hull}_1^{P_\gamma}(\alpha^* \cup i_{\theta+1,\gamma}(t))$

$= P_\gamma$ and $P_\gamma \triangleleft Q_\delta$, we have

$$\rho_1(P_\gamma) = i_{\theta+1,\gamma}(\alpha) = j_{\tau+1,\delta} \circ i_D(\alpha)$$
$$= \alpha^* .$$

There is no $v <_{lex} t$ such that $t = h_1^{P_{\theta+1}}(\bar{u}, v)$ for some $\bar{u} \in \alpha^{<\omega}$. [Otherwise we have $g_\theta = h_1^{P_\theta}(\bar{u}, \sigma_\theta(v))$ with $\sigma_\theta(v) <_{lex} g_\theta$, contrary to the solidity of g_θ in P_θ.] Since $i_{\theta+1,\gamma}$ is Σ_1 elementary, there are no $v <_{lex} i_{\theta+1,\gamma}(t)$ and $\bar{u} \in (\alpha^*)^{<\omega}$ such that $h_1^{P_\gamma}(\bar{u}, v) = i_{\theta+1,\gamma}(t)$. But $P_\gamma = \mathrm{Hull}_1^{P_\gamma}(\alpha^* \cup i_{\theta+1,\gamma}(t))$, so

$$i_{\theta+1,\gamma}(t) = p_1(P_\gamma)$$

$$= i^N_{\langle 0\rangle^\frown s}(p_1(N)).$$

This implies

$$P_{\theta+1} = N,$$

as desired.

Subclaim 3B. $\boxtimes$

But now $E_\alpha \in Q_\tau$, so $N \in Q_\tau$,

i.e. $P_{\theta+1} \in Q_\tau = P_\theta$. Contradiction.

Claim 3. $\boxtimes$

Let us describe the ~~next~~ successor step in the construction of $\mathcal{S} = \mathcal{S}_{\nu,\ell}$. We have $\mathcal{U} = \mathcal{U}_{\nu,\ell}$, and $\mathcal{S} \upharpoonright \gamma+1$, where γ is stable in $\mathcal{S}$. We assume that the construction of $\mathcal{S}$ does not end at γ, meaning it is not the case that either (i) $M^{\mathbb{C}}_{\nu,\ell} \trianglelefteq (M^{\mathcal{S}}_\gamma, \Sigma^{\mathcal{S}}_\gamma)$ or (ii) $(M^{\mathcal{S}}_\gamma, \Sigma^{\mathcal{S}}_\gamma) \trianglelefteq M^{\mathbb{C}}_{\nu,\ell}$ and $[rt(\gamma), \gamma]_{\mathcal{S}}$ does not drop in model or degree.

(In case (i), we start defining $S_{\nu,\ell+1}$, and in case (ii) $S_{\nu,\ell}$ is our last tree on (M, H, α).)

$\left(E^{\mathcal{S}}_{\gamma}\right)^{-}$ will be the first extender on the $M^{\mathcal{S}}_{\gamma}$ sequence but not on the $M^{\alpha}_{\nu,l}$ sequence, as in [CPMP], unless γ is "special".

<u>Def. 3</u> γ is <u>special</u> iff

(1) γ is stable and $st^{*}(\gamma) = \theta+1$ exists, and

(2) $P_{\gamma} \| i_{\theta+1,\gamma}(\alpha_\theta) = M_{\nu,l} \| i_{\theta+1,\gamma}(\alpha_\theta)$, and

(3) either

(a) $\gamma = \theta+1$, or

(b) $\theta+1 <_{\mathcal{S}} \theta+2 \leq_{\mathcal{S}} \gamma$, and

(i) $E^{\mathcal{S}}_{\theta+1}$ is the order 0 total measure on λ_θ in $P_{\theta+1}$,

(ii) for all $\eta \in [\theta+1,\gamma)_S$, $\mathrm{crit}(i_{\eta,\gamma}) \leq i_{\theta+1,\eta}(\alpha_\theta)$ and η is $\mathcal{U}_{\nu,l}$-accessible, and

(iii) γ is not $\mathcal{U}_{\nu,l}$-accessible.

We say γ is <u>very special</u> if γ is special and (3)(b) holds.

<u>Remark</u> If γ is very special, then $\theta+2 <_s \gamma$. For otherwise, $e^s_{\theta+2,\gamma} = \emptyset$ occurs (trivially) as an $e^{\mathcal{U}}_{ap}$.

<u>Claim 4</u> Lemma 9.6.5 of [CPMP] holds in the following form:

(a) If γ is not special, then the least disagreement between $(P_\gamma, \Sigma_\gamma)$ and $m^{\mathcal{Q}}_{\nu,\ell}$ is an extender disagreement, and it is passive on the $m^{\mathcal{Q}}_{\nu,\ell}$ side, and

(b) if γ is special, then letting $st^*(\gamma) = \theta+1$ and $\alpha = i_{\theta+1,\gamma}(\alpha_\theta)$,

$$(P_\gamma, \Sigma_\gamma) \| \alpha = m^{\mathcal{Q}}_{\nu,\ell} \| \alpha$$

and α is passive in $M^{\mathcal{Q}}_{\nu,\ell}$.

<u>Proof</u> Deferred for now. The proof follows that of [CPMP, 9.6.5]. We shall indicate why a least extender disagreement must be passive on the $M^{\mathcal{Q}}_{\nu,\ell}$-side below.

$\boxtimes$

If γ is not special, then we set

$$E^{\delta}_{\gamma} = E^{+}, \text{ where } E \text{ is such that}$$

$$P_{\gamma} \| lh(E) = M^{\Phi}_{\nu,\ell} \| lh(E) \text{ and}$$

$$E = \dot{F}^{P_{\gamma}|lh(E)} \text{ and } E \neq \emptyset,$$

and

$$\varepsilon_{\gamma} = \cancel{}\, lh(E^{\delta}_{\gamma}).$$

(For telling us where to go back to, it would be equivalent to set $\varepsilon_{\gamma} = \lambda(E^{\delta -}_{\gamma})$. This version of ε_{γ} also measures agreement with future models.)

If γ is special and $\text{st}^{*}(\gamma) = \theta + 1 = \gamma$,

we set

$$E^{\delta}_{\gamma} = \text{order } 0 \text{ total measure on } \lambda_{\theta} \text{ in } P_{\theta+1}$$

and

$$\varepsilon_{\gamma} = \alpha_{\theta}.$$

(Again, $\varepsilon_{\gamma} = \lambda_{\theta}$ would be equivalent as an "exchange ordinal".)

Finally, if γ is very special, then letting $\theta+1 = ST^*(\gamma)$, we have

$$\theta+1 <_S \theta+2 \leq_S \gamma$$

and $i_{\theta+1,\gamma}$ is given by an extender with domain $P_{\theta+1} \| \alpha_\theta = P_\theta \| \alpha_\theta$. Thus, since α_θ is regular in $P_{\theta+1}$, $i_{\theta+1,\gamma}(E^{P_\theta}_{\alpha_\theta})$ makes sense. We set

$$E^\Delta_\gamma = i_{\theta+1,\gamma}(E^{P_\theta}_{\alpha_\theta})^+$$

in this case. We need to be a bit careful about ε_γ. We set

$$\begin{aligned}\varepsilon_\gamma &= \sup\{\varepsilon_\eta \mid \eta <_S \gamma\},\\ &= \sup\{\mathrm{lh}(G) \mid G \text{ is used in } [\theta+1,\gamma]_S\}.\end{aligned}$$

($\varepsilon_\gamma = \sup\{\lambda(G^-) \mid G$ used in $[\theta+1,\gamma]_S\}$ would have the same effect.) It is possible here that $\varepsilon_\gamma < \lambda(E^\Delta_\gamma)$. We have

$\varepsilon_\gamma \leq \lambda(E^{\mathcal{S}}_\gamma)$ by clause (2)(i) of Def. 2. Notice that $E^{\mathcal{S}}_\gamma$ is generated by $E^{\mathcal{S}}_\gamma \restriction \varepsilon_\gamma$, ~~which makes it ~~ ~~This~~ Also, this choice for ε_γ turns γ into a "dead node", in that no later extender in $\mathcal{S}$ will be applied to P_γ. Notice also that the ISC fails for $E^{\mathcal{S}}_\gamma$, since $E^{\mathcal{S}}_\gamma = E^{P_\theta}_{\alpha_\theta}$-then-$E_{\lambda_{\theta+1,\gamma}}$. All this is parallel to the SZ move — we are making that move here.

This completes the definition of $E^{\mathcal{S}}_\gamma$. Let us use it now to extend $\mathcal{S}$.

We let

$$pd_{\mathcal{S}}(\gamma+1) = \text{least } \beta \text{ s.t. } \operatorname{crit}(E) < \varepsilon_\beta .$$

Let $\beta = pd_{\mathcal{S}}(\gamma+1)$. If β is stable, then so is $\gamma+1$, and $M^{\mathcal{S}}_{\gamma+1} = \operatorname{Ult}(N, E^+)$ for the longest possible $N \trianglelefteq M^{\mathcal{S}}_\beta$ as usual. Similarly, if β is unstable but E^+ is not used in $\mathcal{U}$, then $\gamma+1$ is stable and $M^{\mathcal{S}}_{\gamma+1} = \operatorname{Ult}(N, E^+)$ for the longest possible $N \trianglelefteq M^{\mathcal{S}}_\beta$. (This is the case when $\cancel{\gamma}$ is very special.)

Finally, suppose β is unstable and E^+ is used in $\mathcal{U}$. ~~Letting~~ $\cancel{e^{\mathcal{S}}_\beta = e^{\mathcal{U}}_\tau}$, ~~we have~~ $\cancel{e^{\mathcal{S}}_{\gamma+1}}$ We let then $M^{\mathcal{S}}_{\gamma+1} = \operatorname{Ult}(M^{\mathcal{S}}_\beta, E^+)$. Letting τ be such that $M^{\mathcal{U}}_\tau = M^{\mathcal{S}}_\beta$, we have then that

$$\begin{aligned} e^{\mathcal{S}}_{\gamma+1} &= e^{\mathcal{S}}_\beta {}^\frown \langle E^+ \rangle \\ &= e^{\mathcal{U}}_\tau {}^\frown \langle E^+ \rangle \\ &= e^{\mathcal{U}}_{\mu+1} \end{aligned}$$

where $E^+ = E^{\mathcal{U}}_\mu$. [We must see $\tau = pd_{\mathcal{U}}(\mu+1)$.

Let $\kappa = \operatorname{crit}(E)$. Since $\lambda_F \le \kappa$ for all $F \in \operatorname{ran}(e^{\mathcal{S}}_\beta) = \operatorname{ran}(e^{\mathcal{U}}_\tau)$, $\tau \le pd_{\mathcal{U}}(\mu+1)$. So we must see that $\kappa < \lambda(E^{\mathcal U}_\gamma)$. If $\varepsilon_\beta = \lambda(E^{\mathcal S}_{\beta+1}) < \lambda_\beta$, then $M_{\infty,e} \| \mathrm{lh}(E^{\mathcal S}_{\beta+1}) = P_{\beta+1} \| \mathrm{lh}(E^{\mathcal S}_{\beta+1}) = P_\beta \| \mathrm{lh}(E^{\mathcal S}_{\beta+1})$ $= Q_\gamma \| \mathrm{lh}(E^{\mathcal S}_{\beta+1}) = Q_{\gamma+1} \| \mathrm{lh}(E^{\mathcal S}_{\beta+1})$, so $E^{\mathcal S}_{\beta+1} = E^{\mathcal U}_\gamma$, ~~$\lambda(E^{\mathcal S}_{\beta+1})$~~ so $\kappa < \lambda(E^{\mathcal U}_\gamma)$. If $\varepsilon_\beta = \lambda_\beta$, then $P_\beta \| \lambda_\beta = Q_\gamma \| \lambda_\beta = M_{\infty,e} \| \lambda_\beta = Q_{\gamma+1} \| \lambda_\beta$, so $\kappa < \varepsilon_\beta \le \lambda(E^{\mathcal U}_\gamma)$.]

Thus $\gamma+1$ is unstable in $\mathcal S$ in this case, and we set

$$P_{\gamma+2} = M^{\mathcal S}_{\gamma+2} = \mathrm{cHull}_1^{P_{\gamma+1}}\left(i_{\beta,\gamma+1}(x_\beta) \cup i_{\beta,\gamma+1}(q_\beta)\right).$$

Let $\sigma_{\gamma+1} : P_{\gamma+2} \to P_{\gamma+1}$ be the anticollapse.

One can easily check the induction hypotheses on p. 421 of [CPMP], just as is done there.

At limit steps λ we extend $\mathcal S \restriction \lambda$ to $\mathcal S \restriction \lambda+1$ by choosing the branch chosen by $\mathcal J = (\mathrm{id}, \pi)\mathcal S \restriction \lambda$. See [CPMP] for further detail.

<u>Remark</u> The new case here for the lifting procedure occurs when γ is special and (b)(ii) holds. Let $\mathrm{st}^*(\gamma) = \theta+1$, so that $E^*_\gamma = i_{\theta+1,\gamma}(E^{P_\theta}_{\alpha_\theta})^+$.

We have the diagram

$$\begin{tikzcd}
P_\gamma \arrow[rr, "\pi_\gamma"] & & P^*_\gamma \\
P_\xi \arrow[u, "i_{\xi,\gamma}"] \arrow[rr, "\pi_\xi"] & & P^*_\xi \arrow[u, "i^*_{\xi,\gamma}"'] \\
P_{\theta+2} \arrow[u] \arrow[rr] & & P^*_{\theta+2} \arrow[u] \\
P_{\theta+1} \arrow[u] \arrow[r, "\sigma_\theta"'] \arrow[rr, bend right, "\pi_{\theta+1}"'] & P_\theta \arrow[r, "\pi_\theta"'] & P^*_{\theta+1} = P^*_\theta \arrow[u]
\end{tikzcd}$$

Let

$$F_\xi = i_{\theta+1,\xi}(E^{P_\theta}_{\alpha_\theta})$$

for all $\xi \in [\theta+1, \gamma]_S$, and

$$\beta_\xi = \mathrm{lh}(F_\xi) = i_{\theta+1,\xi}(\alpha_\theta)$$

for such ξ. The structure $(P_{\theta+1}\|\kappa_\theta, E^{P_\theta}_{\alpha_\theta}) = P_\theta | \alpha_\theta$ is a premouse, and $(P_{\theta+1}\|\alpha_\theta, F_{\theta+1}) = (P_{\theta+1}\|\beta_{\theta+1}, E^{P_\theta}_{\alpha_\theta})$.

Let

$$\bar{P}_\xi = (P_\xi \| \beta_\xi, F_\xi),$$

and

$$\psi_\xi = \pi_\xi \upharpoonright \bar{P}_\xi .$$

$\bar{P}_\xi$ satisfies all the premouse conditions, except that the ISC fails if $\xi > \theta+1$. $\bar{\psi}_\xi$ is Σ_0 elementary in the language of coherent structures, and cardinal-preserving. Let

$$F^*_\xi = i^*_{\theta+1,\xi}(\pi_\theta(E^{P_\theta}_{\alpha_\theta}))$$

and

$$\beta^*_\xi = \mathrm{lh}(F^*_\xi),$$

and

$$\bar{P}^*_\xi = (P^*_\xi \| \beta^*_\xi, F^*_\xi)$$

$$= P^*_\xi | \beta^*_\xi .$$

(Since $F^*_{\theta+1} = \pi_\theta(E^{P_\theta}_{\alpha_\theta})$ is on the $P^*_{\theta+1} = P^*_\theta$ sequence.)

Then

$$\psi_\xi : \bar{P}_\xi \longrightarrow \bar{P}^*_\xi$$

is nearly elementary in the language of coherent structures. Thus we can use it to copy: we let

$E^{\mathcal{T}}_\gamma = F^*_\gamma$, and $\pi_{\gamma+1} : P_{\gamma+1} \longrightarrow P^*_{\gamma+1}$

is given by

$$\pi_{\gamma+1}\left([a,f]^{P_\beta}_{F_\gamma}\right) = [\psi_\gamma(a), \pi_\beta(f)]^{P^*_\beta}_{F^*_\gamma} .$$

End Remark. $\boxtimes$

Let us prove the part of Claim 4 that involves extender disagreements.

<u>Subclaim 4A</u> Suppose that $P_\gamma | \eta$ is active, $P_\gamma || \eta = M_{\nu,l} || \eta$, and $\Sigma_{P_\gamma || \eta} = (\Omega_{\nu,l})_{M_{\nu,l} || \eta}$, and $P_\gamma | \eta \neq M_{\nu,l} | \eta$; then $M_{\nu,l} | \eta$ is passive.

<u>Proof</u> Suppose not. Letting $\langle \nu, l \rangle$ be lex minimal as a counterexample, we get that $l = 0$ and $\eta = \nu$. (See [CPMP].)

Let

$$F = \dot{F}^{M_{\nu,0}}$$

and

$$F^* = B^{\mathbb{C}}(F) = \text{background extender for } F.$$

Let

$$j = i^{V}_{F^*},$$

$$\kappa = \text{crit}(F) = \text{crit}(j)$$

It is shown in [CPMP] that F^* backgrounds F^+, and $j(M^{\mathbb{C}}_{\nu,0}) | \langle \nu, 0 \rangle = M^{\mathbb{C}}_{\langle \nu,0 \rangle} || \nu$

and $\text{lh}(F) = o(M^{\mathbb{C}}_{\nu,0})$ is a cardinal in $j(M^{\mathbb{C}}_{\nu,0})$.

The agreement between $M^{\mathbb{C}}_{\nu,0}$ and $j(M^{\mathbb{C}}_{\nu,0})$ implies

$$\Delta \restriction \gamma+1 = j(\Delta) \restriction \gamma+1,$$

and on the $\mathcal{U}$-side, letting μ be least s.t. $\text{lh}(E^{\mathcal{U}}_{\mu}) \geq \text{lh}(F)$, so that $\text{lh}(E^{\mathcal{U}}_{\mu}) > \text{lh}(F)$,

$$j(\mathcal{U}) \restriction \mu+2 = (\mathcal{U} \restriction \mu+1)^\frown \langle F^+ \rangle.$$

As usual, we have

$$j^{\mathcal{S}} \upharpoonright M_\kappa^{\mathcal{S}} = i^{j(\mathcal{S})}_{\kappa, j(\kappa)}$$

and

$$j \upharpoonright M_\kappa^{\mathcal{U}} = i^{j(\mathcal{U})}_{\kappa, j(\kappa)}$$

and $M_\kappa^{\mathcal{S}}$ agrees with $M_\kappa^{\mathcal{U}}$ to their common value for κ^+. Let

$$\bar{E}_j = E_j \cap \left([j(\kappa)]^{<\omega} \times M_\kappa^{\mathcal{U}} | \kappa^{+M_\kappa^{\mathcal{U}}}\right).$$

We can compute the extenders in $e^{j(\mathcal{U})}_{\kappa, j(\kappa)}$ from $\bar{E}_j$; namely, given $s = e^{j(\mathcal{U})}_{\kappa, j(\kappa)} \upharpoonright n$, then $e^{j(\mathcal{U})}_{\kappa, j(\kappa)}(n) = H^+$, where H is the first initial segment κ of the factor embedding such that $\kappa \notin M^{j(\mathcal{U})}_{j(\kappa)} \| j(\kappa)$. Now let

$$G = e^{\mathcal{S}}_{\kappa, j(\kappa)}(0),$$

or in other words, $G = E^{j(\mathcal{S})}_\eta$ where $\mathrm{pd}_{j(\mathcal{S})}(\eta+1) = \kappa$ and $\eta+1 <_{j(\mathcal{S})} v(\kappa)$. We have

Subclaim 4A.1 For any ξ, the Jensen ISC holds for $(E_\xi^{\mathcal{J}})^-$ iff ξ is not very special.

Pf If ξ is not very special, then $(E_\xi^{\mathcal{J}})^-$ is on the sequence of P_ξ, which is a premouse, so the ISC holds. If ξ is very special, then $(E_\xi^{\mathcal{J}})^- = i_{\theta+1,\xi}(E_{\alpha_\theta}^{P_\theta})$ where $st(\xi) = \theta+1$, and $E_{\alpha_\theta}^{P_\theta}$ is a missing whole initial segment of it. (I.e. missing from P_ξ). $\boxtimes$

Thus if η is not very special in $j(\mathcal{J})$, then $E_\eta^{j(\mathcal{J})} = H^+$, where the ISC holds for H, and thus H is the least whole initial segment of $\bar{E}_\delta$ that is not in $M_{j(\kappa)}^{j(\mathcal{J})} \,\|\, j(\kappa) = M_{j(\kappa)}^{j(\mathcal{U})} \,\|\, j(\kappa)$. It follows that $H = F$. But F^+ is not used in $j(\mathcal{J})$, since otherwise F would be on the sequence of $M_\gamma^{\mathcal{J}} = M_\gamma^{j(\mathcal{J})}$.

Thus η is very special in $j(\mathcal{J})$.

Let

$$ST^{\#}(\eta) = \theta+1$$

in $j(\mathcal{S})$,

$$D = E^{j(\mathcal{S})}_{\theta+1},$$

$$\sigma = e^{j(\mathcal{S})}_{\theta+1,\eta},$$

and

$$H^{+} = E^{j(\mathcal{S})}_{\eta}.$$

The first missing (from $\mathcal{M}^{j(\mathcal{S})}_{j(\kappa)} \| j(\kappa) = \mathcal{M}^{j(\mathcal{U})}_{j(\kappa)} \| j(\kappa)$) initial segment of H is $e^{j(\mathcal{U})}_{\kappa, j(\kappa)}(0)^{-}$, that is, it is F. So

$$F = E^{\mathcal{M}^{j(\mathcal{S})}_{\theta}}_{\alpha_\theta}.$$

But F was on the $\mathcal{M}_{\eta,\theta}$ sequence and its least disagreement with $\mathcal{M}^{\mathcal{S}}_{\gamma}$, so

$$\theta+1 = \gamma,$$

and letting τ be such that $e^{j(\mathcal{S})}_{\theta} = e^{j(\mathcal{U})}_{\tau}$, in fact

$$e^{\mathcal{S}}_{\theta} = e^{j(\mathcal{S})}_{\theta} = e^{j(\mathcal{U})}_{\tau} = e^{\mathcal{U}}_{\tau},$$

$$\mathcal{U} \restriction \tau+1 = j(\mathcal{U}) \restriction \tau+1$$

and

$$E^{j(\mathcal{U})}_{\tau} = F^{+}$$

and $\tau+1$ is the successor of κ in $[\kappa, j(\kappa)]_{j(\mathcal{U})}$. (In other words, $\tau = \mu$.)

Our rules for $j(\mathcal{S})$ then require

$$F^+ = F\text{-then-}D$$

where $D = E^{j(\mathcal{S})}_{\theta+1}$. The nodes in $[\theta+2, \eta]^{\mathcal{T}}_{j(\mathcal{S})}$ are all stable, so the extenders in $\mathrm{ran}(s)$ are all applied to stable non-root nodes, and hence no $s(n) = E^{j(\mathcal{S})}_{\xi}$ for ξ special in $j(\mathcal{S})$. Thus all $s(n)$ have the Jensen ISC. But

$$H = i_{\langle D\rangle^\frown s}(F)$$

$$= \langle F^+\rangle^\frown s,$$

and hence

$$s = e^{j(\mathcal{U})}_{\eta+1,\xi}$$

See p. 33a

for some ξ. Thus η is $j(\mathcal{U})$-accessible, so η is not $j(\mathcal{S})$-special, contradiction.

Subclaim 4.A.1 $\boxtimes$

The remainder of Claim 4, namely that no $(P_\gamma, \Sigma_\gamma)$ has a strategy disagreement with $M^{\mathcal{Q}}_{\nu,e}$, *Proof* Since $\eta+1 <_{j(\kappa)} j(\kappa)$ and $\operatorname{crit}(i^{j(\kappa)}_{\eta+1, j(\kappa)}) \geq \varepsilon_\eta$, we have

$$H \restriction \varepsilon_\eta = \bar{E}^{j(\kappa)}_{i_{\kappa, j(\kappa)}} \restriction \varepsilon_\eta = \bar{E}_j \restriction \varepsilon_\eta$$

$$= E^{j(\kappa)}_{d_{\kappa, j(\kappa)}} \restriction \varepsilon_\eta .$$

Since $H = i'_{\langle 0\rangle^\frown s}(F)$ and $lh(s(n)) \leq \varepsilon_\eta$ for all n, the recovery of L^+ for missing whole initial segments of factor embeddings, ~~produces~~ applied to $H \restriction \varepsilon_\eta$, produces $\langle F^+\rangle^\frown s$. It follows that $\langle F^+\rangle^\frown s \trianglelefteq e^{j(\kappa)}_{\kappa, j(\kappa)}$.

Note here that $\varepsilon_\eta < \lambda(H)$ is possible, so $H \ntrianglelefteq E^{j(\kappa)}_{\kappa, j(\kappa)}$ is possible. But ε_η is large enough that we can recover S from $H \restriction \varepsilon_\eta$, which is what we need here.

can be proved as in [CPMP]. We shall not go through the (long) proof here.

<u>Claim 5</u> There is a $\langle \nu, \ell \rangle \leq_{lex} \langle \nu_0, 0 \rangle$ such that $\mathcal{S}_{\nu,\ell}$ exists, has last pair $(P_\beta, \Sigma_\beta) \trianglelefteq (M_{\nu,\ell}, \Omega_{\nu,\ell})^\Phi$, and $[0,\beta]_{S_{\nu,\ell}}$ does not drop.

<u>Proof</u> This is Claim 3 in the proof of 9.6.2 in [CPMP]. It's a Dodd-Jensen argument; the proof is there. $\boxtimes$

We now fix $\langle \nu, \ell \rangle$ as in Claim 5, and $\mathcal{S} = \mathcal{S}_{\nu,\ell}$, $\tilde{\mathcal{T}} = \mathcal{T}_{\nu,\ell}$, $\mathcal{U} = \mathcal{U}_{\nu,\ell}$, etc. Let $lh(\mathcal{S}) = \theta + 1$. We have that $[rT(\theta), \theta]_S$ does not drop in model or degree. (Here $rT(\theta) = sT^*(\theta)$ if $sT^*(\theta)$ is defined, and $rT(\theta) =$ largest unstable $\leq_S \theta$ if $sT^*(\theta)$ is undefined and $sT(\theta)$ is a successor ordinal, and $rT(\theta) = sT(\theta)$ otherwise.)

We are now at the bottom of p. 427 of [CPMP], and have adopted the notation there.

<u>Claim 6</u> Suppose $[0,\eta]_{\mathcal{U}}$ does not drop in model or degree, and let $j = j_{0\eta}$; then

(a) $\forall \beta < \alpha_0$, $Th_1^{Q_\eta}(j(\beta) \cup j(q)) \in Q_\eta$,

(b) $\sup j''\rho = \rho(Q_\eta)$,

(c) $Th_1^{Q_\eta}(\rho(Q_\eta) \cup j(q)) \in Q_\eta$.

<u>Proof</u> It's Claim 4 in [CPMP]. Part (c) simplifies since we have assumed $q \neq r_0$ $\boxtimes$

<u>Claim 7</u> For some unstable ξ, $rt(\theta) = \xi + 1$.

<u>Proof</u> This is the Dodd-Jensen proof in Claim 5 of [CPMP], but we need to take care with failures of ISC in $\mathcal{S}$. That is done just as in the termination proof. (Cf. Subclaim 4.A.1)

Some details: We follow the proof of Claim 5 in 9.6.2's proof. If not we have $0 \leq_S \theta$ and $[0, \theta]_\Delta$ does not drop. Using Dodd-Jensen we get

$$(P_\theta, \Sigma_\theta) = (M_{\nu,\ell}, \Omega_{\nu,\ell})^{\mathcal{C}} = (Q_\delta, \Lambda_\delta),$$

$$[0, \delta]_{\mathcal{U}} \cap D^{\mathcal{U}} = \emptyset,$$

and

$$i_{0\theta} = j_{0\delta}.$$

[PMP] then uses the Jensen ISC to conclude that $e_\theta^\Delta = e_\delta^{\mathcal{U}}$, and gets a contradiction from that. This works for us unless we have some $\eta + 1 \leq_\Delta \theta$ such that η is very special, so assume that. Let

$$\gamma + 1 = st^{\mathcal{U}}(\eta)$$

be the unique stable root $\leq_\Delta \eta$, and

$$E^{\delta}_{\eta} = H^{+}.$$

Because η is special (very)

$$H = i_{\gamma+1,\eta}(F)$$

where

$$F = E^{\beta_\gamma}_{\alpha_\gamma}.$$

Moreover, we have τ such that

$$e^{\delta}_{\gamma} = e^{u}_{\tau}$$

and

$$E^{u}_{\tau} = F^{+} = F\text{-then-}D$$

where $D = E^{\delta}_{\gamma+1}$. Let $\beta_0 = pd_{\delta}(\eta+1)$, and notice that $\beta_0 \leq_{\delta} \gamma$, because $\operatorname{crit}(H) = \operatorname{crit}(F) \in \operatorname{ran} i_{0\gamma}$. Thus

$$e^{\delta}_{\theta} = e^{\delta}_{\beta_0} {}^\frown \langle H^{+} \rangle {}^\frown t$$

for some t. But $E_{i_{0\theta}} = E_{\delta_0 s}$, and the extenders $e_{\beta_0}(u)^-$ satisfy the ISC, so

$$e^{\delta}_{\beta_0} {}^\frown \langle F^{+} \rangle \trianglelefteq e^{u}_{\delta}.$$

(F is the first missing initial segment of H.)

Thus we have β_1 such that

$$e^{\mathcal{U}}_{\beta_1} {}^\frown \langle F^+ \rangle \trianglelefteq e^{\mathcal{U}}_{\gamma},$$

so that $\tau+1 \leq_{\mathcal{U}} \gamma$ and $\beta_1 = \mathrm{pd}_{\mathcal{U}}(\tau+1)$ and $e^{\mathcal{S}}_{\beta_0} = e^{\mathcal{U}}_{\beta_1}$. But now, letting

$$s = e^{\mathcal{S}}_{\gamma+2,\eta}$$

we have

$$H = i_{\gamma+1,\eta}(F) = i_{\langle D \rangle ^\frown s}(F)$$
$$= F\text{-then-}D\text{-then-}s$$
$$= F^+\text{-then-}s$$

The extenders in $\mathrm{ran}(s)^-$ all satisfy the ISC so since H is an initial segment of $E^{\mathcal{S}}_{\beta_0,\theta} = E^{\mathcal{U}}_{\beta_1,\gamma}$, we get that

$$e^{\mathcal{U}}_{\beta_1} {}^\frown \langle F^+ \rangle {}^\frown s \trianglelefteq e^{\mathcal{U}}_{\gamma}.$$

See the proof on p. 33a

But s is not of the form $e^{\mathcal{U}}_{\rho,\sigma}$ because η is special. Contradiction.

Claim 7. $\boxtimes$

Now fix ξ s.t. $st^*(\theta) = \xi+1$. Let τ be such that $e_\xi^{\mathcal{S}} = e_\tau^{\mathcal{U}}$ and $P_\xi = Q_\tau$. Since $\xi+1 \leq_{\mathcal{S}} \theta$ and $(\xi+1, \theta]_{\mathcal{S}}$ does not drop,

$$P_{\xi+1} \,\|\, \alpha_\xi = M^a_{\theta,\ell} \,\|\, \alpha_\xi = Q_\tau \,\|\, \alpha_\xi .$$

Letting
$$F = E_{\alpha_\xi}^{P_\xi}$$
we have by the rules of $\mathcal{U}$ (since α_ξ is passive in $M^a_{\theta,\ell}$ by ~~[illegible]~~ Claim 4)

$$F^+ = E_\tau^{\mathcal{U}} = F\text{-then-}D$$

and by the rules of $\mathcal{S}$
$$E_{\xi+1}^{\mathcal{S}} = D,$$
and
$$\xi+1 <_{\mathcal{S}} \xi+2 \leq_{\mathcal{S}} \theta .$$

(Note $\xi+1$ is a dead node, so $\xi+2 \leq_{\mathcal{S}} \theta$.)

<u>Claim 8</u> $\rho(P_{\xi+1}) \in \{\lambda_\xi, \alpha_\xi\}$.

<u>Proof</u> Otherwise $\rho(P_{\xi+1}) < \lambda_\xi$. But then we have a finite $p \subseteq \alpha_\xi$ such that

$$Th_1^{P_{\xi+1}}(\rho \cup \{p, t\}) \notin P_{\xi+1},$$

where $\rho = \rho(P_{\xi+1})$ and $t = \sigma_\xi^{-1}(q_\xi)$.

Applying σ_ξ, we get

$$Th_1^{P_\xi}(\rho \cup \{p, q_\xi\}) \notin P_{\xi+1}$$

so

$$Th_1^{P_\xi}(\rho \cup \{p, q_\xi\}) \notin P_\xi$$

because $P_\xi \| \lambda_\xi = P_{\xi+1} \| \lambda_\xi$ and λ_ξ is a limit cardinal in P_ξ, and $\rho < \lambda_\xi$. But then $Th_1^{P_\xi}(\max(p)+1 \cup \rho \cup q_\xi) \notin P_\xi$, contrary to $\max(p)+1 < \alpha_\xi$ and Claim 6(a).

$\boxtimes$

<u>Claim 9</u> $(P_\theta, \Sigma_\theta) \trianglelefteq (Q_\delta, \Lambda_\delta)$ and $[\xi+1, \theta]_S$ does not drop.

<u>Proof</u> $(P_\theta, \Sigma_\theta) \trianglelefteq M^{\mathcal{C}}_{\nu,l} \trianglelefteq (Q_\delta, \Lambda_\delta)$ because $\mathcal{S}_{\nu,l}$ was the last $\mathcal{S}$-tree, and $\langle \nu, l \rangle \leq_{lex} \langle \nu_0, 0 \rangle$. Similarly, $[\xi+1, \theta]_S$ does not drop.

$\boxtimes$

Now let

$$\eta = \text{least } \gamma \text{ s.t. } \gamma \leq_S \theta \text{ and either } \gamma = \theta \text{ or } \operatorname{crit}(i_{\gamma\theta}) > i_{\xi+1,\gamma}(\alpha_\xi),$$

and

$$\alpha^* = i_{\xi+1,\eta}(\alpha_\xi).$$

Claim 10 η is $\mathcal{U}$-accessible.

Proof Suppose not, and let β be least s.t. $\beta \in [\xi+2, \eta]_S$ and β is not $\mathcal{U}$-accessible; then β is very special. (and $\beta > \xi+2$). $\beta = \theta$ is impossible, because $S_{\nu,\mu}$ cannot end at a very special node, by its definition. So $\beta <_S \theta$. But as we observed, the fact that $\varepsilon_\beta = \sup\{ h(G^-) \mid G$ is used in $[0,\beta)_S \}$

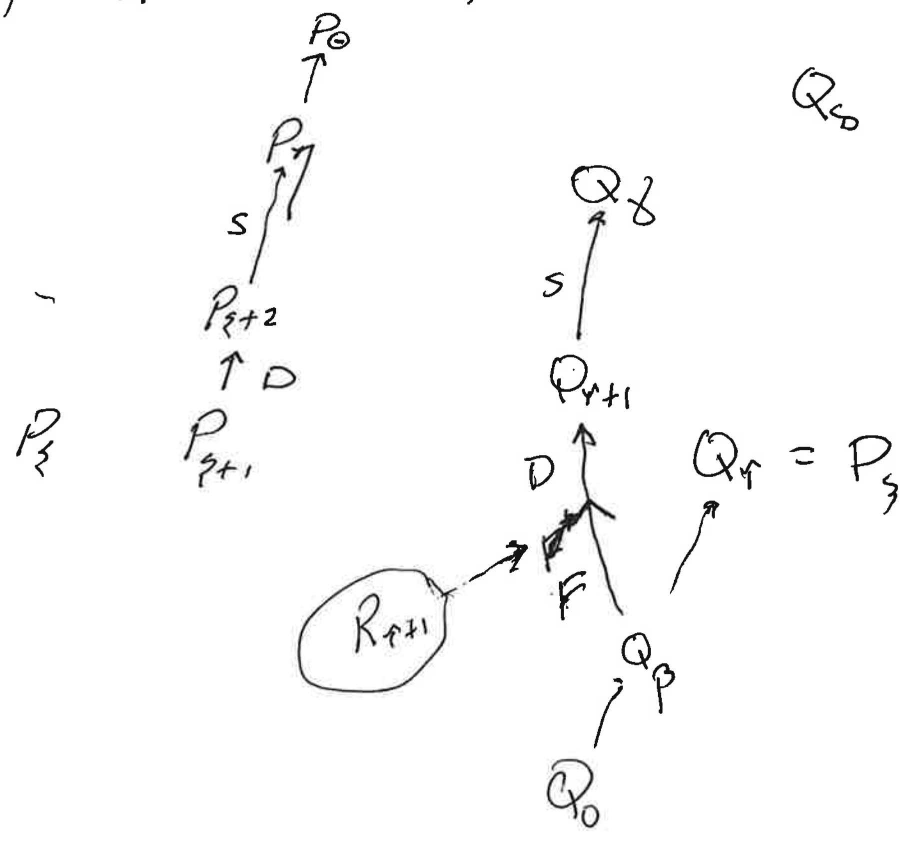
implies that $\beta \not\leq_\delta \mu$ for all μ. $\boxtimes$

Thus we have $\gamma \leq \delta$ such that

$$e^{\delta}_{\xi+2,\eta} = e^{\mathcal{U}}_{\tau+1,\gamma},$$

so $e^{\mathcal{U}}_\gamma = e^{\mathcal{U}}_\beta {}^\frown \langle F^+ \rangle {}^\frown s$, where $s = e_{\tau+1,\gamma}$.

Here $\beta \leq_{\mathcal{U}} \tau$. The picture is



Let

$$t = \sigma_\xi^{-1}(q_\xi).$$

Since $P_{\xi+1} = \mathrm{Hull}_1^{P_{\xi+1}}(\alpha_\xi \cup t)$,

$$P_\eta = \mathrm{Hull}_1^{P_\eta}(\alpha^* \cup i_{\xi+1,\eta}(t)).$$

<u>Claim 11</u> $P_\theta = Q_\delta$.

<u>Proof</u> Otherwise $P_\theta \triangleleft Q_\delta$, so P_θ is 1-sound. But then $\theta = \eta$, as otherwise $\rho_1(P_\theta) = \rho_1(P_\eta) \leq \alpha^*$ and $\alpha^* < \mathrm{crit}(i_{\eta,\theta})$, so $\mathrm{crit}(i_{\eta,\theta}) \notin \mathrm{Hull}_1^{P_\theta}(\rho_1(P_\theta) \cup p_1(P_\theta))$. So $P_\eta \triangleleft Q_\delta$, so $P_\eta \triangleleft Q_\delta \| (\alpha^*)^{+,Q_\delta} \trianglelefteq Q_\gamma \| (\alpha^*)^{+,Q_\gamma}$. (Note $\mathrm{lh}(E_\delta^{\mathcal{U}}) > \alpha^*$ if $\delta > \gamma$.) Thus

$$P_\eta \triangleleft Q_\gamma.$$

But η was $\mathcal{U}$-accessible, with $e^{\triangleleft}_{\xi+2,\eta} = e^{\mathcal{U}}_{\xi+1,\gamma}$, so $P_\eta \triangleleft Q_\gamma$ is impossible by Claim 3.

$\boxtimes$

<u>Claim 12</u> $\rho_1(P_{\xi+1}) = \alpha_\xi$.

<u>Proof</u> If not, then $\rho_1(P_{\xi+1}) = \alpha_\xi$ ~~$\mathrm{crit}(i_{\xi+1,\theta})$~~ $= \mathrm{crit}(i_{\xi+1,\theta})$, so

$$\rho_1(P_\theta) = \rho_1(Q_\delta) = \lambda_\xi .$$

Let $\nu+1$ be least in $[0,\delta]_U$ such that $\nu+1 > \tau$, and $G = E_\nu^U$; then

$$\rho(Q_\delta) \notin (\mathrm{crit}(G), lh(G)] .$$

Since $lh(G) \geq lh(E_\tau^U) = \alpha_\xi$, we have $\mathrm{crit}(G) \geq \lambda_\xi$. Thus $\nu \neq \tau$. By minimality of $\nu+1$, $pd_U(\nu+1) \leq \tau$, so $\mathrm{crit}(G) < \hat{\lambda}(E_\tau^U) = \lambda_\xi$. [Note $\mathrm{crit}(G) = \lambda_\xi$ is impossible because λ_ξ is not measurable in Q_ν for any $\nu \geq \tau+1$.] Contradiction.

☒

<u>Claim 13</u> $\rho(P_\eta) = \rho(P_\theta) = \alpha^*$.

<u>Proof</u> This follows from Claim 12 and the fact that $\mathrm{crit}(i_{\eta,\theta}) \geq \alpha^*$ if $\eta \neq \theta$.

☒

~~Claim 14~~

<u>Claim 14</u> $\gamma \leq_{\mathcal{U}} \delta$ and $[\gamma, \delta]_{\mathcal{U}}$ does not drop.

<u>Proof</u> Assume not; in particular, $\gamma < \delta$. Note that α_ξ is passive in $R_{\xi+1}$, so α^* is passive in Q_γ; moreover $Q_\gamma \| \alpha^* = M_{\nu,\ell} \| \alpha^* = P_\eta \| \alpha^*$. Thus $\hat{\lambda}(E^{\mathcal{U}}_\gamma) > \alpha^*$. Let ν be least such that $\gamma \leq \nu$ and $\nu+1 \leq_{\mathcal{U}} \delta$; then $\hat{\lambda}(E^{\mathcal{U}}_\nu) > \alpha^* = \rho(Q_\delta)$, and hence $\alpha^* < \mathrm{crit}(E^{\mathcal{U}}_\nu)$. But $\varepsilon_\gamma^{\mathcal{U}} \subseteq \alpha^*$, so then $\gamma \leq \mathrm{pd}_{\mathcal{U}}(\nu+1)$. By the minimality of ν, $\gamma = \mathrm{pd}_{\mathcal{U}}(\nu+1)$. But then $\gamma <_{\mathcal{U}} \delta$, contradiction.

Thus $\gamma \leq_{\mathcal{U}} \delta$. If $\gamma <_{\mathcal{U}} \delta$ and $[\gamma, \delta]_{\mathcal{U}}$ drops, then $\rho_1(Q_\delta) > \alpha^*$, contradiction.

$\boxtimes$

We have already shown that $[0, \gamma]_{\mathcal{U}}$ does not drop. Thus $[0, \delta]_{\mathcal{U}}$ does not drop.

Our earlier diagram becomes

$$\begin{tikzcd}
 & P_\theta \arrow[r, phantom, "="] & Q_\delta & \\
 & P_\eta \arrow[u] & Q_\gamma \arrow[u] & \\
 & P_{\xi+2} \arrow[u, "s"] & Q_{\tau+1} \arrow[u, "s"'] & \\
P_\xi & P_{\xi+1} \arrow[u, "D"'] & R_{\tau+1} \arrow[u, "D"'] & Q_\tau = P_\xi \\
 & & Q_\beta \arrow[u, "F"] \arrow[ur] & \\
 & & Q_0 \arrow[u] &
\end{tikzcd}$$

$F = E^{P_\xi}_{\alpha_\xi}$

$F^+ = E^{u}_{\eta}$

$= F\text{-then-}D$

<u>Claim 15</u> $i_{\xi+1,\theta}(t) = j_{0\delta}(q)$

<u>Proof</u> This is Claim 10 in the proof of 9.6.2 in [CPMP]. The proof here is the same.

In sum:

<u>Subclaim 14A</u> $j_{0,\delta}(g) \leq_{lex} i_{\xi+1,\theta}(t)$.

<u>Proof</u> Let $g = \{g_0, \ldots g_n\}$ and $t = \{t_0 \ldots t_n\}$ where $g_0 > \cdots > g_n$ and $t_0 > \cdots > t_n$. Suppose $j_{0\delta}(g \restriction k) = i_{\xi+1,\theta}(t \restriction k)$ and $i_{\xi+1,\theta}(t_k) < j_{0,\delta}(g_k)$. Since g is solid

$$A = Th_1^{Q_0}(g_k \cup \{g \restriction k\}) \in Q_0.$$

It follows that

$$B = Th_1^{Q_\delta}(j_{0\delta}(g_k) \cup \{j(g \restriction k)\}) \in Q_\delta.$$

But
$$B = Th_1^{P_\theta}(j_{0\delta}(g_k) \cup \{i_{\xi+1,\theta}(t \restriction k)\}),$$

so from B we can compute $Th_1^{P_\theta}(i_{\xi+1,\theta}(t_k)+1, i_{\xi+1,\theta}(t \restriction k))$. It follows that

$$Th_1^{P_\theta}(\alpha^* \cup i_{\xi+1,\theta}(t)) \in Q_\delta.$$

Thus $Th_1^{P_\eta}(\alpha^* \cup i_{\xi+1,\eta}(t)) \in Q_\delta$,

contrary to Claim 3. $\boxtimes$

<u>Subclaim 14B</u> $i_{\xi+1,\theta}(t) \leq_{lex} j_{0\delta}(q)$.

<u>Proof</u> This is a Dodd-Jensen argument.

The diagram is

$$
\begin{tikzcd}
P^*_\theta & & P_\theta \arrow[ll, "\pi_\theta"'] & = & Q_\delta \\
P^*_{\xi+1} = P^*_\xi \arrow[u, "i^*_{\xi+1,\theta}"] & P_\xi \arrow[l, "\pi_\xi"'] & P_{\xi+1} \arrow[l, "\sigma_\xi"'] \arrow[u, dash, "i_{\xi+1,\theta}"'] & & \\
P^*_1 = P^*_0 \arrow[u, "i^*_{0\xi}"] & P_0 \arrow[l, "id"'] \arrow[u, "i_{0\xi}"'] & P_1 \arrow[l, "\sigma_1"'] & & Q_0 = P_0 = P^*_0 \arrow[uu, "j_{0\delta}"']
\end{tikzcd}
$$

The Dodd-Jensen calculation is

$$\pi_\theta \circ j_{0\delta}(q) \geq_{lex} i^*_{0,\theta}(q)$$

and

$$i^*_{0,\theta}(q) = i^*_{\xi+1,\theta} \circ \pi_\xi \circ i_{0\xi}(q)$$

$$= i^*_{\xi+1,\theta} \circ \pi_{\xi+1}(t)$$

$$= \pi_\theta \circ i_{\xi+1,\theta}(t),$$

so

$$\pi_\theta \circ j_{0\delta}(q) \geq_{lex} \pi_\theta \circ i_{\xi+1,\theta}(t)$$

so

$$j_{0\delta}(q) \geq_{lex} i_{\xi+1,\theta}(t).$$

$\boxtimes$

This proves Claim 15.

$\boxtimes$

We can now finish the proof of parameter solidity. We have, for

$$\rho = p_1(Q_0)$$

that

$$Th_1^{Q_0}(\rho \cup q) \in Q_0$$

because $q \neq p_1(Q_0)$. Thus

$$Th_1^{Q_\delta}(\sup j_{0\delta}{}''\rho \cup j_{0\delta}(q)) \in Q_\delta .$$

But $\sup j_{0\delta}{}''\rho = \rho(Q_\delta) = \rho(P_\theta) = \alpha^*$.

So $$Th_1^{P_\theta}(\alpha^* \cup i_{\xi+1,\theta}(t)) \in P_\theta .$$

So $$Th_1^{P_\eta}(\alpha^* \cup i_{\xi+1,\eta}(t)) \in P_\eta$$

(no new subsets of α^* in $P_\theta - P_\eta$). But $P_\eta = Hull_1^{P_\eta}(\alpha^* \cup i_{\xi+1,\eta}(t))$, contradiction.

Parameter solidity. $\boxtimes$

References

[CPMP] J. Steel, A comparison process for mouse pairs, LNL vol. 51, CUP 2022.