A stationary-tower-free proof of the derived model theorem

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In this note, we give a proof of one direction of a version of Woodin's *derived model theorem*:

Theorem 0.1 (Woodin) *Let* λ *be a limit of Woodin cardinals, let* G *be* V*-generic over* $\mathrm{Col}(\omega, <\lambda)$*, let* $\mathbb{R}^* = \bigcup\{\mathbb{R} \cap V[G \restriction \alpha] \mid \alpha < \lambda\}$*, and let* $\mathrm{Hom}^* = \{p[T] \cap \mathbb{R}^* \mid \exists\alpha < \lambda(T \in V[G \restriction \alpha] \wedge V[G \restriction \alpha] \models T$ *is* $<\lambda$ *absolutely complemented* $)\}$*; then*

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$,

(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A$ *is Suslin and co-Suslin in* $L(\mathbb{R}^*, \mathrm{Hom}^*)\}$.

Woodin proved the theorem in perhaps 1986 or 1987, using stationary tower forcing and (through the work of Martin and the author) iteration trees. The proof we give here uses only iteration trees. Stationary tower forcing is replaced by "genericity iterations", as it can be in certain related contexts as well. We believe that the unity of method in the resulting proof gives it some interest.

The first stationary-tower-free proof of the special case $L(\mathbb{R}^*) \models \mathsf{AD}^+$ was discovered by the author in the early 90's. That proof uses fine-structural mice and Woodin-style genericity iterations to replace the stationary tower forcing. The fine-structural mice were needed because Woodin-style genericity iterations require ω_1+1-iterable structures, and we do not know how to prove that kind of iterability for countable $M \prec V$. In the mid 90's, Neeman found a new kind of genericity iteration that requires only $\omega+1$ iterabilty, which

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we do know how to prove for countable $M \prec V$. This gave the proof that $L(\mathbb{R}^*) \models \mathsf{AD}^+$ with the greatest conceptual economy; only the tools of [3] were used. Neeman's work can be found in [4] and[5]. Our proof here uses Neeman's genericity iterations to prove the full 0.1.

The basic structure of our proof is that of Woodin's original proof. That proof, along with related material, is exposited in [6]. Larson's monograph [2] is an excellent source on stationary tower forcing, including some of the material in [6], although it does not prove Theorem 0.1 itself.

Woodin actually proved a stronger version of Theorem 0.1. (See [6, p. 28] for a statement of this version.) We do not know whether our proof here gives that stronger result as well. Both of these results are definitive, in that the Suslin-co-Suslin sets of any model of AD^+ can be realized as some Hom* as above, while the $L(P(\mathbb{R}))$ of any model of AD^+ can be realized as a derived model in the sense of the stronger result.

1 Universally Baire to weakly homogeneous

Let us say that an iteration tree $\mathcal{T}$ is 2^ω- *closed* iff for all α, $\mathcal{M}_\alpha^{\mathcal{T}} \models$ " $\mathrm{Ult}(V, E_\alpha^{\mathcal{T}})$ is closed under 2^ω-sequences". We say that $\mathcal{T}$ is *above* μ if $\mathrm{crit}(E_\alpha^{\mathcal{T}} > \mu$ for all α. The following lemma is essentially due to K. Windszus. (See [1].)

Lemma 1.1 *Let $\pi\colon M \to V_\theta$ be elementary, where M is countable and transitive and let $\mu \in M$. Put*

$$W = \{\mathcal{T} \mid \mathcal{T} \textit{ is a } 2^\omega\textit{-closed iteration tree on } M \textit{ of length } \omega+1,\ \mathcal{T} \textit{ is above } \mu \textit{ and } \mathcal{M}_\omega^{\pi\mathcal{T}} \textit{ is wellfounded}\}$$

Then W is $\pi(\mu)$-homogeneously Suslin.

Proof sketch. By a Skolem hull argument, we can find a transitive N of cardinality 2^ω, and

$$\sigma\colon M \to N \text{ and } \psi\colon N \to V_\theta$$

such that $\pi = \psi \circ \sigma$, and for all $\mathcal{T}$ on M of length $\omega+1$,

$$\mathcal{M}_\omega^{\pi\mathcal{T}} \text{ is wellfounded } \Leftrightarrow \mathcal{M}_\omega^{\sigma\mathcal{T}} \text{ is wellfounded.}$$

We can now define our homogeneous tree U projecting to W. On the first coordinate, branches of U attempt to build a length $\omega+1$ iteration tree $\mathcal{T}$ on M which is 2^ω-closed and above μ. (A node of length k must specify the k-th element of $[0,\omega]_T$.) On the second coordinate, they attempt to build a ψ-realization map $\tau\colon \mathcal{M}_\omega^{\sigma\mathcal{T}} \to V_\theta$. A node of U approximates the τ it is building with a map $\tau_i\colon \mathcal{M}_i^{\mathcal{T}} \to V_\theta$, where i is largest such that the node has determined that $iT\omega$. These maps must commute with the tree embeddings of $\sigma\mathcal{T}$, so that along infinite branches of U they will fit together into the desired realization τ.

It is easy to see that $p[U]$ is the set of iteration trees $\mathcal{T}$ on M such that $\mathcal{T}$ is of length $\omega+1$, above μ, and 2^ω closed, and $\mathcal{M}_\omega^{\sigma\mathcal{T}}$ is ψ-realizable. But by our choice of N, this set is W.

Finally, we get a homogeneity measure μ on U for the space associated to a finite tree $\mathcal{T}$ on M which has i distinguished as the last element of $[0,\omega]_T$ specified so far. Let

$$\phi_k\colon \mathcal{M}_k^{\sigma\mathcal{T}} \to \mathcal{M}_k^{\pi\mathcal{T}},$$

be the copy map, and put

$$A \in \mu \Leftrightarrow \langle i_{k,i}^{\pi\mathcal{T}}(\phi_k) \mid k \in [0,i]_T\rangle \in i_{0,i}^{\pi\mathcal{T}}(A).$$

It is not hard to show this works. □

Although we shall not need it directly, the following well known result is an easy corollary.

Theorem 1.2 (Woodin) *Let δ be Woodin; then every δ^+-universally Baire set of reals is $<\delta$ -weakly homogeneous.*

Proof. Let (T,S) be a pair trees which project to complements after the collapse of δ. Let $\gamma<\delta$; we wish to show $p[T]$ is γ -weakly homogeneous. Let $\pi\colon M \to V_\theta$, where M is countable transitive, and $\pi(\mu)=\gamma$. Let Wb be the set defined from π, M, and μ as in Windszus' theorem. Let $\pi(\bar{T},\bar{S}) = (T,S)$. Since the existential real quantification of a homogeneously Suslin set is weakly homogeneously Suslin, it will be enough to prove:

Claim. For any real x, $x \in p[T]$ iff $\exists\mathcal{T}(\mathcal{T} \in W \wedge x \in p[i_{0,\omega}^{\mathcal{T}}(\bar{\mathcal{T}}]$.

*Proof.*If $\mathcal{T}$ is as on the right hand side, then as $\mathcal{T} \in W$, there is a realization map $\sigma\colon \mathcal{M}_\omega^{\mathcal{T}} \to V_\theta$. But then σ embeds $i_{0,\omega}^{\mathcal{T}}(\bar{T})$ into T, and therefore $x \in p[T]$, as desired.

Now let $x \in p[T]$. By Neeman's genericity theorem ([4]), we can find $\mathcal{T} \in W$ such that, letting $i = i_{0,\omega}^{\mathcal{T}}$ and $N = \mathcal{M}_\omega^{\mathcal{T}}$, x is $\mathrm{Col}(\omega, i(\delta))$-generic over N. Since $\mathcal{T} \in W$, we have a realization $\sigma\colon N \to V_\theta$. But then if $x \in p[i(\bar{S})]$, we get $x \in p[S]$ using σ, a contradiction. Since $i((\bar{T},\bar{S}))$ is absolutely complementing over N, we must then have $x \in i(\bar{T})$, as desired. □

The claim completes the proof. □

2 Hom is a basis for $(\Sigma^2_1)^{L(\mathbb{R}^*,\mathrm{Hom}^*)}$

We shall use the notation of [6] for towers of measures, homogeneity systems, and the like. See section 1 of that paper. In particular, if $A = p[T]$ for some $< \lambda$-absolutely complemented T, and we are given $\mathbb{R}^*$ as the reals of a symmetric collapse below λ, then $A^* = p[T] \cap \mathbb{R}^*$. The notation is justified because A^* is independent of the particular T chosen.

If $Y \subseteq \mathrm{meas}(Z^{<\omega})$, then we write TW_Y for the set of all towers of measures $\bar{\mu}$ such that each $\mu_i \in Y$. A function $f\colon \mathrm{TW}_Y \to \mathrm{TW}_R$ is Lipschitz just in case $f(\bar{\mu}) \restriction n$ is determined by $\bar{\mu} \restriction n$, for all $\bar{\mu}$ and n. We need the following lemma, which combines [3] with a result of Woodin (see [6, Lemma 1.5]).

Lemma 2.1 *Let δ be Woodin, and let $Y \subseteq \mathrm{meas}_{\delta^+}(Z^{<\omega})$ be such that $|Y| < \delta$. Then for any $\gamma < \delta$ there is some W and $R \subseteq \mathrm{meas}_\gamma(W^{<\omega})$, and a Lipschitz*

$$f\colon TW_Y \to TW_R$$

such that whenever G is V-generic over a poset of size $< \gamma$, then for all $\bar{\mu} \in (TW_Y)^{V[G]}$,

$$\bar{\mu} \text{ is wellfounded } \Leftrightarrow f(\bar{\mu}) \text{ is illfounded.}$$

Proof. We should note that f induces a map from $(\mathrm{TW}_Y)^{V[G]}$ to $(\mathrm{TW}_R)^{V[G]}$, which we have also called f, because the forcing is small.

We use Woodin's proof that the set of wellfounded $\bar{\mu} \in \mathrm{TW}_Y$ is δ^+-homogeneous. Fot working in V, pick for each illfounded tower $\bar{\mu} \in \mathrm{TW}_Y$ sets $A_i^{\bar{\mu}} \in \mu_i$ such that they witness the countable incompleteness of $\bar{\mu}$, in that $\neg\exists f \forall i (f \restriction i \in A_i)$. For ν a finite tower from Y, and $i < \mathrm{lh}(\nu)$, let $B_i^\nu = \bigcap\{A_i^{\bar{\mu}} \mid \nu \subseteq \bar{\mu}\}$. Because $|Y| < \delta$, $B_i^\nu \in \nu_i$ for all i. For ν a finite tower of length k from Y, put

$$(\nu, t) \in T \Leftrightarrow \forall i < k (t \restriction i \in B_i^\nu).$$

It is easy to check that T is a tree, that $p[T]$ is the set of wellfounded $\bar{\mu} \in \mathrm{TW}_Y$, both in V and in generic extensions by posets of size $< \delta$, and that T is δ^+-homogeneous. (The homogeneity measure on T_ν is the last measure in ν.)

T is a tree on $W \times U$, for some set W of size $< \delta$. Let S be he Martin-Solovay tree projecting to $\mathrm{TW}_Y \setminus p[T]$ which we get from the homogeneity of T. By [3], S is γ-homogeneous; let $\bar{\nu} \mapsto f(\bar{\nu})$ be a homogeneity system for S consisting of γ-additive measures. Then f is the desired Lipschitz function. $\square$

The heart of the matter is the following theorem.

Theorem 2.2 (Woodin) *Let G be* $\mathrm{Col}(\omega, < \lambda)$*-generic over V, where λ is a limit of Woodin cardinals. Let $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$, where $\alpha < \lambda$. Let φ be a sentence in the language of set theory with two additional unary predicate symbols, and suppose that*

$$\exists B \subseteq \mathbb{R}^*[B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \wedge (HC^*, \in, A^*, B) \models \varphi];$$

then

$$\exists B[B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]} \wedge (HC^{V[G\restriction\alpha]}, \in, A, B) \models \varphi].$$

Proof. We may as well assume $A \in \mathrm{Hom}_{<\lambda}^V$.

Claim 1. For some $B \in L(\mathbb{R}^V, \mathrm{Hom}_\lambda^V)$, $(\mathrm{HC}, \in, A, B) \models \varphi$.

Proof. Fix a $< \lambda$- absolutely complemented pair (S, U) such that $A = p[S]$. Let

$$\pi \colon M \to V_\theta,$$

where θ is sufficiently large and M is countable transitive, with $\pi((\bar{S}, \bar{U}, \bar{\lambda}) = (S, U, \lambda)$. Working in $V^{\mathrm{Col}(\omega,\mathbb{R})}$, we can use the genericity iterations of [4] to form an $\mathbb{R}$-genericity iteration of M, below $\bar{\lambda}$, that is, a sequence

$$I = \langle \mathcal{T}_n \mid n < \omega \rangle$$

such that the $\mathcal{T}_n$ are length $\omega + 1$ iteration trees whose composition

$$\mathcal{T} = \oplus_n \mathcal{T}_n$$

is a normal iteration tree on M, with

$$M_\omega = \lim_n M_n,$$

the direct limit along the main branch of $\mathcal{T}$ (where M_n is the base model of $\mathcal{T}_n$, and the last model of $\mathcal{T}_{n-1}$ if $n > 0$), being such that $\mathbb{R}^V$ is the reals of a symmetric collapse over M_ω below λ_ω, the image of $\bar{\lambda}$. Let

$$i_{n,k} \colon M_n \to M_k$$

be the canonical embedding, for $0 \leq n \leq k \leq \omega$, and $\lambda_k = i_{0,k}(\lambda_0)$. We write

$$\mathrm{Hom}_I^* = \bigcup \{p[T] \cap \mathbb{R}^V \mid \exists x \in \mathbb{R}^V (M_\omega \models T \text{ is } < \lambda_\omega \text{ absolutely complemented})\},$$

so that $L(\mathbb{R}^V, \mathrm{Hom}_I^*)$ is a derived model of M_ω at λ_ω whose set of reals is $\mathbb{R}^* = \mathbb{R}^V$. Because our individual genericity iterations $\mathcal{T}_n$ have length $\omega + 1$, M is iterable enough that we can do them, realizing the M_n and M_∞ in V_θ in the process. Thus we have realizing maps

$$\sigma_k \colon M_k \to V_\theta,$$

for all $k \leq \omega$, such that

$$\sigma_n = \sigma_k \circ i_{n,k}$$

whenever $n \leq k \leq \omega$. ($\sigma_0 = \pi$.) Finally, we arrange that there is an increasing sequence ordinals δ_k, $k < \omega$, with sup λ_ω, such that

$$\delta_k < \text{crit}(i_{k,\omega}),$$

together with M_k-generic objects g_k for $\text{Col}(\omega, \delta_k)$, such that

$$\mathbb{R}^V = \bigcup_{k<\omega} \mathbb{R} \cap M_k[g_k],$$

and $g_k \in M_n[g_n]$ if $k < n$. If $k \leq n \leq \omega$, then $i_{k,n}$ lifts to an embedding

$$i_{k,n} \colon M_k[g_k] \to M_n[g_k],$$

moreover, $\mathbb{R}^V = \bigcup_{k\leq\omega} M_\omega[g_k]$.

Since $\sigma_\omega \circ i_{0,\omega}((\bar{S}, \bar{U})) = (S, U)$, we easily get that $\mathbb{R}^V \cap p[i_{0,\omega}(\bar{S}]) = A$. Written another way, $i_{0,\omega}(\bar{A})^* = A$. The claim will then follow from the elementarity of $i_{0,\omega}$, provided we can show Hom_I^* is a Wadge initial segment of $\text{Hom}_{<\lambda}^V$. Since Hom_I^* is closed downward under Wadge reduction, it suffices to show:

Subclaim 1.1. $\text{Hom}_I^* \subseteq \text{Hom}_{<\lambda}^V$.

Proof. Let $C \in \text{Hom}_I^*$, and fix k and a pair $(T, W) \in M_k[g_k]$ such that (T, W) is $< \lambda_k$ absolutely complementing in $M_k[g_k]$, and $C = p[i_{k,\omega}(T)] \cap \mathbb{R}^V$. Working in $M_k[g_k]$, and letting ρ be the least Woodin cardinal $> \delta_k$, we have a sequence $\langle \bar{\mu}_\eta \mid \rho < \eta < \lambda_k \rangle$ such that

$$M_k[g_k] \models \bar{\mu}_\eta \text{ is a } \delta(\eta)^+\text{-complete homogeneity system such that } S_{\bar{\mu}_\eta} = p[T],$$

where $\delta(\eta)$ is the least Woodin cardinal $> \eta$, and $S_{\bar{\mu}} = \{x \in \mathbb{R} \mid \bar{\mu}_x \text{ is a wellfounded tower }\}$. Note each component measure $(\bar{\mu}_\eta)_u$, for $u \in \omega^{<\omega}$, is actually in M_k (essentially), because g_k was generic for a partial order of size $< \delta(\eta)$. We may therefore define

$$(\bar{\nu}_\eta)_u = \sigma_k(\bar{\mu}_\eta)_u),$$

for all $\eta < \lambda_k$ and $u \in \omega^\omega$. It is not hard to see that $\bar{\nu}_\eta$ is a $\sigma_k(\delta(\eta))$-complete homogeneity system in V, for each η. Recalling that there is a $\xi < \lambda$ such that $\text{Hom}_\xi = \text{Hom}_{<\lambda}$, we see that

$$S_{\bar{\nu}_\eta} \in \text{Hom}_{<\lambda}^V$$

for all sufficiently large η. We may assume this holds for all η by re-indexing. We shall complete the proof by showing that $S_{\bar{\nu}_\eta} = p[i_{k,\omega}(T)] \cap \mathbb{R}^V$ for all η.

Subclaim 1.1a. If $\delta_k < \eta < \gamma < \lambda_k$, then $S_{\bar{\nu}_\eta} = S_{\bar{\nu}_\gamma}$.

Proof. Fix η and γ, and let $x \in \mathbb{R}$. Working in M_k, we can cover the sets of possible values for the $(\bar{\mu}_\xi)_u$ with sets Y_ξ such that $|Y_\xi| \leq \delta_k$ for all ξ. We have by Lemma 2.1 in M_k a sequence of Lipschitz maps

$$f_\xi \colon \mathrm{TW}_{Y_\xi} \to \mathrm{TW}_{Z_\xi}$$

having the properties of that lemma, with the measures in Z_ξ being each ξ-complete. Now let $j \colon M_k \to N$ come from a genericity iteration of M_k which is above δ_k and below ρ, using the Neeman method, so that $x \in N[g_k][h]$ for some h on $\mathrm{Col}(\omega, j(\rho))$. Let $\tau \colon N \to V$ be a realizing map, so that $\sigma_k = \tau \circ j$. By Lemma 2.1, we have that exactly one of the following is true

(1) $j(\bar{\mu}_\eta)_x$ and $j(\bar{\mu}_\gamma)_x$ are wellfounded, while $j(f_\eta)(j(\bar{\mu}_\eta)_x)$ and $j(f_\eta)(j(\bar{\mu}_\gamma)_x)$ are both illfounded,

(2) $j(\bar{\mu}_\eta)_x$ and $j(\bar{\mu}_\gamma)_x$ are illfounded, while $j(f_\eta)(j(\bar{\mu}_\eta)_x)$ and $j(f_\eta)(j(\bar{\mu}_\gamma)_x)$ are both wellfounded.

Now illfoundedness of the towers above passes upward to their (pointwise) images under τ. Thus in case (2), both $(\bar{\nu}_\eta)_x$ and $(\bar{\nu}_\gamma)_x$ are illfounded. In case (1), both $\sigma_k(f_\eta)((\bar{\nu}_\eta)_x)$ and $\sigma_k(f_\gamma)((\bar{\nu}_\gamma)_x)$ are illfounded, and since $\sigma_k(f_\eta)$ and $\sigma_k(f_\gamma)$ flip wellfoundedness of towers, we have that $(\bar{\nu}_\eta)_x$ and $(\bar{\nu}_\gamma)_x$ are both wellfounded. Since x was arbitrary, subclaim 1.1a is proven. □

Now fix η such that $\delta_k < \eta < \lambda_k$, and let $x \in \mathbb{R}^V$. We wish to show $x \in p[i_{k,\omega}(T)]$ iff $(\bar{\nu}_\eta)_x$ is wellfounded. Fix $k < n < \omega$ so that $x \in M_n[g_n]$. It will suffice to show that if $x \in p[i_{k,n}(T)]$, then $(\bar{\nu}_\eta)_x$ is wellfounded, and if $x \in p[i_{k,n}(W)]$, then $(\bar{\nu}_\eta)_x$ is illfounded.

Since $i_{k,n}$ lifts to $M_k[g_k]$, we can set

$$\langle \bar{\tau}_\xi \mid \xi < \lambda_n \rangle = i_{k,n}(\langle \bar{\mu}_\xi \rangle \mid \xi < \lambda_k \rangle,$$

and

$$\langle h_\xi \mid \xi < \lambda_n \rangle = i_{k,n}(\langle f_\xi \rangle \mid \xi < \lambda_k \rangle.$$

For $\xi > \delta_n$, we define $\bar{\rho}_\xi$ by

$$(\bar{\rho}_\xi)_u = \sigma_n((\bar{\tau}_\xi)_u).$$

The proof of subclaim 1.1a gives

Subclaim 1.1b. If $i_{k,n}(\rho) < \xi < \gamma < \lambda_n$, then $S_{\bar{\rho}_\xi} = S_{\bar{\rho}_\gamma}$.

This is proved just as in 1.1a; given an arbitrary real z, we iterate M_n above δ_k and below $i_{k,n}(\rho)$ to obtain N such that z is $N[g_k]$-generic for the collapse of the image of $i_{k,n}(\rho)$, and use a realizing map $\sigma\colon N \to V_\theta$ extending σ_n to draw the required conclusion.

By 1.1a, we may and do assume $\eta > \rho$. The commutativity of the realizing maps gives

$$\bar{\nu}_\eta = \bar{\rho}_{i_{k,n}(\eta)},$$

so that by 1.1b, it will be enough to show that for $\gamma > \delta_n$, $x \in p[i_{k,n}(T)]$ implies $(\bar{\rho}_\gamma)_x$ is wellfounded, and $x \in p[i_{k,n}(W)]$ implies $(\bar{\rho}_\gamma)_x$ is illfounded.

But if $x \in p[i_{k,n}(T)]$, then the elementarity of $i_{k,n}\colon M_k[g_k] \to M_n[g_k]$ gives that $(\bar{\tau}_\gamma)_x$ is wellfounded. (If not, the tree searching for an $x \in p[i_{k,n}(T)]$ such that $(\bar{\tau}_\gamma)_x$ is illfounded would have an infinite branch, and therefore have an infinite branch in $M_n[g_k]$, contrary to the elementarity of $i_{k,n}$.) It follows that $g_\gamma((\bar{\tau}_\gamma)_x)$ is illfounded, and hence $\sigma_n(g_\gamma)((\bar{\rho}_\gamma)_x))$ is illfounded, and hence $(\bar{\rho}_\gamma)_x$ is wellfounded. A completely symmetric argument shows that if $x \in p[i_{k,n}(W)]$, then $(\bar{\rho}_\gamma)_x$ is illfounded.

This proves subclaim 1, and hence Claim 1. □

Let us write Hom $=$ $\mathrm{Hom}^V_{<\lambda}$, and Hom $\restriction \alpha$ for the collection of sets in Hom having Wadge rank $< \alpha$. By Claim 1, we have a lexicographically least pair $\langle \alpha, \beta \rangle$ such that there is a $B \in L_\beta(\mathrm{Hom} \restriction \alpha)$ such that $(\mathrm{HC}, \in, A, B) \models \varphi$. Let $\langle \alpha_0, \beta_0 \rangle$ be this pair. Let $C \in \mathrm{Hom} \restriction \alpha_0$ be such that some such B is ordinal definable over $L_{\beta_0}(\mathrm{Hom} \restriction \alpha_0)$ from the parameter (A, C). We can eliminate the need for the ordinals by minimizing them, and as a result we can fix B such that $(\mathrm{HC}, \in, A, B) \models \varphi$, and a formula ψ such that

$$x \in B \Leftrightarrow L_{\beta_0}(\mathrm{Hom} \restriction \alpha_0) \models \psi[x, (A, C)].$$

We now show B is $\mathrm{Hom}^V_{<\lambda}$, completing the proof.

Case 1. $\mathrm{Hom} \restriction \alpha_0 = \mathrm{Hom}$.

Proof.

Let (T, S) be a $< \lambda$ absolutely complementing pair of trees such that $p[T] = (A, C)$. Let

$$\pi\colon M \to V_\eta$$

be elementary, with M countable transitive, η large, and

$$\pi((\bar{T}, \bar{S}, \bar{\lambda})) = (T, S, \lambda).$$

Let

$$\begin{aligned} \theta(u, v, w) \quad = \quad & \text{“}u \text{ is a limit of Woodin cardinals, } v \in \mathbb{R}, \text{ and} \\ & w \text{ is a tree on } \omega \times \lambda, \text{ and if } \mathbb{R}^*, \mathrm{Hom}^* \text{ are derived} \\ & \text{from } \mathrm{Col}(\omega, < u), \text{ then there is a } \beta \text{ such that} \\ & \exists B \in L_\beta(\mathrm{Hom}^*)((\mathrm{HC}, \in, (p[w])_0 \cap \mathbb{R}^*, B) \models \varphi, \\ & \text{and for the least such } \beta,\ L_\beta(\mathrm{Hom}^*) \models \psi[v, p[w]]. \end{aligned}$$

Let μ be chosen large enough that any $\pi(\mu)$-weakly homogeneous set of reals is in Hom. Let W be the set defined from M, π, and μ as in Theorem 1.1, so that W is $\pi(\mu)$-homogeneous. Let ρ be the least Woodin cardinal of M which is $> \mu$.

Claim 2. For any real x, the following are equivalent:

(a) $x \in B$,

(b) $\exists \mathcal{T} \in W \exists g$ (g is $\mathcal{M}^{\mathcal{T}}_\omega$-generic over $\mathrm{Col}(\omega, i^{\mathcal{T}}_{0,\omega}(\rho))$, and $x \in \mathcal{M}^{\mathcal{T}}_\omega[g]$, and $\mathcal{M}^{\mathcal{T}}_\omega[g] \models \theta[i^{\mathcal{T}}_{0,\omega}(\bar{\lambda}), x, i^{\mathcal{T}}_{0,\omega}(\bar{T})]$).

Proof. Let $x \in \mathbb{R}^V$. Let $\mathcal{T} \in W$ and g on $\mathrm{Col}(\omega, i^{\mathcal{T}}_{0,\omega}(\rho))$ be such that $x \in \mathcal{M}^{\mathcal{T}}_\omega[g]$. (There are such $\mathcal{T}$ and g by [4].) It is enough to show that

$$x \in B \Leftrightarrow \mathcal{M}^{\mathcal{T}}_\omega[g] \models \theta[i^{\mathcal{T}}_{0,\omega}(\bar{\lambda}), x, i^{\mathcal{T}}_{0,\omega}(\bar{T})]).$$

Setting $M_0 = M$, $M_1 = \mathcal{M}^{\mathcal{T}}_\omega$, and $g_1 = g$, we can continue to form an $\mathbb{R}$-genericity iteration I, with models $\langle M_i \mid i \leq \omega \rangle$ and embeddings $i_{k,n} \colon M_n \to M_k$, as in the proof of Claim 1. We have also realizations $\sigma_k \colon M_k \to V_\theta$, as in the proof of Claim 1. Since $i_{1,\omega}$ is elementary on $M_1[g]$, it will be enough to show that $M_\omega[g] \models \theta[i_{0,\omega}(\bar{\lambda}), x, i_{0,\omega}(\bar{T})]$. However, since $\sigma_\omega \circ i_{0,\omega}((\bar{T}, \bar{S})) = (T, S)$, we have that $p[i_{0,\omega}(\bar{T})] \cap \mathbb{R}^V = (A, C)$. Moreover, as in Subclaim 1.1, Hom^*_I is a Wadge initial segment of Hom.

Let $\gamma = \mathrm{OR} \cap M_\omega$. By the hypothesis of 2.2, and the elementarity of σ_ω, there is a $D \in L_\gamma(\mathbb{R}^V, \mathrm{Hom}^*_I)$ such that $(\mathrm{HC}, \in, A, D) \models \varphi$. Since Hom^*_I is a Wadge initial segment of Hom, our case hypothesis then gives that $\mathrm{Hom}^*_I = \mathrm{Hom}$. We then get that $\beta_0 < \gamma$. It follows that β_0 is the ordinal β referred to by the formula θ when it is interpreted in $M_\omega[g]$ at the relevant parameters. From this we easily get the equivalence displayed above, and thus we have proved Claim 2. □

According to Claim 2, B is defined by existential real quantification from $\pi(\mu)$-homogeneously Suslin sets. It follows that $B \in \mathrm{Hom}$, as desired. This finishes Case 1. □

Case 2. $\mathrm{Hom} \restriction \alpha_0 \neq \mathrm{Hom}$.

Proof. Let $D \in \mathrm{Hom}$ have Wadge rank α_0. We use the argument of Case 1, but replacing the parameter (A, C) by (A, C, D). We take θ to be the natural formula defining B from this parameter, as the first witness to φ constructed in $L(\{X \mid X <_w D\})$. The rest goes as in Case 1. □

This finishes the proof of Theorem 2.2. □

The derived model theorem, Theorem 0.1, follows easily from Theorem 2.2. See [6] for some details.

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