Notes on work of Gappo and Sargsyan

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0 Introduction

T. Gappo and G. Sargsyan have recently shown

Theorem 0.1 (Gappo, Sargsyan [2]). *Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair* (P, Σ) *such that* P *is countable,* $\mathrm{Code}(\Sigma)$ *is* Hom_∞*, and* $P \models \mathsf{ZFC}+$ *"there is a Woodin limit of Woodin cardinals; then the Chang model* $L({}^\omega OR)$ *satisfies* AD.

The proof relies heavily on the main theorem of Sargsyan's [3]. Below we shall embellish their proof slightly. Let $F(\alpha, X)$ iff $X \subseteq P_{\omega_1}({}^\omega\alpha)$ and contains a club in $P_{\omega_1}({}^\omega\alpha)$. We shall show

Theorem 0.2. *Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair* (P, Σ) *such that* P *is countable,* $\mathrm{Code}(\Sigma)$ *is* Hom_∞*, and* $P \models \mathsf{ZFC}+$ *"there is a measurable Woodin cardinal. Let* $F(\alpha, X)$ *iff* X *contains a club in* $P_{\omega_1}({}^\omega\alpha)$*; then*

(1) $L({}^\omega OR)[F] \models \mathsf{AD}$*, and*

(2) $L({}^\omega OR)[F] \models$ *"for all* α*,* $\{X \mid F(\alpha, X)\}$ *is an ultrafilter".*

We don't see how to reduce the mouse-existence hypothesis in 0.2 to anything close to that in 0.1.

The proof of 0.2 relies on a corresponding embellishment of the main result of [3]. Below we shall trace through the proofs given by Sargsyan and Gappo-Sargsyan in [3] and [2], and indicate where a few extra steps yield a proof of 0.2.

Woodin had already shown[1] that the conclusions of 0.1 and 0.2 follow from the hypothesis that there are arbitrarily large Woodin limits of Woodin cardinals. This is weaker than the hypothesis of 0.2; we do not know what its relationship to the hypothesis of 0.1 is. Woodin's arguments go through long game determinacy, and seem to be fairly different from those given here. The Gappo-Sargsyan proofs yield natural enlargements of the Chang models $L({}^\omega\mathrm{OR})$ and $L({}^\omega\mathrm{OR})[F]$, and connect those enlargements to the derived models of hod pairs. This connection to the theory of hod pairs gives the proofs a special interest.

[1] See [7].

1 A Chang model over the derived model of a hod mouse

Here we trace through [3]. We shall rely on the basic theory of least branch hod pairs in [5], and the results on full normalization in [6].

Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC such that P is countable. Suppose that

$$P \models \mathsf{ZFC} + \text{“}\delta \text{ is a regular limit of Woodin cardinals”}.$$

Let g be $\mathrm{Col}(\omega, < \delta)$-generic over P, $\mathbb{R}^*_g = \mathbb{R} \cap P[g]$, and $\mathrm{Hom}^*_g = \{p[T] \cap \mathbb{R}^*_g \mid \exists \alpha < \delta(P[g \restriction \alpha] \models T \text{ is absolutely complemented}\}$. So $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$ is the derived model of P at δ, and

$$L(\mathbb{R}^*_g, \mathrm{Hom}^*_g) \models \mathsf{AD}_{\mathbb{R}}$$

by [5, 11.3.9]. The iteration strategies $\Sigma_{P|\alpha}$ for $\alpha < \delta$ extend in a canonical way to trees that are countable in $P[g]$, and the extensions $(\Sigma_{P|\alpha})^g$ are Wadge cofinal in Hom^*_g.[2] By [5, 11.3.2], HOD in the sense of the derived model is an initial segment of a nondropping iterate of P; in fact, letting

$$\theta^g = \text{Wadge ordinal of } \mathrm{Hom}^*_g,$$

we have

$$\mathrm{HOD}^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)} | \theta^g = \bigcup_{\alpha<\delta} M_\infty(P|\alpha^{+,P}, \Sigma^g_{P|\alpha^{+,P}})^{P[g]}.$$

The proof goes by forming in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$ the direct limit system $\mathcal{F}_0$ consisting of all lbr hod pairs (N, Λ) such that N is countable and OD-full, and Λ is OD-fullness preserving. The direct limit of this system is $\mathrm{HOD}|\theta$, and the nondropping iterates of pairs $(P|\alpha^{+,P}, \Sigma_{P|\alpha^{+,P}})$, for $\alpha < \delta$, are cofinal it. Using Boolean-valued comparisons, one can show that the iterates in P are cofinal, so that

$$M_\infty(P|\alpha^{+,P}, \Sigma_{P|\alpha^{+,P}})^P = M_\infty(P|\alpha^{+,P}, \Sigma^g_{P|\alpha^{+,P}})^{P[g]}.$$

Let us look now a stronger direct limit system.

Lemma 1.1. *Let (Q, Λ) and (R, Ψ) be nondropping iterates of (P, Σ); then they can be coiterated by iterating away least extender disagreements to a common (S, Ω).*

Proof. By [6], the strategy of an lbr hod pair is positional, and hence no strategy disagreements show up as we coiterate (Q, Λ) and (R, Ψ). □

Working now in P, where δ is inaccessible, we get that if (Q, Λ) and (R, Ψ) be nondropping iterates of (P, Σ) via trees of size $< \delta$ based on $P|\delta$, then they can be coiterated by least extender disagreement to a common (S, Ω) using trees of size $< \delta$ based on $Q|\delta$ and $R|\delta$. So working in P, we can form a direct limit system

$$\mathcal{F}(P, \Sigma, \delta) = \{(Q, \Lambda) \mid (Q, \Lambda) \text{ is a nondropping iterate of } (P, \Sigma) \\ \text{via a tree of size } < \delta \text{ based on } P|\delta\}$$

[2]See [5, 11.1.1, 11.3.4].

with the order being

$$(Q,\Lambda) \prec (N,\Phi) \text{ iff } (N,\Phi) \text{ is a nondropping iterate of } (Q,\Lambda)$$

and the maps being the iteration maps, and set

$$M_\infty(P,\Sigma,\delta) = \text{ direct limit of } \mathcal{F}(P,\Sigma,\delta).$$

Note that the models (Q,Λ) of the system are proper classes from the point of view of P, although the iterations between them have size $< \delta$. One could also allow nondropping iterations that are countable in $P[g]$, and obtain the system $\mathcal{F}(P,\Sigma,\delta)^g$. Boolean-valued comparisons show that $\mathcal{F}(P,\Sigma,\delta)$ is cofinal in $\mathcal{F}(P,\Sigma,\delta)^g$, so they have the same direct limit.

When possible, we shall suppress P,Σ, and δ, and write $\mathcal{F}$ and $\mathcal{F}_g$ for the two systems. If $(Q,\Lambda) \prec^{\mathcal{F}_g} (R,\Omega)$, then

$$\pi_{Q,R} \colon (Q,\Lambda) \to (R,\Omega)$$

and

$$\pi_{Q,\infty} \colon (Q,\Lambda) \to (M_\infty(P,\Sigma,\delta),\Psi)$$

are the maps of the system $\mathcal{F}_g$. Here Ψ is the tail strategy determined by Σ. We don't need to mention the strategies in the subscript of $\pi_{Q,R}$ or $\pi_{Q,\infty}$ because Σ is positional. Let

$$\delta_\infty = \pi_{P,\infty}(\delta).$$

It is not hard to see that $\mathcal{F}_0$ is a subsystem of $\mathcal{F}$, and

$$\mathrm{HOD}^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}|\theta^g = \begin{cases} M_\infty(P,\Sigma,\delta)|\delta_\infty & \text{if } \delta \text{ is a limit of cutpoints in } P,\\ M_\infty(P,\Sigma,\delta)|\pi_{P,\infty}(\kappa) & \text{if } \kappa \text{ is the least} < \delta \text{ strong cardinal of } P. \end{cases}$$

We are most interested in the second case, where $\theta^g = \pi_{P,\infty}(\kappa)$ and $M_\infty|\delta_\infty$ properly extends $\mathrm{HOD}^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}$.

Definition 1.2. Let (P,Σ) be an lbr hod pair with scope HC, and $P \models \mathsf{ZFC} +$ "δ is a regular limit of Woodin cardinals." Let g be $\mathrm{Col}(\omega,<\delta)$-generic over P; then working in $P[g]$, we set

$$\mathcal{C}_g(P,\delta) = L(\mathbb{R}^*_g, \mathrm{Hom}^*_g, M_\infty, {}^\omega\omega_2),$$

where $M_\infty = M_\infty(P,\Sigma,\delta)$.

Note $\omega_2^{P[g]} = \delta^{+,P}$. The set ${}^\omega\omega_2$ is computed in $P[g]$; that is, all ω-sequences from $P[g]$ are in it.

Now suppose δ is also measurable in P, via the normal measure D. Let g be $\mathrm{Col}(\omega,<\delta)$-generic over P. It is well known that in $P[g]$, D induces a supercompactness measure μ_D that is defined on all $A \subseteq P_{\omega_1}(\mathbb{R}^*_g)$ such that A is definable from parameters in $P \cup {}^\omega\mathrm{OR}$.[3] In $P[g]$

[3] $\mu_D(A) = 1$ iff $\exists X \in D \forall \alpha \in X(\mathbb{R} \cap P[g \upharpoonright \alpha] \in A)$. The definability of A and the homogeneity of the forcing imply that either $\mu_D(A) = 1$ or $\mu_D(P_{\omega_1}(\mathbb{R}) \setminus A) = 1$.

there is for each $\alpha < \omega_2$ a definable[4] surjection of $\mathbb{R}$ onto ${}^\omega\alpha$, and hence a definable surjection π_α of $P_{\omega_1}(\mathbb{R})$ onto $P_{\omega_1}({}^\omega\alpha)$. So we can define

$$\mu_D^\alpha(B) = 1 \Leftrightarrow \mu_D(\pi_\alpha^{-1}(B)) = 1,$$

and μ_D^α is defined on all $B \subseteq P_{\omega_1}({}^\omega\alpha)$ such that B is definable in $P[g]$ from parameters in $P \cup {}^\omega\mathrm{OR}$. μ_D is fine and normal on its domain[5], and thus the μ_D^α are all fine and normal on their domains.

Definition 1.3. Let (P, Σ) be an lbr hod pair with scope HC, and $P \models \mathsf{ZFC}+$ "δ is a measurable limit of Woodin cardinals, as witnessed by the normal measure D on δ". Let g be $\mathrm{Col}(\omega, <\delta)$-generic over P; then working in $P[g]$, we set

$$\mathcal{C}_g(P, D)^+ = L(\mathbb{R}_g^*, \mathrm{Hom}_g^*, M_\infty, {}^\omega\omega_2)[F_D],$$

where $M_\infty = M_\infty(P, \Sigma, \delta)$, and $F_D(\alpha, B)$ iff $\alpha < \omega_2$ and $\mu_D^\alpha(B) = 1$.

Notice that $\mathbb{R}_g^* = \mathbb{R} \cap P[g]$, so $\mathbb{R}_g^*$ is OD in $P[g]$. Building on this, it is not hard to see that every set in $\mathcal{C}_g(P, D)^+$ is definable in $P[g]$ from parameters in $P \cup {}^\omega\mathrm{OR}$. Thus for all $\alpha < \delta^{+,P}$,

$$\mathcal{C}_g(P, D)^+ \models \{B \mid F_D(\alpha, B)\} \text{ is a fine, normal ultrafilter on } P_{\omega_1}({}^\omega\alpha).$$

Sargsyan [3] proves part (a) of the following theorem. The proof of (b) is nearly the same.

Theorem 1.4. *[Sargsyan [3]] Assume* AD^+ *and let* (P, Σ) *be an lbr hod pair with scope HC. Suppose that* $P \models$ *"δ is a regular limit of Woodin cardinals, and g is* $\mathrm{Col}(\omega, <\delta)$*-generic over* P*; then*

(a) $P(\mathbb{R}_g^*) \cap \mathcal{C}_g(P) = \mathrm{Hom}_g^*$*, and thus* $\mathcal{C}_g(P) \models \mathsf{AD}_\mathbb{R}$*, and*

(b) if δ is measurable via the normal measure D, then $P(\mathbb{R}_g^*) \cap \mathcal{C}_g(P, D)^+ = \mathrm{Hom}_g^*$*, so that* $\mathcal{C}_g(P, D)^+ \models \mathsf{AD}_\mathbb{R}$.

Proof. For definiteness, we prove (b). Let $A \subseteq \mathbb{R}_g^*$ and $A \in \mathcal{C}_g(P, D)^+$. Writing $M_\infty = M_\infty(P, \Sigma, \delta)$, we have that A is definable over $\mathcal{C}_g(P, D)^+$ from some ordinal α, some real x_0, some $t\colon \omega \to \delta^{+,P}$, M_∞, and some Hom_g^* set. Let us regularize the parameters.

We assume toward contradiction that $A \notin \mathrm{Hom}_g^*$, and take α least such that some $A \notin \mathrm{Hom}_g^*$ is definable over $\mathcal{C}_g(P, D)^+$ from such parameters. Then α is definable over $\mathcal{C}(P, D)^+$, so can assume $\alpha = 0$.

The iteration strategies $\Sigma_{P|\gamma}^g$ are Wadge cofinal in Hom_g^*, so by enlarging our real x_0 we may assume the Hom_g^* parameter is $\Sigma_{P|\gamma_0}^g$, where $\gamma_0 < \delta$. So we can fix a formula φ such that

$$z \in A \text{ iff } \mathcal{C}_g(P, D)^+ \models \varphi[\mathrm{Hom}_g^*, M_\infty, F_D, x_0, \Sigma_{P|\gamma_0}^g, t, z],$$

[4]From parameters in P.

[5]For example, if $x \mapsto A_x$ is $\mathrm{OD}(P \cup {}^\omega\mathrm{OR})^{P[g]}$ and $\mu_D(A_x) = 1$ for all $x \in \mathbb{R}_g^*$, then for μ_D a.e. σ, $\forall x \in \sigma(\sigma \in A_x)$.

where $M_\infty = M_\infty(P, \Sigma, \delta)$. [6]

Since

$$\delta_\infty^{+,M_\infty} = \delta^{+,P},$$

we may assume $\mathrm{ran}(t) \subseteq \delta_\infty$. Since $\mathrm{cof}(\delta_\infty) = \delta = \omega_1$ in $P[g]$, $\mathrm{ran}(t)$ is then bounded in δ_∞. $\mathcal{F}(P, \Sigma, \delta)$ is countably directed, so we have $(P, \Sigma) \prec (Q, \Sigma_Q) \in \mathcal{F}(P, \Sigma, \delta)$ such that $\mathrm{ran}(t) \subseteq \mathrm{ran}(\pi_{Q,\infty})$. We assume γ_0 was chosen large enough that for $\gamma_1 = \pi_{P,Q}(\gamma_0)$,

$$\mathrm{ran}(t) \subseteq \pi_{Q,\infty}``\gamma_1.$$

Let x_1 be a real that codes x_0, $Q|\gamma_1$, the function

$$s(i) = \pi_{Q,\infty}^{-1}(t(i)),$$

and the embedding

$$\pi_0 = \pi_{P,Q} \restriction P|\gamma_0.$$

Letting γ_2 be the least Woodin of Q strictly above γ_1, we may assume that P-to-Q includes a genericity iteration such that $x_1 \in Q[h_0]$, where h_0 is $\mathrm{Col}(\omega, \gamma_2)$-generic over Q and $h_0 \in P[g]$. Thus

$$x_0, s, \pi_0 \in Q[h_0].$$

The following sublemma captures the absoluteness of our definition of A that is behind everything.

Sublemma 1.5. Let $(Q, \Sigma_Q) \prec^{\mathcal{F}} (R, \Sigma_R)$, where $\mathrm{crit}(\pi_{Q,R}) > \gamma_2$, and let $D_R = \pi_{P,R}(D)$. Let $h_1 \in P[g]$ be generic over $R[h_0]$ for some poset of size $< \delta$; then for any real $z \in R[h_0][h_1]$,

$$z \in A,$$

if and only if

$$\begin{aligned}&R[h_0][h_1] \models 1 \text{ forces in } \mathrm{Col}(\omega, <\delta) \text{ that}\\ &\qquad C_{\dot g}(R, D_R)^+ \models \varphi[\mathrm{Hom}^*_{\dot g}, F_{D_R}, M_\infty(R, \Sigma_R, \delta), x_0, (\Sigma^{\dot g}_{R|\gamma_1})^{\pi_0}, \pi_{R,\infty}(s), z].\end{aligned}$$

Proof. In the formula being forced by $\mathrm{Col}(\omega, <\delta)$, $\dot g$ is the name for the $\mathrm{Col}(\omega, <\delta)$ generic. The other objects $(\Psi, D_R, M_\infty(R, \Psi, \delta), x_0, \pi_0, s, z)$ belong to $R[h_0][h_1]$, and we should have written checks for their forcing names.

To prove the sublemma, let $i\colon R \to S$ come from an $\mathbb{R}^*_g$-genericity iteration of R with $\mathrm{crit}(i)$ above the size of the forcing that gave us h_1, so that i lifts to

$$i\colon R[h_0][h_1] \to S[h_0][h_1],$$

[6] M_∞ is a proper class of $\mathcal{C}_g(P, D)^+$, so φ must use it as a predicate. A would have to be definable from $M_\infty|\delta^{++,P}$ anyway.

and we have k that is $\mathrm{Col}(\omega, <\delta)$-generic over $S[h_0][h_1]$ such that

$$\mathbb{R}^*_g = \mathbb{R} \cap S[h_0][h_1][k].$$

i comes from a normal stack of normal trees $\langle \mathcal{T}_\alpha \mid \alpha < \delta \rangle$ is constructed in $P[g]$ from an enumeration $\langle x_\alpha \mid \alpha < \delta \rangle$ of $\mathbb{R}^*_g$. $\mathcal{T}_\alpha$ makes x_α generic for the collapse of the next Woodin cardinal going up. For each $\alpha < \delta$, $\langle \mathcal{T}_\xi \mid \xi < \alpha \rangle$ is countable in $P[g]$, so that its last model $(S_\alpha, \Psi_\alpha) \in \mathcal{F}(P, \Sigma, \delta)$. $(S, \Psi_{\vec{\mathcal{T}},S})$ is not itself in $\mathcal{F}(P, \Sigma, \delta)$ because the iteration leading to it is too long.

So $S_0 = R$, $S_\delta = S$, and for $\alpha < \delta$, S_α is the base model of $\mathcal{T}_\alpha$. Let

$$i_{\alpha,\beta} \colon S_\alpha \to S_\beta$$

be the iteration map, so that $i_{\alpha,\beta} = \pi^{\mathcal{F}_P}_{S_\alpha, S_\beta}$ when $\alpha < \beta < \delta$, and $i_{0,\delta} = i$. We can arrange that all critical points of $\mathcal{T}_\alpha$ are strictly greater than ν, where ν is the least inaccessible strictly greater than $i_{0,\alpha}(\alpha)$. This has the consequence that

$$\text{for } D\text{-a.e. } \alpha(i(\alpha) = \alpha).$$

Thus $i(\delta) = \delta$, and for any $X \in P \cap S$, $X \in D$ iff $X \in D_S$.

The most difficult point in Sargsyan's argument is

$$M_\infty(R, \Sigma_R, \delta) = M_\infty(S, \Sigma_S, \delta),$$

and

$$\pi^{\mathcal{F}_R}_{R,\infty}\text{``}(s) = \pi^{\mathcal{F}_S}_{S,\infty}\text{``}(s).$$

Let us assume this and finish the proof of 1.5.

Let l be a re-arrangement of $\langle h_0, h_1, k \rangle$ as a $\mathrm{Col}(\omega, <\delta)$-generic over S. Since $\mathrm{crit}(i) > \gamma_2$ and $i \colon R[h_0][h_1] \to S[h_0][h_1]$ is elementary, it is enough to show

$$z \in A \text{ iff } C_l(S, D_S)^+ \models \varphi[\mathrm{Hom}^*_l, F_{D_S}, M_\infty(S, \Sigma_S, \delta), x_0, (\Sigma^l_{S|\gamma_1})^{\pi_0}, \pi_{S,\infty}(s), z].$$

But note that $\mathbb{R} \cap P[g] = \mathbb{R} \cap S[l]$, and since $\delta^{+,P} = \delta^{+,S}$,

$$({}^\omega\omega_2)^{P[g]} = ({}^\omega\omega_2)^{S[l]}.$$

The strategies $\Sigma^l_{S|\alpha}$, for $\alpha < \delta$ a cardinal of S, are each projective in some tail of a strategy of the form $(\Sigma^g_{P|\gamma}$ for $\gamma < \delta$, where $P|\gamma$ iterates past $S|\alpha$ without dropping. Every $(\Sigma^g_{P|\gamma}$ is projective in its tails corresponding to nondropping iteration, by pullback consistency. Thus we have

$$\mathrm{Hom}^*_g = \mathrm{Hom}^*_l.$$

Since D and D_S agree on the sets $P[g] \cap S[l]$ and $M_\infty(P, \Sigma, \delta) = M_\infty(S, \Sigma_S, \delta)$, we get

$$C_g(P, D)^+ = C_l(S, D_S)^+.$$

This implies that for all $z \in \mathbb{R} \cap P[g]$,

$$C_g(P, D)^+ \models \varphi[\mathrm{Hom}^*_g, M_\infty, F_D, x_0, \Sigma^g_{P|\gamma_0}, t, z]$$

if and only if

$$C_l(S, D_S)^+ \models \varphi[\mathrm{Hom}^*_l, M_\infty(S, \Sigma_S, \delta), x_0, (\Sigma^l_{S|\gamma_1})^{\pi_0}, \pi_{S,\infty}``s, z].$$

(Note here that $t = \pi^{\mathcal{F}_R}_{R,\infty}``(s) = \pi^{\mathcal{F}_S}_{S,\infty}``(s)$.) This equivalence yields Sublemma 1.5.

For the proof that $M_\infty(R, \Sigma_R, \delta) = M_\infty(S, \Sigma_S, \delta)$ and $\pi_{R,\infty}\mathcal{F}_R``(s) = \pi^{\mathcal{F}_S}_{S,\infty}``(s)$, the reader should see [3, Theorem 3.8]. We shall just sketch the main points.

The proof uses some elementary facts about the iteration trees that result from comparing iterates of a single mouse pair that go beyond Lemma 1.1. These facts have some general interest, so we prove them here. There are also versions of these facts proved in [4].

Definition 1.6. Suppose (M, Ω) is a mouse pair. We say that (Q, Ψ) is *γ-sound over* (M, Ω) iff (Q, Ψ) is a nondropping iterate of (M, Ω) with iteration map i, and $Q = \mathrm{Hull}^Q(\gamma \cup \mathrm{ran}(i))$.

The next two lemmas can be applied to mouse pairs in $\mathcal{F}(P, \Sigma)^g$, but they are more general than that, so we leave the background hypotheses somewhat vague. Like Lemma 1.1, they are consequences of full normalization.

Lemma 1.7. *Let (Q, Ψ) and (R, Φ) be γ-sound over (M, Ω), and suppose $Q|\gamma = R|\gamma$; then $(Q, \Psi) = (R, \Phi)$.*

Proof. There are normal trees $\mathcal{T}$ and $\mathcal{U}$ leading from (M, Ω) to (Q, Ψ) and (R, Φ) respectively. Letting E and F be the extenders of their main branches, $\nu(E) \leq \gamma$ and $\nu(F) \leq \gamma$ by γ-soundness. But $E \restriction \gamma = F \restriction \gamma$ because $\mathcal{T}$ and $\mathcal{U}$ come from iterating disagreements with $Q|\gamma = R|\gamma$. Thus $E = F$, so $(Q, \Psi) = (R, \Phi)$. □

Let α be a regular cardinal of N. We say that a tree $\mathcal{S}$ on N is *based on* $N|\alpha$ if $\mathcal{S} = \mathcal{T}^+$ for some $\mathcal{T}$ on $N|\alpha$.[7] It is easy to see that a normal tree $\mathcal{S}$ is based on $N|\alpha$ iff for all $\xi + 1 < \mathrm{lh}(\mathcal{S})$, either the partial iteration map $\hat{i}^{\mathcal{S}}_{0,\xi}$ is undefined at α, or $\mathrm{lh}(E^{\mathcal{S}}_\xi) < \hat{i}_{0,\xi}(\alpha)$.

If $\mathrm{crit}(E^{\mathcal{S}}_\xi) > \nu$ for all ξ, then we say $\mathcal{S}$ is *above* ν.

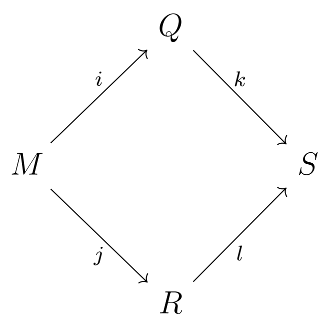
Lemma 1.8. *Let (M, Ω) be a mouse pair, and (Q, Ψ) and (R, Φ) nondropping iterates of (M, Ω) via stacks based on $M|\alpha$ and above ν, where α is a cardinal of M. Let $i = i^{\mathcal{T}}$ and $j = i^{\mathcal{U}}$ be the iteration maps, and let $\mathcal{W}$ and $\mathcal{V}$ be the normal trees on Q and R with common last model (S, Λ) that come from iterating away least extender disagreements. Then*

(1) $\mathcal{W}$ is based on $Q|i(\alpha)$ and above ν, and $\mathcal{V}$ is based on $R|j(\alpha)$ and above ν, and

(2) if all measures in the branch extenders of M-to-Q and M-to-R concentrate on η, then all measures in the branch extender of M-to-S concentrate on η.

[7] $\mathcal{T}^+$ is the lift of $\mathcal{T}$ under the identity map. See [5, 4.5.19].

Proof. We start with (2). Let



be the comparison maps. Thus $k \circ i = l \circ j$. Let $\eta^* = k \circ i(\eta) = l \circ j(\eta)$, and $E = E_{k \circ i} = E_{l \circ j}$. Then $\mathrm{Ult}(M, E \upharpoonright \eta^*)$ is a model of $\mathcal{W}$, namely the first model N on the main branch of $\mathcal{W}$ such that $\mathrm{crit}(i^{\mathcal{W}}_{N,S}) > \eta^*$. Similarly, $\mathrm{Ult}(M, E \upharpoonright \eta^*)$ is a model of $\mathcal{V}$. Since we were iterating away extender disagreements, $S = \mathrm{Ult}(M, E \upharpoonright \eta^*)$, as desired.

For (1), suppose E is an extender of minimal length used in $\mathcal{W}$ or $\mathcal{V}$ that is bad, in the sense that either $\mathrm{crit}(E) \leq \nu$, or $E = E^{\mathcal{W}}_\mu$ and $\hat{i}^{\mathcal{W}}_{0,\mu} \circ i(\alpha) < \mathrm{lh}(E)$, or $E = E^{\mathcal{V}}_\eta$ and $\hat{i}^{\mathcal{V}}_{0,\eta} \circ j(\alpha) < \mathrm{lh}(E)$. By the symmetry, we may assume that $E = E^{\mathcal{W}}_\mu$. Note that since $\hat{i}^{\mathcal{W}}_{0,\mu}(i(\alpha))$ is defined and $i(\alpha)$ is a cardinal of Q, the branch $[0, \mu]_W$ of $\mathcal{W}$ does not drop at all.

Let

$$\mathcal{X} = X(\mathcal{T}, \mathcal{W} \upharpoonright \mu + 1)$$

and

$$\mathcal{Y} = X(\mathcal{U}, \mathcal{V} \upharpoonright \mu^* + 1)$$

be the full normalizations, where μ^* is least such that $\mathrm{lh}(E^{\mathcal{V}}_{\mu^*}) \geq \mathrm{lh}(E)$. $\mathcal{W} \upharpoonright \mu + 1$ and $\mathcal{V} \upharpoonright \mu^* + 1$ are based on $Q|i(\alpha)$ and $R|j(\alpha)$ respectively, and above ν, by our choice of E. It follows that $\mathcal{X}$ and $\mathcal{Y}$ are based on $M|\alpha$ and above ν.

The last models of $\mathcal{X}$ and $\mathcal{Y}$ are $M^{\mathcal{W}}_\mu$ and $M^{\mathcal{V}}_{\mu^*}$, and

$$M^{\mathcal{W}}_\mu || \mathrm{lh}(E) = M^{\mathcal{V}}_{\mu^*} || \mathrm{lh}(E).$$

Let γ be least such that E is on the sequence of $M^{\mathcal{X}}_\gamma$, that is, least such that either $\mathrm{lh}(E^{\mathcal{X}}_\gamma) \geq \mathrm{lh}(E)$ or $\gamma + 1 = \mathrm{lh}(\mathcal{X})$. $\mathcal{X}$ and $\mathcal{Y}$ are normal and have last models that agree to $\mathrm{lh}(E)$, so

$$\mathcal{X} \upharpoonright \gamma + 1 = \mathcal{Y} \upharpoonright \gamma + 1.$$

E is on the sequence of $M^{\mathcal{X}}_\gamma = M^{\mathcal{Y}}_\gamma$, and E is not on the sequence of $M^{\mathcal{Y}}_\infty = M^{\mathcal{V}}_{\mu^*}$ because it was part of a disagreement, so

$$E = E^{\mathcal{Y}}_\gamma.$$

So if $\mathrm{crit}(E) \le \nu$, then $\mathcal{Y}$ uses an extender with critical point $\le \nu$, contradiction.

Thus $\mathrm{crit}(E) > \nu$, $[0, \mu]_W \cap D^{\mathcal{W}} = \emptyset$, and $i^{\mathcal{W}}_{0,\mu} \circ i(\alpha) < \mathrm{lh}(E)$. We claim $\gamma + 1 = \mathrm{lh}(\mathcal{X})$. For otherwise $[0, \infty]_X$ uses an extender F such that $\mathrm{lh}(F) > \mathrm{lh}(E)$, and $\mathrm{lh}(F) < i^{\mathcal{X}}_{0,\infty}(\alpha)$ because $\mathcal{X}$ is based on $M|\alpha$[8], so $\mathrm{lh}(E) < i^{\mathcal{X}}_{0,\infty}(\alpha) = i^{\mathcal{W}}_{0,\mu} \circ i(\alpha)$, contradiction.

But then

$$\begin{aligned} i^{\mathcal{Y}}_{0,\gamma}(\alpha) &= i^{\mathcal{X}}_{0,\gamma}(\alpha) \\ &= i^{\mathcal{X}}_{0,\infty}(\alpha) \\ &= i^{\mathcal{W}}_{0,\mu} \circ i(\alpha) < \mathrm{lh}(E). \end{aligned}$$

Since $E = E^{\mathcal{Y}}_{\gamma}$, we have that $\mathcal{Y}$ is not based on $M|\alpha$, contradiction. □

Let us drop the iteration strategies from our notation when it is clear how to fill them in. We shall define an isomorphism $j \colon M_\infty(S)|\delta^S_\infty \to M_\infty(R)|\delta_\infty$. Thus $\delta^S_\infty = \delta_\infty$, and since $M_\infty(R)$ and $M_\infty(S)$ are both δ_∞-sound, $M_\infty(R) = M_\infty(S)$ by Lemma 1.7.

Let $x \in M_\infty(S)|\delta^S_\infty$, and pick W_0 and x_0 such that

$$x = \pi^{\mathcal{F}_S}_{W_0,\infty}(x_0).$$

Let $\mathcal{U}_0$ be the stack from S to W_0[9], and pick a regular cardinal α of R such that

(i) $i_{0,\alpha}(\alpha) = \alpha$,

(ii) $\mathcal{U}_0$ is based on $S|\alpha = S_\alpha|\alpha$, and

(iii) $x_0 \in W_0|\pi_{S,W_0}(\alpha)$.

Let N_0 be the last model of $\mathcal{U}_0$ when it is regarded as an iteration tree on S_α, and let

$$k_0 \colon N_0 \to W_0$$

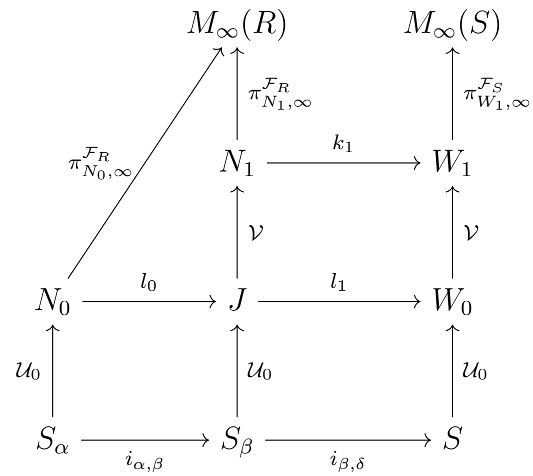
come from copying $i_{\alpha,\delta} \colon S_\alpha \to S$ via the iteration map of $\mathcal{U}_0$. It is important that k_0 is itself an iteration map, via the stack of lifts of the $\mathcal{T}_\eta$ for $\alpha \le \eta < \delta$. Note also $\mathrm{crit}(k_0) > \alpha$, so $k_0(x_0) = x_0$. We now set

$$j(x) = \pi^{\mathcal{F}_R}_{N_0,\infty}(x_0).$$

We must see that $j(x)$ is independent of our choices for $\mathcal{U}_0$ and α. (These determine W_0, x_0, N_0, and k_0.) Suppose $\mathcal{U}_1$ and β are chosen instead, with associated W_1, x_1, N_1, and k_1. We may assume that $\mathcal{U}_1 = \langle \mathcal{U}_0, \mathcal{V} \rangle$ for some $\mathcal{V}$ on W_0 and $\alpha < \beta$. The relevant diagram is

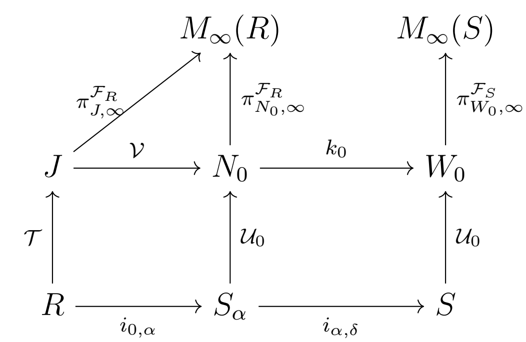
[8] Let $F = E^{\mathcal{X}}_{\xi}$ be applied to $M^{\mathcal{X}}_{\beta}$; then $\mathrm{crit}(F) < \lambda(E^{\mathcal{X}}_{\beta}) < i^{\mathcal{X}}_{0,\beta}(\alpha)$, so $\mathrm{lh}(F) < i^{\mathcal{X}}_{0,\xi+1}(\alpha)$.

[9] $\mathcal{U}_0$ can be taken to be a single normal tree, but we don't need that.



Here $k_0 = l_1 \circ l_0$. Since l_0 is an iteration map, $\pi^{\mathcal{F}_R}_{N_0,\infty} = \pi^{\mathcal{F}_R}_{N_1,\infty} \circ i^{\mathcal{V}} \circ l_0$. The commutativity of the diagram implies that $\pi^{\mathcal{F}_R}_{N_0,\infty}(x_0) = \pi^{\mathcal{F}_R}_{N_1,\infty}(x_1)$, as desired.

We must also see that j is surjective. The relevant diagram is below. Let $y \in M_\infty(R)|\delta_\infty$ and $y = \pi^{\mathcal{F}_R}_{J,\infty}(y_1)$. Let $\mathcal{T}$ be the tree from R to J, and let α be a regular cardinal of R such that $\alpha < \delta$, $i_{0,\alpha}(\alpha) = \alpha$, and $\mathcal{T}$ is based on $R|\alpha$. Let $(\mathcal{V}, \mathcal{U}_0)$ be the coiteration trees on J and S_α, with common last model N_0. Note that $\mathcal{U}_0$ is based on $S_\alpha|\alpha$ by Lemma 1.8. Since $\mathrm{crit}(i_{\alpha,\delta}) > \alpha$, we can regard $\mathcal{U}_0$ as a tree on S, with last model W_0. Letting $k_0 \colon N_0 \to W_0$ come from lifting $i_{\alpha,\beta}$, we have $\mathrm{crit}(k_0) > \alpha$. Let $x_0 = i^{\mathcal{V}}(y_1) = k_0(i^{\mathcal{V}}(y_1))$, and $x = \pi^{\mathcal{F}_S}_{W_0,\infty}(x_0)$. One can easily check that $j(x) = y$. Here is the relevant diagram:



This completes the proof of Lemma 1.5. □

We can now finish the proof of Theorem 1.4. Let ν be the least Woodin cardinal of R strictly greater than γ_2. Let τ be a term such that whenever h_1 is $\mathrm{Col}(\omega,\nu)$-generic over R,

$$z \in \tau_{h_1} \text{ iff } R[h_0][h_1] \models 1 \text{ forces in } \mathrm{Col}(\omega, <\delta) \text{ that } C_{\dot g}(R, D_R)^+ \models \varphi[\mathrm{Hom}^*_{\dot g}, F_{D_R}, M_\infty(R, \Sigma_R, \delta), x_0, (\Sigma^{\dot g}_{R|\gamma_1})^{\pi_0}, \pi_{R,\infty}(s), z].$$

Then $z \in A$ iff there is an iteration map $i \colon R \to Q$ according to $\Sigma^g_{R|\nu}$ and a generic h for $\mathrm{Col}(\omega, i(\nu))$ such that $z \in i(\tau)_h$. Thus A is projective in $\Sigma^g_{R|\nu}$, so that $A \in \mathrm{Hom}^*_g$, as desired. □

2 Proof of Theorem 0.2

Suppose that there are arbitrarily large Woodin cardinals, and let (P, Σ) be an lbr hod pair such that P is countable, $\mathrm{Code}(\Sigma)$ is Hom_∞, and

$$P \models \mathsf{ZFC} + \text{“}\delta \text{ is a measurable Woodin cardinal.}$$

Let $F(\alpha, X)$ iff X contains a club in $P_{\omega_1}({}^\omega\alpha)$; we wish to show that $L({}^\omega\mathrm{OR})[F] \models \mathsf{AD}$, and $L({}^\omega\mathrm{OR})[F] \models$ “for all α, $\{X \mid F(\alpha, X)\}$ is an ultrafilter”. Suppose not, and let α_0 be the least bad level of $L({}^\omega\mathrm{OR})[F]$; that is, let α_0 be least such that either

(1) $L_{\alpha_0}({}^\omega\mathrm{OR})[F] \models \neg\mathsf{AD}$, or

(2) there is some $\eta < \alpha_0$ and $X \subseteq P_{\omega_1}({}^\omega\eta)$ such that X is definable over $L_{\alpha_0}({}^\omega\mathrm{OR})[F]$ and neither $F(\eta, X)$ nor $F(\eta, P_{\omega_1}({}^\omega\eta) \setminus X)$.

We may assume without loss of generality that $\alpha_0 < \omega_2$ and CH holds. For letting G be $\mathrm{Col}(\omega_1, \alpha_0)$-generic over V, our hypotheses still hold in $V[G]$, and because no new countable sequences of ordinals are added and stationarity in $P_{\omega_1}(Z)$ is preserved, $(L({}^\omega\mathrm{OR})[F])^V = (L({}^\omega\mathrm{OR})[F])^{V[G]}$, and α_0 is still the least bad level of $L({}^\omega\mathrm{OR})[F]^{V[G]}$.

By CH, we can fix $A \subseteq \omega_1$ such that A codes $L_{\alpha_0}({}^\omega\mathrm{OR})[F]$ as well as the relevant clubs. (That is, if $\eta, X \in L_{\alpha_0}({}^\omega\mathrm{OR})[F]$ and $F(\eta, X)$, then X contains a club that is coded into A.)

We now construct a genericity iteration of (P, Σ) analogous to the iteration that occurs in the proof that iterable mice with measurable Woodin cardinals can compute $(\Sigma^2_1)^{V^{\mathrm{Col}(\omega_1,\mathbb{R})}}$ truth.[10] Let $\mathbb{B}$ be the δ-generator extender algebra of P, and let D be the order zero measure of P on δ. We iterate P by Σ so as to make A generic over the image of $\mathbb{B}$, iterating away extenders that induce axioms not satisfied by A when we encounter them, and using the current image of D to continue if there are no such extenders.

The result is an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$ on (P, Σ) with associated iteration map

$$i \colon P \to Q = M^{\mathcal{T}}_{\omega_1}$$

such that

(1) $i(\delta) = \omega_1$,

(2) A is $i(\mathbb{B})$-generic over Q,

(3) for club many $\eta < \omega_1$,

  (a) $\eta = i^{\mathcal{T}}_{0,\eta}(\eta) = \mathrm{crit}(i^{\mathcal{T}}_{\eta,\omega_1})$, and $E^{\mathcal{T}}_\eta = i^{\mathcal{T}}_{0,\eta}(D)$,

  (b) $A \cap \eta$ is $i^{\mathcal{T}}_{0,\eta}(\mathbb{B})$-generic over Q, and

(4) $\mathbb{R}^V = \mathbb{R}^{Q[A]} = \mathbb{R}^{Q[g]}$, for some $\mathrm{Col}(\omega, < i(\delta))$-generic g over Q.

[10] See the proof of Theorem 5.9 in [1] for the details of this construction.

By (3)(a), $i(D)$ agrees with the club filter on ω_1 for sets in Q. Every real in V is coded into $A \cap \eta$ for some $\eta < \omega_1$, so by (3)(b), every real in V is generic over Q for a poset of size $< \omega_1^V = i(\delta)$ in Q. This and Solovay's factoring lemmas yield (4).

Fixing g as in (4), let us consider the generalized derived model $C_g(Q, i(D))^+$ of Theorem 1.4(b). Note that $\alpha_0 < \omega_2^{Q[g]}$ because A is Q-generic and codes a collapse of α_0 to ω_1. Moreover, for $\eta < \alpha_0$, $F_\eta \cap L_{\alpha_0}({}^\omega\mathrm{OR})[F]$ is the club filter. But $i(D)$ is generated by clubs, so $(F_{i(D)})_\eta \cap C_g(Q, i(D))$ is generated by clubs for all $\eta < \omega_2^{Q[g]}$; moreover these $(F_{i(D)})_\eta \cap C_g(Q, i(D))$ are all total over $C_g(Q, i(D))$. It follows that

$$L_{\alpha_0}({}^\omega\mathrm{OR})[F] = (L_{\alpha_0}({}^\omega\mathrm{OR})[F_{i(D)}])^{C_g(Q,i(D))}.$$

This implies that α_0 is not bad, a contradiction. □

The proof of 0.1 involves more work, in that one cannot move δ all the way out to ω_1^V in an iteration.[11] One must instead move δ into some properly chosen club $C \subseteq \omega_1$, and argue that this is good enough. See [2].

3 Some questions

The proofs of 0.1 and 0.2 give corresponding generic absoluteness theorems. Let $I(\alpha) = L_\alpha({}^\omega\mathrm{OR})$ and $J(\alpha) = L_\alpha({}^\omega\mathrm{OR})[F]$. Let

$$\mathrm{Th}_1^{\mathcal{C}} = \{\varphi \mid \exists\alpha(L_\alpha({}^\omega\mathrm{OR}), \in, I \restriction \alpha) \models \varphi)\},$$
$$\mathrm{Th}_1^{\mathcal{C}+} = \{\varphi \mid \exists\alpha(L_\alpha({}^\omega\mathrm{OR})[F], \in, J \restriction \alpha) \models \varphi)\}.$$

Corollary 3.1. *Under the hypotheses of 0.1, $(Th_1^{\mathcal{C}})^V = (Th_1^{\mathcal{C}})^{V[G]}$, for all G set generic over V. Under the hypotheses of 0.2, $(Th_1^{\mathcal{C}+})^V = (Th_1^{\mathcal{C}+})^{V[G]}$, for all G set generic over V.*

Woodin [7] shows the generic absoluteness of the full first order theories of $\mathcal{C}$ and $\mathcal{C}^+$, and obtains indiscernibles for the models. We don't see how to do that using the methods above. Can this be done?

Woodin also showed that if $A \subseteq {}^\omega\mathrm{OR}$ and $A \in \mathcal{C}^+$, then G_A is determined (in V, not in $\mathcal{C}^+$). It should be possible to show this for $A \subseteq {}^\omega\mathrm{OR}$ in $C_g(P, D)^+$, but we do not see a proof.

Another question is whether θ^g is regular in $\mathcal{C}_g(P, D)^+$, perhaps under stronger hypotheses on the hod pair (P, Σ). So far as we know, Woodin's [7] does not answer the corresponding question for the pure Chang models $\mathcal{C}$ and $\mathcal{C}^+$.

More generally: what is the first order theory of $\mathcal{C}_g(P, D)^+$? How does it depend on (P, Σ)?

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[11]It δ is regular but not measurable, its images under iteration have cofinality ω.

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