Hod pairs with indiscernible Woodins

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§0. We shall give some applications of the method of comparing lbr hod pairs developed in [IJ]. First, we have a general upper bound on the internal definability of the wellorder of $\mathbb{R}$.

<u>Theorem 1</u> Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC such that $P \models ZFC$; then

$$P \models <_{\mathbb{R}} \text{ is } \Sigma^2_3,$$

where $<_{\mathbb{R}}$ is the order of construction on reals.

We don't know whether Σ^2_3 can be improved to Σ^2_2 in Theorem 1. If

$P \models$ "there is a measurable Woodin",

then $P \models$ "$\mathbb{R}$ has no Σ^2_1 wellorder".

More precisely

Def 2 (AD^+)

(a) (M_{emw}, Σ_{emw}) is the minimal pure extender pair (M, Σ) with scope HC such that M is active and $M \models crit(\dot{F}^M)$ is Woodin.

(b) (M_{hmw}, Σ_{hmw}) is the minimal lbr hod pair (M, Σ) s.t. M is active and $M \models crit(\dot{F}^M)$ is Woodin

Theorem 2 ($AD^+_{\mathbb{R}}$) The following are equivalent:

(a) (M_{emw}, Σ_{emw}) exists,

(b) (M_{hmw}, Σ_{hmw}) exists.

That (b) $\to$ (a) is easily proved; one simply does the pure extender pair construction inside M_{hmw}.

(Remark "Woodinness" goes down as $L[\vec{E}]$ would take some work!)

Our proof that (a) $\Rightarrow$ (b) is much more indirect. We use (a) and AD^+ to get a hod pair (N, Λ) such that $M_{\omega}^{\#} \in N$ and $\Sigma_{M_{\omega}^{\#}} \restriction N$ is definable over N. (This is just the proof that LEC $\Rightarrow$ HPC in [2].) Working then in N, where (a) now holds, we get divergent models in N of A containing the reals (of N!). We then use the comparison by least disagreement method of [1] to show that $M_{\omega}^{\#} \trianglelefteq N$.

Remark 3 Sargsyan used divergent models to show that there is a ~~hod pair (P, Σ) such that premouse~~ proper class $P \models$ "δ is a Woodin limit of Woodins and $P \mid \delta$ is a lpm". His proof used comparison by least disagreement below a Woodin limit of Woodins.

Remark 4 Woodin [3] showed that the existence of divergent models (of AD^+ containing all reals) is consistent relative to a Woodin limit of Woodin cardinals. What will give it the strength of measurable Woodins in our context, where V itself is a mouse, is that one of the two models will have the property that its mice are $\omega_2 + 1$ iterable in V.

The hypothesis of Theorem 2 is AD^+, but the conclusion holds in hod mice by the same proof.

Corollary 3 Assume AD^+, λ is a limit of Woodins and let (P, Σ) be a hod pair s.t. $P \models ZFC$; then equivalent are

(i) $P \models (M_{emw}, \Sigma_{emw})$ exists in $D(P, < \lambda)$

(2) $(M_{hmw}, \Sigma_{hmw}) \triangleleft (P, \Sigma)$.

Remark Weaker hypotheses on P should suffice.

It is known that under AD^+,

$$MSC \Rightarrow LEC \Rightarrow HPC.$$

(Cf. [ZJ.]) The proof does not transfer large cardinal properties from pure extender pairs to hod pairs however. The results above doing this are proved using divergent models, so as far as we can see, they are not part of a general method. In fact, we don't see how to answer!

<u>Question I</u> Assume AD^+ and there is a pure extender pair (P, Σ) such that $P \models \exists \kappa(\kappa$ is Woodin $\wedge$ κ is $\kappa+2$-strong$)$. Must there be an lbr hod pair (P, Σ) such that $P \models \exists \kappa(\kappa$ is Woodin $\wedge$ $\kappa+1$-strong (i.e. measurable)$)$?

Returning to the wellorder of $\mathbb{R}$ in a hod mouse, we have the following.

<u>Theorem 4</u> Assume $AD_{\mathbb{R}} + HPC$; then the following are equivalent

(a) $HOD \models$ there is no Σ^2_1 wellorder of $\mathbb{R}$,

(b) $HOD \models <_{HOD} \cap (\mathbb{R} \times \mathbb{R})$ is not Σ^2_1,

(c) (M_{emw}, Σ_{emw}) exists

(d) $M_{hmw} \trianglelefteq HOD$

(e) $HOD \models$ there are divergent models of $AD_{\mathbb{R}} + HPC$ containing all the reals.

The full statement of (e) is: HOD satisfies ~~"$\exists A, B \in \Delta$~~ "there are transitive M, N satisfying $ZF + AD_{\mathbb{R}} + HPC$ such that $\mathbb{R} \subseteq M \cap N$, and $P(\mathbb{R})^M \not\subseteq N$ and $P(\mathbb{R})^N \not\subseteq M$.

<u>Remark 5</u> (c) can be formulated as a first-order property of HOD: "my derived model below Θ satisfies (M_{emw}, Σ_{emw}) exists". This of course implies $M_{emw} \in HOD$ and $\Sigma^+_{emw} \cap HOD|\Theta \in HOD$. (Where Σ^+_{emw} is the canonical extension using $<\Theta$-uBness) (Here $\Theta = \Theta^V$, not Θ^{HOD}.)

<u>Remark 6</u> It seems plausible that "HOD $\models$ there are divergent models of AD^+ + MSC + $V = L(P(\mathbb{R}))$"
and hence HPC

is strictly weaker than indiscernible Woodins, and is roughly at the level of LSA instead.
You might have M and N diverging, with $M, N \models LSA$ and $P(\mathbb{R}) \cap M \cap N = ~~Bad~~\ (\text{Suslin-co-Suslin})^M = (\text{Suslin-co-Suslin})^N$.
Then the divergence between M, N does not yield incomparable ω_1-iterable psuedo-hod-pairs, one of which is truly a hod pair. (At least, not immediately.)

<u>Remark 7</u> Continuing with Remark 6, let's drop the demand that the divergent models be inside a hod mouse. Woodin and Sargsyan showed: the following are equiconsistent

(1) ZFC + $\exists$ Woodin limit of Woodins

(2) ZFC + there are divergent models of AD^+ + MSC + $\Theta_0 = \Theta$ containing $\mathbb{R}$.

It seems plausible that (2) becomes weaker if we drop "$\Theta_0 = \Theta$", and becomes roughly at the strength of LSA. Two divergent models might satisfy $(\text{Suslin-co-Suslin})^M \subseteq (\text{Suslin-co-Suslin})^N$. That probably implies $M \models LSA$, using strong mouse capturing. (This needs to be proved.) But if ~~$M \models LSA$~~ $(\text{Suslin-co-Suslin})^M \subseteq N$, then we aren't getting incomparable ω_1-iterable pseudo-hod-pairs, so the argument for $Con(2) \to Con(1)$ seems to break down.

Question 2 What is the consistency strength of $ZFC +$ "there are divergent models of $ZF + AD^+$ containing all reals and ordinals".

Theorem 4 tells us exactly when the HOD's of determinacy models have a Σ^2_1 wellorder of $\mathbb{R}$. The hod's in pure extender mice have a Σ^2_2 wellorder (cf. [4], Cor. 3.14). So we get

Remark This requires that the mouse is self-iterable, in that it knows how to ω_1+1-iterate its countable initial segments. Otherwise the wellorder may not be OD.

Theorem 5 Assume AD^+ + MSC ; then

$$HOD \models \text{" } <_{HOD} \cap (\mathbb{R}\times\mathbb{R}) \text{ is } \Sigma^2_2 \text{ "}.$$

Again, we don't know whether $AD_{\mathbb{R}}$ + HPC implies HOD $\models$ "$\mathbb{R}$ has a Σ^2_2 wellorder".

One has (letting NLE = "no ω_1+1 - iteration strategy for a pure extender mouse with a long extender")

$$AD^+ + HPC \Longrightarrow NLE,$$

and the MSC conjecture is that NLE $\Rightarrow$ MC (capturing by pure ext mice). Maybe the Σ^2_2 wellorder of $\mathbb{R}$ in HOD is a much deeper fact than the Σ^2_3 def., and the true situation under AD^+ is

$$HPC \to NLE \to MC \to LEC \to HPC,$$

and the key to it all is to prove NLE $\to$ MC.

Remark We don't know whether the long extender mice at the level of κ^{+n} - supercompactness satisfy "$\mathbb{R}$ has a Σ^2_2 - wellorder". The question is whether they are sufficiently self - iterable. Probably at sufficiently closed levels they are, but I don't think that has been proved.

§1. Proof of Theorem I.

We are assuming AD^+, and that (P, Σ) is an lbr hod pair with scope HC s.t. $P \models ZFC$. We work in P. Let $<_P$ be the wellorder of $\mathbb{R}$ by construction stage. We can define $<_P$ using the comparison-to-a-background method of [2], provided P reconstructs its countable levels via a background construction, but even then, the definition won't be Σ^2_n for any $n < \omega$. Instead we use the comparison method of [1].

Let (N, Λ) be a mouse pair with scope HC such that Λ has very strong hull condensation, and ~~hence~~ fully normalizes well. We are in P now, where ω_1 is not measurable. Λ is an (ω_1, ω_1)-strategy, and may not extend to an ω_1+1-strategy. "Has very strong hull condensation and fully normalizes well" doesn't follow from being a mouse pair with scope HC in our current context. We define a possibly partial ~~...~~ extension of Λ to normal trees of length $< \omega_2$ by: for $\mathcal{T}$ of limit length $< \omega_2$, $\mathcal{T}$ on N,

$$\Lambda^*(\mathcal{T}) = b \quad \text{iff}$$

(1) $\mathcal{T}$ is by Λ^*

and

(2) letting $M = L_1[TC(\{N, \mathcal{T}, b\})]$, for club many $X \prec M$, with $\pi_X(\langle \bar{\mathcal{T}}_X, b_X \rangle) = \langle \mathcal{T}, b \rangle$ being the anti-collapse map,

$$\Lambda(\mathcal{T}_X) = b_X .$$

Put another way, $\mathcal{U}$ is by Λ^* iff club many of its countable hulls are by Λ!

<u>Def 1.1</u> A <u>good pair</u> is a mouse pair (N, Λ) with scope HC s.t. Λ has very strong hull condensation and fully normalizes well.

<u>Lemma 1.2</u> Let $N \lhd P|\omega_1^P$, and $\Lambda = \Sigma_N \restriction HC^P$. Then $P \models$ "(N, Λ) is a good pair, and $\Lambda^* = \Sigma_N \restriction H_{\omega_2}$".

<u>Proof</u> AD^+ implies that mouse pairs have strong hull condensation and fully normalize well in V. This easily implies that $(N, \Sigma_N \restriction HC^P)$ is a good pair in P. But Σ_N is total in P, so hull condensation implies that $(\Sigma_N \restriction HC^P)^* = \Sigma_N \restriction H_{\omega_2}^P$.

$\boxtimes$

Working in P, let (M, Ω) and (N, Λ) be good pairs. We attempt to compare them using the method of $[S/J]$. Let us assume that (M, Ω) is more than

good, in fact, $M \lhd P$ and $\Omega = \Sigma_M \restriction HC$. (13)
Suppose also that there are reals x and y such that $x, y \in M \cap N$ and

$$x <_M y \quad \text{and} \quad y <_N x$$

in the orders of construction. Thus (M, Ω) and (N, Λ) cannot be successfully compared.

The attempted comparison produces ~~all~~ systems $\hat{\mathcal{W}}_\alpha = \langle \mathcal{W}_\xi \mid \xi \leq z(\alpha) \rangle$ and $\hat{\mathcal{V}}_\alpha = \langle \mathcal{V}_\xi \mid \xi \leq z(\alpha) \rangle$ for $\alpha \leq \omega_1$ by induction on α. $\mathcal{W}_\xi$ is a tree on M by Ω^* and $\mathcal{V}_\xi$ is a tree on N by Λ^*, ~~until~~ and $(\mathcal{W}_\xi, \mathcal{V}_\xi)$ is a slow comparison, until we reach a $\xi < \omega_2$ such that $\Lambda^*(\vec{\mathcal{V}}_\xi)$ is undefined. The proof in §1J shows that this must eventually happen.

In more detail, notice first that $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$ are defined for all $\alpha < \omega_1$. For Ω and Λ have scope HC, Moreover, if $\alpha < \omega_1$ is a limit ordinal then the completions $\hat{\mathcal{W}}_\alpha$ of $\bigcup_{\beta<\alpha} \hat{\mathcal{W}}_\beta$ and $\hat{\mathcal{V}}_\alpha$ of $\bigcup_{\beta<\alpha} \hat{\mathcal{V}}_\beta$ must exist and have length $z(\alpha) < \omega_1$. For otherwise there are common part trees $\mathcal{W}^-_{\omega_1}$ and $\mathcal{V}^-_{\omega_1}$ of length ω_1, and *since* $\Omega^*(\mathcal{W}^-_{\omega_1})$ *is defined*, we get the contradiction in [IJ], Lemma 10.6. (Distinct extenders used in the branch $\Omega^*(\mathcal{W}^-_{\omega_1})$ have distinct ancestors in the system $\bigcup_{\beta<\alpha} \hat{\mathcal{W}}_\beta$. But there are ω_1 many extenders used in $\Omega^*(\mathcal{W}^-_{\omega_1})$, and only countably many extenders used in $\bigcup_{\beta<\alpha} \hat{\mathcal{W}}_\beta$.)

Thus $\hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}_\alpha$ are defined for all $\alpha < \omega_1$. We let $\hat{\mathcal{W}}^-_{\omega_1,0} = \bigcup_{\alpha<\omega_1} \hat{\mathcal{W}}_\alpha$ and $\hat{\mathcal{V}}^-_{\omega_1,0} = \bigcup_{\alpha<\omega_1} \hat{\mathcal{V}}_\alpha$.
Now we use Ω^* and Λ^* to generate the completion stages $\hat{\mathcal{W}}_{\omega_1,\eta}$ and $\hat{\mathcal{V}}_{\omega_1,\eta}$

by induction on $\eta < \omega_2$. They have the form

$$\hat{\mathcal{W}}_{\omega_1,\eta} = \langle \mathcal{W}_\xi \mid \xi \le z_\eta(\omega_1) \rangle$$

and

$$\hat{\mathcal{V}}_{\omega_1,\eta} = \langle \mathcal{V}_\xi \mid \xi \le z_\eta(\omega_1) \rangle.$$

The arguments in Σ/J show that we can do this as long as $\Omega^*(\mathcal{W}^-_{z_\eta(\omega_1)})$ and $\Lambda^*(\mathcal{V}^-_{z_\eta(\omega_1)})$ are defined, where those are the common part trees from completion stages $< \eta$. Ω^* is total on $H\omega_2$, so if $\Lambda^*(\mathcal{V}^-_{z_\eta(\omega_1)})\downarrow$ for all η, then $\hat{\mathcal{W}}_{\omega_1,\eta}$ and $\hat{\mathcal{V}}_{\omega_1,\eta}$ exist for all $\eta < \omega_2$. But this is impossible by C/J, Lemma 10.6: let $\mathcal{W}^-_{\omega_2}$ be the common part tree on the $\mathcal{W}$ side; then there are ω_2 many extenders used in $\Sigma_M(\mathcal{W}^-_{\omega_2})$, and they have distinct ancestors in $\bigcup_{\beta<\omega_1} \hat{\mathcal{W}}_\beta$, which is impossible as there are only ω_1 many such possible ancestors.

Thus the comparison process produces some $\vartheta^-_{z_\eta(\omega_1)}$ by Λ^* such that $\Lambda^*(\vartheta^-_{z_\eta(\omega_1)})$ is undefined.

This leads to the following Σ^2_3 definition of $<_P \cap (\mathbb{R}\times\mathbb{R})$ in P. Working in P

$x <_P y$ iff $\exists$ good pair (M,Ω) s.t. $[\, x <_M y$
$\wedge$ $\forall$ good pair (N,Λ) s.t.
$y <_N x$ $\forall$ system $\langle \hat{W}_{\omega_1,\beta} \mid \beta<\eta\rangle$
$\langle \hat{\vartheta}_{\omega_1,\beta} \mid \beta<\eta\rangle$ according to Ω^* and
Λ^* $\exists b\,(\Omega^*(W^-_{z_\eta(\omega_1)}) = b)\,]$.

Being good just involves quantifiers over HC.
To say $\langle \hat{W}_{\omega,\beta} \mid \beta<\eta\rangle$ is by Ω^*
and $\langle \hat{\vartheta}_{\omega_1,\beta} \mid \beta<\eta\rangle$ is by Λ^* is ~~to~~ Σ^2_1 about
(HC,Ω,Λ). It is in the antecedent of a "$\forall$ system" conditional, so it gets absorbed into.
So the rhs is Σ^2_3, as desired.

Theorem I. $\boxtimes$

§2. Proof of Theorem 2.

We show first that a certain instance of divergent models implies (M_{hmw}, Σ_{hmw}) exists.

<u>Lemma 2.1</u> Assume ZFC + λ is a limit of Woodin cardinals. Suppose (M, Ω) and (N, Λ) are such that

(a) (M, Ω) and (N, Λ) are good pairs,

(b) Ω is $\mathrm{Hom}_{<\lambda}$, and

(c) Ω is not definable over $(HC, \in, \Lambda)$ and Λ is not definable over $(HC, \in, \Omega)$;

then $(M_{hmw}, \Sigma_{hmw}) \trianglelefteq (M, \Omega)$.

<u>Proof</u> We attempt to compare (M, Ω) with (N, Λ) by the method of [?]. By (c), the comparison cannot terminate at any countable stage. As in the proof of Theorem 1, that gives us some completion stage $\eta < \omega_2$ such that $\hat{W}_{\omega_1, \beta}$ and $\hat{V}_{\omega_1, \beta}$ are

defined for all $\beta < \gamma$, but

$\Lambda^*(\mathcal{V}^-_{z_\eta(\omega_1)})$ does not exist. Because

Ω is $\mathrm{Hom}_{<\lambda}$, it extends to $D(V, <\lambda)$ (uniquely), so it extends to all trees in V_λ uniquely, so $\Omega^*(\mathcal{W}^-_{z_\eta(\omega_1)})$ does exist.

We'll adopt the notation of $\Sigma_1 J$: set

$$\mathcal{W}^- = \mathcal{W}^-_{z_\eta(\omega_1)}$$

$$\mathcal{V}^- = \mathcal{V}^-_{z_\eta(\omega_1)},$$

where recall these are the common part trees, in that

$$\mathcal{S} \trianglelefteq \mathcal{W}^- \text{ iff } \mathcal{S} \trianglelefteq \mathcal{W}_\alpha \text{ for all suff. lg. } \alpha < z_\eta(\omega_1)$$

$$\mathcal{S} \trianglelefteq \mathcal{V}^- \text{ iff } \mathcal{S} \trianglelefteq \mathcal{V}_\alpha \text{ for all suff lg } \alpha < z_\eta(\omega_1),$$

Let $\theta = lh(\mathcal{W}^-) = lh(\mathcal{V}^-)$. θ is a limit ordinal.

Let

$$b = \Omega^*(\mathcal{W}^-).$$

Unlike $\Sigma_1 J$, § 11 we don't have $\Lambda^*(\mathcal{V}^-)$ to work with.

<u>Claim 9</u> $cf(z_\eta(\omega_1)) = \omega_1$.

<u>Proof</u> Otherwise $cf(z_\eta(\omega_1)) = \omega$.

But then we can find a continuous elem chain of countable elem. submodels $X_\alpha \prec (H_{\omega_3}, \in)$ s.t. everything relevant (i.e. the $\hat{\mathcal{W}}$'s and $\hat{\mathcal{T}}$'s, etc.) is in X_0, ~~This~~ and the transitive closure of ~~$\mathcal{W}_{\omega_1,\beta}$~~ $\mathcal{T}^-_{z_\eta(\omega_1)}$ is $\subseteq \bigcup_{\alpha<\omega_1} X_\alpha$.

So we get transitive collapses H_α of X_α with $i_{\alpha\beta} : H_\alpha \to H_\beta$, $\pi_\alpha : H_\alpha \cong X_\alpha \prec H_{\omega_3}$ and $i_{\alpha\beta} = \pi_\beta^{-1} \circ \pi_\alpha$.

Let $\mathcal{T}_\alpha = \pi_\alpha^{-1}(\mathcal{T}^-)$;

then $\mathcal{T}_{\omega_1} = \mathcal{T}^-$, and $i_{\alpha\beta} : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ is a tree embedding, and <u>continuous</u>, in that $i_{\alpha\beta}$ " $lh(\mathcal{T}_\alpha)$ is cofinal in $lh(\mathcal{T}_\beta)$. (20)

(Because $\mathrm{cf}(lh(\mathcal{U}^-)) = \omega$ if $\mathrm{cf}(\tau_\eta(\omega_1)) = \omega$.) Let

$$c_\alpha = \Lambda(\mathcal{T}_\alpha).$$

Since $i_{\alpha\beta}$ is cofinal and Λ has (very strong) hull condensation

$$i_{\alpha\beta} \text{"} c_\alpha \subseteq c_\beta$$

for $\alpha < \beta < \omega_1$. But this means that setting

$$c = \bigcup_{\alpha<\omega_1} i_{\alpha\omega_1} \text{"} c_\alpha$$

we have

$$\Lambda^{\#}(\mathcal{U}^-) = c,$$

contradiction.

Claim I. $\boxtimes$

Since $\operatorname{cof}(\tau_\eta(\omega_1)) = \omega_1$, we have that either $\eta = 0$, η is a succ. ord., or $\operatorname{cof}(\eta) = \omega_1$. (20)

Case 1 $\eta = 0$.

In this case $\mathcal{W}^-$ and $\mathcal{V}^-$ have length $\theta = \omega_1$, and $\mathcal{A} \trianglelefteq \mathcal{V}^-$ iff $\mathcal{A} \trianglelefteq \mathcal{V}_\xi$ for all suff lg $\xi < \omega_1$ $(= \tau_0(\omega_1))$, and similarly for $\mathcal{W}^-$. Let

$$\pi : H \to V$$

where H is countable and transitive and everything relevant is in $\operatorname{ran}(\pi)$. Let

$$\pi(\nu) = \omega_1.$$

So $\nu = \tau_0(\nu)$ and $(\bar{\mathcal{W}}^-, \bar{\mathcal{V}}^-) = (\mathcal{W}^-_\nu, \mathcal{V}^-_\nu)$ are the common part trees at ν, where the overbars stand for $\pi^{-1}(\mathcal{W}^-, \mathcal{V}^-)$. Let

$$c_0 = \operatorname{Lh}(\bar{\mathcal{V}}^-).$$

It will suffice to show $M_{hmw} \not\trianglelefteq M$ implies $c_0 \in H$, for then $\pi(c_0)$ is a cofinal, hence club, branch of $\mathcal{V}^-$, which easily implies $\pi(c_0) = \Lambda^*(\mathcal{V}^-)$.

Claim 2. If $M_{hmw} \not\trianglelefteq M_b^{\mathcal{W}^-}$, then $c_0 \in H$.

Proof. Let $\varepsilon = \delta(\mathcal{W}^-) = \delta(\mathcal{V}^-) = \omega_1$. Since $i^{\mathcal{W}^-}_{\nu b}(\nu) = \varepsilon$ and $\nu = \operatorname{crit}(i_{\nu b})$, there is an extender E on the $M_b^{\mathcal{W}^-}$-sequence such that $\operatorname{crit}(E) = \varepsilon$. But $M_{hmw} \not\trianglelefteq M_b^{\mathcal{W}^-}$, so

$$M_b^{\mathcal{W}^-} \models \varepsilon \text{ is not Woodin.}$$

Let $Q = Q(b, \mathcal{W}^-)$ be the Q-structure. $Q \in M_b^{\mathcal{W}^-}$. Let $\bar{Q}$ be such that

$$\pi(\bar{Q}) = i^{\mathcal{W}^-}_{\nu b}(\bar{Q}) = Q.$$

Then $\bar{Q} \in M_\nu^{\mathcal{W}^-}$, and $\bar{Q}$ is essentially a subset of ν. Since b does not drop after ν (all its finitely many drops are in $\mathrm{ran}(\pi)$), $\bar{Q} \triangleleft M_\xi^{\mathcal{W}^-}$, so

$$\bar{Q} \triangleleft M(\mathcal{W}^-) = M(\nu^-),$$

so $\bar{Q} \triangleleft M_\nu^{\nu^-} = M_{c_0}^{\nu_0}$. Thus

$$\bar{Q} = Q(c_0, \bar{\nu}^-).$$

Since $\bar{Q} \in H$, $c_0 \in H$.

Claim 2. $\boxtimes$

Claim 2 finishes the proof in Case 1.

Lemma 2.1, Case 1. $\boxtimes$

<u>Case 2</u> $\mathrm{cof}(\eta) = \omega_1$.

Again, let $\varepsilon = \delta(\mathcal{W}^-) = \delta(\eta^-)$. Let

$$\pi(\bar{\mathcal{W}}^-, \bar{\nu}^-, \bar{\varepsilon}, \bar{\theta}, \bar{b}) = \langle \mathcal{W}^-, \nu^-, \varepsilon, \theta, b \rangle,$$

where $\pi : H \to V$, H is countable transitive, and $\pi(\bar{\nu}) = \omega_1$. Let also

$$\pi(\bar{\eta}) = \eta,$$

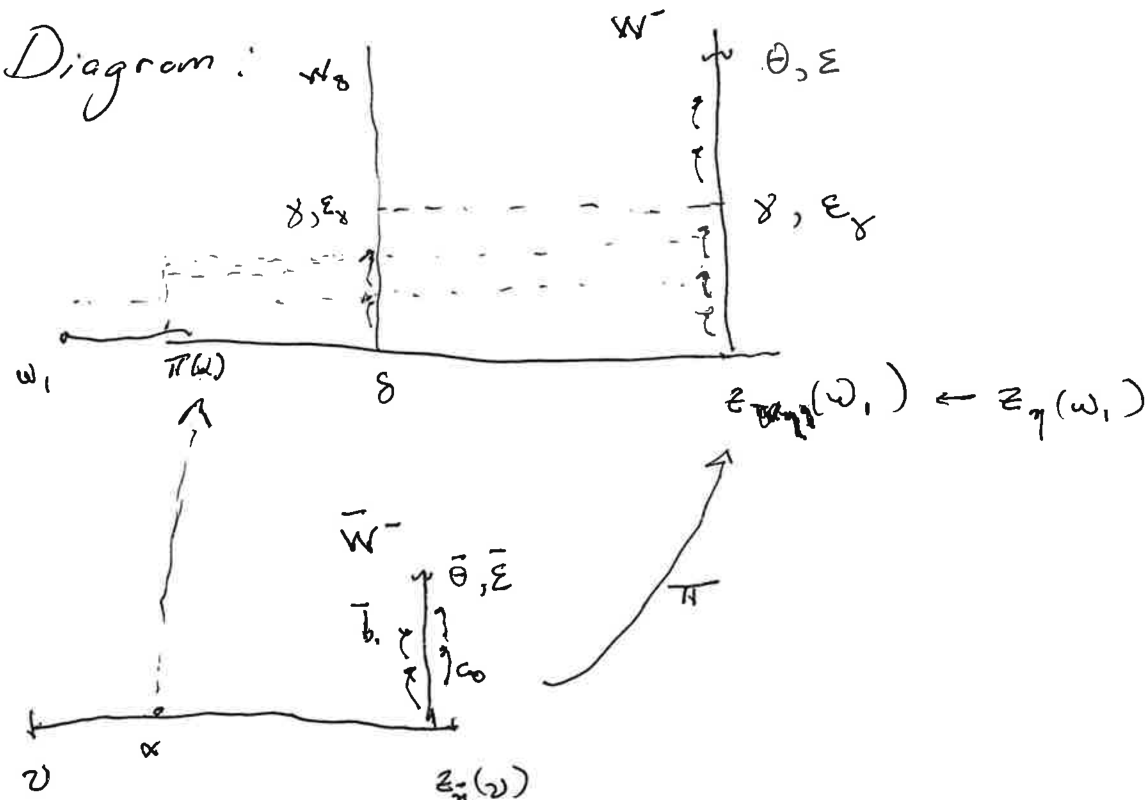
and $$\gamma = \sup \pi'' \bar{\theta}$$

$$\varepsilon_\gamma = \sup \pi'' \bar{\varepsilon}.$$

Notice that $b \cap \gamma = [0, \gamma)_{W^-}$ and $\varepsilon_\gamma = \sup\{ lh(E) \mid E \text{ used in } b \cap \gamma \}$. Let

$$\delta = \sup \pi'' z_{\bar{\eta}}(v).$$

Diagram:



Let $$c_0 = \Lambda(\bar{v}^-).$$

We'd like to see $c_0 \in H$, for in that case we can set $c = \pi(c_0)$, and check that $\Lambda^*(v^-) = c$.

Let $$b_\gamma = b \cap \gamma = [0, \gamma)_{W^-} = [0, \gamma)_{W_\delta}$$

and $$c_\gamma = [0, \gamma)_{V^-} = [0, \gamma)_{V_\delta}.$$

<u>Claim 3</u> $\pi'' c_0 \subseteq c_\gamma$.

<u>Proof</u> Let $d = (\pi^{-1})'' c_\gamma$, and note that d is a cofinal branch of $\bar{\mathcal{V}}^-$. π induces a tree embedding from $\bar{\mathcal{V}}^- \frown d$ into $\mathcal{V}^- \restriction \gamma+1$; call that Φ_π . For club many countable hulls $\mathcal{U}$ of $\mathcal{V}^- \restriction \gamma+1$ we have that $\mathcal{U}$ is by Λ , and a factoring

$$\begin{tikzcd} \bar{\mathcal{V}}^- \frown d \arrow[r, "\Gamma"] \arrow[rr, bend right, "\Phi_\pi"'] & \mathcal{U} \arrow[r, "\Psi"] & \mathcal{V}^- \restriction \gamma+1 \end{tikzcd}$$

where Γ, Ψ are tree embeddings. Since $\mathcal{U}$ is by Λ and (N, Λ) has hull condensation, $\Lambda(\bar{\mathcal{V}}^-) = d$. So $d = c_0$, as desired.

$\boxtimes$

<u>Claim 4</u> $z_{\bar\eta}(\mathcal{V}) <^{\mathcal{V}}_A \delta$.

Proof The arguments of $\Sigma 1 J$ show that there is a unique cofinal branch d of the extender tree $\hat{W}^{\nu, ext}$ such that for

$$p_d = p\text{-map of } \left(\Phi_d^{\nu}\right)_{\tau_{\bar{\eta}}(\nu)}$$

we have

$$\hat{p}_d(e_{\bar{b}}^{\bar{W}^-}) = e_{b_\gamma}^{W^-}$$

and

$$p_d \upharpoonright Ext(\bar{W}^-) = \pi \upharpoonright Ext(\bar{W}^-).$$

(In other notation, $p_d(e_{\bar{b}}) = e_{\bar{b}} \circledast d$.)

So d fits into e_{b_γ}.

On the $\bar{\nu}^-$ side, we get by the same proof a unique cofinal branch d_1 of $\hat{\nu}^{\nu, ext}$ such that for

$$q_{d_1} = p\text{-map of } \left(\Psi_{d_1}^{\nu}\right)_{\tau_{\bar{\eta}}(\nu)},$$

we have

$$q_{d_1}(e_{c_0}^{\bar{\nu}^-}) = e_{c_\gamma}^{\nu^-}$$

and

$$q_{d_1} \upharpoonright Ext(\bar{\nu}^-) = \pi \upharpoonright Ext(\bar{\nu}^-).$$

d_1 fits into $e^{\nu^-}_{c_\gamma}$. Since d and d_1 come from the same π, as initial segments of its tail factors that are missing from $M^{\nu\nu^-}_\gamma$ and $M^{\nu^-}_\gamma$ (and below where they agree), we get

$$d = d_1 .$$

(See [1], §11, Claim 4.2C.). Thus

$$z_{\vec{\eta}}(\vec{\nu}) <^{\vec{\nu}}_A \delta$$

as witnessed by

$$d = a(\vec{\nu}, z_{\vec{\eta}}(\vec{\nu}), \delta).$$

Subcase 1 $\quad M_b^{W^-} \models \varepsilon$ is Woodin

Pf. In this case $M_{\bar{b}}^{\bar{W}^-} \models \bar{\varepsilon}$ is Woodin.

Since $\bar{\varepsilon}$ is regular, the canonical emb

$$t_0 =_{df} (\vec{\Phi}_d)_{\varepsilon_\gamma(\partial)}^{\bar{\gamma}} : M_{\bar{b}}^{\bar{W}^-} \longrightarrow M_{b_\gamma}^{W_\gamma} = M_\gamma^{W^-}$$

is continuous at $\bar{\varepsilon}$, with

$$t_0(\bar{\varepsilon}) = \varepsilon_\gamma .$$

t_0 is also the canonical emb from

$M_{\bar{b}}^{\bar{W}^-}$ to $Ult(M_{\bar{b}}^{\bar{W}^-}, E_\pi \restriction \varepsilon_\gamma)$, and

$i_{\gamma b}^{W^-}$ is the factor map into $\pi(M_{\bar{b}}^{\bar{W}^-}) = M_b^{W^-}$.

So
$$\begin{aligned} \varepsilon &= \pi(\bar{\varepsilon}) \\ &= i_{\gamma b} \circ t_0(\bar{\varepsilon}) \\ &= i_{\gamma b}(\varepsilon_\gamma). \end{aligned}$$

But then ε_γ is Woodin in $M_\gamma^{W^-}$ and the crit of some extender there (maybe the top one).

So we reached a ~~contr~~ Woodin.

Subcase 2 $M_b^{\mathcal{W}^-} \models \mathcal{E}$ is not Woodin

Pf let $Q = Q(b, \mathcal{W}^-)$, so we have $Q \in M_b^{\mathcal{W}^-}$ and $Q \triangleleft M_b^{\mathcal{W}^-}$.

Let $\pi(\bar{Q}) = Q$. ~~We have~~ INma

$$\bar{Q} \ntrianglelefteq M_{c_0}^{\bar{\mathcal{W}}^-}$$

since otherwise H has the Q-structure for c_0, so $c_0 \in H$, and we're done.

Let

$$\hat{Q} = \mathrm{Ult}(\bar{Q}, E_\pi \restriction \mathcal{E}_\gamma)$$

where the ult uses only fcns in $\bar{Q}$; then with t_0 as above

$$\hat{Q} \in M_\gamma^{\mathcal{W}^-} - M_\gamma^{\mathcal{T}^-}$$

by Schlutzenberg's lemma. ($\hat{Q} \subseteq \mathcal{E}_\gamma$, essentially). But $P(\mathcal{E}_\gamma)^{M_\theta^{\mathcal{W}^-}} \subseteq P(\mathcal{E}_\gamma)^{M_\gamma^{\mathcal{T}^-}}$ so $\hat{Q} \notin M_\theta^{\mathcal{W}^-}$, so $(\gamma, \theta)_{\mathcal{W}}$ drops, contradiction. ☒

The two subcases combined prove Lemma 2.1 in Case 2.

$\boxtimes$

Case 3 η is a successor ordinal.

This is a lot like case 2, but a little simpler. We omit the details. Again we get $M_{hmw} \trianglelefteq M_b^{\nu^-}$ from our hypothesis that $\Lambda^*(\nu^-)$ is undefined.

$\boxtimes$

This proves Lemma 2.1.

$\boxtimes$

Corollary 2.2 Assume AD^+, and let (P, Σ) be an lbr hod pair such that $P \models \lambda$ is a limit of Woodins + ZFC; ~~then~~ Suppose $P \models (N, \Lambda)$ is a good pair and $\mathrm{code}(\Lambda) \notin \mathrm{Hom}_{<\lambda}$. Then $(M_{hmw}, \Sigma_{hmw}) \trianglelefteq (P, \Sigma)$.

Cor

Proof It's really a corollary to the proof of 2.1. Working in P, let $M \triangleleft P|\omega_1^P$ be such that $N \in M$, and is coded by a real in N. Let $\Omega = \Sigma_M$. We compare (M, Ω) with (N, Λ). The comparison cannot terminate with the N-side shorter because $\mathrm{code}(\Omega) \in \mathrm{Hom}_{<\lambda}$ and $\mathrm{code}(\Lambda) \notin \mathrm{Hom}_{<\lambda}$. It cannot terminate with the M-side shorter because the real coding N is not in N. So it does not terminate. The proof of 2.1 now shows that $(M_{hmw}, \Sigma_{hmw}) \trianglelefteq (M, \Omega)$.

$\boxtimes$

<u>Corollary 2.3</u> Assume AD^+, and let (P, Σ) be an lbr hod pair such that $P \models ZFC + \lambda$ is a limit of Woodins. Suppose that P satisfies:

"there are models M and N of $AD^+ + HPC$ such that $\mathbb{R} \subseteq M \cap N$ and ~~SM~~ $(\text{Suslin-co-Suslin})^M \not\subseteq N$ and $(\text{Suslin-co-Suslin})^N \not\subseteq M$.

Then $(M_{hmw}, \Sigma_{hmw}) \trianglelefteq (P, \Sigma)$.

<u>Proof</u> At least one of the diverging models has a Suslin-co-Suslin set $B \subseteq \mathbb{R}$ s.t. $B \notin Hom_{<\lambda}$. Say N does. By HPC in N, we have $B \leq_w \text{code}(\Psi)$ for some (Q, Ψ) that is a hod pair in the sense of N. But then

$$P \models (Q, \Psi) \text{ is a good pair s.t. } \text{code}(\Psi) \notin Hom_{<\lambda}.$$

So we can apply 2.2.

$\boxtimes$

We can use the existence of (M_{emw}, Σ_{emw}) to show that there are nonstandard good pairs, and thereby prove Corollary 3.

<u>Lemma 2.4</u> Assume AD^+, and let (P, Σ) be an lbr hod pair s.t. $P \models ZFC + \lambda$ is a limit of Woodin cardinals. Suppose that $D(P, <\lambda) \models$ "(M_{emw}, Σ_{emw}) exists"; then $(M_{hmw}, \Sigma_{hmw}) \trianglelefteq (P, \Sigma)$.

<u>Proof</u> The hypothesis implies that $M_{emw} \in P$ and $\Sigma_{emw} \cap P$ is $\mathrm{Hom}_{<\lambda}$ in P.

P is a model of CH, so working in P, M_{emw} "knows" Σ^2_1 truth, in that the universal Σ^2_1 set of natural numbers is definable over M_{emw}: for φ a Σ^2_1 sentence

$$\varphi \Longleftrightarrow M_{emw} \models \psi[\ulcorner \varphi \urcorner]$$

for some formula ψ.

This implies that $Th(M_{emw})$ is not a Σ^2_1 set of natural numbers, by Tarski's theorem that $Th(M_{emw})$ is not definable over M_{emw}.

That in turn implies there is $B \subseteq \mathbb{R}$ s.t. $B^\#$ exists and $L(B,\mathbb{R}) \models AD^+ + HPC$ and $L(B,\mathbb{R}) \models (M_{emw}, \Sigma_{emw})$ exists, but

$$\cancel{L(B,\mathbb{R})}\ (M_{emw}, \Sigma_{emw}) \neq (M_{emw}, \Sigma_{emw})^{L(B,\mathbb{R})}.$$

Fixing such a B, $(\Sigma_{emw})^{L(B,\mathbb{R})}$ is Suslin-co-Suslin (in fact Δ^2_1) in $L(B,\mathbb{R})$, so we have (N, Λ) s.t.

$$L(B,\mathbb{R}) \models (N, \Lambda) \text{ is an lbr hod pair}$$
$$\text{and } \Sigma_{emw}^{L(B,\mathbb{R})} \leq_w \Lambda.$$

But $code(\Lambda) \notin Hom_{<\lambda}$, as otherwise $\Sigma_{emw}^{L(B,\mathbb{R})} \in Hom_{<\lambda}$, so it extends to an ω_1+1 strategy, contrary to $\cancel{M_{emw}}\ M_{emw}^{L(B,\mathbb{R})} \neq M_{emw}$.

So (N, Λ) is a good pair s.t. $code(\Lambda) \notin Hom_{<\lambda}$, and Corollary 2.2 finishes the proof.

2.4 $\boxtimes$

References

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