The core model induction

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This is a set of notes on the proceedings of the joint Muenster-Irvine-Berlin-Gainesville-Oxford seminar in core model theory, held in cyberspace April–June 2006.

The plan now is to eventually turn this set of notes into a reasonable paper.

*Ralf Schindler and John Steel, July 2007*
# Contents

1  Why an induction?  
   1.1  CMIP theory  
   1.2  Counterexamples to uncountable iterability  
   1.3  Universally Baire iteration strategies  

2  The projective case  
   2.1  Introduction  
   2.2  The failure of $\square$  
   2.3  Successive cardinals with the tree property  
   2.4  Pcf theory  
   2.5  $L(\mathbb{R})$ absoluteness  
   2.6  The unique branches hypothesis  
   2.7  All uncountable cardinals are singular  
   2.8  An ideal  
   2.9  How on earth should we generalize this?  

3  The core model induction through $L(\mathbb{R})$  
   3.1  Core model theory for one $J$-Woodin  
   3.2  Mouse witness conditions  
   3.3  The witness dichotomy  
   3.4  Scales in $L(\mathbb{R})$  
   3.5  The inadmissible case  

4  The core model induction through $L(\mathbb{R})$, continued  
   4.1  Review  
   4.2  Scales at the end of a gap  
   4.3  The Plan  
   4.4  Fullness-preserving iteration strategies
4.5 An sjs-guided iteration strategy in $V[g]$ .................. 47
4.6 Back to $V$ ................................................. 54
4.7 Hybrid strategy-mice and operators ......................... 56
4.8 AD$^L(\mathbb{R})$ from a homogeneous ideal .............. 60

5 A model of AD plus $\Theta_0 < \Theta$ .................. 65
  5.1 $\Gamma_0$ and $M_0$ .................................. 66
  5.2 The HOD of $M_0$ up to its $\Theta$ .......................... 69
  5.3 A model of AD plus $\Theta_0 < \Theta$ ................. 70

6 The core model induction beyond $L(\mathbb{R})$ .... 73
  6.1 More strength from homogeneous ideals ................. 73
  6.2 A maximal model of AD$^+ + \theta_0 = \theta$ ............ 74
  6.3 The Plan .................................................... 76
  6.4 HOD$^{M_0}$ as viewed in $j(M_0)$ ....................... 77
  6.5 HOD below $\theta_0$ ...................................... 78
    6.5.1 Quasi-iterability ................................... 78
    6.5.2 HOD$|^{\theta_0}$ as a direct limit of mice ......... 82
  6.6 A fullness preserving strategy for $H_0^+$ ............ 85
  6.7 Branch condensation .................................... 92
  6.8 Conclusion ............................................. 98
Chapter 1

Why an induction?

The core model induction is a method whereby the theory of CMIP ([11]) is extended so as to produce inner models with more than one Woodin cardinal under various hypotheses. We begin by indicating where the theory of CMIP is deficient.

The main deficiency has to do with our inability to prove iterability for $K^c$, especially for uncountable iteration trees. For trees of size greater than the completeness of the background extenders in the $K^c$ construction, there is an obstacle in the mathematics itself; one cannot hope to prove the iterability needed in ZFC alone. In order to prove iterability for a level $N$ of $K^c$ beyond $M^{#1}_1$, one must assume that $V$ is closed under certain mouse operators weaker than $N$. One is led, seemingly inevitably, into an induction.

Our goal in this lecture is to explain these points in greater detail. We begin with a short review of CMIP.

1.1 CMIP theory

Countable iterability and $K^c$

Definition 1.1.1 A premouse $M$ is countably iterable iff every countable elementary submodel of $M$ is $(\omega, \omega_1, \omega_1 + 1)$-iterable.

The $\omega$ in $(\omega, \omega_1, \omega_1 + 1)$ iterability refers to the degrees of ultrapowers allowed. We shall drop reference to it in the future, with the understanding that one takes ultrapowers of the largest degree possible, and the caveat that the author has not always considered the issue carefully. The $\omega_1$ refers to the fact one can stack normal trees $\omega_1$ times. In the sequel, an iteration
CHAPTER 1. WHY AN INDUCTION?

tree in general is almost always a stack of normal trees in which maximal
ultrapowers are taken, even shifting between normal trees.

In the sequel we shall sometimes speak of partial iteration strategies,
with the obvious meaning. We make the convention here that an iteration
strategy \( \Sigma \) is only defined on those iteration trees \( T \) such that \( T \) has limit
length and is itself a play by \( \Sigma \). If \( \Sigma \) is defined on all such trees, it is total.

Countable iterability is enough to compare countable mice. A countably
iterable premouse must have the solidity and universality properties of iterable
mice, as these are first order. Finally, if \( \mathcal{M} \) and \( \mathcal{N} \) agree to a common
cutpoint \( \eta \), project to \( \eta \), and are countably iterable above \( \eta \), then one is an
initial segment of the other.

Let \( \Omega \) be strongly inaccessible. A \( K^c \)-construction below \( \Omega \) is determined
by a sequence
\[
\langle N_\alpha \mid \alpha \leq \Omega \rangle,
\]
essentially what is called by Jensen an ms-array. The precise details depend
on what background condition one demands for the last extender of an \( N_\alpha \);
for official purposes, we follow CMIP. As long as each \( C_k(N_\alpha) \) is countably
iterable, the construction does lead to fine-structural pathology, and one can
set
\[
K^c = \mathcal{N}_\Omega.
\]
If \( \mu \) is a normal measure on \( \Omega \), then again assuming only countable iterability,
one has
\[
(a^+)^{K^c} = a^+, \text{ for } \mu\text{-a.e. } \alpha,
\]
for any maximal \( K^c \) construction (i.e. one which adds extenders wherever
it can, perhaps with some \( \mu \)-measure zero set of forbidden critical points for
total extenders),

**Question 1.** Must every \( C_k(N_\alpha) \) occurring in a \( K^c \)-construction be countably iterable?

The good (i.e. affirmative) answer is known for \( N_\alpha \) which are tame, or even
domestic. It is not known significantly beyond that, for example, it is open
below a Woodin limit of Woodins. One seems to need some form of UBH,
even to handle countable trees, and certainly to handle the length \( \omega_1 \) trees
which one must face as part of countable iterability.

**Question 2.** Can one prove that \( K^c \) computes successor cardinals correctly
on a stationary class without assuming that \( \Omega \) is measurable?
1.1. CMIP THEORY

More generally, can one develop a theory of true $K$ without assuming the measurable? One test problem is the equiconsistency of one Woodin cardinal with the existence of a saturated ideal on $\omega_1$, which should be a byproduct of a good theory.

Core model inductions generally solve the problem of the measurable by producing sufficiently closed inner models $M$ admitting $M$-measures over some $\Omega$. One then constructs $K^c$ and $K$ inside $M$.

**Uncountable iterability and $K$**

In order to construct $K$ from $K^c$, one needs that $K^c$ is $(\omega, \Omega + 1)$-iterable. Assuming this much iterability, and that $\Omega$ is measurable, one can make the

**Definition 1.1.2** $K$ is the transitive collapse of the intersection of all thick hulls of $K^c$.

One then has

1. $K$ is $(\omega, \Omega + 1)$-iterable.

2. (Generic absoluteness) $K^V = K^V[g]$, whenever $g$ is $V$-generic over a poset of size $\omega_1$.

3. (Weak covering a.e.) $\alpha^+K = \alpha^+$ for $\mu$-a.e. $\alpha$, for any normal $\mu$ on $\Omega$.

4. (Weak covering) For any $K$-cardinal $\kappa \geq \omega_2^V$, $\text{cof}(\kappa) \geq |\alpha|$. Thus $\alpha^+K = \alpha^+$, whenever $\alpha$ is a singular cardinal of $V$. (Mitchell, Schimmerling. This may use a little more iterability for $K^c$ than we have stated.)

5. (Inductive definition) $K \cap HC$ is definable over $(J_{\omega_1}(R), \in, I)$, where $I$ is the collection of all countably iterable countable premice.

6. (Embeddings of $K$) Any elementary $j: K \rightarrow K$ is the identity. $K$ is the unique universal weasel which embeds into all other iterable universal weasels. $K^c$ is an iterate of $K$.

See Schimmerling’s forthcoming Handbook of Set Theory article for a survey of what’s known about $K$.

In order to prove $(\omega, \Omega + 1)$ iterability for $K^c$, one needs to have an iteration strategy on countable iteration trees which is sufficiently absolutely definable that it can be extended to uncountable trees. The main technique is based on $Q$-structures.
CHAPTER 1. WHY AN INDUCTION?

Mouse operators and \(Q\)-structures

We shall often consider relativised mice, that is, mice built by starting with some set \(x\), then constructing from an extender sequence with all critical points \(> \text{rk}(x)\). We tacitly assume here that \(x\) is transitive, and usually assume that it is equipped with a wellorder. We could always code such an \(x\) as \(\text{sup}(A) \cup \{A\}\), for \(A\) a set of ordinals. Let us call such an \(x\) self-wellordered, or swo. The main exception to this rule is the case of mice over the reals, where \(x = V_{\omega+1}\) must be considered without a wellorder.

Definition 1.1.3 An r-premouse is a premouse over some swo’d set \(x\).

The theory of r-premice and mice is a trivial variant of the lightface theory.

Definition 1.1.4 A mouse operator on \(Z\) is a function \(\mathcal{N}\) assigning to each swo \(x \in Z\) a countably iterable \(x\)-premouse \(\mathcal{N}(x)\) such that \(\mathcal{N}(x)\) is pointwise definable from members of \(x \cup \{x\}\). We say that \(\mathcal{N}\) is first order just in case there is a theory \(T\) in the language of r-premice (so having a symbol \(\dot{x}\) for \(x\)) such that for all \(x \in Z\), \(\mathcal{N}(x)\) is the least countably iterable \(x\)-mouse satisfying \(T\).

Countable iterability goes down under Skolem hulls, so we get:

Lemma 1.1.5 (Condensation) Let \(\mathcal{N}\) be a first order mouse operator on \(Z\), and suppose \(\pi: P \rightarrow \mathcal{N}(x)\) is fully elementary in the language of relativised mice; then \(P = \mathcal{N}(\pi^{-1}(x))\).

A very important first order mouse operator is the \(Q\)-structure function:

Definition 1.1.6 Let \(\mathcal{M}\) be an r-premouse. A \(Q\)-structure for \(\mathcal{M}\) is an r-premouse \(Q\) such that

(a) \(\mathcal{M} \leq^* Q\) (i.e. \(\mathcal{M}\) is a cutpoint initial segment of \(Q\)),

(b) \(Q\) defines a minimal failure of \(o(\mathcal{M})\) to be Woodin via the extenders of \(\mathcal{M}\), and

(c) \(Q\) is countably iterable above \(o(\mathcal{M})\).

A comparison argument shows there is at most one \(Q\) structure for \(\mathcal{M}\). We denote it \(Q(\mathcal{M})\) if it exists. It is easy to see that \(T \mapsto Q(\mathcal{M})\) is a
first order mouse operator on its domain. Let us call this operator $Q^t$. An important special case is $M = M(T)$, for $T$ an iteration tree of limit length on some r-premouse. In this case we write $Q(T)$ for $Q^t(M(T))$. Thus $Q(T)$, when it exists, defines a minimal failure of $\delta(T)$ to be Woodin via extenders of the common part model $M(T)$.

One can sometimes use $Q$-structures to determine an iteration strategy on an r-premouse:

**Definition 1.1.7** Let $T$ be an iteration tree of limit length on an r-premouse, and $b$ a cofinal branch of $T$. Then $Q(b, T)$ is the first initial segment of $M^T_b$ defining a failure of $\delta(T)$ to be Woodin. We let $Q(b, T)$ be undefined if there is no such initial segment.

**Definition 1.1.8** $\Sigma^t$ is the following partial iteration strategy (for arbitrary tame r-premice):

$$\Sigma^t(T) = \text{unique } b \text{ such that } Q(b, T) = Q(T).$$

We write $\Sigma^t_M$ for the restriction of $\Sigma^t$ to iteration trees based on $M$.

One can show:

1. If $T$ is an iteration tree on a tame r-premouse $M$, then there is at most one cofinal $b$ such that $Q(T)$ and $Q(b, T)$ are defined, and $Q(T) = Q(b, T)$, moreover

2. if in addition $M \models "\text{There are no Woodin cardinals}"$, or even just projects below its bottom Woodin, (above the set over which it is built, in each case) then for any cofinal $b$ of $T$, $Q(b, T)$ exists, so that

3. if in addition, $M$ is $\omega_1 + 1$-iterable, then $\Sigma^t$ is its unique $\omega_1 + 1$-iteration strategy.

Tameness is important here; the natural extension of these notions to nontame r-premice would involve $Q$-phalanxes. For pretty much the rest of these notes, we shall stick to the tame case. (The superscript in $\Sigma^t$ is for “tame”.)

We shall use mouse operators to keep track of the complexity of the $Q$-structures determining $\Sigma^t_M$:

**Definition 1.1.9** Let $N$ be a mouse operator, and let $M$ be a tame premouse. We say that $\Sigma^t_M$ is $N$-guided on $Y$ just in case whenever $T \in Y$
is a tree of limit length played by $\Sigma_M^t$, then $\Sigma_M^t(T)$ is defined, and letting $b = \Sigma_M^t(T)$, we have $Q(b, T) \leq N(M(T))$.

Of course, $\Sigma^t_M$ is $Q^t$-guided on its domain, simply by definition.

The following basic condensation property of $\Sigma^t$ is often used.

Lemma 1.1.10 (Condensation for $\Sigma^t$) Let $T$ be an iteration tree played according to $\Sigma^t_M$, and let $\pi: S \to V_\theta$ be such that $\pi(P) = M$ and $\pi(U) = T$; then $U$ is played according to $\Sigma^t_P$.

Proof. This follows immediately from condensation for the mouse operator $Q^t$. \hfill \Box

The following lemma encapsulates the main way one extends iteration strategies for countable trees on tame mice so as to act on uncountable trees.

Lemma 1.1.11 ($Q$-reflection) Let $\mathcal{M}$ be tame, and $Z$ be transitive and rudimentarily closed, with $HC \subseteq Z$. Let $N$ be a first order mouse operator defined on all $M(T)$ such that $T \in Z$ is played according to $\Sigma^t_M$. Suppose that whenever $P$ is countable and elementarily embeddable into $\mathcal{M}$, then $\Sigma^t_P$ is an $N$-guided strategy on $HC$. Then $\Sigma^t_M$ is an $N$-guided strategy on $Z$.

Proof. Let $T \in Z$ be a tree of limit length on $\mathcal{M}$ which is played by $\Sigma^t$, so that $N(M(T))$ exists by hypothesis. We must show that there is a cofinal branch $b$ of $T$ such that $Q(b, T) \leq N(M(T))$. Let $\pi: S \to V_\theta$, where $S$ is countable transitive, $\theta$ is large, $\pi(P) = M$, and $\pi(U) = T$. Let $g$ be $P$-generic for the collapse of $P$ and $U$ to be countable. By 1.1.10, $U$ is by $\Sigma^t$, and by mouse condensation,

$$\pi^{-1}(N(M(T))) = N(M(U)).$$

But $\Sigma^t_P$ is $N$-guided on $HC$, so letting $b = \Sigma^t_P(U)$, $Q(b, U) \leq N(M(U))$. Since $S[g]$ is $\Sigma^t_1$ absolute in $V$, we get

$$S[g] \models \exists c(Q(c, U) \leq N(M(U))),$$

which then implies that $b \in S[g]$ by the uniqueness of such $c$. This is true for all $g$, and hence $b \in S$. That is, $S \models \exists b(Q(b, U) \leq N(M(U)))$, and hence $V_\theta \models \exists b(Q(b, T) \leq N(M(T)))$, and we are done. \hfill \Box

Here is a variant of the $Q$-reflection lemma.
Lemma 1.1.12 Let $\mathcal{M}$ be tame. Suppose that

(a) For any countable $\mathcal{P}$ embeddable into $\mathcal{M}$, $\Sigma^t_{\mathcal{P}}$ is an $\omega_1$ iteration strategy for $\mathcal{P}$, and

(b) For any tree $T \in V_\Omega$ on $\mathcal{M}$ of limit length which is played by $\Sigma^t$, $Q(T)$ exists.

Then $\Sigma^t_{\mathcal{M}}$ is an $\Omega$-iteration strategy for $\mathcal{M}$.

Lemma 1.1.12 is just the special of 1.1.11 in which $\mathcal{N}$ is the operator $Q^t$ itself, and $Z = V_\Omega$.

The existence of $K$

First, we have the result of CMIP:

Theorem 1.1.13 Let $\Omega$ be measurable, and suppose there is no premouse of height $\Omega$ with a Woodin cardinal; then $K^c$ is $(\omega, \Omega + 1)$-iterable via $\Sigma^t$, and hence $K$ exists.

This is an easy consequence of the $Q$-reflection lemmas. Hypothesis (a) of 1.1.12 holds because $K^c$ must be tame, and hypothesis (b) holds because $L_\Omega[|\mathcal{M}(T)|] = \delta(T)$ is not Woodin, and $L[\mathcal{M}(T)]$ is trivially iterable above $\delta(T)$.

One can use the $Q$-reflection lemmas to go beyond one Woodin cardinal. In the case of finitely many Woodins, the central result is:

Theorem 1.1.14 Let $\Omega$ be measurable. Suppose that for all $x \in V_\Omega$, $M^*_n(x)$ exists and is $(\omega, \Omega + 1)$-iterable. Then exactly one of the following holds:

1. for all $x \in V_\Omega$, $M^*_{n+1}(x)$ exists and is $(\omega, \Omega + 1)$-iterable,

2. for some $x \in V_\Omega$, $K^c(x)$ is n-small, has no Woodin cardinals, and is $(\omega, \Omega + 1)$-iterable. (Hence $K(x)$ exists, is n-small, and has no Woodin cardinals.)

Proof. It is easy to see that (1) and (2) are mutually exclusive.

Assume that (1) fails, and let $x$ be such that there is no fully iterable $M^*_{n+1}(x)$. Then there must be an $\eta < \delta$ such that the maximal $K^c(x)$ construction done with all background extenders having critical point above
\[ \eta \text{ fails to reach } M_{n+1}^{\sharp}(x). \] The reason is that any fully sound premouse projecting to \( \omega \) which is reached by such a construction is \( \eta \)-iterable, for we can carry out the proof of countable iterability in \( V^{\text{Col}(\omega, \xi)} \), with \( \xi < \eta \).

Our background extenders prolong to \( V^{\text{Col}(\omega, \xi)} \), and the strategy we get in \( V^{\text{Col}(\omega, \xi)} \) pulls back to \( V \) by uniqueness.

So let \( K^c(x) \) come from a construction which does not reach \( M_{n+1}^{\sharp}(x) \).

**Claim.** \( K^c(x) \models \text{there are no Woodin cardinals}. \)

**Proof.** Otherwise, let \( \delta \) be the largest Woodin of \( K^c(x) \). We can then compare \( K^c(x) \) with \( M^d_\delta(K^c(x)|\delta) \), using the \( \Omega + 1 \)-iterability of the latter to pick branches on both sides. (There is another Lowenheim-Skolem argument to see that the \( \mathcal{Q} \)-structures provided by the iterates of \( M^d_\delta(K^c(x)|\delta) \) are realized by branches on the \( K^c(x) \) side.) Because \( K^c(x) \) is universal, the \( M^d_\delta(K^c(x)|\delta) \) side comes out strictly shorter, which implies that \( K^c(x) \) did indeed reach \( M_{n+1}^{\sharp}(x) \).

To see that \( K^c(x) \) is \((\omega, \Omega + 1)\)-iterable, we simply apply the \( \mathcal{Q} \)-reflection lemma, with our mouse operator \( \mathcal{N} \) given by \( \mathcal{N}(y) = M^d_d(y) \). It is clear from the fact that that \( K^c(x) \) has no Woodin cardinals and does not reach \( M_{n+1}^{\sharp}(x) \) that for any countable \( \mathcal{P} \) embeddable into \( K^c(x) \), \( \Sigma^\mathcal{P}_1 \) is \( \mathcal{N} \)-guided on \( HC \). Thus \( \Sigma^d \) is an \( \mathcal{N} \)-guided strategy on all of \( V_\omega \), as desired.

### 1.2 Counterexamples to uncountable iterability

Before we look at some applications of 1.1.14 and other similar arguments, let us collect some examples which show that we cannot expect that the full iterability of \( K^c \) will be provable in \( ZFC \). The theme here is: Woodin cardinals are the enemy of full iterability.

1. (H. Woodin) \( M_1 \models \text{“I am not } \delta^+ + 1 \text{ iterable, where } \delta \text{ is my Woodin cardinal”}. \)

2. (I. Neeman) \( M_1 \models \text{“I am not } \delta + 1 \text{ iterable, where } \delta \text{ is my Woodin cardinal”}. \)

3. (J. Steel) Let \( M \) be an iterable extender model satisfying \( ZFC^- + \text{“} \delta \text{ is Woodin”} \). Then \( M \models \text{“I am not } \delta^+ + 1 \text{ iterable, where } \delta \text{ is my least Woodin cardinal”} \).

The original argument here is Woodin’s (1), of which the other two are variants. In each case it is a genericity iteration which cannot be executed
1.3. UNIVERSALLY BAIRE ITERATION STRATEGIES

in the model in question. Genericity iterations are a very important source of logically complicated iterations.

In contrast to (1)-(3), we have that if $\xi$ is strictly less than the Woodin cardinal of $M_1$, then $M_1$ knows its own iteration strategy restricted to set length iteration trees based on $M_1|\xi$. This is because the $\mathcal{Q}$ structures for such a tree $T$ is an initial segments of $L[\mathcal{M}(T)]$.

(4) (Woodin) Suppose there is a set-length iterable proper class model with a Woodin cardinal; then every set has a sharp. More generally, if there is a set-length iterable proper class model with $n + 1$ Woodin cardinals, then for all sets $x$, $M_n^+(x)$ exists and is set-length iterable.

This shows pretty clearly why we need $M_n^\sharp$-closure in order to build a fully iterable $K^c$ reaching $M_{n+1}$. If we find a fully iterable $M_{n+1}$, then $V$ was $M_n^\sharp$-closed.

(5) (J. Steel) Assume boldface $\Delta^1_2$ determinacy; then for a Turing cone of $x$, $L[x] \models K^c$ is not $\Omega_x$-iterable, where $\Omega_x$ is the least $x$-indiscernible, and $K^c$ is built up to that point.

This comes out the proof in CMIP that $\Delta^1_2$ determinacy implies for all reals $x$, $M_1^+(x)$ exists and it $\omega_1$-iterable. (The result itself is due to Woodin.)

(6) (J. Steel) Let $M$ be a fully iterable, tame extender model, and suppose that $M \models \delta$ is Woodin. Then for no $\kappa \geq \delta$ do we have $(\kappa^+)^M = \kappa^+$.

So within the tame mice, an iterable $K$ with weak covering and a Woodin cardinal is impossible. Unpublished results of Woodin show that it is possible just a bit past tame (i.e. below the $\text{AD}_R$ hypothesis).

1.3 Universally Baire iteration strategies

In this section, we show

**Theorem 1.3.1 (Steel, Woodin 1990)** Suppose $\Omega$ is measurable, and every set of reals in $L(\mathbb{R})$ is $< \Omega$-weakly homogeneous; then for all $x \in V_\Omega$, $M_2^\omega(x)$ exists and is $\Omega + 1$-iterable.
The proof pre-dated the core model induction. It is interesting because it shows that the main thing the core model induction is giving us is something like a universally Baire representation of $\Sigma^t$ restricted to countable trees. Granted such a representation, there is no need for an induction.

Our proof needs weak homogeneity, rather than just universal Baireness, however. It seems to be open whether theorem 1.3.1 remains true if “weakly homogeneous” is replaced by “universally Baire”.

An immediate corollary of 1.3.1 is

**Corollary 1.3.2** If every set in $L(\mathbb{R})$ is weakly homogeneous, then $\text{AD}^{L(\mathbb{R})}$ holds.

The existence of a strongly compact implies all sets of reals in $L(\mathbb{R})$ are weakly homogeneous by work of Woodin from the mid 80’s. So

**Corollary 1.3.3** If there is a strongly compact cardinal, then $\text{AD}^{L(\mathbb{R})}$ holds.

We now prove 1.3.1. Our presentation simplifies the original proof somewhat.

**Lemma 1.3.4** Let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of distinct reals. Put

$$R(y, n, m) \iff (y \in \text{WO} \text{ and } x_{y}(n) = m).$$

Then $R$ is not $\omega_1$-universally Baire.

**Proof.** Let $(T, U)$ be an $\omega_1$-absolutely complementing pair such that $p[T] = R$. Let $\pi : M \rightarrow V_\theta$, where $M$ is transitive and countable, $\theta$ is large, and $\pi((R, S)) = (T, U)$. We have $\pi(\langle x_\alpha \mid \alpha < \omega_1^M \rangle) = \langle x_\alpha \mid \alpha < \omega_1 \rangle$, and we have that whenever $x_\beta \in M$, then $\beta < \omega_1^M$.

Now let $g$ be $M$-generic for $\text{Col}(\omega, \omega_1^M)$, and let $y \in (M[g] \cap \text{WO})$ be such that $|y| = \omega_1^M$. Put

$$(n, m) \in x \iff (y, n, m) \in p[R].$$

Since $p[R] \subseteq p[T]$, we have $x \subseteq x_{\omega_1^M}$. But if $(n, m) \notin x$, then $(y, n, m) \notin p[S]$ because $R$ and $S$ are absolute complements over $M$, and since $p[S] \subseteq p[U]$, we get that $(n, m) \notin x_{\omega_1^M}$. Thus $x_{\omega_1^M} \in M[g]$. This is true for all $g$, so $x_{\omega_1^M} \in M$. But then $x_{\omega_1^M} = x_\beta$ for some $\beta < \omega_1^M$, a contradiction. \qed
Lemma 1.3.5 Under the hypotheses of the theorem, \( \mathbb{R}^2 \) is \(<\Omega\)-universally Baire. Moreover, if \((T,U)\) is a pair of \(<\Omega\)-absolutely complementing pair of trees witnessing this, then

\[
V[G] \models p[T] = \mathbb{R}^2,
\]

whenever \( G \) is \( V \)-generic for a poset of size \(<\Omega\).

Proof. \( \mathbb{R}^2 \) exists because we have a measurable cardinal. \( \mathbb{R}^2 = \bigcup_{n<\omega} T_n \), where \( T_n \) is the type with real parameters of the first \( n \) indiscernibles. Since \( T_n \in L(\mathbb{R}) \), it is \( \Omega \)-weakly homogeneous. But the class of \( \Omega \)-weakly homogeneous sets is closed under countable unions.

In order to show the “moreover” part, it suffices to produce just one pair \((T,U)\) of \( \Omega \)-absolutely complementing trees with the property, since all such pairs determine the same set of reals in every size \(<\Omega\) generic extension. Let \( T \) be \( \Omega \)-weakly homogeneous with \( p[T] = \mathbb{R}^2 \), and let \( U \) be the Martin-Solovay tree for \( \mathbb{R} \setminus p[T] \). Supposing toward contradiction that \( V[G] \models p[T] \neq \mathbb{R}^2 \), for some \( G \) is \( V \)-generic for a poset of size \(<\Omega\), we get in \( V \)

\[
\pi: M \rightarrow V_\theta, \text{ such that } \pi((R,S)) = (T,U),
\]

where \( M \) is countable transitive. Letting \( g \) be \( M \)-generic, we have

\[
M[g] \models p[R] \neq \mathbb{R}^2.
\]

It is easy to see, using that \((R,S)\) are absolute complements over \( M \), \( p[R] \subseteq p[T] \), and \( p[S] \subseteq p[U] \), that \( p[R] \cap M[g] = \mathbb{R}^2 \cap M[g] \). This yields that all the properties of \( \mathbb{R}^2 \) hold of \( p[R] \) in \( M[g] \), except possibly the witness condition.

The witness condition asserts that if \( \exists \psi \in P \), where \( \psi \) involves indiscernibles and real parameters \( \bar{x} \), then there is a term \( \tau \) involving indiscernibles and real parameters \( \bar{y} \) such that \( \psi(\tau) \in \mathbb{R}^\ast \). To verify it of \( p[R]\mid M[g] \), it suffices to show that if \( x \in \mathbb{R} \cap M[g] \), then

\[
\exists y \in \mathbb{R}(\psi, x, y) \in p[T] \Rightarrow \exists y \in \mathbb{R} \cap M[g](\psi, x, y) \in p[R].
\]

So assume we have such an \( x \). We now use the full Martin-Solovay construction, which gives a tree \( W \) such that \( p[W] = \forall^\mathbb{R} p[T] \) holds in all size \(<\Omega\) extensions of \( V \). Let \( Q \) be such that \( \pi(Q) = W \). Suppose toward contradiction that it is not the case that \( \exists y \in \mathbb{R} \cap M[g](\psi, x, y) \in p[R] \). We then get that from the elementarity of \( \pi \) that \( (\psi, x) \in p[Q] \), which implies that \( (\psi, x) \in p[W] \), so that \( \forall y(\psi, x, y) \notin p[T] \), a contradiction. \( \Box \)
It is in verifying the witness condition part of \( p[T] = (\mathbb{R}^*)^V[G] \) that we need weak homogeneity, and not just universal Baireness. The resulting generic absoluteness of the theory of \( L(\mathbb{R}) \) is used below.

Now fix \( \Omega \) as in the hypotheses of 1.3.1. Fix also \( x \in V_\Omega \). We are done if we can show that every maximal-above-some-\( \eta \) \( K^c(x) \) construction reaches \( M^c_\Omega(x) \), so fix a \( K^c(x) \) which does not. It follows that there is a cardinal cutpoint \( \xi \) of \( K^c(x) \) above which \( K^c(x) \) has no Woodin cardinals.

Claim. \( K^c(x) \) is \( (\omega, \Omega + 1) \)-iterable above \( \xi \).

Proof. We shall need the following basic facts: Let \( Q \) be a countable, \( \omega \)-small mouse with no Woodin cardinals over \( z \), and \( Q \) project to \( z \); then

(a) If \( Q \) is \( \omega_1 + 1 \)-iterable, then \( \Sigma^*_Q \cup HC \in L(\mathbb{R}) \), and

(b) If \( L(\mathbb{R}) \models Q \) is \( \omega_1 \)-iterable, then \( Q \) is \( \Omega \)-iterable in \( V \).

For (a), see [15, §7]. Part (b) uses that any such strategy is determined by choosing the unique \( b \) such that \( Q(b, T) \) is \( \omega_1 \)-iterable in \( L(\mathbb{R}) \) (again, [15, §7]), together with the generic absoluteness of the theory of \( L(\mathbb{R}) \) given by 1.3.5.

To prove the Claim, is enough by 1.1.12 to see that whenever \( T \in V_\Omega \) is an iteration tree of limit length on \( K^c(x) \) above \( \xi \), then \( Q(T) \) exists. Let \( T \) be given, and let \( \pi : M \to V_\emptyset \) be elementary, where \( M \) is countable transitive, and \( \pi(\mathcal{U}) = T \). By 1.1.10, \( \mathcal{U} \) is by \( \Sigma^*_P \), where \( \pi(P) = K^c(x) \). It follows that \( Q(\mathcal{U}) \) exists, and has an \( \omega_1 \) iteration strategy in \( L(\mathbb{R}) \). Let \( g \) be generic over a poset of size \( < \pi^{-1}(\Omega) \) over \( M \), and make \( \mathcal{U} \) and some initial segment of \( P \) on which \( \mathcal{U} \) is based countable. Then \( Q(\mathcal{U}) \) is definable from \( \mathcal{U} \) over \( (L(\mathbb{R}))^{M[g]} \) as the unique \( \omega_1 \)-iterable \( Q \)-structure for \( \mathcal{U} \). We use here the correctness of \( (L(\mathbb{R}))^{M[g]} \) which follows from \( (\mathbb{R}^*)^{M[g]} = \mathbb{R}^* \cap M[g] \). It follows that \( Q(\mathcal{U}) \) in \( M \), and is \( \omega_1 \)-iterable in \( L(\mathbb{R})^M \). But then \( Q(\mathcal{U}) \) is \( \pi^{-1}(\Omega) \)-iterable in \( M \), and from this we easily get that \( \pi(Q(\mathcal{U})) = Q(T) \), so that indeed \( Q(T) \) exists. \( \Box \)

But now, letting \( y = K^c(x)\mid_\xi \), we see that \( K(y) \) exists. Weak covering and the \( L(\mathbb{R}) \)-definability of \( K(y) \) imply that in some \( V[G] \) for \( G \) \( \Omega \)-generic over \( V \), we have an uncountable sequence of distinct reals in \( L(\mathbb{R}) \). (See [15, 7.3.7.4].) But 1.3.5 then implies this sequence is \( \omega_1 \)-universally Baire in \( V[G] \), contrary to our first lemma. \( \Box \)

Exercises.
1.3. UNIVERSALLY BAIRE ITERATION STRATEGIES

(1) Show that if every OD($\mathbb{R}$) set of reals is weakly homogeneous, then there is a nontame mouse. (We do not know whether the hypothesis is consistent. Replacing OD($\mathbb{R}$) with definability by a certain long game quantifier, one gets a consistent hypothesis which yields a nontame mouse.)

(2) Let $c: HC \rightarrow \mathbb{R}$ be some nice coding of hereditarily countable sets by reals. Let $\Sigma$ be an $\omega_1$-iteration strategy for a countable premouse $M$, and suppose that

$$I = \{c(T) \mid T \text{ is according to } \Sigma\}$$

is $\kappa$-weakly homogeneous. Show that $\Sigma$ extends to a $\kappa$-iteration strategy for $M$. 
Chapter 2

The projective case

In this chapter we show how to prove that (an initial segment of) $V$ is closed under all $X \mapsto \mathcal{M}_n^\#(X)$ $(n < \omega)$ from various hypotheses.

2.1 Introduction.

We now show how to get inner models with finitely many Woodin cardinals from various hypotheses via the first $\omega$ steps of the core model induction. Given a hypothesis, say $\varphi$, we basically aim to prove that in the light of $\varphi$ the second alternative of the Witness Dichotomy (cf. Chapter 3) cannot hold true. However, we need not have a measurable cardinal around; and even if so, its presence is often of no real help, as we may have to work more “locally.” Hence whereas the key method is always the same, each $\varphi$ comes with its set of details with respect to the application of the core model induction technique.

Definition 2.1.1 Let $X$ be a set of ordinals, and let $n < \omega$. Then $\mathcal{M}_n^\#(X)$ denotes the least active $X$-premouse which satisfies “there are $n$ Woodin cardinals” and which is countably iterable “above X.”

Here’s a list of theorems which involve covering for $K$ plus some other stuff.

Theorem 2.1.2 (Schimmerling for $n = 1$; Steel, Woodin, independently, for $n > 1$) If PFA holds, then for every $X$ and for every $n < \omega$, $\mathcal{M}_n^\#(X)$ exists.
Theorem 2.1.3 (Schimmerling, Steel) If $\square_\kappa$ fails, where $\kappa$ is a singular strong limit cardinal, then for every $X \subset \kappa$ and for every $n < \omega$, $M^\#_n(X)$ exists.

Steel has actually shown that the hypothesis of Theorem 2.1.3 implies that $AD$ holds in $L(\mathbb{R})$ (cf. [13]).

Theorem 2.1.4 (Foreman, Magidor, Schindler) Suppose that $(\delta_n : n < \omega)$ is a strictly increasing sequence of cardinals with supremum $\delta$, a strong limit cardinal, such that for every $n < \omega$, both $\delta_n$ and $\delta_n^+$ have the tree property. Then for all bounded $X \subset \delta$ and for all $n < \omega$, $M^\#(X)$ exists.

This result is published in [1]. It is not known if its conclusion can be strengthened significantly. The same remark applies to each of the following two results which are produced in [2].

Theorem 2.1.5 (Gitik, Schindler) Suppose that $\kappa$ is a singular cardinal of uncountable cofinality such that the set

$$\{\alpha < \kappa : 2^\alpha = \alpha^+\}$$

is stationary as well as costationary in $\kappa$. Then for all bounded $X \subset \kappa$ and for all $n < \omega$, $M^\#_n(X)$ exists.

Theorem 2.1.6 (Schindler) Suppose that $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_{\omega_1}$. Then for every bounded $X \subset \aleph_{\omega_1}$ and for all $n < \omega$, $M^\#_n(X)$ exists.

The following theorem involves computing the complexity of $K \cap HC$.

Theorem 2.1.7 (Steel, Woodin) Suppose that the (lightface) theory of $L(\mathbb{R})$ cannot be changed by set-sized forcing. Then for every $X$ and for all $n < \omega$, $M^\#_n(X)$ exists.

Woodin has shown that the hypothesis of Theorem 2.1.7 implies that $AD^{L(\mathbb{R})}$ holds.

Theorem 2.1.8 (Steel) Suppose that there is a non-overlapping iteration tree $T$ on $V$ with cofinal well-founded branches $b \neq c$. Then for every bounded subset $X \in \mathcal{M}(T)^1$ of $\delta(T)$ and for all $n < \omega$, $M^\#_n(X)$ exists.

\footnote{the common part model}
2.2. THE FAILURE OF □.

This result is shown in [12]. It is actually shown there that its hypothesis
gives the existence of $M_\omega^\#$.

The following is a result for which we have to work in ZF rather than
ZFC.

**Theorem 2.1.9 (Busche, Schindler)** Suppose that all uncountable cardinals are singular. Then for every set $X$ of ordinals and every $n < \omega$, $M_\#^n(X)$ exists.

D. Busche has shown that the hypothesis of Theorem 2.1.9 in fact proves
that $AD$ holds in the $L(\mathbb{R})$ of a set-forcing extension of $HOD$ (cf. his talk).

Let us also consider a result which involves an ideal.

**Theorem 2.1.10 (Steel, with CH; Woodin, in general)** If there is a
homogeneous presaturated ideal on $\omega_1$, then for all $X \subseteq \omega_1$ and for all $n < \omega$, $M_\#^n(X)$ exists (and is $\omega_2$-iterable).

This list could (and should!) be extended.

A few open problems:

1. Does Wadge determinacy for projective sets imply PD?

2. If all projective sets are Lebesgue measurable and have the Baire
   property, and if all $\Pi^1_{2n+1}$ relations can be uniformized by $\Pi^1_{2n+1}$ functions,
   does PD hold?

2.2 The failure of □.

We first prove Theorem 2.1.2 by a “global” induction. We then prove Theorem 2.1.3 by “localizing” the argument.

**Lemma 2.2.1 (Todorcevic)** If PFA holds, then for all $\kappa \geq \omega_1$, □$_\kappa$ fails.

**Lemma 2.2.2 (Schimmerling, Zeman)** Let $L[E]$ be a countably iterable extender model. If $\kappa$ is not subcompact in $L[E]$, then $L[E] \models \Box_\kappa$.

By a “coarse premouse” we mean a model of the form

$$(M; \in, U),$$
where $\mathcal{M}$ is transitive, $(\mathcal{M}; \in) \models \text{ZFC}^-$ and there is a largest cardinal, say $\Omega$, and $(\mathcal{M}; \in U) \models \text{"U is a measure on } \Omega\text{"}$. In this context, “weasels” will have height $\Omega$, and the Witness Dichotomy (cf. below) holds inside every coarse premouse. Such coarse premice will be the “local universes” we work in.

**Lemma 2.2.3 (Mitchell, Schimmerling, Steel)** Let $\mathcal{P} = (\mathcal{M}; \in, U)$ be a coarse premouse. Suppose $X \in V^\mathcal{P}_\Omega$ to be such that $K(X)^\mathcal{P}$ exists. (I.e., $K^c(X)^\mathcal{P}$ has no Woodin cardinals “above $X$” and is $(\omega, \Omega+1)$-iterable “above $X$” in $\mathcal{P}$.) Then for every $\beta \in (\text{Card}(\text{TC} \{X\}), \Omega)$,

$$\mathcal{P} \models \text{cf}(\beta^{K(X)}) \geq \text{Card}(\beta).$$

In particular, $\beta^{+K(X)} = \beta^+$ in $\mathcal{P}$ for every $\beta \in (\text{Card}(\text{TC} \{X\}), \Omega)$ which is singular in $\mathcal{P}$.

**Proof** of Theorem 2.1.2 by induction on $n$. Let $X \in V$. Let $\theta$ be “large” (we want that there is a singular $\kappa$ between $\text{Card}(\text{TC} \{X\})$ and $\theta$). Let $L_p(H_\theta)$ be the “lower part closure” of $H_\theta$. We must have that $\lambda^{+L_p(H_\theta)} < \lambda^+$ for all $\lambda > \text{Card}(H_\theta)$ by Lemmas 2.2.1 and 2.2.2. This gives us some $L_p(H_\theta)$-cardinal $\mu$ such that

$$\mathcal{P} = (H^L_{\mu^+}(H_\theta); \in, U)$$

is a coarse premouse (cf. [8]).

Now fix $n$, and let us suppose that if $n > 0$, then $V$ (and hence $\mathcal{P}$) is closed under $Y \mapsto M^\#_{n-1}(Y)$. By Lemmas 2.2.1, 2.2.2, and 2.2.3, $K(x)^\mathcal{P}$ cannot exist, i.e., alternative (2) of the Witness Dichotomy (cf. below) cannot hold true in $\mathcal{P}$, so that $M^\#_n(x)$ exists from the point of view of $\mathcal{P}$. By letting $\theta$ vary, we see that $M^\#_n(x)$ exists and is iterable with respect to set-sized trees. □

**Proof** of Theorem 2.1.3 by induction on $n$. Fix $n$, and let us suppose that if $n > 0$, then $V$ (and hence $\mathcal{P}$) is closed under $Y \mapsto M^\#_{n-1}(Y)$.

**Definition 2.2.4** Let $X \subset \kappa$. We let $M^\#_n(X)$ be the “least” countably iterable active $X$-premouse which is closed under $Y \mapsto M^\#_n(Y)$. 
2.3. SUCCESSIVE CARDINALS WITH THE TREE PROPERTY

$M_n^{##}(X)$, if it exists, is actually also a coarse premouse in the sense of the definition above.

We first show that $M_n^{##}(X)$ exists for every $X \subseteq \kappa$. If $X$ is a bounded subset of $\kappa$, then this follows from a covering argument. Now let $X \subseteq \kappa$ be arbitrary. By Lemma 2.2.2,

$$\kappa^{+LP}(X) < \kappa^+,$$

so that we may take a nice hull of $H_{\kappa^+}$ which is cofinal in $\kappa^{+LP}(X)$ and has size $< \kappa$. This gives $\pi : H \rightarrow H_{\kappa^+}$. Let $X = \pi^{-1}(X)$. Then $M_n^{##}(X) \in H$, by a lift up argument. But then $M_n^{##}(X) \in H_{\kappa^+}$.

This argument can also be used to show that if $M_{n+1}^#(X)$ exists for every bounded $X \subseteq \kappa$, then $M_{n+1}^#(X)$ exists for every $X \subseteq \kappa$. It therefore suffices to prove that $M_{n+1}^#(X)$ exists for every bounded $X \subseteq \kappa$.

Fix such an $X$, and suppose that $K(X)$ exists in every $M_n^{##}(Z)$, $Z \subseteq \kappa$. For each $\alpha \leq \kappa$, there is a cone $C$ of $Z \subseteq \kappa$ such that $K(X)^{M_n^{##}(Z)}$ up to its $\alpha^+$ is independent from the choice of $Z \in C$. But then there is some $Z \in C$ such that $K(X)^{M_n^{##}(Z)}$ doesn’t satisfy weak covering at $\kappa$ inside $M_n^{##}(Z)$. Contradiction!

This gives $M_{n+1}^#(X)$.

For the construction of stable K’s like in the preceding argument the following result is useful.

**Lemma 2.2.5 (Schindler)** Suppose that $\mathcal{P}$ is a coarse premouse, and that $K(X)^\mathcal{P}$ exists for some $X \in \mathcal{P}$. Then inside $\mathcal{P}$ the following holds true. For all $\kappa \geq \aleph_2 \cdot \text{Card}(TC(\{X\}))$, if $\kappa$ is a cardinal of $K(X)$, then $K(X)^{||\kappa^+K(X)}$ is just the stack of collapsing mice for $K(X)^{||\kappa}$.

2.3 Successive cardinals with the tree property

**Proof** of Theorem 2.1.4. The key fact here is that if $\eta$ has the tree property and if $M$ is an inner model such that $\eta$ is inaccessible in $M$ and $\text{cf}(\eta^{+M}) = \text{cf}(\eta^{++M}) = \eta$, then there is an elementary embedding $\pi : M \rightarrow N$ such that $N$ is transitive and $\pi$ is discontinuous at $\eta^{+M}$.

We prove by induction on $n$ that for every bounded subset $X$ of $\delta$ and for every $n < \omega$, $M_n^{##}(X)$ exists. Fix $X$ and $n$. First, for all bounded $Y \subseteq \delta$, we easily get $M_n^{##}(Y)$ by the key fact. We may then argue in $M_n^{##}(H_\delta)$, ...
where \( \theta < \delta \) is such that there is some pair \( \delta_n, \delta_n^+ \) “between” \( X \) and \( \theta \), to get that \( K(X) M_n^\#(H_\theta) \) cannot exist. The reason is that if \( \sigma : K(X) \rightarrow W \) is an elementary embedding, then \( \sigma \) is continuous at all successor cardinals of \( K(X) \).

\[ \square \]

### 2.4 Pcf theory

**Proof of Theorem 2.1.5.** From the hypothesis of Theorem 2.1.5, we get that for every club \( C \subset \kappa \) there is some strictly increasing sequence \( (\kappa_n : n < \omega) \) with supremum \( \kappa \) in \( C \) such that \( \text{pcf}(\{\kappa_n^+ : n < \omega\}) \geq \kappa^{++} \). The idea is that if the core model exists, then this can’t be the case by a covering argument.

We prove by induction on \( n \) that for every bounded subset \( X \subset \kappa \) and for every \( n < \omega \), \( M_n^\#(X) \) exists. Fix \( X \) and \( n \). First, for all bounded \( Y \subset \kappa \), we get \( M_n^\#(Y) \) by a covering argument. (This is easy; notice that the hypothesis implies that SCH fails cofinally often below \( \kappa \).) Now suppose that \( K(X) M_n^\#(Y) \) exists for all \( Y \) above \( X \). We may then use Lemma 2.2.5 to define a stable \( K(X) \) of height \( \kappa \) in a straightforward way.

We may now pick the club \( C \subset \kappa \) in such a way that we’ll have the following for the strictly increasing sequence \( (\kappa_n : n < \omega) \) which is given to us. For every \( f \in \prod_{n<\omega} \kappa_n^+ \), a covering argument gives us some \( M \subset K(X) \) projecting to \( \kappa^+ \) such that for each \( n < \omega \), \( f(\kappa_n^+) \) (which is \( < \kappa_n^+ \)) is contained (as a subset) in \( \text{Hull}^M(\kappa_n \cup \{p\}) \) (for some parameter \( p \)). But this clearly implies that \( \text{cf}(\prod_{n<\omega} \kappa_n^+) = \kappa^+ \). Contradiction! \[ \square \]

The **proof** of Theorem 2.1.6 is similar.

### 2.5 \( L(\mathbb{R}) \) absoluteness

**Proof of Theorem 2.1.7.** By applying the hypothesis to adding \( \omega_1 \) Cohen reals, we see that there can’t be a well-ordering of \( \mathbb{R} \) in \( L(\mathbb{R}) \). The idea now is that if \( K(X) \) were to exist, then such a well-ordering would have to exist after all.

The key fact here will be that if \( x \) is a real and if \( K(x) \) exists, then \((K(x) \cap \text{HC}) \) is definable over \( (J_{\omega_1}(\mathbb{R}); \in, \exists) \) in the parameter \( x \), where \( \exists \) is the collection of all countably iterable countable premice (cf. Chapter 3).

We prove by induction on \( n \) that for every \( X \) and for every \( n < \omega \), \( M_n^\#(X) \) exists. Fix \( X \) and \( n \). Let \( \theta \) be such that there is a singular strong limit cardinal \( \kappa \) “between” \( X \) and \( \theta \). Set \( W = Lp(H_\theta) \), if \( M_n^\#(H_\theta) \) doesn’t
2.6. THE UNIQUE BRANCHES HYPOTHESIS

exist; if it does, but \( M_{n+1}^\#(X) \) does not, then we set \( W = K(X)^{M_{n+1}^\#(H_\theta)} \). It suffices to derive a contradiction from the hypothesis that \( \kappa^+ W = \kappa^+ \).

Well, if \( \kappa^+ W = \kappa^+ \), then there is a set-sized forcing \( P \) such that if \( G \) is \( P \)-generic over \( V \), then \( \omega_1^{V[G]} = \kappa^+ \) and in \( V[G] \), there is a real \( z \) such that a well-ordering of the reals may be defined over \( W|\kappa^+ W[z] \). But then the desired contradiction follows from the key fact above. □

2.6 The unique branches hypothesis

Proof of Theorem 2.1.8. Let \( T \) be a non-dropping iteration tree on \( V \) with cofinal well-founded branches \( b \neq c \). Set \( \delta = \delta(T) \), which is an ordinal of cofinality \( \omega \).

Lemma 2.6.1 (Woodin) The set \( P(\delta) \cap M_b^T \cap M_c^T \) has size \( \delta \).

The reason for this is that every \( X \subset \delta \) in \( M_b^T \cap M_c^T \) is coded by some bounded subset of \( \delta \) in a fashion which comes out of the proof that \( \delta \) is Woodin in \( M_b^T \cap M_c^T \).

Another fact is that \( \delta \) is either singular or measurable in \( M_b^T \) as well as in \( M_c^T \).

Let us now fix \( n \), where we assume that \( M(T) \) is closed under \( Y \mapsto M_b^\#(Y) \). Let us consider \( X \in M(T) \). If \( M_{n+1}^\#(X) \) does not exist, then we may produce \( W = K(X)^{M(T)} \) by knitting together \( K(X)^{M(T)}|\Omega \) for \( M(T) \)-measurables \( \Omega < \delta \) (there are cofinally in \( \delta \) many such). We may also let \( P \upharpoonright W \) be the stack of all \( \delta \)-sound premice end-extending \( W \) which project to \( \delta \) and are countably iterable above \( \delta \). We also have versions of \( P \) inside \( M_b^T \) and \( M_c^T \), respectively; call them \( P_b \) and \( P_c \). W.l.o.g., \( P_b \supseteq P_c \). Because \( \delta \) is Woodin in \( M_b^T \cap M_c^T \), \( \delta \) is Woodin in \( P_b \). A covering argument shows that \( P_b \cap \text{OR} = \delta^+M_b^T \), and similarly for \( c \).

Let \( \mu \leq \delta \) be least such that \( \pi^T_{b\delta}(\mu) \geq \delta \). If \( \pi^T_{b\delta}(\mu) > \delta \), or else if \( \pi^T_{b\delta} \) is discontinuous at \( \mu \), then we may then produce inside \( M_b^T \) (by a “lift up”) a \( \delta \)-sound premouse \( Q \) end-extending \( W \) which project to \( \delta \), is countably iterable above \( \delta \), and in fact kills the Woodinness of \( \delta \). This is nonsense.

So \( \pi^T_{b\delta}(\mu) = \delta \) and \( \pi^T_{b\delta} \) is continuous at \( \mu \). We then use Lemma 2.6.1 to get a contradiction.

\footnote{By this we just mean \( J_{\kappa^+ W}[E, z] \), where \( W \) is constructed from \( E \).}
2.7 All uncountable cardinals are singular.

Proof of Theorem 2.1.9: Let $X$ be a set of ordinals. Let $\kappa$ be a cardinal above $\text{Card}(TC(\{X\})) \cdot \aleph_2$. Let $Y \subset \kappa^+$ be cofinal such that $\text{otp}(\kappa^+ L[X]) = \omega$. Then $L[X,Y] \models \text{``X\# exists,''}$ because $\text{cf}(\kappa^+ L[X]) = \omega$ in $L[X,Y]$.

Now let $n > 0$ and let us assume inductively that $M_{n-1}^\#(X)$ exists for every set $X$ of ordinals. We leave to the reader the argument which gives that $M_n^\#(X)$ exists for every set $X$ of ordinals.

Fix $X$, a set of ordinals. Let $\kappa$ be a cardinal above $\text{Card}(TC(\{X\})) \cdot \aleph_2$. Let $\theta$ be a cardinal above $\kappa$ such that every subset of $\kappa^+$ is generic for the Vopenka algebra over $H_\theta^{\text{HOD}}$. Consider $M_{n-1}^\#(H_\theta^{\text{HOD}})[X] = M_{n-1}^\#(H_\theta^{\text{HOD}}[X])$.

We want to see that $K(X)^{M_{n-1}^\#(H_\theta^{\text{HOD}}[X])}$ cannot exist. Deny, and let us write $K$ for it. Let $Y \subset \kappa^+ K$ be cofinal such that $\text{otp}(Y) = \omega$. Notice that $K = K(X)^{M_{n-1}^\#(H_\theta^{\text{HOD}}[X,Y])}$, but we have a failure of weak covering at $\kappa$.

This argument gives that $M_n^\#(X)$ exists in $\text{HOD}[X]$. But we can prove that $M_n^\#(X)$ is in fact iterable with respect to set-sized trees.

2.8 An ideal.

Proof of Theorem 2.1.10: We prove Theorem 2.1.10, albeit under the additional hypothesis that CH holds.

Let $I$ be a homogeneous presaturated ideal on $\omega_1$. If we force with $I$, we get a generic $V$-ultrafilter $G$ and some

$$\pi: V \to M,$$

where $M$ is transitive, $\text{crit}(\pi) = \omega^V_1$, $\pi(\omega^V_1) = \omega^V_2$, and $<\omega^V_2 M \cap V[G] \subset M$ (so that $\omega^V_2 = \omega^V_1[G]$). This easily implies that $x\#$ exists for every real $x$, because set-forcing cannot add sharps. Also, if $X \subset \omega_1$, then $X$ is coded by a real in $M$, so $X\#$ exists in $M$ by elementarity; but then $X\#$ exists in $V$, again because set-forcing cannot add sharps. We have therefore verified Claim 0:
2.8. AN IDEAL.

Claim 2n. For every $X \subseteq \omega_1$, $M_\omega^n(X)$ exists and is $\omega_2^V$-iterable (in $V$ as well as in $M$ and $V[G]$). Moreover, for every $x \in \mathbb{R}^M = \mathbb{R}^{V[G]}$, $(M_\omega^n(x))^M$ is $\omega_1^{V[G]} + 1$ iterable in $V[G]$.

Claim 2n+1. For every $X \subseteq \omega_1$, $M_\omega^{\#}(X)$ exists and is $\omega_2^V$-iterable (in $V$ as well as in $M$ and $V[G]$).

Claim 2n is used to prove Claim 2n+1, which in turn is used to prove Claim 2n+2. Claim 2n $\Rightarrow$ Claim 2n+1 is a simplified variant of the argument which is to come and which shows Claim 2n+1 $\Rightarrow$ Claim 2n+2, so we leave it to the reader.

Let us suppose that Claim 2n+1 holds.

By the above argument for sharps, it is enough to prove that $M_\omega^n(x)$ exists for every real $x$. This can be seen as follows. Let $A \subseteq \omega_1$. If the first part of Claim 2n+2 holds for all $X \subseteq \omega$, then $M \models "M_{n+1}(A)$ exists and is $\omega_2$-iterable.” By the second part of Claim 2n+2, $(M_{n+1}^\#(A))^M$ is the unique least $M$ in $V[G]$ which is not $n+1$ small and which is $\omega_1^{V[G]}$-iterable. (We may compare any such $M$ with $(M_{n+1}^\#(A))^M$ inside $M$!) Therefore, $(M_{n+1}^\#(A))^M$ is definable in $V[G]$, and hence $(M_{n+1}^\#(A))^M \in V$ by homogeneity. Using the first part of Claim 2n, $(M_{n+1}^\#(A))^M$ is $\omega_1$-iterable in $V$ and thus also $\omega_2$-iterable in $V$, as $H_{\omega_2}^V$ is closed under the $M_{n+1}^\#$ operation.

Let $N = M_{n-1}^{\#}(\mathbb{R})$, which is an $\mathbb{R}$-premouse. (Here we use CH.) Let $H$ be $Col(\omega_1, \mathbb{R})$-generic over $N$. Let $x \in \mathbb{R}$. It suffices to prove that $K(x)^N[H]$ cannot exist. Suppose it did, and write $K = K(x)^N[H]$.

We claim that $\pi(K) \in V$. Well, this is where the homogeneity of $I$ is used. $K$ is definable in $N[H]$, and hence in $N$, as $Col(\omega_1, \mathbb{R})$ is homogeneous. So $\pi(K)$ is definable in $\pi(N)$. However, $\pi(N)$ is definable in $M$ as well as in $V[G]$. Therefore, $\pi(K) \in V$ by the homogeneity of $I$.

The fact that $\pi(K) \in V$ easily implies that $\omega_1^V$ is inaccessible in $K$. Now let $E$ be the extender at $\omega_1^V$, $\omega_2^V$ derived from $\pi \upharpoonright K$. Notice that for every $\alpha < \omega_2^V$, $E \upharpoonright \alpha \in M$ because of $<\omega_2^V M \cap V[G] \subseteq M$. We claim that in fact for every $\alpha < \omega_2^V$, $E \upharpoonright \alpha \in \pi(K)$, which gives a contradiction.

In order to verify this claim, we need to see that

$$\pi(N) \models ((\pi(K), \Ult(\pi(K)), E \upharpoonright \alpha), \alpha) \text{ is } \pi(\Omega)\text{-iterable}$$

where $\Omega$ is the largest measurable cardinal of $N$. By reflection, we’d otherwise have, in $\pi(N)$, $\sigma: \bar{K} \rightarrow \Ult(\pi(K), E \upharpoonright \alpha)$ with $\sigma \upharpoonright \alpha = \text{id}, \bar{K}$ is
countable, and
\[ \pi(N) = (\pi(K), K, \alpha) \] is not \( \omega_1 \)-iterable.

Now notice that because \( \pi(K) \in V \), \( \pi(\pi(K)) \) makes sense. We have the factor map \( k: \text{Ult}(\pi(K), E \upharpoonright \alpha) \to \pi(\pi(K)) \) with \( k \upharpoonright \alpha = \text{id} \), so that
\[ \sigma: K \to \pi(\pi(K)) \]
is such that \( \tau \circ \sigma \upharpoonright \alpha = \text{id} \).

Set \( \psi = k \circ \sigma \). We have that \( \psi, K \in M \). Let \( \psi = [\xi \mapsto \psi(\xi)]_G, K = [\xi \mapsto K_\xi]_G, \) and \( \alpha = [\xi \mapsto \alpha_\xi]_G \). We need to see that for \( G \)-almost all \( \xi \),
\[ N \models ((K, K_\xi), \alpha_\xi) \]
is \( \omega_1 \)-iterable.

By absoluteness, for \( G \)-almost all \( \xi \), in \( \pi(N) \) there is some \( \psi': K_\xi \to \pi(K) \) such that \( \psi' \upharpoonright \alpha_\xi = \text{id} \). But for any such \( \xi \) we then have in \( N \) some \( \tilde{\psi}: K_\xi \to K \) such that \( \tilde{\psi} \upharpoonright \alpha_\xi = \text{id} \).

\[ \square \]

### 2.9 How on earth should we generalize this?

In general, we aim to show that (an initial segment of) \( V \) is closed under more powerful “mouse operators” \( X \mapsto M(X) \). However, we shouldn’t try keeping track of the first order theory of \( M(X) \). Rather, what we want is that \( M(X) \) has a Woodin cardinal \( \delta \) as well as a term \( \tau \in M(X)^{\text{Col}(\omega, \delta)} \) which captures a complicated set \( A \) of reals in the sense that if \( \pi: M(X) \to M^* \) is a non-dropping iteration of \( M(X) \) and \( g \) is \( \text{Col}(\omega, \pi(\delta)) \)-generic over \( M^* \), then \( \pi(\tau)^g = M^* \cap A \). (For this to make sense we must have that \( M^* \) is countable and \( g \in V \).) It is easy to verify (using the extender algebra) that for any \( \Sigma^1_{2n+2} \) set of reals \( A \), \( M^*_2 \) has a term (at its largest Woodin cardinal) capturing \( A \).
Chapter 3

The core model induction through $L(\mathbb{R})$

3.1 Core model theory for one $J$-Woodin

What we have done so far is start with some transitive set closed under the mouse operator $J(x) = M^+_n(x)$, and use core model theory to show that transitive set is closed under the “one $J$-Woodin” operator $J^w(x) = M^2_{n+1}(x)$. \footnote{There was an intermediate step, in which we obtained closure under the operator $J^f(x) = M^2_{n}(x)$.} We have done this by showing that the alternative to this closure is that there is a $J$-closed core model, then invoking whatever strong hypothesis we are considering to rule out such a core model.

This is the basic pattern of the core model induction at all its successor steps. The limit steps are devoted to constructing an operator $J$ which is suitable as a basis for the next successor step. In order to go further, we must abstract what it is about $J$ which makes it suitable. We begin the abstraction process in this section.

Definition 3.1.1 Let $A \in H_\nu$; then a mouse operator over $A$ on $H_\nu$ is a function $J$ such that for some $Q$-formula $\psi$,

$$J(B) = \text{least } \mathcal{P} \sqsubseteq \mathcal{L}p(B) \text{ such that } \mathcal{P} \models \psi[A,B]$$

for all self-wellordered $B \in H_\nu$ such that $A \in B$. ($J$ must be defined at all such $B$.) We call $J$ a $(\nu,A)$-mo. 

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Assuming that they are defined on all of $H_\nu$, the $M^\#_n$ operators are $(\nu,0)$-mouse operators, for each $n \leq \omega$. The weakest operator beyond all the $M^\#_n$ operators, for $n$ finite, is $x \mapsto \bigcup M^\#_n(x)$. This would be the operator we would construct at step $\omega$ of the core model induction, as a basis for the steps $\omega + n$, for $n < \omega$. A core model induction which remains in $L(\mathbb{R})$ will never reach the $M^\#_\omega$ operator.

A $(\nu,A)$-mo may not be a first order mouse operator in the sense we defined earlier, because we are allowing a name for $A$, and not just $B$, in the theory which fixes $J(A \cup \{A\})$. A $(\nu,0)$-mo is a first order mouse operator, but the converse may fail because we have imposed a special restriction on the domains. The domain of a $(\nu,A)$-mo is the “cone above $A$” in $H_\nu$. We should say here that we are going to use this notion also in the case that $\nu$ is singular, and in that case, our convention is that $H_\nu = \bigcup_{\mu < \nu} H_\mu = \{ x \mid \text{TC}(x) < \nu \}$.

**Definition 3.1.2** Letting $J$ be a $(\nu,A)$-mo, we define the one $J$-Woodin operator $J^w$ by

$$J^w(B) = M^J_1(B) = \text{least } \mathcal{P} \leq \text{Lp}(B) \text{ s.t. } \exists \delta \mathcal{P} \models \delta \text{ is Woodin, and } \mathcal{P} = J(\mathcal{P}|\delta).$$

So if $J^0$ is the sharp operator, and $J^{n+1} = (J^n)^w$, then $J^n$ is the $M^\#_n$ operator.

**Definition 3.1.3** Let $J$ be a $(\nu,A)$-mo, and let $\mathcal{M}$ be a premouse over $A$; then $\mathcal{M}$ is $J$-closed above $\eta$ iff whenever $\eta < \xi < \nu$ and $\xi$ is a cardinal of $\mathcal{M}$, then $J(\mathcal{M}|\xi) \subseteq \mathcal{M}$. We say $\mathcal{M}$ is $J$–closed iff $\mathcal{M}$ is $J$-closed above 0.

**Definition 3.1.4** A $(\nu,A)$-mo $J$ relativises well iff

1. for any rudimentary function $f$, there is a formula $\theta(u,v,w)$ such that whenever $B,C \in \text{dom}(J)$ and $B = f(C)$, and $N$ is a transitive model of ZFC$^-$ such that $J(C) \in N$, then $J(B) \in N$ and $J(B)$ is the unique $x \in N$ such that $N \models \theta[x,A,J(C)]$, and

2. if $B \in \text{dom}(J)$ and $\eta$ is a cutpoint of $J(B)$, then $J(J(B)|\eta)$ is not a proper initial segment of $J(B)$. 

Suppose that $n \leq \omega$, and that the $M_n^\sharp$ operator is defined on $H_\nu$. It is then a $(\nu,0)$-mo which relativises well. There are many more examples.

The next two lemmas motivate definition 3.1.4.

**Lemma 3.1.5** Let $J$ be a $(\nu, A)$-mo which relativises well, and let $M$ be a $J$-closed premouse over $A$ such that $M|\eta \models \text{ZFC}^-$ for arbitrarily large $\eta < \alpha(M)$; then

(a) for all $x \in \text{dom}(J) \cap M$, $J(x) \in M$,

(b) $J \upharpoonright M$ is definable over $M$.

**Lemma 3.1.6** Let $J$ be a $(\nu, A)$-mo which relativises well; then for any $B \in \text{dom}(J)$, $J(B)$ has a $J$-guided $\nu$-iteration strategy.

**Proof.** Let $T$ be a $J$-guided tree on $J(B)$ of limit length $< \nu$. Note that $J(M(T))$ exists. Taking a countable Skolem hull of $V$ as usual, with $\mathcal{P}, \mathcal{U}, \mathcal{R}$ the images of $J(B), T, J(M(T))$ under collapse, we get that $Q(b, U) \subseteq \mathcal{R}$, where $b$ is the good branch of $U$. This is because clause (2) of 3.1.4 is a first order property of $J(B)$ (relative to $A, B$). This property (relative to the collapses of $A$ and $B$) holds in $\mathcal{P}$, and by inspecting it, we see that it also holds in $Q(b, U)$, and that therefore $Q(b, U) \subseteq \mathcal{R}$. \hfill $\Box$

The following is our basic core model dichotomy theorem.

**Theorem 3.1.7** ($K^\omega$-existence dichotomy) Let $\Omega$ be ineffable, and $J$ an $(\Omega, B)$-mo which relativises well. Suppose $K^\omega(B)$, as constructed with $\tau$-complete background extenders from $V_\Omega$, is tame, and suppose $\xi < \Omega$ is the strict sup of the Woodin cardinals of $K^\omega(B)$, with $\xi = 0$ if there are none; then

(a) $K^\omega(B)$ is $J$-closed above $\xi$, and

(b) if there is no $\tau$-iterable $M_\omega^\sharp$, then either

(i) $M_1^\sharp(B) \subseteq K^\omega(B)$ (so that $M_1^\sharp(B)$ exists and is $\tau$-iterable, or

(ii) $K^\omega(B)$ is $(\omega, \Omega + 1)$-iterable, so that $K(B)$ exists.

**Proof.** For (a): Compare $J(K^\omega(B)|\gamma)$ with $K^\omega(B)$. The $J(K^\omega(B)|\gamma)$ side provides the $Q$-structures identifying the good branch on the $K^\omega(B)$ side, so
this can be done. We have the branches at $\Omega$ by ineffability. The $J(K^c(B)|\gamma)$ side comes out shorter because $K^c(B)$ is universal.

For (b): If $K^c(B)$ has a Woodin cardinal, then it has a largest one $\delta$. (This is where we use that $M^2_\omega$ does not exist.) But then $J(K^c(B)|\delta) \preceq K^c(B)$ by (a), so $K^c(B)$ reaches a one-$J$-Woodin level. We leave it as an exercise to show that $M^1_\delta(B) \preceq K^c(B)$.

So we may assume $K^c(B)$ has no Woodin cardinals, and fails to reach $M^1_\delta(B)$. In this case, $J$ guides an $(\omega, \Omega)$-iteration strategy for $K^c(B)$.

One can show that under the hypotheses of 3.1.7, if $J$ relativises well, then $K^c(B)$ is fully $J$-closed. We shall not need this fact, however.

We sometimes write $K^{c,J}(B)$ for $K^c(B)$, and $K^{J}(B)$ for $K(B)$, for such $J$-closed $K^c(B)$ or $K(B)$. When dealing later with hybrid mouse operators $J$, one must close the analog of $K^{c,J}$ under $J$ by explicitly telling the model how $J$ acts on its levels. At the current level of abstraction, $K^{c,J}(B)$ is an ordinary $B$-mouse, simply equal to $K^c(B)$, and the notation just reminds us it is $J$-closed, and can define $J$ restricted to itself.

3.1.7 uses the hypothesis that $M^2_\omega$ does not exist in a fairly inessential way. If we had defined $K^{c,J}(B)$ by explicitly closing its levels under $J$, we could have avoided this hypothesis. We chose to defer a discussion of the explicit-closure construction until we are dealing with hybrid $J$, so that we need it. The price is that we have to prove that our $K^c$ is $J$-closed, and in that proof, the assumption that $K^c$ has a largest Woodin, if it has any, shows up.

By applying our $K^{J}$-dichotomy inside suitably chosen $J$-closed inner models, we get a “stable $K^c$” dichotomy theorem which requires no large cardinal hypothesis on $V$ whatsoever. The following version will suffice for our global-strength applications.

Suppose $\kappa$ is uncountable, $\kappa = |V_\alpha|$, $J$ is a $(\kappa, A)$-mo, and $B \in \text{dom}(J)$. We say then that $K^{J}(B)^{V_\kappa}$ exists iff for any $\alpha < \kappa$ such that $B \in V_\alpha$, there is a model $N = N_\alpha$ of ZFC such that $V_\alpha \in N$, and an $\Omega > \alpha$ such that $\Omega$ is subtle in $N$, and

$$N \models K^{c,J}(B) \text{ is } (\omega, \Omega + 1)$-iterable.$$

In this case, $(K^{J}(B))^{N}$ exists, of course, by CMIP. More importantly, $(K^{J}(B)^{N_\alpha})^{N_\alpha} \upharpoonright \alpha = (K^{J}(B)^{N_\beta} \upharpoonright \alpha$, for all $\beta > \alpha$, because the inductive definition of $K$ is local. When $(K^{J}(B)^{V_\kappa}$ exists, then it is the limit of these approximating $K$’s. Note that $(K^{J}(B)^{V_\kappa}$ has all the local properties of $K(B)$; for exam-
3.2. MOUSE WITNESS CONDITIONS

ple, it computes successors of singular cardinals \( \xi \) such that \( \text{rk}(B) < \xi < \kappa \) correctly.

**Theorem 3.1.8 (Stable-\( K^J \) dichotomy)** Let \( \kappa > \omega \) be such that \( \kappa = |V_\kappa| \), and let \( J \) be a \((\kappa, A)\)-mo which relativises well. Suppose that there is no \( \kappa \)-iterable \( M^\kappa_2 \), and let \( B \in \text{dom}(J) \); then either

(a) \( M^J_1(B) \) exists, and is \( \kappa \)-iterable, or

(b) \( K^J(B)^{V_\kappa} \) exists, or

(c) for some \( C \in \text{dom}(J) \), \( K^J(C)^{V_\kappa} \) exists, and is the minimal \( J \)-closed model of height \( \kappa \) over \( C \).

**Proof.** It is an easy exercise to abstract this from the proofs we have already given. \( \square \)

In order to use these dichotomy theorems to obtain more than one Woodin cardinal, we need also

**Lemma 3.1.9** Let \( J \) be a \((\nu, A)\)-mo which relativises well, and suppose that \( J^w \) is also a \((\nu, A)\)-mo; then \( J^w \) relativises well.

### 3.2 Mouse witness conditions

The induction variable in a core model induction represents the degree of correctness of the mice one can construct, or what is roughly equivalent, the complexity of their iteration strategies. The description of how these two measures move in tandem is at the heart of the method.

The useful measure of correctness and complexity here is descriptive set theoretic. We saw in section 1 that we should expect to obtain uncountable iterability from sufficiently generically absolute \( \omega_1 \)-iterability for countable structures. An \( \omega_1 \)-iteration strategy for a countable premouse is a set of reals, so our measure of “sufficient absoluteness” should be a measure of definability for sets of reals. For this reason, even when one is trying to show some very large transitive set \( X \) is closed under mouse operators, one generally moves to a generic extension \( V'[g] \) with \( X \subseteq \text{HC}^{V'[g]} \), and shows \( \text{HC}^{V'[g]} \) is appropriately closed. In the ensuing discussion, we shall basically assume we are already in such a \( V[g] \).
Our plan then is to construct mice which are correct for some given level \( \Gamma \) of the Wadge hierarchy, via an induction on those levels. For now, we shall remain within \( L(\mathbb{R}) \), but eventually we shall consider \( \Gamma \) which are beyond \( L(\mathbb{R}) \). Descriptive set theory is used to organize the induction, and in particular, the next \( \Gamma \) to consider is the next scaled pointclass.

There will actually be two hypotheses of the existence correct mice, or *mouse witnesses*. In the first, the witnessing mouse is coarse-structural, and in the second, it is an honest fine-structural mouse. We begin with the coarse-structural witness condition.

**Definition 3.2.1** Let \( U \subseteq \mathbb{R} \), and \( k < \omega \). Let \( N \) be countable and transitive, and suppose \( \delta_0, ..., \delta_k, S, \) and \( T \) are such that

1. \( N \models ZFC \land \delta_0 < ... < \delta_k \) are Woodin cardinals,
2. \( N \models S, T \) are trees which project to complements after the collapse of \( \delta_k \) to be countable, and
3. there is an \( \omega_1+1 \)-iteration strategy \( \Sigma \) for \( N \) such that whenever \( i : N \rightarrow P \) is an iteration map by \( \Sigma \) and \( P \) is countable, then \( p[i(S)] \subseteq U \) and \( p[i(T)] \subseteq \mathbb{R} \setminus U \).

Then we say that \( N \) is a coarse \((k, U)\)-Woodin mouse, as witnessed by \( S, T, \Sigma, \delta_0, ..., \delta_k \).

Our inductive hypothesis is

\((W^*_{\alpha})\) Let \( U \) be a subset of \( \mathbb{R} \), and suppose there are scales \( \vec{\phi} \) and \( \vec{\psi} \) on \( U \) and \( \mathbb{R} \setminus U \) respectively such that \( \vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R}) \), where \( \vec{\phi}^* \) and \( \vec{\psi}^* \) are the sequences of prewellorders associated to the scales. Then for all \( k < \omega \) and \( x \in \mathbb{R} \) there are \( N, \Sigma \) such that

1. \( x \in N \), and \( N \) is a coarse \((k, U)\)-Woodin mouse, as witnessed by \( \Sigma \), and
2. \( \Sigma \upharpoonright \text{HC} \in J_\alpha(\mathbb{R}) \).

We emphasize that in \( W^*_{\alpha} \), it is the sequences \( \vec{\phi}^*, \vec{\psi}^* \) which are in \( J_\alpha(\mathbb{R}) \), not just the individual prewellorders in the sequences.

In the end, the mice we construct to verify \( W^*_{\alpha} \) will not be particularly coarse; they will either be ordinary mice constructed from fine extender sequences, or *hybrid* mice, constructed from a fine extender sequence and
3.2. MOUSE WITNESS CONDITIONS

an iteration strategy. In either case they will have a fine structure, and be suitable for building core models. We shall use the core model theory of [11] to construct them.

One can think of $W^*_\alpha$ as asserting, for the given $U$, that there is a mouse operator $x \mapsto M_x$, defined on $x \in \mathbb{R}$ such that $M_x$ is a $(k, U)$-Woodin mouse over $x$.

We now derive some useful consequences of $W^*_\alpha$.

**Lemma 3.2.2** If $W^*_\alpha$ holds, then $J_\alpha(\mathbb{R}) \models \text{AD}$.

**Proof.** By the reflection theorems of Kechris-Solovay or Kechris-Woodin, it is enough to show that $U$ is determined whenever $U$ and $\mathbb{R} \setminus U$ admit scales in $J_\alpha(\mathbb{R})$. So fix such a $U$, and let $N$ be a coarse $(1, U)$-Woodin mouse, as witnessed by $S, T, \delta_0, \delta_1$, and $\Sigma$. We have that

$N \models p[S]$ is homogeneously Suslin,

and hence $p[S]$ is determined in $N$. Let

$N \models \tau$ is a winning strategy for $p[S]$.

We may assume without loss of generality that $N$ believes $\tau$ wins for I. We claim that in $V$, $\tau$ wins the game with payoff $U$ for I. For suppose $y$ is a play for II defeating $\tau$; then we can iterate $N$ by $\Sigma$, yielding $i: N \to P$, with $y$ generic over $P$ at $i(\delta_0)$. Since $\tau(y) \notin U$, and $i(S), i(T)$ are absolute complements over $P$, $\tau(y) \in p[i(T)]$. By absoluteness of wellfoundedness, $P \models \exists y \tau(y) \in p[i(T)]$. This contradicts the elementarity of $i$. \hfill \Box

From Woodin’s mouse set argument, we get “capturing on a cone”:

**Lemma 3.2.3** Suppose $W^*_\alpha$ holds; then for a Turing cone of reals $x$, the following are equivalent, for all reals $y$:

(a) $y$ is $\text{OD}^{J_\beta(\mathbb{R})}(x)$, for some $\beta < \alpha$,

(b) there is a (fine-structural) $x$-mouse $M$ such that $y \in M$ and $M$ has an $\omega_1$-iteration strategy in $J_\alpha(\mathbb{R})$.

\footnote{Henceforth, this means that the sequence of associated prewellorders is in the model.}
CHAPTER 3. THE CORE MODEL INDUCTION THROUGH $L(\mathbb{R})$

Proof. See [16].

We now want to prove a lightface result on the existence of fine-structural mouse witnesses. We shall call this result $W_\alpha$. For a technical reason having to do with the real parameters which may enter into the definition of a scale, we are only able to prove $W_\alpha$ in the case that $\alpha$ is a limit ordinal and $\alpha$ begins a (perhaps trivial) gap.

To any $\Sigma_1$ formula $\theta(v)$ we associate formulae $\theta^k(v)$ for $k \in \omega$, such that $\theta^k$ is $\Sigma_k$, and for any $\gamma$ and any real $x$,

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \iff \exists k < \omega J_{\gamma}(\mathbb{R}) \models \theta^k[x].$$

Our fine-structural witnesses are as follows.

**Definition 3.2.4** Suppose $\theta(v)$ is a $\Sigma_1$ formula (in the language of set theory expanded by a name for $\mathbb{R}$), and $z$ is a real; then a $\langle \theta, z \rangle$-witness is an $\omega$-sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable $z$-mouse $N$ in which there are $\delta_0 < \ldots < \delta_9$, $S$, and $T$ such that $N$ satisfies the formulae expressing

(a) ZFC,

(b) $\delta_0, \ldots, \delta_9$ are Woodin,

(c) $S$ and $T$ are trees on some $\omega \times \eta$ which are absolutely complementing in $\mathcal{V}^{\text{Col}(\omega, \delta_0)}$, and

(d) For some $k < \omega$, $p[T]$ is the $\Sigma_{k+3}$-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where $\gamma$ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.

We should note that this is different from the notion of $\langle \theta, z \rangle$-witness defined in [15]. The witnesses in that sense are mice with infinitely many Woodin cardinals, and so they are too crude for our purposes here.

**Remark 3.2.5** The more general notion would be that of a $\Sigma^2_1$-witness. Given a $\Sigma^2_1$ formula $\exists A \phi(A, v)$ and real $z$, a mouse witness to $\exists A \phi(A, z)$ is a mouse $N$ satisfying the conditions of 3.2.4, with (d) changed to: $N \models \phi[p[T], z]$. The definition given above amounts to considering only $\Sigma^2_1$ formulae of the form $\exists A( A$ codes some $J_\gamma(\mathbb{R})$ such that $J_\gamma(\mathbb{R}) \models \theta[z]$). This is all we need for an induction which stays in $L(\mathbb{R})$.

The next lemma justifies “witness”.

**Lemma 3.2.6** If there is a $(\theta, z)$-witness, then $L(\mathbb{R}) \models \theta[z]$. 
3.3. THE WITNESS DICHOTOMY

Proof. This is an exercise using genericity iterations. The key is that if \( N, T, U \) are as in 3.2.4, then the iteration strategy for \( N \) lets us interpret \( T \) on arbitrary reals in an unambiguous way: if \( i: N \to N_0 \) and \( j: N \to N_1 \) are iteration maps, and \( x \) is generic over both \( N_0 \) and \( N_1 \), then \( x \in p[i(T)] \) iff \( x \in p[j(T)] \). We get this from comparing \( N_0 \) with \( N_1 \), and using Dodd-Jensen plus the existence of \( U \).

Further details are in [13, 1.10]. \( \square \)

What we show in our core model induction is just that the converse of 3.2.6 holds for \( L(\mathbb{R}) \), at least for \( \alpha \) a limit ordinal. More precisely, we show that for \( \alpha \) a limit,

\[(W_\alpha) \text{ If } \theta(v) \text{ is } \Sigma_1, \ z \in R, \text{ and } J_\alpha(R) \models \theta(z), \text{ then there is a } \langle \theta, z \rangle\text{-witness } \mathcal{N} \text{ whose associated iteration strategy, when restricted to countable iteration trees, is in } J_\alpha(R). \]

Lemma 3.2.7 Let \( \alpha \) be a limit ordinal, and suppose that \( W_\alpha^* \) holds; then \( W_\alpha \) holds.

\[ \text{Proof. This is very close to the proof of the lightface capturing theorem of [16]. See also [13, 1.11].} \ \square \]

Note also that we get lightface capturing from \( W_\alpha \).

Lemma 3.2.8 Assume \( W_\alpha \) holds; then if \( x, y \in \mathbb{R} \) and \( x \) is ordinal definable from \( y \) over some \( J_\gamma(R) \), where \( \gamma < \alpha \), then there is a \( y \)-premouse \( \mathcal{M} \) such that \( x \in \mathcal{M} \), and \( J_\alpha(R) \models \mathcal{M} \text{ is } \omega_1\text{-iterable.} \]

3.3 The witness dichotomy

Although our induction hypothesis \( W_\alpha^* \) is often interpreted in some \( V[g] \), one must nevertheless periodically go back to \( V \), where the proposition from which one is mining logical strength holds true. We shall see that \( W_\alpha \) gives us mice in \( V \) over terms \( \tau \) such that \( \tau^g \in V[g] \). Those mice can be used in \( V \) to form operators \( J \) to which our dichotomy theorems 3.1.7 and 3.1.8 on the existence of \( K^J \) can be applied. What we show in this part of the argument is

Theorem 3.3.1 (Witness dichotomy) Let \( \nu \) be an uncountable cardinal, and let \( g \) be \( \text{Col}(\omega, < \nu)\)-generic over \( V \), let \( R^g = \bigcup_{\alpha<\nu} R^V[g^\alpha] \) be the reals of the symmetric collapse, and suppose that \( L(R^g) \models \text{DC} \); then either
(a) for all $\alpha$, $W^*_\alpha$ holds in $L(\mathbb{R}^g)$, or

(b) for some $A \in V_\kappa$, and some $(\nu, A)$-hmo $J$, $J^w$ does not exist.

There is no misprint in alternative (b): in general, $J$ will be a hybrid mouse operator, or “hmo”. We shall give the definition later. The assertion that $J^w$ does not exist means just that there is some $B \in \text{dom}(J)$ such that there is no countably iterable $M^J(B)$.

Alternative (a) is equivalent to the assertion that AD holds in $L(\mathbb{R}^g)$. One direction is the Mouse Set Theorem, the other is 3.2.2.

Notice that as stated, even when $\nu = \omega_1$ the Witness Dichotomy does not apply directly to $L(\mathbb{R})^V$. However, one can show that in this case, if $W^*_\alpha$ holds in $L(\mathbb{R}^g)$, then it holds in $L(\mathbb{R})^V$. In fact, if AD$^L(\mathbb{R})$ holds after adding any number of Cohen reals, then it held in $V$.

Combining 4.1.2 with 3.1.8, we get

Theorem 3.3.2 Let $\kappa > \omega$ be such that $|V_\kappa| = \kappa$. Let $g$ be Col$(\omega, < \kappa)$-generic over $V$, let $\mathbb{R}^g = \bigcup_{\alpha < \kappa} V^{[g][\alpha]}$ be the reals of the symmetric collapse, and suppose that $L(\mathbb{R}^g) \models \text{DC}$; then either

(a) for all $\alpha$, $W^*_\alpha$ holds in $L(\mathbb{R}^g)$, or

(b) for some $A \in V_\kappa$, and some $(\kappa, A)$-hmo $J$, $K^J(A)_{V_\kappa}$ exists.

The assertion that $K^J(A)_{V_\kappa}$ exists is to be understood in the same stable-$K$ sense that we had for pure mouse operators.

Hybrid $K^J(A)_{V_\kappa}$ behaves much like a pure $K^J(A)$, so that, for example, its existence is incompatible with the failure of square at a singular $\mu$ such that $\text{rk}(A) < \mu < \kappa$. So we get

Corollary 3.3.3 Let $\kappa > \omega$ be such that $|V_\kappa| = \kappa$, and suppose that for arbitrarily large $\alpha < \kappa$, one of the following holds:

(1) $\alpha$ is a singular cardinal and $\Box_\alpha$ fails,

(2) $\alpha$ and $\alpha^+$ have the tree property,

(3) there is a generic almost-huge embedding with critical point $\alpha$ given by some ideal in $V_\kappa$.

Let $g$ be Col$(\omega, < \kappa)$-generic over $V$, let $\mathbb{R}^g = \bigcup_{\alpha < \kappa} V^{[g][\alpha]}$ be the reals of the symmetric collapse, and suppose that $L(\mathbb{R}^g) \models \text{DC}$. Then for all $\alpha$, $W^*_\alpha$ holds in $L(\mathbb{R}^g)$. 
Corollary 3.3.4 (Woodin) If PFA holds, and \( \kappa \) is inaccessible, and \( g \) is \( \text{Col}(\omega, < \kappa) \)-generic over \( V \), then \( \text{AD}^{L(\mathbb{R})} \) holds in \( V[g] \).

Remark 3.3.5 From 3.3.3 one gets that the following are equiconsistent over ZFC:

(a) there are infinitely many Woodin cardinals,

(b) there is an uncountable \( \kappa \) such that \( |V_\kappa| = \kappa \) and

(i) for arbitrarily large \( \alpha < \kappa \), there is a generic almost-huge embedding with critical point \( \alpha \) given by some ideal in \( V_\kappa \), and

(ii) \( L(\mathbb{R}^g) \models \text{DC} \), where \( \mathbb{R}^g \) is the set of reals of a symmetric collapse below \( \kappa \).

The result would be cleaner if one could omit (ii) from (b). In fact, we do not know whether the hypothesis that \( L(\mathbb{R}^g) \models \text{DC} \) can be omitted from 3.3.3. This would amount to proving it from the other hypotheses, as the \( W^*_\alpha \)'s collectively imply \( L(\mathbb{R}^g) \models \text{DC} \).

The Witness Dichotomy summarizes the part of a core model induction which does not involve \( K \) directly; the part which keeps track of mouse-correctness, and builds appropriate mouse operators at limit steps. One then combines 4.1.2 with our \( K^J \) dichotomies 3.1.7, 3.1.8, and some strong proposition which implies we must be in the “\( J^w(B) \) exists” case of 3.1.7 or 3.1.8. The upshot is that \( W^*_\alpha \) holds in \( L(\mathbb{R}^g) \), for all \( \alpha \).

We shall prove the Witness Dichotomy in the succeeding sections.

3.4 Scales in \( L(\mathbb{R}) \)

One proves the Witness Dichotomy 4.1.2 by considering the least \( \alpha \) for which \( W^*_\alpha \) fails, and analyzing the situation well enough to obtain alternative (b). The \( W^*_\alpha \) assert that, given \( U \subseteq \mathbb{R} \), as soon as a scale on \( U \) appears, an iteration strategy for a coarse mouse having a forcing term for such a scale appears. Thus it is useful to know how scales appear. Under appropriate determinacy hypotheses, there is in the Wadge hierarchy of \( L(\mathbb{R}) \) (and beyond) a tight correspondence between the appearance of scales on sets which did not previously admit them, and certain failures of reflection. This correspondence is analyzed in detail in [9] and [14]. Our proof of 4.1.2 breaks into cases which reflect that analysis.
Definition 3.4.1 An ordinal \( \beta \) is critical just in case there is some set \( U \subseteq \mathbb{R} \) such that \( U \) and \( \mathbb{R} \setminus U \) admit scales in \( J_{\beta + 1}(\mathbb{R}) \), but \( U \) admits no scale in \( J_\beta(\mathbb{R}) \).

(Once again, we are identifying a scale with the sequence of its prewellorders here.) Clearly, we need only show that \( W^*_\beta \) holds whenever \( \beta \) is critical, in order to conclude that \( W^*_\alpha \) holds for all \( \alpha \).

It follows from [9] that if \( \beta \) is critical, then \( \beta + 1 \) is critical. Moreover, if \( \beta \) is a limit of critical ordinals, then \( \beta \) is critical if and only if \( J_\beta(R) \) is not an admissible set. Letting \( \beta \) be critical, we then have the following possibilities

1. \( \beta = \eta + 1 \), for some critical \( \eta \);
2. \( \beta \) is a limit of critical ordinals, and either
   a. \( \text{cof}(\beta) = \omega \), or
   b. \( \text{cof}(\beta) > \omega \), but \( J_\beta(\mathbb{R}^\beta) \) is not admissible;
3. \( \alpha = \sup(\{ \eta < \beta \mid \eta \text{ is critical} \}) \) is such that \( \alpha < \beta \), and either
   a. \( [\alpha, \beta] \) is a \( \Sigma_1 \) gap, or
   b. \( \beta - 1 \) exists, and \( [\alpha, \beta - 1] \) is a \( \Sigma_1 \) gap.

In each case, we prove 4.1.2 by constructing a \((\nu, A)\)-hmo \( J \) such that, were \( J^w \), \((J^w)^w \), etc., to exist, then collectively they would yield mice which verify \( W^*_\beta \). Thus, we must find a way of feeding truth at the bottom of the Levy hierarchy of \( J_\beta(\mathbb{R}) \) into mice. In cases 1 and 2(a), this is fairly easy: \( \Sigma^1_{\text{iter}}(\mathbb{R}) \) is the class of countable unions of sets belonging to \( J_\beta(\mathbb{R}) \), so we can just put together countably many mice given by our induction hypothesis. We shall not give any details on these cases in this paper.

We call case 2(b) the inadmissible case, and we shall give it a fairly thorough treatment in the next section.

Case 3, the end-of-gap (in scales) case, is the most subtle. In this case, the set coding truth at the bottom of the Levy hierarchy over \( J_\beta(\mathbb{R}) \) which we feed into our mice will be an iteration strategy \( \Sigma \) for a mouse \( \mathcal{M} \) with a Woodin cardinal which is \( \text{Lp}^\alpha \)-full, in a sense we shall explain. The structures which witness the truth of \( W^*_\beta \) will be hybrid \( \Sigma \)-mice, mice over \( \mathcal{M} \) constructed from an extender sequence as usual, while simultaneously closing under \( \Sigma \). We shall go further into case 3 in the section after next.
3.5. The inadmissible case

In this section we prove 4.1.2 in the case that \( \alpha \) begins a \( \Sigma_1 \) gap in \( L(\mathbb{R}^\theta) \), \( J_\alpha(\mathbb{R}^\theta) \) is inadmissible, and \( \alpha \) has uncountable cofinality in \( L(\mathbb{R}^\theta) \). We assume \( W_\alpha^* \) holds in \( L(\mathbb{R}^\theta) \), and \( W_{\alpha+1}^* \) fails in \( L(\mathbb{R}^\theta) \). We shall show that there is in \( V \) a pure \((\nu, A)\)-mo \( J \) such that \( J^w \) does not exist.

Since \( \alpha \) is a limit ordinal, we have \( W_\alpha \) by Lemma 3.2.7.

Let \( \phi(v_0, v_1) \) and \( x \in \mathbb{R}^\theta \) determine the failure of admissibility, so that \( \phi \) is \( \Sigma_1 \),

\[
\forall y \in \mathbb{R}^\theta \exists \beta < \alpha J_\beta(\mathbb{R}^\theta) = \phi[x, y],
\]

and letting \( \beta(x, y) \) be the least such \( \beta \),

\[
\alpha = \sup\{\beta(x, y) \mid y \in \mathbb{R}^\theta\}.
\]

(Since \( \alpha \) begins a gap, \( J_\alpha(\mathbb{R}^\theta) \) is the \( \Sigma_1 \) hull of its reals, so the parameter from which a failure of admissibility is defined can be taken to be a real.) Let \( x = \tau^0|\mu \), where \( \tau \in H_\mu \), with \( \mu < \kappa \). Let \( p \in g \restriction \mu \) force in \( \text{Col}(\omega, < \mu) \) over \( V \) all the properties of \( \tau \) which we have listed as properties of \( x \) in \( V[g] \) so far.

Our plan is to construct, in \( V \), \((\nu, \tau)\)-mo \( J^0 \) which is strong enough that were the \( n \)-J-Woodin operators \( J^n \), given by \( J^{n+1} = (J^n)^w \), to exist as \((\nu, \tau)\)-mo’s, then collectively they would verify \( W_{\alpha+1}^* \). (After extending them to act on \( \mathbb{R}^\theta \), that is.) \( J \), and hence \( J^w \), etc., will relativise well. Since \( W_{\alpha+1}^* \) fails, we then have alternative (b) of 4.1.2, with \( J = J^n \) for some \( n < \omega \).

Let us call \( A \) suitable if \( A \in V_\nu \), \( A \) is transitive and self-wellordered, and \( \tau \in A \). So our \( J \) is to be defined on all suitable \( A \).

Let \( A \) be suitable, and let \( \mathcal{M} \) be an \( A \)-premouse, and \( G \times H \) be \( \mathcal{M} \)-generic over \( \text{Col}(\omega, < \mu) \times \text{Col}(\omega, A) \); then \( \mathcal{M}[G][H] \) can be regarded as a \( z \)-premouse, where \( z = z(G, H) \) is a real obtained in some simple fashion from \( G, H \), and \( A \), and which in turn codes \( G, H \), and \( A \) in some simple fashion. (See [14].) Also, there is a term \( \sigma = \sigma_A \) defined uniformly from \( A, \tau \) in \( \mathcal{M} \) such that whenever \( G \times H \) is generic as above, then \( \sigma^{G \times H} \in \mathbb{R} \) and

\[
(\sigma^{G \times H})_0 = \tau^G,
\]

and

\[
\{ (\sigma^{G \times H})_i \mid i > 0 \} = \{ \rho^{G \times H} \mid \rho \in L_1(A) \text{ and } \rho^{G \times H} \in \mathbb{R} \}.
\]
CHAPTER 3. THE CORE MODEL INDUCTION THROUGH $L(R)$

Here $(w)_i$ is the $i^{th}$ real coded into the real $w$, in some fixed simple way, and $L_1(A)$ is the first level of Gödel’s $L$ over $A$. For $n < \omega$, let $\phi^*_n$ be the $\Sigma_1$ formula

$$\phi^*_n(v) = \exists \alpha(J_\alpha(R) \models \forall i \in \omega(i > 0 \Rightarrow \phi((v)_0, (v)_i)) \land (\alpha + \omega n) \text{ exists} )$$

Now let $\psi$ be the natural sentence in the language of $A$-premice (having therefore a name for $A$) such that for any $A$-premouse $M$:

$$M \models \psi$$

iff whenever $G \times H$ is $M$-generic over $\text{Col}(\omega, \mu) \times \text{Col}(\omega, \nu)$ and $p \in G$, then for any $n$ there is a $\gamma < o(M)$ such that

$$M[z(G,H)][\gamma] \text{ is a } \langle \phi^*_n, \sigma^{G \times H} \rangle \text{-witness}.$$  

**Definition 3.5.1** For any suitable $A$, $J^0(A)$ is the shortest initial segment of $L_p(A)$ which satisfies $\psi$, if it exists, and is undefined otherwise.

**Lemma 3.5.2** For any suitable $A$, $J^0(A)$ exists, and moreover, $J^0(A)$ is ordinal definable from $A$ over $J(\mathbb{R}_0)$, for some $\gamma < \alpha$.

**Sketch of Proof.** Since $W_\alpha$ is a closure condition on the mice over reals in $\mathbb{R}_0$, there is some work to be done in going back to $V$, as is done in this lemma.

Fix $(h \times H) \in V[g]$ which is $V$-generic over $\text{Col}(\omega, < \mu) \times \text{Col}(\omega, A)$, and such that $p \in h$. For $q \leq p$, let $h_q$ be the finite variant of $h$ such that $q \in h_q$. Clearly, $h_q \times H$ is $V$-generic for $\text{Col}(\omega, < \mu) \times \text{Col}(\omega, A)$.

Since $p$ forced what it did over $V$, we have in $V[g]$ a $\langle \phi^*, \sigma^{h_q \times H} \rangle$-witness $N_q$, for each $q \leq p$. Here $N_q$ is a mouse over $z(h_q, H)$ which has an iteration strategy in $J_\alpha(\mathbb{R}_0)$. (We use $\text{col}^{L(\mathbb{R}_0)}(\alpha) > \omega$ at this point, as we have to sup over the stages $\gamma < \alpha$ at which $\Sigma_1$ witnesses for $\phi(\tau^{h_q}, \rho^H)$, for $p \in L_1(A)$, have iteration strategies.)

Note $z(h_q, H)$ is Turing equivalent to $z(h_r, H)$, for $q, r \leq p$. So the $N_q$ are all essentially mice over the same real $z$ (though the parameters $\tau^{h_q}$ and $\tau^{h_r}$ for the $\Sigma_1$ sentences witnessed by these mice have nothing to do with one another). So we can, in $L(\mathbb{R}_0)$, take the union of the countably many mice over $z$ corresponding to different $q$’s. Call this union $N$.

Let $\mathcal{P}$ be the structure constructed over $A$ from the extender sequence of $N$. One can show that in $L(\mathbb{R}_0)$, $\mathcal{P}$ is an iterable mouse over $A$ such that
3.5. THE INADMISSIBLE CASE

$P[h \times H] = N$. This is done by showing, by induction on $\eta$, that $P|\eta$ is an iterable mouse, that $P|\eta \in V$ (so that $h \times H$ is generic over $P|\eta$), and that $(P|\eta)[h \times H] = N|\eta$. See the proof of Theorem 3.9 of [14] for a detailed version of this “inverting a generic extension” argument.

$P$ is the desired $J^0(A)$.

\[\begin{array}{l}
\text{Lemma 3.5.3 } J^0 \text{ relativises well.}
\end{array}\]

\[\begin{array}{l}
\text{Lemma 3.5.4 Suppose that } J^n \text{ is defined for all } n < \omega; \text{ then } W^*_{\alpha+1} \text{ holds in } L(\mathbb{R}^\alpha).
\end{array}\]

Proof sketch. Let $U$ be a set of reals in $J_{\alpha+1}(\mathbb{R}^\alpha)$, and $k < \omega$; we seek a coarse $(k,U)$-Woodin mouse. Suppose that $U$ is $\Sigma_n$-definable over $J_\alpha(\mathbb{R}^\alpha)$ from the real parameter $z$.

Let $P = J^{k+n+3}(\tau,\rho)$. We show that $P[\bar{g}]$ is the desired witness.

Let $\delta_0$ be the largest Woodin cardinal of $P$, and $\delta_1$ the next-to-largest. Let $W$ be the universal $\Sigma_1^{J_\alpha}(\mathbb{R}^\alpha)$ set of reals, and $\theta$ a $\Sigma_1$ formula which defines it over $J_\alpha(\mathbb{R}^\alpha)$. Let $\Sigma$ be the canonical iteration strategy for $P$, and hence for $P[\bar{g}]$. There is a term $W \in P[\bar{g}]$ such that whenever

$$i: P[\bar{g}] \rightarrow Q[\bar{g}]$$

is an iteration map by $\Sigma$, and $h$ is Col$(\omega, i(\delta_1))$- generic over $Q[\bar{g}]$, and $y \in \mathbb{R} \cap Q[\bar{g}][h]$, then

$$y \in W \iff y \in i(\bar{W})^h.$$  

Roughly speaking, the term $\bar{W}$ asks: if we Levy collapse $\delta_0$ via $l$, and then using $J^0(\bar{P}[\bar{g}][h][l]')[\delta_0]$ as our oracle for the theory of the first level of $L(\mathbb{R})$ at which $\phi(x,\sigma)$ is seen to be true for all terms $\sigma \in L_1(\bar{P}[\bar{g}][h]')[\delta_0]$, do we see that $\theta(y)$ has been verified before that level? Since any real $t$ can be obtained as such a $\sigma$ after an iteration of $Q[\bar{g}][h]$ above $i(\delta_1)$ and below $i(\delta_0)$, and since the ordinals we called $\beta(x,t)$ were cofinal in $\alpha$, $\bar{W}$ behaves as advertised.

Since $\alpha$ is inadmissible and begins a gap, the $\Sigma_1$ theory of $J_\alpha(\mathbb{R}^\alpha)$ can be computed from the $\Sigma_1^1$ theory of $(\mathbb{R},W,x)$. Let $\delta$ be the $k^{th}$ Woodin.

\[\begin{array}{l}
\text{There is a lightface } \Sigma_1 \text{ partial map of } \mathbb{R}^\alpha \text{ onto } J_\alpha(\mathbb{R}^\alpha).\end{array}\]
cardinal (from the bottom) of $P[\bar{g}]$. Using the Woodins above $\delta$ to answer one-real-quantifier questions as above, we get a term $\dot{U}$ in $P[\bar{g}]$ such that if $h$ is $P$-generic over $\text{Col}(\omega, \delta)$ and $y$ is a real in $P[\bar{g}][h]$, then

$$y \in U \iff y \in \dot{U}^h.$$ 

Moreover, letting $\gamma = (\delta^+_0)^P$, and $\pi : Q[\bar{g}] \to P[\bar{g}]|\gamma$ and $\pi(\dot{Z}) = \dot{U}$, and $h$ is $Q[\bar{g}]$ generic over $\text{Col}(\omega, \pi^{-1}(\delta))$, then again, $\dot{Z}^h = U \cap Q[\bar{g}][h]$.\footnote{This fact about Skolem hulls follows from the construction. It comes down to the fact that an elementary submodel of an iterable structure is still iterable.} In $P[\bar{g}]$ we can then construct the absolutely complementing trees $S$ and $T$ required by 4.1.1: $T_y$ tries to build $\pi, Q, h$ as above with $y \in \dot{Z}^h$, and $S_y$ tries to build $\pi, Q, h$ as above with $y \notin \dot{Z}^h$. \hfill $\square$

This completes our proof of 4.1.2 in the inadmissible, uncountable cofinality case. \hfill $\square$

Case (1) and the remainder of case (2) in the proof of 4.1.2 are similar to case we have just done.
Chapter 4

The core model induction through $L(\mathbb{R})$, continued

4.1 Review

Here is a short review of some key points:

1. Mouse witnesses

   A countable coarse mouse $\mathcal{N}$, together with an iteration strategy $\Sigma$ for $\mathcal{N}$, can capture a set of reals $U$ as follows:

   **Definition 4.1.1** Let $U \subseteq \mathbb{R}$, and $k < \omega$. Let $\mathcal{N}$ be countable and transitive, and suppose $\delta_0, ..., \delta_k, S, T$ are such that

   (a) $\mathcal{N} \models ZFC \wedge \delta_0 < ... < \delta_k$ are Woodin cardinals,

   (b) $\mathcal{N} \models S, T$ are trees which project to complements after the collapse of $\delta_k$ to be countable, and

   (c) there is an $\omega_1+1$-iteration strategy $\Sigma$ for $\mathcal{N}$ such that whenever $i: N \rightarrow P$ is an iteration map by $\Sigma$ and $P$ is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R} \setminus U$.

   Then we say that $\mathcal{N}$ is a coarse $(k, U)$-Woodin mouse, as witnessed by $S, T, \Sigma, \delta_0, ..., \delta_k$.

   Core model inductions running through $L(\mathbb{R})$ show:
(Wₙ*) Let $U$ be a subset of $\mathbb{R}$, and suppose there are scales $\vec{\phi}$ and $\vec{\psi}$ on $U$ and $\mathbb{R} \setminus U$ respectively such that $\vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R})$, where $\vec{\phi}^*$ and $\vec{\psi}^*$ are the sequences of prewellorders associated to the scales. Then for all $k < \omega$ and $x \in \mathbb{R}$ there are $N, \Sigma$ such that

1. $x \in N$, and $N$ is a coarse $(k, U)$-Woodin mouse, as witnessed by $\Sigma$, and
2. $\Sigma \upharpoonright \text{HC} \in J_\alpha(\mathbb{R})$.

The more general notion, applying beyond $L(\mathbb{R})$, would be $\Sigma^2_1$-capturing. Via the Mouse Set Conjecture, $W^*_\alpha$ implies the version of itself in which the capturing mouse $\mathcal{N}$ is fine structural. This is important for the induction. Within $L(\mathbb{R})$, MSC is a theorem (cf. [16]).

2. The witness dichotomy

The “core-model free” portion of a core model induction can be considered as a proof of:

**Theorem 4.1.2 (Witness dichotomy, WD)** Let $\nu$ be an uncountable cardinal, and let $g$ be $\text{Col}(\omega, < \nu)$-generic over $V$, let $\mathbb{R}^g = \bigcup_{\alpha < \nu} \mathbb{R}^V[g \upharpoonright \alpha]$ be the reals of the symmetric collapse, and suppose that $L(\mathbb{R}^g) \models \text{DC}$; then either

1. for all $\alpha$, $W^*_\alpha$ holds in $L(\mathbb{R}^g)$, or
2. for some $A \in V_\kappa$, and some $(\nu, A)$-hmo $J$, $J^w$ does not exist.

Alternative (b) is a statement about $V$. One then uses core model theory in $V$ to rule out (b) under whatever hypothesis whose strength is being mined. We have stated some $K^J$-dichotomy theorems which help do this in a general way, although we do not have truly comprehensive general theorems there as yet.

3. Structure of WD proof

One proves the Witness Dichotomy 4.1.2 by considering the least $\beta$ for which $W^*_{\beta+1}$ fails, and analyzing the situation well enough to obtain alternative (b). This $\beta$ is critical:

**Definition 4.1.3** An ordinal $\beta$ is critical just in case there is some set $U \subseteq \mathbb{R}$ such that $U$ and $\mathbb{R} \setminus U$ admit scales in $J_{\beta+1}(\mathbb{R})$, but $U$ admits no scale in $J_\beta(\mathbb{R})$. 
Letting $\beta$ be critical, we have the following possibilities

(1) $\beta = \eta + 1$, for some critical $\eta$;

(2) $\beta$ is a limit of critical ordinals, and either
   (a) $\text{cof}(\beta) = \omega$; or
   (b) $\text{cof}(\beta) > \omega$, but $J_\beta(\mathbb{R}^\theta)$ is not admissible;

(3) $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical} \})$ is such that $\alpha < \beta$, and either
   (a) $[\alpha, \beta]$ is a $\Sigma_1$ gap, or
   (b) $\beta - 1$ exists, and $[\alpha, \beta - 1]$ is a $\Sigma_1$ gap.

In each case, we prove WD by constructing a $(\nu, A)$-hmo $J$ such that, were $J^w$, $(J^w)^w$, $\ldots$, etc., to exist, then collectively they would yield mice which verify $W^*_\beta + 1$. Thus, we must find a way of feeding truth at the bottom of the Levy hierarchy of $J_\beta(\mathbb{R})$ into mice.

The parameter $A$ over which $J$ is defined codes a term $\tau$ in $V$ such that $\tau^\mathbb{R}$ is a real about which a new $\Sigma_1$ statement has become true in $J_\beta(\mathbb{R}^\theta)$. There may be auxiliary information coded in $A$ as well.

We proved WD in case (2)(b) last time, and the time before. (Busche intertwined the proof of WD in this case with a proof that “All uncountable cardinals are singular” implies alternative (b) of WD cannot hold.)

The proof in cases (1) and (2)(a) is like that in case (2)(b), but easier.

In these notes, we prove WD in case (3).

## 4.2 Scales at the end of a gap

For the rest of this note, we fix $\alpha$ and $\beta$ as in (3). (We can ignore the distinction between (3)(a) and (3)(b).) We have that $J_\alpha(\mathbb{R}^\theta)$ is admissible, and $\Sigma_1\text{-proyects to } \mathbb{R}^\theta$. We have $W^*_\beta$, and hence $W_\alpha$.

Here are the basic facts about scales and reflection we shall need. As to reflection, we have

**Theorem 4.2.1 (Martin [5])** Assume $W^*_\beta$, where $\beta$ is critical and case 3 holds at $\beta$. Then for any $x, y \in \mathbb{R}^\theta$, if $x \in OD^\gamma(y)$ for some $\gamma < \beta$, then $x \in OD^\gamma(y)$ for some $\gamma < \alpha$.

As to scale existence, we have
Theorem 4.2.2 ([9]) Assume $W^*_\beta$, where $\beta$ is critical and case 3 holds at $\beta$; then

1. every set of reals $A \in J_\beta(\mathbb{R}^9)$ admits a scale $\vec{\psi}$ such that each prewellorder $\leq_{\psi_i}$ belongs to $J_\beta(\mathbb{R}^9)$, and

2. letting $n$ be least such that $\rho_n(J_\beta(\mathbb{R}^9)) = \mathbb{R}$, and $U$ be any boldface $\Sigma^J_\beta(\mathbb{R}^9)$ set of reals, we have $U = \bigcup_{n<\omega} U_n$, where each $U_n \in J_\beta(\mathbb{R}^9)$.

In part (1), the sequence of prewellorders may not belong to $J_\beta(\mathbb{R}^9)$. Part (2) implies that the boldface pointclass $\Sigma^J_\beta(\mathbb{R}^9)$ is in fact the class of countable unions of sets of reals in $J_\beta(\mathbb{R}^9)$, and has the scale property.

Motivated by 4.2.2, we make the

Definition 4.2.3 A self-justifying system (sjs) is a countable set $A \subseteq P(\mathbb{R})$ which is closed under complements (in $\mathbb{R}$), and such that every $A \in A$ admits a scale $\vec{\psi}$ such that $\leq_{\psi_i} \in A$ for all $i$.

So for any set of reals $A \in J_\beta(\mathbb{R}^9)$, there is a self-justifying system $A \subseteq J_\beta(\mathbb{R}^9)$ such that $A \in A$.

4.3 The Plan

The set coding truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^9)$ which we feed into our mice will be an iteration strategy $\Sigma$ for a mouse $\mathcal{N}$ with a Woodin cardinal $\delta$. There will be a sjs $\{A_i \mid i < \omega\}$ such that $U = \bigcup_i A_{2i}$ is universal at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^9)$, and such that for each $i$, there is a term $\tau_i \in \mathcal{N}$ which captures $A_i$, in that

$$\pi(\tau_i)^h = A_i \cap \mathcal{P}[h],$$

whenever $\pi : \mathcal{N} \to \mathcal{P}$ is an iteration map by $\Sigma$, and $h$ is $\mathcal{P}$-generic for $\text{Col}(\omega, \pi(\delta))$. Thus $\mathcal{N}$ is close to being a coarse $(1, U)$-Woodin mouse, as witnessed by $\Sigma$.

From $(\mathcal{N}, \Sigma)$ we shall construct hybrid $\Sigma$-mice, mice over $\mathcal{N}$ constructed from an extender sequence as usual, while simultaneously closing under $\Sigma$. The condensation properties of $\Sigma$ will imply that these hybrid mice behave like ordinary mice. The upshot is that we have a hybrid mouse operator $J^0$. $J^0$ will be a $(\nu, A)$-hmo, where $A$ codes $\mathcal{N}$. Setting $J^{n+1} = (J^n)^\nu$, we can
4.4. FULLNESS-PRESERVING ITERATION STRATEGIES

capture truth at the higher levels of the Levy hierarchy over $J_\beta(\mathbb{R})$ via the $J^n(B)\upharpoonright g$’s, for $B \in H_\nu$.

So either some $(J^n)^w$ fails to exist (be defined on a cone), or we have $W_{\beta+1}^*$.

4.4 Fullness-preserving iteration strategies

Definition 4.4.1 For any self-wellordered (swo) $A \in H_\nu$, let $Lp^\alpha(A)$ be the “union” of all $A$-mice $N$ projecting to $\text{sup}(A)$ such that $J_\alpha(\mathbb{R}) \models N$ is $\omega_1$-iterable.

Note $Lp^\alpha(A)$ is an initial segment of $Lp(A)$, since the iteration strategy witnessing $N \in Lp^\alpha(A)$ is unique, so that its restriction to $V$ is in $V$.

Definition 4.4.2 Let $A \in H_\nu$ be swo. An $A$-premouse $N$ is suitable iff $\text{card}(N) < \nu$ and

(a) $N \models$ there is exactly one Woodin cardinal. We write $\delta^N$ for the unique Woodin cardinal of $N$.

(b) Letting $M_0 = N\upharpoonright \delta^N$, and $M_{i+1} = Lp^\alpha(M_i)$, we have that $N = \bigcup_{i \in \omega} M_i$. That is, $N$ is the $Lp^\alpha$ closure of $N\upharpoonright \delta^N$, up to its $\omega$th cardinal above $\delta^N$.

(c) If $\xi < \delta^N$ is a cardinal of $N$, then $Lp^\alpha(N\upharpoonright \xi) \models \xi$ is not Woodin.

We say an iteration tree $U$ on a premouse $N$ lives below $\eta$ if $U$ can be regarded as an iteration tree on $N\upharpoonright \eta$. If $U$ is normal, then as usual we write $\delta(U)$ for $\text{sup}\{\text{lh}(E^U_\alpha) \mid \alpha < \text{lh}(U)\}$, and $M(U)$ for $\bigcup\{M^U_\alpha \mid \text{lh}(E^U_\alpha) \mid \alpha < \text{lh}(U)\}$.

Definition 4.4.3 Let $U$ be a normal iteration tree of length $< \nu$ on a suitable $N$, and suppose $U$ lives below $\delta^N$; then $U$ is short iff for all limit $\xi \leq \text{lh}(U)$, $Lp^\alpha(M(U \upharpoonright \xi)) \models \delta(U \upharpoonright \xi)$ is not Woodin. Otherwise, we say $U$ is maximal.

Just to emphasize, a non-normal iteration tree is neither short nor maximal. Similarly, a tree on $N$ which cannot be regarded as a tree on $N\upharpoonright \delta^N$ is neither short nor maximal.

Definition 4.4.4 Let $\Sigma$ be a $\nu$-iteration strategy on a suitable $N$; then $\Sigma$ is fullness-preserving iff whenever $P$ is an iterate of $N$ by $\Sigma$, via a tree which lives below $\delta^N$, then
(1) if \( \mathcal{N} \)-to-\( \mathcal{P} \) does not drop, then \( \mathcal{P} \) is suitable, and

(2) if \( \mathcal{N} \)-to-\( \mathcal{P} \) drops, then \( J_\alpha(\mathbb{R}) \models \mathcal{P} \) is \( \omega_1 \)-iterable.

It is not hard to see that in case (2) of 4.4.4, we have that for all \( \xi \), \( Lp^\alpha(\mathcal{P} | \xi) \models \xi \) is not Woodin, and thus no initial segment of \( \mathcal{P} \) is suitable.

Of course, we should really speak of \( \alpha \)-suitability, etc., but \( \alpha \) has been fixed.

**Lemma 4.4.5** Suppose \( \Sigma \) is a fullness-preserving iteration strategy for \( \mathcal{N} \), and \( T \) is an iteration tree living below \( \delta^\mathcal{N} \), played by \( \Sigma \), which has a last normal component tree \( \mathcal{U} \) having base model \( \mathcal{P} \) and of limit length. Let \( b \) be the branch of \( \mathcal{U} \) chosen by \( \Sigma \); then

(1) if \( \mathcal{N} \)-to-\( \mathcal{P} \) drops, then \( \mathcal{U} \) is short, and \( Q(b, \mathcal{U}) \) is a proper initial segment of \( Lp^\alpha(\mathcal{M}(\mathcal{U})) \), and

(2) if \( \mathcal{N} \)-to-\( \mathcal{P} \) does not drop, so that \( \mathcal{P} \) is suitable, then

(a) for all \( \xi < \text{lh}(\mathcal{U}) \), \( \mathcal{U} \upharpoonright \xi \) is short,

(b) if \( \mathcal{U} \) is short, then \( Q(b, \mathcal{U}) \) exists and is a proper initial segment of \( Lp^\alpha(\mathcal{M}(\mathcal{U})) \), and

(c) if \( \mathcal{U} \) is maximal, then \( b \) does not drop, and \( i^\mathcal{U}_b(\delta^\mathcal{P}) = \delta(\mathcal{U}) \).

We shall omit the straightforward proof of this lemma.

According to this lemma, a fullness-preserving strategy is guided by \( Q \)-structures in \( Lp^\alpha \), unless, for the current normal component \( \mathcal{U} \), there is no such \( Q \)-structure. That is case (2)(c) above, and then from (2)(c) we see that \( \mathcal{U} \) has no normal continuation. Moreover, although \( Lp^\alpha \) cannot tell us what \( b \) is, it can identify \( \mathcal{M}_b(\mathcal{U}) \), since

\[
\mathcal{M}_b(\mathcal{U}) = (Lp^\alpha) - \text{closure of } (\mathcal{M}(\mathcal{U}))
\]

up to its \( \omega^{\text{th}} \) cardinal. This important insight is due to Woodin. It means that \( Lp^\alpha \) can “track” a fullness-preserving iteration strategy, in that it can find the models of an evolving iteration tree, although it cannot always find the branches and embeddings.\(^1\)

\(^1\)For infinite stacks of normal trees, more work is needed even to find the models using only \( Lp^\alpha \) as a guide. Using “quasi-iterations”, Woodin has solved this problem. We shall not need quasi-iterations for our proof of \( AD^{L(\mathbb{R})} \), but they are needed in adapting Ketchersid’s work.
We wish to describe a condensation property for iteration strategies. For the notion of a finite support in an iteration tree, see [10]. Let $T$ be an iteration tree on $\mathcal{N}$, and

$$\sigma : \beta \to \text{lh}(T)$$

an order preserving map such that $\text{ran}(\sigma)$ is support-closed. Then there is a unique iteration tree $S$ on $\mathcal{N}$ of length $\beta$ such that there are maps

$$\pi_\gamma : \mathcal{M}^S_\gamma \to \mathcal{M}^T_{\sigma(\gamma)}$$

for $\gamma < \beta$, which commute with the tree embeddings, with

$$\pi_\gamma(E^S_\gamma) = E^T_{\sigma(\gamma)}$$

for all $\gamma < \beta$, and $\pi_{\gamma+1}$ determined by the shift lemma. (Support-closure is just what we need to keep this process going.)

**Definition 4.4.6** Let $S$ and $T$ be iteration trees related as above; then we say that $S$ is a hull of $T$, as witnessed by $\sigma$ and the $\pi_\gamma$, for $\gamma < \text{lh}(T)$.

**Definition 4.4.7** An iteration strategy $\Sigma$ condenses well iff whenever $T$ is an iteration tree played according to $\Sigma$, and $S$ is a hull of $T$, then $S$ is according to $\Sigma$.

It is clear that if $\Sigma$ is the unique iteration strategy$^2$ on $\mathcal{N}$, then $\Sigma$ condenses well. More generally, if $\mathcal{N}$ is an initial segment of $\mathcal{M}$, and $\Gamma$ is the unique iteration strategy for $\mathcal{M}$, and $\Sigma$ is the strategy for $\mathcal{N}$ which is determined by $\Gamma$, then $\Sigma$ condenses well. One can think of an iteration strategy which condenses well as the “trace” of a unique iteration strategy on a stronger mouse.

**4.5 An sjs-guided iteration strategy in $V[g]$**

In this and the next section, we shall prove

**Theorem 4.5.1** There is, in $V$, a suitable $\mathcal{N}$ and a fullness-preserving $\nu$-iteration strategy $\Sigma$ for $\mathcal{N}$ such that $\Sigma$ condenses well.

$^2$For some reasonable sort of iteration game.
Proof. We work in $V[g]$ in this section, and obtain $\mathcal{N}$ and $\Sigma$ there. In the next section, we move back to $V$.

Recall that $OD^\gamma(z)$ is the collection of sets which are ordinal definable from $z$ over $J_\gamma(\mathbb{R}^g)$; we write $OD^{<\xi}(z)$ for $\bigcup_{\gamma<\xi} OD^\gamma(z)$.

Let $\langle A_i \mid i \in \omega \rangle$ be a self-justifying system, with each $A_i \in J_\beta(\mathbb{R}^g)$, and $A_0$ the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set. Let

$$x^* = \tau^g$$

be a real such that for all $i$, $A_i$ is $OD^{<\beta}(x^*)$. Here $\tau$ is (essentially) a bounded subset of $\nu$, and of course $\tau \in V$. The suitable $\mathcal{N}$ we seek will be a $\tau$-mouse.

We need some concepts and results, due to Woodin, which are explained in more detail in [16] and [17]. First

**Lemma 4.5.2** (Woodin) Let $\mathcal{N}$ be a suitable premouse over some $z \in HC$ which simply codes $x^*$, and let $\mu \geq \delta^\mathcal{N}$ be a cardinal of $\mathcal{N}$, let $A \subseteq R^g$ be $OD^{<\beta}(z)$; then there is a term $\sigma \in \mathcal{N}$ such that whenever $h$ is $\mathcal{N}$-generic for $\text{Col}(\omega, \mu)$, then

$$\sigma^h = A \cap \mathcal{N}[h].$$

**Definition 4.5.3** For $\mathcal{N}$, $z$, $\mu$, and $A$ as in 4.5.2, $\tau^{\mathcal{N}}_{A, \mu}$ is the unique standard term $\sigma$ such that $\sigma^h = A \cap \mathcal{N}[h]$ for all $\text{Col}(\omega, \mu)$-generics $h$ over $\mathcal{N}$. We write $\tau^{\mathcal{N}}_A$ for $\tau^{\mathcal{N}}_{A, \delta}$, where $\delta = \delta^\mathcal{N}$.

See [16] for further explanation. Woodin proved the following key condensation result:

**Theorem 4.5.4** (Term-relation condensation) [Woodin] Let $\mathcal{N}$ be a suitable premouse over $z \in HC$, and let $\mathcal{B}$ be a self-justifying family of subsets of $\mathbb{R}^g$ containing the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set, and such that each $B \in \mathcal{B}$ is $OD^{<\beta}(z)$. Suppose

$$\pi : M \rightarrow \mathcal{N}$$

is $\Sigma_0$-elementary and such that

$$\forall B \in \mathcal{B} \forall \mu \geq \delta^\mathcal{N} \tau^{\mathcal{N}}_{B, \mu} \in \text{ran}(\pi).$$

Then
4.5. **AN SJS-GUIDED ITERATION STRATEGY IN V[G]**

(a) $\mathcal{M}$ is suitable, and for all $B \in \mathcal{B}$,

$$\pi(\tau_{B,\mu}^\mathcal{M}) = \tau_{B,\mu}^\mathcal{N},$$

where $\pi(\bar{\mu}) = \mu$,

(b) $\text{ran}(\pi)$ is cofinal in $\delta^\mathcal{N}$, and

(c) if $\delta^\mathcal{N} \subseteq \text{ran}(\pi)$, then $\pi = \text{identity}$.

**Proof.** For part (a), see [17].

For part (b): Let $\gamma = \sup(\text{ran}(\pi) \cap \delta^\mathcal{N})$. Let $\psi: \mathcal{P} \to \mathcal{N}$ be the transitive collapse of the set of points definable over some $\mathcal{N}|\pi(\mu)$ from the $\tau_{j,l}$ for $j,l < \omega$ and ordinals $< \gamma$. Using the regularity of $\delta(T)$ in $\mathcal{N}$, we get that $\psi \upharpoonright \gamma = \text{identity}$, and $\psi(\gamma) = \delta(T)$. From part (a), we then have that $\mathcal{P}$ is suitable. But $\mathcal{P}|\gamma = \mathcal{N}|\gamma$, so we have then that $\mathcal{N}|\gamma$ is $\text{Lp}^\alpha$-Woodin. The minimality condition in the suitability of $\mathcal{N}$ then implies $\gamma = \delta(T)$, as desired.

Part (c) follows at once from the $\text{Lp}^\alpha$-fullness of $\mathcal{M}$. \qed

**Definition 4.5.5** If $\mathcal{N}$ is suitable, and $T$ is a maximal normal iteration tree on $\mathcal{N}$, then $\mathcal{M}(T)^+$ is the unique suitable $\mathcal{M}$ such that $\mathcal{M}(T) = \mathcal{P}|\delta^\mathcal{P}$.

**Definition 4.5.6** Let $\mathcal{N}$ be suitable $z$-premouse, where $z \in HC$ and codes $x^*$, and $A \subseteq R^g$ be $OD^{<\beta}(z)$. We say $\mathcal{N}$ is weakly $A$-iterable just in case for all $n < \omega$, there is a fullness-preserving winning strategy $\Sigma$ for $\Pi$ in the iteration game $G(\omega, n, \omega_1)$\(^4\) such that whenever

$$i: \mathcal{N} \to \mathcal{P}$$

is an iteration map produced by an iteration according to $\Sigma$, then

$$i(\tau_{A,\mu}^\mathcal{N}) = \tau_{A,i(\mu)}^\mathcal{P}$$

for all cardinals $\mu \geq \delta^\mathcal{N}$ of $\mathcal{N}$.

\(^3\)The reason is essentially that $\text{Lp}^\alpha$-fullness is a $\Pi_1^1(\text{ODr})$ statement, true of reals coding $\mathcal{N}|\eta$ added by collapsing $\eta$ and Skolemized by the $\tau$s.

\(^4\)The output of this game is a linear stack of $n$ normal iteration trees, the first one being on $\mathcal{N}$. 


We should remark that if $\mathcal{N}$ is weakly $A$-iterable, and $\Sigma, \Gamma$ are iteration strategies for $\mathcal{G}(\omega, n, \omega_1)$ and $\mathcal{G}(\omega, k, \omega_1)$ witnessing this with $n \leq k$, then $\Sigma$ and $\Gamma$ can only disagree at some maximal normal component $U$, and then their disagreement has no effect on the remainder of either game, since they agree that $\mathcal{M}(\mathcal{U})^+$ will be the base model for the next normal component. In particular, any model reached using $\Sigma$ is itself weakly $A$-iterable.

We rely heavily on the following basic result of Woodin.

**Theorem 4.5.7 (Woodin)** Let $z \in HC^V[g]$, and let $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$; then there is a suitable, weakly $A$-iterable $z$-premouse.

The reader can find a proof of 4.5.7, in the weak gap case, outlined in [18]. (See lemma 1.12.1 there.) We shall prove the full result later.

**Remark 4.5.8** It is quite easy to derive Martin’s reflection theorem 4.2.1 from 4.5.7. Of course, Martin’s result is a one-liner in the weak gap case, while 4.5.7 still has content there.

**Remark 4.5.9** Going further in the same direction, it is easy to combine 4.5.7 with the main result of Neeman’s [6] (see also [7]), and obtain thereby a proof that $J_\beta(\mathbb{R}^g) \models AD$. In the weak gap case, this follows immediately from $J_\alpha(\mathbb{R}^g) \models AD$, which we get from $W^*_\alpha$, together with $J_\alpha(\mathbb{R}^g) \prec_1 J_\beta(\mathbb{R}^g)$, which is the weak gap case hypothesis. However, in the strong gap case, we have here a highly nontrivial “determinacy transfer” theorem known as the Kechris-Woodin transfer theorem. The original proof of Kechris and Woodin (cf. [3]) was purely descriptive set theoretic; no mice got involved.

Theorem 4.5.7, together with our self-justifying system, yields a fullness-preserving strategy that condenses well, as we now show.

**Definition 4.5.10** Let $\mathcal{N}$ be a suitable $z$-premouse, and $A$ a collection of $OD^{<\beta}(z)$ sets of reals; then we say $\mathcal{N}$ is weakly $A$-iterable iff for all finite $F \subseteq A$, $\mathcal{N}$ is weakly $\oplus F$-iterable, where $\oplus F$ is the join of the sets of reals in $F$.

**Corollary 4.5.11 (Woodin)** Let $A$ be a countable collection of $OD^{<\beta}(z)$ sets of reals, where $z \in HC^V[g]$ and codes $x^*$; then there is a suitable, weakly $A$-iterable $z$-premouse.
4.5. An SJS-Guided Iteration Strategy in $V[G]$

Proof. For each $F \subseteq A$ finite, we have by theorem 4.5.7 a suitable, weakly $\oplus F$-iterable $N_F$. Let $\Sigma_F$ be a fullness-preserving strategy for II in $G(\omega, 1, \omega_1)$ for $N_F$. We now simultaneously coiterate all the $N_F$, using $\Sigma_F$ to iterate $N_F$.

Claim. The coiteration ends successfully at some countable ordinal.

Proof. Let $M$ be the $L^\alpha$-closure of $\langle N_F | F \in [A]^{<\omega} \rangle$.

We claim that $\omega_1^M < \omega_1$. For otherwise, we define $f: \omega_1 \rightarrow \alpha$ be letting $f(\gamma)$ be the least $\xi$ such that there is in $J_\xi(\mathbb{R})$ an $\omega_1$-iteration strategy for a mouse $\mathcal{P}$ over $\langle N_F | F \in [A]^{<\omega} \rangle$ such that $\mathcal{P}$ projects to $\langle N_F | F \in [A]^{<\omega} \rangle$ and $\alpha(\mathcal{P}) \geq \gamma$. Since $\alpha$ is admissible, there is a $\xi < \alpha$ such that $\text{ran}(f) \subset \xi$.

But then there is an uncountable sequence of distinct reals definable over $J_\alpha(\mathbb{R})$, which cannot happen since $J_\alpha(\mathbb{R}) = AD$.

$M$ can track the coiteration generated by the $\Sigma_F$ until some maximal tree $U_F$ on $N_F$ is produced. (Note that coiterations always generate normal trees.) But as soon as that happens, the coiteration is over. For let $\mathcal{P}_G$ be the next model selected by $\Sigma_G$ to continue $U_G$, for all $G \in [A]^{<\omega}$. As $U_F$ is maximal, $\mathcal{P}_F = M(U_F)^+$ is suitable. If $N_G$-to-$\mathcal{P}_G$ drops, then because $\Sigma_G$ is fullness-preserving, $M(U_F)$ has a $\mathcal{Q}$-structure in $L^\alpha(M(U_F))^5$, a contradiction. But then $\mathcal{P}_G$ is suitable, and the minimality condition in suitability easily implies $\mathcal{P}_G = \mathcal{P}_F$, for all $G$.

The usual regressive function argument shows the coiteration cannot be tracked in $M$ for $\omega_1^M + 1$ steps. Thus it must terminate successfully at some stage $\leq \omega_1^M$. This proves the claim. \[ \square \]

The proof of the claim also shows that if $\mathcal{P}_F$ is the last model on the tree $U_F$ produced in the successful coiteration by $\Sigma_F$, then no branch $N_F$-to-$\mathcal{P}_F$ drops, and $\mathcal{P}_F = \mathcal{P}_G$ for all $F, G$. (Some branch doesn’t drop by general coiteration theory, and then the proof of the claim gives the rest.) It is clear that the common last model $\mathcal{P}$ is suitable, and weakly $A$-iterable. \[ \square \]

Definition 4.5.12 Let $N$ be a suitable $z$-premouse, where $z \in HC^V$, let $A$ be a collection of $OD^{<\beta}(z)$ subsets of $\mathbb{R}$, and let $\Sigma$ be an $\omega_1$-iteration strategy for $N$. We say $\Sigma$ is guided by $A$ just in case $\Sigma$ is fullness preserving, and whenever $T$ is a countable (necessarily normal) iteration tree by $\Sigma$ of limit length, and $b = \Sigma(T)$, then

\[ \text{Some initial segment of } \mathcal{P}_G \text{ is a } \mathcal{Q}\text{-structure for } M(U_F) \text{ because of the drop. This } \mathcal{Q}\text{-structure cannot lie beyond } L^\alpha(M(U_F)), \text{ as otherwise } \mathcal{P}_G \text{ would have a suitable initial segment.} \]
(a) if $T$ is short, then $Q(b, T)$ exists and $Q(b, T) \in L^\alpha(M(T))$, and
(b) if $T$ is maximal, then
\[
i_b(\tau^N_{A, \mu}) = \tau^M_{b, \mu}
\]
for all $A \in A$ and cardinals $\mu \geq \delta^N$ of $N$.

Notice that in case (b) above, $b$ does not drop and $M_b = M(T)^+$, as $\Sigma$ is fullness-preserving.

**Theorem 4.5.13 (Woodin)** Let $z \in HC^{\mathcal{V}[g]}$, and let $A$ be a countable, self-justifying system of $OD^{<\beta}(z)$ sets which contains the universal $\Sigma^I_{\omega}(R^p)$ set. Then there is a suitable $z$-premouse $N$, and a unique fullness-preserving $\omega_1$-iteration strategy for $N$ which is guided by $A$; moreover, this strategy condenses well.

**Proof.** By 4.5.11, we have a suitable $z$-premouse $N$ which is weakly $A$-iterable. Let $A = \{A_k \mid k < \omega\}$, and for each $k < \omega$, let $\Gamma_k$ be a fullness-preserving $\omega_1$-iteration strategy witnessing that $N$ is weakly $A_0 \oplus \ldots \oplus A_k$-iterable. The desired strategy $\Sigma$ will be a sort of limit of the $\Gamma_k$.

So long as all $\Gamma_k$ agree, $\Sigma$ simply plays according to their common prescription. So suppose $T$ is a normal tree of limit length which has been played according to all $\Gamma_k$, but there are $k$ and $l$ such that $\Gamma_k(T) \neq \Gamma_l(T)$. Since the $\Gamma_k$ are fullness-preserving and guided by $L^\alpha$ $Q$-structures when these exist, $T$ must be maximal, and letting
\[
b_k = \Gamma_k(T)
\]
for all $k$, and
\[
i_k : N \to M(T)^+ = M^T_{b_k}
\]
be the canonical embedding, we have that $i_k$ moves the term relations for all $A_i$ with $i \leq k$ correctly.

For $k < \omega$, let $\mu_k$ be the $k$th cardinal of $M(T)^+$ which is $\geq \delta(T)$, and set
\[
M_k = M(T)^+|_{\mu_k},
\tau_{j,k} = \tau_{A_j, \mu_k},
\]
and
\[
\gamma_k = \sup\{\xi \mid \xi \text{ is definable over } M_k \text{ from points of the form } \tau_{i,j}, \text{ where } i, j \leq k\}.
\]
Let $M = M(T)^+$. 

**Claim 1.** The $\gamma_k$ are cofinal in $\delta(T)$. 

**Proof.** This follows at once from part (b) of Theorem 4.5.4. \hfill \Box 

The usual uniqueness proof for good branches in iteration trees \(^6\) yields Claim 2. Let $k \leq l$, and let $E$ be an extender of length $\leq \gamma_k$; then $E$ is used in $b_k$ if and only if $E$ is used in $b_l$.

**Proof.** This is a simple consequence of the fact that $\text{ran}(i_k) \cap \text{ran}(i_l)$ is cofinal in $\gamma_k$. \hfill \Box 

Define now $\xi \in b \iff \exists k \forall l \geq k (\xi \in b_l)$. By claim 2, we have $E$ is used in $b$ iff $E$ is used in $b_k$, for some (all) $k$ such that $\text{lh}(E) \leq \gamma_k$.

**Claim 3.** $b$ is cofinal in $\text{lh}(T)$. 

**Proof.** If not, then let $\eta = \bigcup b < \text{lh}(T)$. Fix $k$ such that $\text{lh}(E_{\eta}^T) < \gamma_k$.

All extenders used in $b$ have length $< \text{lh}(E_{\eta}^T)$, so by claim 2, 

$$b \subseteq b_l, \text{ for all } l \geq k.$$ 

This implies that $\eta \in b$. (If not, then $b$ is cofinal in $\eta$, but then all $b_l$ for $l \geq k$ are cofinal in $\eta$, so $\eta \in b_l$ for all $l \geq k$ since branches are closed.) Now let $F$ be the extender applied to $M_{\eta}^T$ along the branch $b_k$. We have $\text{crit}(F) < \text{lh}(E_{\eta})$, and since $F$ is not used in $b$, we must have $\text{lh}(F) > \gamma_k$. But $\text{ran}(i_k) \cap [\text{crit}(F), \text{lh}(F)] = \emptyset$, and $\text{ran}(i_k)$ is cofinal in $\gamma_k$, a contradiction. \hfill \Box 

Now set 

$$T_k^M = \text{Th} M_{k+1}^T(\delta^M \cup \{\tau_{i,j} \mid i, j < k\}),$$ 

and let $T_k^N$ be defined from $N$ and its capturing terms in parallel fashion. Thus we have 

$$i_k(T_k^N) = T_k^M$$ 

because $i_k$ moves the relevant term relations correctly.

**Claim 4.** For all $k$, $i_b(T_k^N) = T_k^M$. 

\(^6\)The “zipper argument.”
Proof. Fix $k$. We regard $T^N_k$ as a subset of $\delta^N$. Since $b$ is cofinal, it is enough to see that $i_b(T^N_k) \cap \text{lh}(E) = T^M_k \cap \text{lh}(E)$ whenever $E$ is used in $b$. But fixing such an $E$, we can find $l \geq k$ such that $E$ is used in $b_l$. It follows that $i_b(X) \cap \text{lh}(E) = i_l(X) \cap \text{lh}(E)$ for all $X \in \mathcal{N}$, and applying this to $X = T^N_k$, we have the desired conclusion. 

By part (c) of 4.5.4, $\mathcal{N}$ is pointwise $\Sigma_0$-definable from ordinals $< \delta^N$ and the $\tau^N_{i,j}$. The parallel fact holds for $\mathcal{M}$. Thus $\mathcal{N}$ is coded by the join of the $T^N_k$, so that $\mathcal{M}^T_k$ is coded by the join of the $i_b(T^N_k)$. It follows from claim 4 that $\mathcal{M}^T_k = \mathcal{M}$ and $i_b$ moves all the term relations correctly. Thus $b$ satisfies all the requirements for the choice of a fullness-preserving, $\mathcal{A}$-guided iteration strategy, and we can set $\Sigma(T) = b$. Since $T$ was maximal, the iteration game we were playing is now over, and $\Sigma$ has won.

We leave it to the reader to show that the strategy $\Sigma$ we have just defined condenses well. The term-condensation lemma 4.5.4 is of course the key. This finishes the proof of 4.5.13. \qed

4.6 Back to $V$

We are finally ready to complete the proof of Theorem 4.5.1. Roughly speaking, 4.5.13 gives us what we want, except that it exists in $V[g]$, and depends on $g$. Slightly re-arranging things so as to ease notation, we may assume

$$g = h \times l,$$

where $h \times l$ is $\text{Col}(\omega, \mu) \times \text{Col}(\omega, < \nu)$-generic over $V$, for $\mu < \nu$, and

$$x^* = \tau^h$$

is the real in $V[g]$ from which we can ordinal-define our sjs $\vec{A}$. So there is in $V[g]$ a suitable $\mathcal{N}$ over $x^*$, with an $\vec{A}$-guided iteration strategy. By considering all possible finite variants of $h$, and comparing the mice associated to each of them, we shall produce a mouse which does not depend on $g$. We shall then show that this mouse has the form $\mathcal{N}[h]$, where $\mathcal{N}$ is a mouse over $\tau$ in $V$.\footnote{The Boolean-valued comparison method is due to Woodin.}

Let $p_0 \in h$ be a condition such that $(p_0, \emptyset)$ forces everything about $\tau$ and $V[g]$ which we have used so far. For each $p \leq p_0$ in $\text{Col}(\omega, \mu)$, let $h_p$ be given by

$$h_p = p \cup h \upharpoonright (\omega \setminus \text{dom}(p)).$$
Here we are identifying $h$ with $\bigcup h : \omega \rightarrow \mu$. So $h_p$ is $V$-generic, and $V[h_p] = V[h]$, for all $p \leq p_0$.

We work in $V[g]$ for a while. For $p \leq p_0$, let $\mathcal{A}_p$ be the self-justifying system of sets which are $\text{OD}^{<\beta} (\tau^{h_p})$ associated to $\tau^{h_p}$. Let

$$z_p = \langle \tau, h_p \rangle,$$

so that the sets in $\mathcal{A}_p$ are all $\text{OD}^{<\beta} (z_p)$. Let

$$\mathcal{A} = \bigcup_{p \leq p_0} \mathcal{A}_p,$$

and notice that since $z_p$ easily computes $z_q$, all sets in $\mathcal{A}$ are $\text{OD}^{<\beta} (z_p)$, for all $p$. Let $\hat{\mathcal{A}}$ be a natural $\text{Col}(\omega, \mu) \times \text{Col}(\omega, < \nu)$-term for $\mathcal{A}$, so that

$$\forall p \leq p_0 \forall q \hat{\mathcal{A}}^{h_p \times l_q} = \mathcal{A},$$

and $(p_0, \emptyset)$ forces that $\hat{\mathcal{A}}$ is a self-justifying system containing the universal $\Sigma^1_{\alpha(R)}$ set. From now on, let’s assume $p_0 = \emptyset$ to save ink. For each $p$, we have by 4.5.13 terms $\hat{N}_p, \hat{\Sigma}_p$ such that

$$(p, \emptyset) \models \hat{\Sigma}_p \text{ is an } \hat{\mathcal{A}}\text{-guided, fullness-preserving strategy for the } \langle \tau, h \rangle \text{ mouse } \hat{N}_p.$$

(Here we use $\hat{h}$ to name the first coordinate of the generic pair.) Since $\hat{N}_p$ is a term for a countable transitive set, we may assume it has support bounded in $\nu$. By increasing $\mu$ if necessary, we may assume $\hat{N}_p$ is a $\text{Col}(\omega, \mu)$-term. Since $\hat{\mathcal{A}}^{h_p \times l_q} = \hat{\mathcal{A}}^{h_q \times l_p}$ for all $p, q$, we get that $\hat{\Sigma}_p^{h_p \times l_q} = \hat{\Sigma}_q^{h_q \times l_p}$ for all $p, q$, and therefore we have for each $p$ a $\text{Col}(\omega, \mu)$-term $\hat{\Gamma}_p$ such that

$$\hat{\Gamma}_p^{h_p} = \hat{\Sigma}_p^{h_p \times l} | V[h_p],$$

for all $p \in \text{Col}(\omega, \mu)$.

We work in $V[h]$ for a while. Let $\hat{\mathcal{N}}_p = \hat{N}^{h_p}$ and $\hat{\Gamma}_p = \hat{\Gamma}^{h_p}$. Now $\hat{\mathcal{N}}_p$ is a $z_p$-mouse, but it can also be regarded as a $z_q$ mouse for any $q$, since $z_p$ and $z_q$ compute each other easily. It therefore makes sense to simultaneously compare all the $\hat{\mathcal{N}}_p$ in $V[h]$, using the $\hat{\Gamma}_p$ to iterate them. We can show the comparison terminates using the argument of the claim in 4.5.11. Let

$$\mathcal{N}_\infty = \text{common iterate of all } \hat{\mathcal{N}}_p.$$
Because the $\Gamma_p$ are fullness-preserving, $\mathcal{N}_p$-to-$\mathcal{N}_\infty$ does not drop for all $p$, and $\mathcal{N}_\infty$ can be regarded as a suitable $z_p$-mouse, for each $p$. These are different presentations so perhaps we should write $\mathcal{N}_p^\beta$, but there is a fixed extender sequence

$$\vec{E}_\infty = \vec{E}^{\mathcal{N}_\infty}, \text{ for all } p.$$ 

Moreover, $\mathcal{N}_\infty$ is weakly $\mathcal{A}$-iterable, and thus by 4.5.13 has a unique $\mathcal{A}$-guided strategy $\Gamma$ which is fullness-preserving and condenses well.

Since the comparison which produced $\mathcal{N}_\infty$ depends only on the set of all $\vec{N}^h_p$, and not any enumeration of this set, we have symmetric terms for $\vec{E}_\infty$ and $\Gamma$; that is $\vec{E}_\infty$ and $\vec{\Gamma}$ such that

$$\vec{E}_\infty^\text{h}_p = \vec{E}_\infty \land \vec{\Gamma}_p = \Sigma$$

for all $p$. It follows from the homogeneity of $\text{Col}(\omega, \mu)$ that any subset of $V$ which is definable in $V[g]$ from $\{h_p \mid p \leq p_0\}$, $\vec{E}_\infty$, $\Gamma$, and elements of $V$ is itself in $V$.

In $V$, we can now inductively build a $\tau$-mouse $\mathcal{N}$. We maintain

$$\mathcal{N}[\eta][h] = \mathcal{N}^\theta[h][\eta],$$

by induction on $\eta$. The first few levels of $\mathcal{N}$ are just initial segments of $L(\tau)$. Given $\mathcal{N}[\eta]$, we get $\mathcal{N}[\eta + 1]$ by letting the next extender be

$$\vec{E}_\eta = (\vec{E}_\infty)_\eta \cap \mathcal{N}[\eta].$$

Note that $\vec{E}_\eta$ is in $V$, and can be defined from $\eta$ over $V$ uniformly in $\eta$. One can show that $\mathcal{N}$ is a $\tau$-mouse, and $\mathcal{N}[h] = \mathcal{N}_\infty$. The proof is given in [14]. It relies on the fact that fine-structure is preserved, level-by-level, by small forcing. That also implies that any iteration tree $T$ on $\mathcal{N}$ can be regarded as a tree $T^*$ on $\mathcal{N}[h] = \mathcal{N}_\infty$, with the same drop and degree structure, and $\mathcal{M}_\xi^T = \mathcal{M}_\xi'[h]$ for all $\xi$. Thus $\Gamma$ induces a $\mu^+$-iteration strategy, which we also call $\Gamma$, on $\mathcal{N}$. Moreover, $\Gamma \in V$. We leave it to the reader to check that $\Gamma$ condenses well in $V$. This proves 4.5.1.

### 4.7 Hybrid strategy-mice and operators

Let $\mathcal{N}$ and $\Sigma$ be as in 4.5.1. We need to use hybrid mice obtained by constructing from some $A$ coding $\mathcal{N}$, $A \in H_\mu$, adding extenders to a coherent sequence we are building, and at the same time closing the model we are
building under $\Sigma$. This is parallel to the method of building $K^c$'s in the inadmissible case by explicitly closing the model under the $\mathcal{M}$-operator, which we did not actually use. (In the present situation, we have no way to argue that a pure extender model over $\mathcal{N}$ must be closed under $\Sigma$.)

Iterability for these hybrid mice includes the provision that $\Sigma$ is moved correctly. (All critical points on the coherent sequence must be $> \sup(A)$, and hence iterations of a hybrid fix $\mathcal{N}$, and move $\Sigma$ to a strategy for non-dropping trees on $\mathcal{N}$. Our requirement is that this strategy is $\Sigma$ itself.) If this is done in a natural way, the resulting model has a fine structure. The key to the fine structure is that $\Sigma$ condenses well. Condensation for $\Sigma$ is also used in the realizability proof that size $\mu$ elementary submodels of levels of $K^c_\Sigma(A)$ are countably iterable in $V[g]$.

Let us call such mice $\Sigma$-hybrid mice.

**Definition 4.7.1** Let $A \in HC^V[g]$ code $\mathcal{N}$ in some specified way; then $P^\Sigma_n(A)^\sharp$ is the minimal iterable $\Sigma$-hybrid mouse over $A$ which is active, and satisfies “there are $n$ Woodin cardinals”.

The following essentially completes the proof of WD in the gap case.

**Lemma 4.7.2** Suppose that in $V$: for all $n < \omega$ and all swo $B \in H_\nu$, $P^\Sigma_n(B)^\sharp$ exists, and is $\nu$-iterable; then $W^*_\beta + 1$ holds in $V[g]$.

**Proof.** We show first that the hybrid operators we are given on $H_\nu^V$ extend to $HC^V[g]$:

Claim 1. In $V[g]$, we have that for all $n < \omega$ and all swo $B \in HC$, $P^\Sigma_n(B)^\sharp$ exists, and has an $\omega_1$-iteration strategy in $J_{\beta+1}(\mathbb{R}^g)$.

**Proof.** In $V[g]$, let $J^n(B) = P^\Sigma_n(B)^\sharp$ We show by induction on $n$ that HC is closed under $J^n$, and that $J^n \in J_{\beta+1}(\mathbb{R}^g)$.

---

8Woodin found the following trick for closing under $\Sigma$ in such a way that the levels of the model we build are all amenable structures, which is important for fine structure: if we are at a level $P$ appropriate for closing further under $\Sigma$, and $T$ is the $P$-least iteration tree of limit length $\alpha$ which is by $\Sigma$, but such that we have not yet told our model what $\Sigma(T)$ is, we let $Q$ be the structure of height $o(P) + \alpha$ obtained by doing $\alpha$ steps of the usual constructible closure starting with $P$, and then take $(Q, B)$ to be the next level of our model, where $B = \{o(P) + \beta \mid \beta \in \Sigma(T)\}$.

9It would be possible to talk only about countable iterability in $V$. Given $\pi: \mathcal{M} \rightarrow \mathcal{Q}$, where $\mathcal{M}$ is countable and $\mathcal{Q}$ is a level of $K^c_\Sigma(A)$, iterability for $\mathcal{M}$ means that the collapse of $\Sigma$ is moved to its pullback $\Sigma^\pi$. By condensation for $\Sigma$, this is what happens along realizable branches of trees on $\mathcal{M}$.
For $n = 0$: Fix $B \in \text{HC}^V[g]$. Let $g = h \times l$, where $h$ is on $\text{Col}(\omega, \mu)$, and $\mu < \nu$ is large enough that $|\mathcal{N}| \leq \mu$, that $B = \rho^h$ for some $\text{Col}(\omega, \mu)$-term $\rho$, and for $\mathcal{A}$ the sjs guiding an extension of $\Sigma$ to $V[g]$, we have that $\langle \tau^N_A | i < \omega \rangle = \sigma^h$, for some $\text{Col}(\omega, \mu)$-term $\sigma$.

In $V[g]$, we can now construct $P_{\Sigma}^V(B)^G$, which is simply an ordinary sharp for $L^\Sigma(B)$, from $P_{\Sigma}^V(\langle \mathcal{N}, \sigma \rangle)^G$[h]. For this, it is clearly enough to show that $L^\Sigma(B)$ is definable over $P[h] = P_{\Sigma}^V(\langle \mathcal{N}, \sigma, \rho \rangle)^G$[h] from $\langle \mathcal{N}, \sigma, \rho, h \rangle$. The only trouble here is that in forming $L^\Sigma(B)$, we may need to apply $\Sigma$ to iteration trees which are in $P[h]$, but not in $P$. For that, we use

**Definition 4.7.3** Let $x$ be countable and transitive; then $x \oplus \mathcal{A}$ is the structure $(R, T)$, where

(a) $R$ is the $\text{Lp}_\alpha$-closure of $x$ through $\omega$ cardinals, call them $\eta_i$ for $i < \omega$, and

(b) for all $i, y$, $T(i, y)$ holds iff $y = \tau^R_{\mathcal{A}, \eta_i}$.

Here we assume the sets in $\mathcal{A}$ are enumerated $A_i$ so that each is repeated infinitely often.

$x \oplus \mathcal{A}$ is a term relation hybrid over $x$. In some contexts, we can replace our strategy hybrids by models formed by adding extenders and closing the levels under $x \mapsto x \oplus \mathcal{A}$. Indeed, this closure operation is equivalent to closing under $\Sigma$, granted the parameter $\langle \mathcal{N}, \sigma, \rho, h \rangle$. This equivalence is expressed by

**Lemma 4.7.4 (Strategy-sjs equivalence)** Let $P, h$ be as above, and let $H(x) = x \oplus \mathcal{A}$ for all $x$; then

(a) $P[h]$ is closed under the function $H$, and $H$ is definable over $P[h]$ from $\Sigma$ and $\langle \mathcal{N}, \sigma, \rho, h \rangle$,

(b) $\Sigma \upharpoonright P[h]$ is definable over $P[h]$ from $H \upharpoonright P[h]$ and $\langle \mathcal{N}, \sigma, \rho, h \rangle$.

**Proof.** In a later installment. \qed

Clearly, 4.7.4 completes the proof that $\text{HC}^V[g]$ is closed under $J^0$. It is easy then to see that $J^0 \in J_{\beta+1}(\mathbb{R}^g)$.

Now let $n = k + 1$. The argument above easily adapts to show that $\text{HC}^V[g]$ is closed under $J^n$. But then, each $J^n(B)$ has an $\omega_1$ iteration strategy which is $\Delta^1_1(J^k)$, as $J^k$ provides the necessary $\mathcal{Q}$-structures. This implies that $J^n$ itself is $\Delta^1_8(J^k)$, and so in $J_{\beta+1}(\mathbb{R}^g)$.
This proves Claim 1.

Now let $U \subseteq \mathbb{R}^g$ be in $J_{\beta+1}(\mathbb{R}^g)$ and $k < \omega$; we seek a coarse $(k, U)$-Woodin mouse. Let $U$ be $\Sigma^J_n(\mathbb{R}^g)$ in the real parameter $z$. Let us also take $z$ so that it codes $\langle \mathcal{N}, \sigma, \rho, h \rangle$, where these are as above. Our desired witness will be

$$P = P_{k+n}^\Sigma(z)^\dagger.$$ 

By Claim 1, $P$ has a unique $\omega_1$-iteration strategy in $J_{\beta+1}(\mathbb{R}^g)$. Let $\Gamma$ be this strategy. Let $\langle A_i \mid i < \omega \rangle$ be our self-justifying system of sets which are $\text{OD}^{<\beta}(z)$. If $j$ is least such that $\rho^J_{j}\mathcal{R}^g = \mathbb{R}^g$, then $\Sigma_j$-truth at $\beta$ is coded in a simple way into

$$W = \oplus_{i<\omega} A_i.$$ 

**Claim 2.** For any $\xi \in P$, there is a term $\hat{W} \in P$ relative to $\text{Col}(\omega, \xi)$ such that whenever $i: P \rightarrow Q$ is an iteration map by $\Gamma$ (constructed in $V[g]$), and $l$ is $Q$-generic over $\text{Col}(\omega, i(\xi))$, then

$$\hat{W}^l = W \cap Q[l].$$

**Proof.** Basically, $\hat{W}$ asks what the $\tau^N_{A_i}$ are moved to in the iteration of $\mathcal{N}$ which makes $P|\mu^+$ generic over the extender algebra of the iterate. This iteration is done inside $P$, using what it knows of $\Sigma$. \hfill $\Box$

**Claim 3.** Let $\delta$ be the $k^{th}$ Woodin cardinal of $P$; then for any $\Sigma^J_n(\mathbb{R}^g)(z)$ set $Y$, there is a term $\hat{Y} \in P$ relative to $\text{Col}(\omega, \delta)$ such that whenever $i: P \rightarrow Q$ is an iteration map by $\Gamma$ (constructed in $V[g]$), and $l$ is $Q$-generic over $\text{Col}(\omega, i(\delta))$, then

$$\hat{Y}^l = Y \cap Q[l].$$

**Proof.** $\hat{Y}$ is constructed from the term $\hat{W}$ given by claim 2, applied at the $k+n^{th}$ Woodin of $P$. The $n$-Woodins above $\delta$ are used to answer the relevant $n$-real-quantifier statements. \hfill $\Box$

We can now see that $P$ is the desired coarse witness. The trees in $\mathcal{P}$ which are moved appropriately by $\Gamma$ are obtained just as in the inadmissible case. \hfill $\Box$
4.8 \( \text{AD}^{\mathbb{L}(\mathbb{R})} \) from a homogeneous ideal

In Chapter 2, §7 we showed that CH plus the existence of a homogeneous, presaturated ideal on \( \omega_1 \) implies PD. In this section we shall join that argument to the Witness Dichotomy, and thereby derive \( \text{AD}^{\mathbb{L}(\mathbb{R})} \) from the same hypothesis. We prove

**Theorem 4.8.1** Assume CH, and suppose there is a homogeneous poset \( \mathbb{P} \) such that whenever \( G \) is \( \mathbb{V} \)-generic over \( \mathbb{P} \), then

\[
\mathbb{V}[G] \models \exists j : \mathbb{V} \rightarrow \mathbb{M} (\text{crit}(j) = \omega_1^\mathbb{V} \land M^\omega \subseteq M).
\]

Then for all \( \alpha \), \( W^*_\alpha \) holds in \( \mathbb{V} \).

**Remark 4.8.2** We are not requiring that \( \mathbb{P} \) be given by an ideal on \( \omega_1 \), so it is easy to obtain a model of the hypothesis by forcing. Namely, let and \( j : \mathbb{V} \rightarrow \mathbb{M} \), with \( \text{crit}(j) = \omega_1^\mathbb{V} \) and \( M^\omega \subseteq M \), be the embedding in \( \mathbb{V}[G] \) which is given by our hypothesis. Let \( \nu = j(\omega_1^\mathbb{V}) \), so that \( \nu \) is a regular cardinal in \( \mathbb{V} \), and \( \nu = \omega_1 \) in \( \mathbb{V}[G] \).

It is clear from the proof of WD that it holds for \( \mathbb{L}(\mathbb{R})^\mathbb{V} \) as well. This is where we shall use it. Suppose that \( W^*_\alpha \) fails to hold in \( \mathbb{V} \) for some \( \alpha \). Let us fix an swo \( A \) and a \( (\omega_1,A) \)-hmo \( J \) such that \( J^\omega \) does not exist. We shall use \( K^J \), and in particular the argument of Chapter 2, §7 to obtain a contradiction.

We do not know whether the hypothesis of 4.8.1 can be shown consistent using less than a superstrong.

**Proof of 4.8.1:** Let \( j : \mathbb{V} \rightarrow \mathbb{M} \), with \( \text{crit}(j) = \omega_1^\mathbb{V} \) and \( M^\omega \subseteq M \), be the embedding in \( \mathbb{V}[G] \) which is given by our hypothesis. Let \( \nu = j(\omega_1^\mathbb{V}) \), so that \( \nu \) is a regular cardinal in \( \mathbb{V} \), and \( \nu = \omega_1 \) in \( \mathbb{V}[G] \).

It is clear from the proof of WD that it holds for \( \mathbb{L}(\mathbb{R})^\mathbb{V} \) as well. This is where we shall use it. Suppose that \( W^*_\alpha \) fails to hold in \( \mathbb{V} \) for some \( \alpha \). Let us fix an swo \( A \) and a \( (\omega_1,A) \)-hmo \( J \) such that \( J^\omega \) does not exist. We shall use \( K^J \), and in particular the argument of Chapter 2, §7 to obtain a contradiction.

We shall begin with the case that \( J \) is an ordinary \( (\omega_1,A) \)-mo, and then indicate the elaborations needed in the hybrid case.

Let \( B \in \mathbb{H}_{\omega_1} \) be swo’d, and in the cone over \( A \), and such that \( M^J_1(B) \) does not exist. Following Chapter 2, §7, we then have that \( (K^{c,J}(B))^N \) does not reach \( M^J_1(B) \), where \( N \) is the background universe \( \mathbb{L}^J(\mathbb{R}) \). A preliminary argument gives indicernibles for \( \mathbb{L}^J(\mathbb{R}) \), and hence enough of a measurable cardinal to get the theory of \( K^J(B) \) going. We then use this theory to obtain a contradiction.
4.8. \( \text{AD}^{L(\mathbb{R})} \) FROM A HOMOGENEOUS IDEAL

It is important for the argument that \( \mathbb{R} \subseteq N \), since the countable-in-\( V[G] \) fragments of \( E_j \) should be in \( j(N) \), where they might be added to \( (K^J(B))^j(N) \). It is also important that \( j(N) \) be ordinal definable in \( V[G] \), so that \( (K^J(B))^j(N) \in V \) by the homogeneity of \( P \). We therefore cannot afford to put a wellorder of \( \mathbb{R} \) into \( N \).

Unfortunately, mouse operators like \( J \) only operate on self-wellordered sets, and not for example on \( \mathbb{R} \), so we must take a little care as to what \( L^j(\mathbb{R}) \) is to be. We should probably modify the notion of "cone over \( A \)" so as to include non-swo’s. For now, let us just simply note that the notion of an \( a \)-premouse makes sense for any transitive set \( a \), self-wellordered or not. The reader should see [14] for a discussion of the elementary properties of such premouse, in the representative special case that \( a = HC \). The main thing is that if \( \mathcal{M} \) is an \( a \)-premouse with top extender \( E \), and \( f: (a \times \xi) \rightarrow E_b \) with \( \xi < \text{crit}(E) \) and \( f \in \mathcal{M} \), then \( \bigcap \text{ran}(f) \in E_b \). This implies that \( i_E \) is the identity on \( a \cup \{a\} \), and that we have Los’ theorem for \( \Sigma_n \) ultrapowers, whenever \( \text{crit}(E) < \rho_n^{\mathcal{M}} = \text{least} \rho \) such that there is a new \( \Sigma_n^{\mathcal{M}} \) subset of \( a \times \rho \). These properties imply that if \( g \) is Col(\( \omega, a \))-generic over \( \mathcal{M} \), then \( \mathcal{M}[g] \) can be regarded as an ordinary premouse over the swo \( \langle a, g \rangle \). This last fact summarizes what it is to be an \( a \)-premouse: you become an ordinary premouse when a wellorder of \( a \) is added generically.

We extend the lower part notation \( L_p(b) \) to arbitrary transitive \( b \) in the obvious way.

**Definition 4.8.3** Let \( C \in H_\mu \); then a extended mouse operator over \( C \) on \( H_\mu \) is a function \( H \) such that for some \( Q \)-formula \( \psi \),

\[
H(b) = \text{least } \mathcal{P} \subseteq L_p(b) \text{ such that } \mathcal{P} \models \psi[C,a]
\]

for all transitive \( b \in H_\mu \) such that \( C \in b \). (\( H \) must be defined at all such \( b \).) We call \( H \) a \( (\mu,C) \)-emo.

**Lemma 4.8.4** There is a \( C \in HC \), and an \( (\omega_1,C) \)-emo \( H \), such that for all \( a \in \text{dom}(H) \) and all \( g \) which are Col(\( \omega, a \))-generic over \( H(a) \), \( H(a)[g] = J(\langle a, g \rangle) \).

One can show that ther operators \( J \) which arise from WD satisfy lemma 4.8.4, but one can also just derive 4.8.4 abstractly from the Turing invariance of \( J \). We defer further detail on the proof of 4.8.4. Fix \( H \) and \( C \) as given there.

The next lemma takes what is a key step in any local core model induction, by extending the domain of our operator \( H \).
Lemma 4.8.5 (Extension Lemma) There is a unique \((\nu,C)\)-emo \(H^*\) such that \(H \subseteq H^*\).

Proof. Uniqueness is a simple Lowenheim-Skolem argument, based on the fact that any such \(H^*\) has condensation.

For existence, we use \(j\), taking \(H^*\) to be simply \(j(H) \upharpoonright V\). We must show this works.

Let \(\psi\) be the sentence which determines \(H\) over the parameter \(C\). We need to see that whenever \(b \in H\) \(\nu\) is transitive, with \(C \in b\), then there is a countably iterable \(b\)-premouse \(P\) such that \(P \models \psi[C,b]\). Fix such a \(b\).

We have that \(b \in HC_\text{M} = HC_{V[G]}\). Working in \(M\), we obtain a countably iterable minimal \(b\)-premouse \(P\) such that \(P \models \psi[C,b]\). We need only show that \(P\) is in \(V\), and is countably iterable there. For this, we need to look more closely at our \(J\).

Remark 4.8.6 At this point, we are using not just the statement of WD, but its proof.

Case 1. \(J\) is the diagonal operator at the bottom of the hierarchy in the inadmissible, uncountable cofinality case (2)(b) of the proof of WD.

Proof. Let \(\beta\) be our inadmissible of uncountable cofinality, with \(W^*_\beta\) holding in \(V\). Inspecting the construction of \(J\), we see that for each \(Z \in \text{dom}(J)\),

\[ J_{\beta}(\mathbb{R}) \models J(Z) \text{ is } \omega_1\text{-iterable.} \]

It follows that the same is true with \(H\) replacing \(J\). (Iterations of \(H(a)\) reduce to iterations of \(H(\langle a, g \rangle)\).) Thus in \(M\),

\[ J_{J(\beta)}(M) \models H(b) \text{ is } \omega_1\text{-iterable.} \]

But \(J_{J(\beta)}(M) = J_{J(\beta)}(V[G])\) is a model of AD. It follows that \(H(b)\) is ordinal definable in \(V[G]\) from \(b\). Thus \(H(b)\) is in \(V\).

Similarly, if \(\Sigma\) is the unique \(\omega_1\)-iteration strategy of \(J_{J(\beta)}(M)\) for \(H(b)\), then \(\Sigma \upharpoonright V\) is in \(V\). In \(V\), it is a \(\nu\)-iteration strategy for \(H(b)\), and thus certainly witnesses countable iterability.

Notice that we have shown in this case that \(j(H)\) is definable over \(J_{J(\beta)}(\mathbb{R})\). Our desired extension of \(H\) is just \(j(H) \upharpoonright V\).

Case 2. \(J = I^w\), for some \((\omega_1,A)\)-mo \(I\).
4.8. $\text{AD}^L(\mathbb{R})$ FROM A HOMOGENEOUS IDEAL

Proof. In this case, we have $H = S^w$, for some $S$. By induction, we may assume that $j(S)$ is definable over $J_{j(\beta)}(\mathbb{R}^M)$. This enables us to define $H(b)$ over $J_{j(\beta)}(\mathbb{R}^M)$, as the unique model of the appropriate theory which is $\omega_1$-iterable via the $Q$-structures provided by $j(S)$. (If $H^1$ and $H^2$ are two such structures, we can compare them in $j(S)((H^1, H^2))$, ending at worst by stepping outside when we reach stage $\omega_1$ in this model.) We can also define an $\omega_1$-strategy for $H(b)$ from $j(S)$. Again, this gives $H(b)$ and $\Sigma \restriction V$ in $V$. Again, we have $H^* = j(H) \restriction V$.

This completes the proof of 4.8.5 in case (2)(b). In cases (1) or (2)(a), the operator at the bottom of the $J_{\beta}(\mathbb{R})$ hierarchy is a countable join $\bigoplus I_n$ of operators belonging to $J_{\beta}(\mathbb{R})$. The $I_n$ can be extended as in 4.8.5 to $j(I_n) \restriction V$, and hence $\bigoplus I_n$ extends to $j(\bigoplus I_n)$. This handles our Case 1 above, and the Case 2 is done in the same way as above.

Since we are ignoring the gap case of $\text{WD}$ for now, this completes our proof of 4.8.5.

To save notation, let us now write $H$ for the operator $H^*$ given by 4.8.5.

Now let $N = L^H_\nu(\mathbb{R})$

be the model $\mathcal{M}_\nu$ obtained by starting with $\mathcal{M}_0 = (\mathcal{H}, \in)$, and letting $\mathcal{M}_{\alpha+1} = H(\mathcal{M}_\alpha)$, with unions taken at limit ordinals.

Lemma 4.8.7 $N^\sharp$ exists.

Pick $\Omega$ an indiscernible of $N$. Inside $N^{\text{Col}(\omega_1, \mathbb{R})}$, we form $K^{c,H}(B)$ up to $\Omega$. (Either explicitly close its levels under $H$, or argue that they must be closed above the largest Woodin by universality.)

Since $M^H_1(B)$ does not exist, we have that

$$K = (K^H(B))^N$$

exists.

Lemma 4.8.8

1. $j(K) \in V$.

2. $\omega^V_1$ is inaccessible in $K$.

Proof. Part (1) follows because $j(K)$ is definable over $j(N)$, and $j(N) = L^H(\mathbb{R}^{V[G]})$ is definable over $V[G]$.

Part (2) follows as otherwise $\omega^V_1$ is collapsed in $j(K)$. But $j(K) \in V$. \qed
By part (2) of the lemma, each fragment $E_j \upharpoonright \alpha \cap j(K)$ of the extender of $j$, for $\alpha < \nu$, is coded by a real in $V[G]$. Hence these fragments are in $j(N)$. If they are in $j(K)$, then $\omega_1^V$ is Shelah in $j(K)$. A standard argument shows that they are indeed in $j(K)$ granted the next lemma.

**Lemma 4.8.9** Let $\omega_1^V < \alpha < \nu$, with $\alpha$ a cardinal of $j(K)$. Then in $j(N)$, the phalanx $(j(K), \text{Ult}(j(K), E_j \upharpoonright \alpha), \alpha)$ is $j(\Omega) + 1$- iterable.

**Proof.** Because $j(N)$ is $j(H)$-closed, it is enough to show the phalanx is countably iterable in $j(N)$. Working in $j(N)$, let $(P, Q, \alpha)$ be a countable phalanx embedding by $(\pi, \sigma)$ into $(j(K), \text{Ult}(j(K), E_j \upharpoonright \alpha), \alpha)$. We have $(P, Q, \alpha)$ embeds into $(j(K), j(j(K)), \alpha)$ by some $(\pi, \tau)$ then.

**Remark 4.8.10** Here is a crucial point at which $j(K) \in V$ is used.

Pulling back to $V$, we have for $G$-a.e. $\xi$, $(P_\xi, Q_\xi, \alpha_\xi)$ embeds by some $(\pi_\xi, \tau_\xi)$ into $(K, j(K), \alpha_\xi)$. But this implies that $(P_\xi, Q_\xi, \alpha_\xi)$ embeds into $(K, K, \alpha_\xi)$ by some $(\pi_\xi, \psi_\xi)$, for $G$-a.e. $\xi$. That in turn means that $(P, Q, \alpha)$ embeds into $(j(K), j(K), \alpha)$ in $j(N)$, and so is countably iterable there.

This finishes our proof of 4.8.1 in the non-gap case. In the gap case, in which our $J$ is a hybrid mouse operator, the main thing we have to see is how to extend the $\omega_1$-iteration strategy $\Sigma$ for a suitable mouse $N$ which we got in $V$, and went into the definition of $J$, to a $\nu$-iteration strategy.

But this is easy; our desired extension is just $j(\Sigma) \upharpoonright V$. $\Sigma$ was guided in $V$ by an sjs $\vec{A}$. Moreover, $\vec{A}$ is OD$_\beta(x)$ in $V$, for some $\beta$. It follows that $j(\Sigma)$ is OD$_{j(\beta)}(x)$ in $M$. Since $R^M = R^V[G]$, $j(\Sigma)$ is OD$_{j(\beta)}(x)$ in $V[G]$. By homogeneity, $j(\Sigma) \upharpoonright V$ is in $V$.

**Remark 4.8.11** For this homogeneous ideal plus CH argument, we could avoid the strategy hybrids, and work instead with the older term-relation hybrids. The reason is that the extension argument for $J$ takes place in the very same universe in which we are trying to prove $W^*_\alpha$ for all $\alpha$. The meaning of our sjs $\vec{A}$ is tied to some real $x$, but this $x$ is in the domain of the embedding $j$ we use to do the extension. Strategy hybrids do seem necessary in getting strength from a failure of $\square$ at a singular.

**Question.** Does AD$^{L(\mathbb{R})}$ follow from the existence of a homogeneous, presaturated ideal on $\omega_1$?
Chapter 5

A model of AD plus $\Theta_0 < \Theta$

In this chapter we propose an organization of the argument which is supposed to produce a transitive model of AD plus $\Theta_0 < \Theta$ containing all the reals and ordinals starting from CH plus there is a homogeneous poset $\mathbb{P}$ such that $\mathbb{P} \models \text{“there is some } j: V \rightarrow M \text{ with critical point } \omega^V_1 \text{ such that } \omega^M \subset M, \text{ where } j \upharpoonright OR \text{ is amenable to } V, \text{ plus } \epsilon.”$

Throughout these notes, we’ll work under the following Hypothesis.

CH holds. Also, there is a homogeneous poset $\mathbb{P}$ such that $\mathbb{P} \models \text{“there is some } j: V \rightarrow M \text{ with critical point } \omega^V_1 \text{ such that } \omega^M \subset M, \text{ where } j \upharpoonright OR \text{ is amenable to } V.”$

We aim to produce a transitive model of AD plus $\Theta_0 < \Theta$ containing all the reals and ordinals. We probably need to add a further hypothesis, $\epsilon$, in order to succeed. (See [4]; more about this in the next talks.)

In what follows, we fix $G$ $\mathbb{P}$-generic over $V$, and we write $j: V \rightarrow M$ for the generic embedding. We have $\text{crit}(j) = \omega^V_1$, $\omega^M \subset M$ in $V[G]$, and $\forall \alpha(j \upharpoonright \alpha \in V)$ by our hypothesis.

We shall (and may!) make the following Assumption.

There is no transitive model of AD plus $\Theta_0 < \Theta$ containing all the reals and ordinals.

This assumption gives us the following two lemmas:

Theorem 5.0.12 (Woodin) There are no incompatible models of $\text{AD}^+$ in the following sense: there are no two transitive models $M, N$ of $\text{AD}^+$ con-
taining all the reals and ordinals such that there are \( A \in \mathcal{P}(\mathbb{R}) \cap M \) and \( B \in \mathcal{P}(\mathbb{R}) \cap N \) with \( A, B \) being Wadge incomparable.

**Theorem 5.0.13 (Woodin)** Let \( M \) be a transitive model of \( \text{AD}^+ \) containing all the reals and ordinals. Then the mouse set conjecture (MSC) holds in \( M \), i.e., in \( M \), if \( z \in \mathbb{R} \), then every real in \( \text{OD}_z \) is captured by\(^1\) some countable \( \omega_1 \)-iterable \( z \)-premouse.

### 5.1 \( \Gamma_0 \) and \( M_0 \).

Here’s a key definition.

**Definition 5.1.1** \( \Gamma_0 = \{ A \subset \mathbb{R} : L(A, \mathbb{R}) \models \text{AD} + \Theta_0 = \Theta \} \).

**Lemma 5.1.2** \( \Gamma_0 \neq \emptyset \).

**Proof.** We have shown that \( L(\mathbb{R}) \models \text{AD} \). As \( L(\mathbb{R}) \models \Theta_0 = \Theta \), we have the lemma. \( \square \)

Our goal is to produce a set \( B \) of reals such that \( L(B, \mathbb{R}) \models \text{AD} + \Theta_0 < \Theta \).

We first need to see that every set of reals in \( L(\Gamma_0, \mathbb{R}) \) is captured by an \( \mathbb{R} \)-mouse, so as to have a neat characterization of \( L(\Gamma_0, \mathbb{R}) \).

**Definition 5.1.3** \( K(\mathbb{R}) = L(\bigcup \{ \mathcal{M} : \mathcal{M} \text{ is a countably iterable}^2 \mathbb{R}\text{-premouse}, \rho_{\omega_1}(\mathbb{R}) = \mathbb{R} \}) \).

**Lemma 5.1.4** \( \Gamma_0 \subset K(\mathbb{R}) \). Moreover, if \( A \in \Gamma_0 \), then \( A \in K(\mathbb{R}) \) is witnessed by an \( \mathbb{R} \)-premouse \( \mathcal{N} \) such that the \( \omega_1 \)-iteration strategies for collapses of countable substructures of \( \mathcal{N} \) are in \( L(A, \mathbb{R}) \).

**Proof.** Fix \( A \in \Gamma_0 \). By CH, there is a generic bijection \( b : \omega \to \mathbb{R}^V \) in \( V[G] \). Let \( A_b \in V[G] \) be the real coding \( A \) relative to \( b \). We have \( L(A, \mathbb{R}) \models \text{AD} + \Theta_0 = \Theta \), so that \( A \) is \( \text{OD}_{\omega_1}(j(A), \mathbb{R}^V[G]) \) for some \( z \in \mathbb{R} \) and

\[
A = j(A) \cap \mathbb{R}^V \in \text{OD}_{\omega_1}(j(A), \mathbb{R}^V[G]).
\]

Hence

\[
A_b \in \text{OD}_{\omega_1}(j(A), \mathbb{R}^V[G]).
\]

\(^1\)i.e., element of
\(^2\)i.e., collapses of countable substructures are \( \omega_1 + 1 \)-iterable
5.1. \( \Gamma_0 \) AND \( M_0 \).

As \( L(j(A), V[G]) \models AD \) + \( \Theta_0 = \Theta \), by Theorem 5.0.13, \( L(j(A), V[G]) \) contains a \( b \)-premouse \( \mathcal{M} \) with \( A_0 \in \mathcal{M} \) and some \( \Sigma \) such that \( L(j(A), V[G]) \models \) “\( \Sigma \) is an \( \omega_1 \)-iteration strategy for \( \mathcal{M} \).” There is then some \( \mathbb{R}^V \)-premouse \( \mathcal{N} \in L(j(A), V[G]) \) such that \( A \in \mathcal{N} \) and \( L(j(A), V[G]) \models \) “\( \Sigma \) is an \( \omega_1 \)-iteration strategy for \( \mathcal{N} \).” With the help of Theorem 5.0.12, \( \mathcal{N} \) is \( \text{OD}_{\mathbb{R}^V} \) in \( V[G] \): for some \( \alpha \), in \( V[G] \), \( \mathcal{N} \) is the unique sound \( \mathbb{R}^V \)-premouse \( \mathcal{N}' \) of height \( \alpha \) such that \( \rho_\omega(\mathcal{N}') = \mathbb{R}^V \), and there is some \( B \subset \mathbb{R}^V \) with \( L(B, V[G]) \models \) “\( \text{AD}^+ \) plus \( \mathcal{N}' \) is \( \omega_1 \)-iterable.” Hence by the homogeneity of \( \mathbb{P} \), \( \mathcal{N} \in V \).

If \( \pi : \tilde{\mathcal{N}} \to \mathcal{N} \) is in \( V \) and countable there, then \( \Sigma \upharpoonright V \in V \) witnesses that \( \tilde{\mathcal{N}} \) is \( < j(\omega^V) \)-, and hence \( \omega_1 + 1 \)-iterable in \( V \). Also, by elementarity, \( L(A, \mathbb{R}) \models \) “\( \tilde{\mathcal{N}} \) is \( \omega_1 \)-iterable” (and this is witnessed by \( \Sigma \upharpoonright \text{HC}^V \)). Therefore, \( A \in K(\mathbb{R}) \), and this is witnessed by an \( \mathbb{R} \)-premouse \( \mathcal{N} \) such that the \( \omega_1 \)-iteration strategies for collapses of countable substructures of \( \mathcal{N} \) are in \( L(A, \mathbb{R}) \). \( \Box \)

**Definition 5.1.5** \( M_0 = L(\Gamma_0, \mathbb{R}) \).

By Lemma 5.1.4, \( M_0 \subset K(\mathbb{R}) \).

**Lemma 5.1.6** Both \( j(\Gamma_0) \) and \( j(M_0) \) are \( \text{OD}^{V[G]} \).

**Proof.** It certainly suffices to see that \( j(\Gamma_0) \) is \( \text{OD}^{V[G]} \). But by Theorem 5.0.12, there is an \( \alpha \) such that \( A \in j(\Gamma_0) \) iff \( A \in \Gamma_0^M \) iff \( A \in \Gamma_0^{V[G]} \) and the Wadge rank of \( A \) in \( L(A, \mathbb{R}) \) is \( < \alpha \).\(^3\) \( \Box \)

**Theorem 5.1.7** \( M_0 \models \text{AD} \).

**Proof.** By a core model induction, guided by [14]. This uses Lemmas 5.1.4 and 5.1.6. \( \Box \)

The core model induction which is used to prove Theorem 5.1.7 is completely parallel to the the core model induction thru \( L(\mathbb{R}) \) which we did earlier in the seminar. We do need to get \( j(K) \in V \) at each step, where \( K \) is the core model of some appropriate local universe \( P \). As in the earlier cases, \( j(K) \in V \) comes from \( j(P) \in \text{OD}^{V[G]} \). In order to see that we seem to need that our putative \( \mathbb{R} \)-mouse beyond \( \Gamma_0 \) is countably iterable “over” \( \Gamma_0 \) (as in the conclusion of Lemma 5.1.4).

The same remark applies to the proof of Lemma 5.1.9 below.

\(^3\)We might have \( \Gamma_0^{V[G]} \setminus \Gamma_0^{(M)} \neq \emptyset \), but \( \Gamma_0^{(M)} \) is a Wadge initial segment of \( \Gamma_0^{V[G]} \) by Theorem 5.0.12.
Also, we don’t know how to show $K(\mathbb{R}) \models AD$, because of this problem of showing that $j(P) \in OD^{V[G]}$ for the relevant local universes $P$.

**Corollary 5.1.8** $\Gamma_0 = M_0 \cap \mathcal{P}(\mathbb{R})$.

**Lemma 5.1.9** If $S$ is an $\mathbb{R}$-premouse such that every countable $\bar{S}$ which embeds into $S$ has an $\omega_1$-iteration strategy in $M_0$, then $S \in M_0$.

**Proof.** We show that $L(S) \models AD$ by a core model induction, guided by [14] and [18]. Notice that $j(S) \in OD^{V[G]}$, which follows using Theorem 5.0.12 and the fact that $j(S)$ is $\omega_1$-iterable “over” $j(M_0)$: for some $\alpha$, in $V[G]$, $j(S)$ is the unique sound $\mathbb{R}^{[\alpha]}$-premouse $S'$ with $\rho(S') = R^{V[G]}$ such that for all countable $\bar{S}$ which embed into $S'$ there is some $B \subset R^{V[G]}$ with $L(B, R^{V[G]}) \models \text{“AD$^+$ plus $\bar{S}$ is $\omega_1$-iterable.”}$ Also, $L(S) \models \Theta_0 = \Theta$, because now $L(S) \models AD$, so that $S \in OD^{L(S)}$ and in $L(S)$, everything is $OD^{L(S)}_z$ for some $z \in \mathbb{R}$.

Here’s a useful characterization of $M_0$, or rather of its sets of reals.

**Lemma 5.1.10** $\mathcal{P}(\mathbb{R}) \cap M_0 = \mathcal{P}(\mathbb{R}) \cap HOD^{\mathcal{P}(\mathbb{R})}_{\mathcal{R}^V}$.

**Proof.** “$\subset$”: Let $A \in \mathcal{P}(\mathbb{R}) \cap M_0$. By Lemma 5.1.4, there is some countably iterable $\mathbb{R}$-premouse $\mathcal{N}$ such that $A \in \mathcal{N}$. By Lemma 5.1.4, if $\mathcal{N} \in V$ is countable in $V$ and embeds into $\mathcal{N}$, then $L(A, \mathbb{R}) \models \text{“}\mathcal{N} \text{ is } \omega_1^{\mathcal{N}}\text{-iterable.”}$ By Lemma 5.1.9, we then have that $\mathcal{N} \in M_0$. But then $M_0 \models \text{“}\mathcal{N} \text{ is countably iterable,” hence } j(M_0) \models \text{“}j(\mathcal{N}) \text{ is countably iterable.”}$ In particular, $j(M_0) \models \text{“}\mathcal{N} \text{ is countably iterable.”}$ But then $A \in OD^{\mathcal{P}(\mathbb{R})}_{\mathcal{R}^V}$.

“$\supset$”: Let $A \subset \mathbb{R}$, $A \in OD^{\mathcal{P}(\mathbb{R})}_{\mathcal{R}^V}$. By Theorem 5.0.13, there is some $\mathbb{R}$-premouse $\mathcal{N}$ such that $A \in \mathcal{N}$ and $j(M_0) \models \text{“}\mathcal{N} \text{ is } \omega_1^{V[G]}\text{-iterable.”}$. We have that $\mathcal{N} \in V$ by Lemma 5.1.6 and the homogeneity of $\mathbb{P}$ also, if $\mathcal{N} \in V$ is countable in $V$ and $\mathcal{N}$ embeds into $\mathcal{N}$ (equivalently in $V$ or in $j(M_0)$), then $j(M_0) \models \text{“}\mathcal{N} \text{ is } \omega_1\text{-iterable,” and hence } M_0 \models \text{“}\mathcal{N} \text{ is } \omega_1\text{-iterable.”}$ We may now use Lemma 5.1.9 to conclude that $A \in \Gamma_0$.

**Lemma 5.1.11** $M_0 \models \text{“}\Theta_0 = \Theta.”$

**Proof.** Let $A$ be a set of reals in $M_0$. There is then an $\mathbb{R}$-premouse $\mathcal{N}$ with $A \in \mathcal{N}$ as in Lemma 5.1.4. But then $\mathcal{N} \in M_0$ by Lemma 5.1.9. So $\mathcal{N}$ is $OD^{M_0}$, and $A$ is $OD^2_{z}M_0$ for some real $z$. It follows that $M_0 \models \text{“}\Theta_0 = \Theta.”$ 

□
5.2. THE HOD OF M₀ UP TO ITS Θ.

Definition 5.1.12 \( \Gamma = (\Sigma_1^2)^{M_0} \).

We now go for a \( \Gamma \)-suitable premouse \( N \). In fact, we want more, cf. Theorem 5.3.1 below. One starting point for this is \( \text{HOD}_{j(M_0)} \). Notice that \( \text{HOD}_{j(M_0)} \) is amenable to \( V \) by Lemma 5.1.6.

5.2 The HOD of \( M_0 \) up to its \( \Theta \).

Definition 5.2.1 \( H_0 = \text{HOD}_{M_0|\Theta M_0} \).

Theorem 5.2.2 (Woodin) Assume AD plus the mouse set conjecture MSC\(^5\). Then \( \text{HOD}|\Theta_0 \) is a direct limit of premice; in particular, it is itself a premouse.

Theorem 5.2.3 (Woodin) Assume AD\(^+\). Let \( A \subset \Theta_0 \); then there is some \( \kappa < \Theta_0 \) such that \( \kappa \) is \( A \)-reflecting\(^6\) in \( \Theta_0 \) via measures, i.e., for all \( \gamma < \Theta_0 \), there is a measure \( \mu \) on \( \kappa \) such that \( i_\mu(\kappa) > \gamma \) and \( i_\mu(A) \cap \gamma = A \cap \gamma \).

Because under AD all measures are OD (by a result of Kunen), this immediately gives:

Corollary 5.2.4 (Woodin) Assume AD\(^+\). If \( S \subset \text{OR} \), then \( \text{HOD}_S \models \Theta_0 \) is a Woodin cardinal.\(^4\)

Lemma 5.2.5 In \( j(M_0) \), \( H_0 \) is full in the following sense: if \( X \subset \Theta_0^{M_0} \) is bounded, \( X \in \text{OD}_{H_0} \), then \( X \in H_0 \).

Proof. Deny. Then by MSC, in \( j(M_0) \) there is some \( \omega_1 \)-iterable \( Q \supset H_0 \) such that \( \rho_\omega(Q) < \Theta^{M_0} \). We have that \( H_0 \in \text{HOD}_{j(M_0)} \), because \( M_0 = L(P(\mathbb{R}) \cap M_0) \) (by Corollary 5.1.8) = \( L(P(\mathbb{R}) \cap \text{HOD}_{j(M_0)}) \) (by Lemma 5.1.10).

But \( Q \) is OD\(^4\)\( H_0 \), so that \( Q \in \text{HOD}_{j(M_0)} \).

Now \( \Theta^{M_0} \) is regular in \( \text{HOD}_{j(M_0)} \). This is because \( \text{HOD}_{j(M_0)} = L(S, \mathbb{R}_V) \) for some set \( S \) of ordinals, \( L(S, \mathbb{R}_V) \models \text{“}\Theta \text{ is regular,”} \) and \( \Theta^{L(S, \mathbb{R}_V)} = \Theta^{M_0} \) (by Lemma 5.1.10). We may thus pick some \( \pi: Q \to Q \) with critical point \( \pi^{-1}(\Theta^{M_0}) \), call it \( \Theta \), such that \( \Theta \) is a cardinal of \( H_0, \pi, Q \in \text{HOD}_{j(M_0)} \), and

\(^4\)I.e., \( H_0 = V^{\text{HOD}_{M_0}} \)

\(^5\)i.e., the conclusion of Theorem 5.0.13

\(^6\)i.e., \( A \)-strong
CHAPTER 5. A MODEL OF AD PLUS $\Theta_0 < \Theta$

\[ \rho_\omega(Q) < \bar{\Theta}. \] Via the Coding Lemma, $\bar{Q}$ is then coded by a set of reals in $\text{HOD}^{j(M_0)}_{\mathbb{R}V}$ (i.e., in $M_0$, cf. Lemma 5.1.10), so that $\bar{Q} \in M_0$.

We claim that $M_0 \forces \text{"Q is countably iterable."}$ Well, if $\sigma : P \rightarrow \bar{Q}$, where $P \in \mathbb{V}$ is countable there, then in $j(M_0)$ there is some $\sigma^* : P \rightarrow Q$ (by the existence of $\pi \circ \sigma$), so that $j(M_0) \forces \text{"P is countably iterable."}$ Pulling this back via $j$ gives $M_0 \forces \text{"P is countably iterable."}$ But now $\bar{Q} \in \text{HOD}^{M_0}$, and we have a contradiction. \[ \square \]

**Definition 5.2.6** Let \( H^+_0 \) be the $C^{\Sigma^2_1}_{\text{HOD}}$-closure of $H_0$ up thru $\omega$ cardinals.

**Lemma 5.2.7** $H^+_0 \forces \text{"$\Theta$ is a Woodin cardinal."}$

**Proof.** Let $A \subset \Theta^{M_0}$, $A \in \text{OD}^{j(M_0)}_{H_0}$. It suffices to see that $\Theta^{M_0}$ is Woodin with respect to $A$, as witnessed by extenders in $H_0$.

We have that $A \in \text{HOD}^{j(M_0)}_{\mathbb{R}V}$, and also $A \cap \gamma \in H_0$ for all $\gamma < \Theta^{M_0}$. Let us pick some $\kappa < \Theta^{M_0}$ such that $\kappa$ is $A$-reflecting as witnessed by measures in $\text{HOD}^{j(M_0)}_{\mathbb{R}V}$ (cf. Theorem 5.2.3). Each such measure is in $M_0$ (cf. Lemma 5.1.10), and it is $\text{OD}^{M_0}$ by Kunen. Moreover, \( (i_\mu \upharpoonright P(\kappa))^{M_0} = (i_\mu \upharpoonright P(\kappa))^{\text{HOD}^{j(M_0)}_{\mathbb{R}V}} \), so that $i_\mu \upharpoonright (P(\kappa) \cap H_0) \in H_0 = \text{HOD}^{M_0}(\Theta^{M_0})$ for each such measure. It thus easily follows that $H^+_0 \forces \text{"$\kappa$ is $A$-reflecting in $\Theta^{M_0}$."}$ \[ \square \]

Using $j$, we may now pull back the true statement $j(M_0) \forces \text{"there is a $\Sigma^2_1$-suitable countable premouse."}$

5.3 A model of AD plus $\Theta_0 < \Theta$.

We need to prove more, namely the following.

**Theorem 5.3.1** There is a $\Gamma$-suitable countable premouse $N$ together with a $\Gamma$-fullness-preserving iteration strategy $\Sigma$ which condenses well.

The proof of this result will be presented in the next two talks. In order to prove Theorem 5.3.1, we need the following information about $\Theta^{M_0}$.

**Lemma 5.3.2** $\Theta^{M_0} < j(\omega^V_1)$. In fact, $\text{cf}(\Theta^{M_0}) = \omega$, and $j'' \Theta^{M_0}$ is cofinal in $j(\Theta^{M_0})$. 

With the help of Lemma 5.1.10, it thus follows that $\Theta = \sup(\gamma)$.

5.3. A MODEL OF AD PLUS $\Theta_0 < \Theta$

We now show that $\text{cf}(\Theta) = \omega$ (which implies that $j''\Theta$ is cofinal in $j(\Theta^M)$).

Suppose that $\text{cf}(\Theta) > \omega$. Then $j''\Theta$ is not cofinal in $j(\Theta^M)$. Let $\gamma = \text{sup}(j''\Theta) < j(\Theta^M)$. Then $\gamma$ is a (limit of) cardinal(s) in $j(H_0)$, and we have some initial segment $Q$ of $j(H_0)$ which defines a counterexample to the Woodinness of $\gamma$.\footnote{This follows from a result of Woodin extending Theorem 5.2.4. There is no Woodin cardinal $< \Theta_0$ in $\text{HOD}_x$.} Let $Q$ the least such premouse.

We have $Q \in \text{HOD}^{j(M_0)}$, and so $L[Q, H_0][\Theta^M] = H_0$ by Lemma 5.2.5. Moreover, $L[Q, H_0] \models "\Theta^M \text{ is a Woodin cardinal} "$ by Lemma 5.2.7.

We get that $Q \in V$, and the map $j \upharpoonright L[Q, H_0]: L[Q, H_0] \to L[j(Q), j(H_0)]$ factors as $j \upharpoonright L[Q, H_0] = k \circ i$, where $i: L[Q, H_0] \to \text{Ult}(L[Q, H_0]; E_j \upharpoonright \gamma)$, $\text{crit}(k) \geq \gamma$, and $\text{sup}j''\Theta^M = \text{sup}j''\Theta^M$ (because $E_j|\gamma$ captures this much of $j$). But then

$$i(\Theta^M) = \text{sup}j''\Theta^M = \gamma,$$

because $\Theta^M$ is regular in $L[Q, H_0]$. Since $k(i(\Theta^M)) = \Theta^{j(M_0)}$,

$$\text{crit}(k) = i(\Theta^M) = \gamma.$$

Write $\text{Ult} = \text{Ult}(L[Q, H_0]; E_j \upharpoonright \gamma)$. Let $g$ be $\text{Col}(\omega, \gamma)$-generic over $\text{Ult}$. In $\text{Ult}[g]$, there is a tree $T$ searching for some premouse $S$ which defines a counterexample to the Woodinness of $\gamma$, $S \supseteq i(H_0)$, such that there is an embedding $S \to i(Q)$. There is a unique branch thru $T$, namely the one which gives $Q$. Therefore, $Q \in \text{Ult}$, which is a contradiction, because $\text{Ult} \models "i(\Theta^M) \text{ is a Woodin cardinal}"$.\hfill $\Box$

Let $N$, $\Sigma$ be a premouse and an $\omega_1$-iteration strategy as in the statement of Theorem 5.3.1. The following Lemmas show how we are going to use $\Sigma$ to produce our model of AD PLUS $\Theta_0 < \Theta$.

**Lemma 5.3.3** $\Sigma \notin M_0$.

**Proof.** Suppose that $\Sigma \in M_0$. We have that $M_0 \models \text{AD}$. Let us work in $M_0$ for the rest of this proof.

Consider the relation $R(z, y)$ iff $z \notin \text{OD}_y$. Let $n \in R$ code $N$. We may define a uniformizing function $F$ for $R$ as follows.

Let $y \in R$. We may then work inside $L[n, y, \Sigma]$ to make $y$ generic over a $\Sigma$-iterate $M$ of $N$. By the properties of $N$, $\Sigma$ we have that if $\delta$ is the
Woodin cardinal of $\mathcal{M}$, then $\mathcal{P}(\mathcal{M}[\delta]) \cap \text{OD}_{\mathcal{M}[\delta]} \subset M$. Hence if $T$ is the tree for a universal $\Sigma^2_1$ set which is derived from a $\Sigma^2_1$ scale on it, then $\mathcal{P}(\mathcal{M}[\delta]) \cap \mathcal{L}[T,\mathcal{M}] \subset \mathcal{M}$. This implies that $\mathcal{P}(\mathcal{M}[y][\delta]) \cap \mathcal{L}[T,\mathcal{M},y] \subset \mathcal{M}[y]$, and hence $\mathcal{P}(\mathcal{M}[y][\delta]) \cap \text{OD}_{\mathcal{M}[\delta][y]} \subset \mathcal{M}[y]$. Therefore, $\mathcal{M}[y]$ contains every real which is $\text{OD}_y$.

But now $\mathcal{L}[n,y,\Sigma]$ contains a real $z$ which codes $\mathcal{P}[y]$. We must then have $z \notin \text{OD}_y$. Moreover, we may let $F(y) = z$, where $(z,\mathcal{M})$ is least in $\mathcal{L}[n,y,\Sigma]$ such that $y$ is generic over the $\Sigma$-iterate $\mathcal{M}$ of $\mathcal{N}$ and $z$ enumerates $\mathcal{M}[y]$. 

$F$ is obviously $\text{OD}_x$ for some real $x$. Then $F(x) \in \text{OD}_x$ as well. But of course we should have $F(x) \notin \text{OD}_x$, as $F$ uniformizes $R$. Contradiction! □

**Theorem 5.3.4** $L(\Sigma,\mathbb{R}) \models \text{AD}$.

**Proof.** By a core model induction, using the fact that $\Sigma$ condenses well. □

**Theorem 5.3.5** $L(\Sigma,\mathbb{R}) \models \text{AD} + \Theta_0 < \Theta$.

**Proof.** Otherwise $\Sigma \in K(\mathbb{R})$ by Lemma 5.1.4. But this contradicts Lemma 5.3.3. □
Chapter 6

The core model induction beyond $L(\mathbb{R})$

6.1 More strength from homogeneous ideals

Our goal in the last 3 lectures will be to show that, granted $\text{CH}$ plus the existence of certain kind of homogenous ideal on $\omega_1$, there is a nontame mouse (and a bit more). We will be following Richard Ketchersid’s thesis [4], and will generally try to use his notation.

More precisely, let $HI$ be the conjunction of:

1. $\text{CH}$,

2. There is a homogeneous poset $\mathbb{P}$ such that whenever $G$ is $\mathbb{P}$-generic, then in $V[G]$ there is a $j: V \rightarrow M$ such that

   (a) $\text{crit}(j) = \omega_1^V$ and $M$ is closed under $\omega$-sequences in $V[G]$, and

   (b) $j \upharpoonright \text{OR} \in V$, and

   (c) enough of the normal measure on $V$ induced by $j$ is in $V$.

We shall spell out the precise statement of clause 2(c) of $HI$ when we need to use it. We suspect that in the end, it can be avoided entirely, but do not know how to do so. Our goal is to show

**Theorem 6.1.1 (Ketchersid, [4])** If $HI$ holds, then there is an inner model of $\text{AD}^+ + \theta_0 < \theta$ containing all real and ordinals, and consequently, there is a nontame mouse.

73
In the presence of $\text{AD}^+$, $\theta_0 < \theta$ is equivalent to “all $\Pi^1_1$ sets are Suslin”. We shall therefore assume throughout our argument that there is no inner model having all reals and ordinals of $\text{AD} + \text{DC}$ “all sets are Suslin” (i.e. no inner model of $\text{AD}_\mathbb{R} + \text{DC}$). We shall use two consequences of this fairly quickly:

**Theorem 6.1.2 (Woodin)** Assume $\text{AD}^+$ there is no inner model having all reals and ordinals of $\text{AD}_\mathbb{R} + \text{DC}$; then the Mouse Set Conjecture (MSC) holds.

**Theorem 6.1.3 (Woodin)** Suppose $N$ and $M$ are models of $\text{AD}^+$ containing all reals and ordinals such that neither $P(\mathbb{R}) \cap N \subseteq M$ nor $P(\mathbb{R}) \cap M \subseteq N$ holds; then $L(P(\mathbb{R}) \cap M \cap N) \models \text{AD}_\mathbb{R} + \text{DC}$.

### 6.2 A maximal model of $\text{AD}^+ + \theta_0 = \theta$

We shall do our core model induction in $V$. It must go well beyond $L(\mathbb{R})$.

We shall use $j : V \to M$ for generic embeddings given by $\mathfrak{H}$. There is an ambiguity here because $j$ depends on our $\mathbb{P}$-generic $G$, but homogeneity and the fact that $j \upharpoonright \text{OR}$ in $V$ imply that $j(x)$ is independent of $G$ in some important cases. (E.g. for $x \in \text{OR}$, but for some other important $x$ as well.)

**Definition 6.2.1**

(a) $\Gamma_0 = \{A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models \text{AD} + \theta_0 = \theta\}$,

(b) $M_0 = L(\Gamma_0, \mathbb{R})$.

We shall obtain an $\text{AD}^+$ model which properly contains $M_0$, and therefore satisfies $\theta_0 < \theta$. Notice that $M_0$ itself may have Wadge initial segments satisfying $\theta_0 < \theta$.

**Lemma 6.2.2**

1. Let $\mathcal{P}$ be an $\mathbb{R}$-premouse projecting to $\mathbb{R}$ such that every countable $Q$ embedded in $\mathcal{P}$ has an $\omega_1$-iteration strategy in $\Gamma_0$; then $\mathcal{P} \in \Gamma_0$.

2. $M_0 \models \text{AD}^+$.

3. $M_0 \models \theta_0 = \theta$.

**Proof.** Part (1) was done in Chapter 5. (Let $\mathcal{P}$ be the least such $\mathbb{R}$-premouse not in $\Gamma_0$. A core model induction, like the one for $L(\mathbb{R})$, shows that $L(\mathcal{P}) \models$
6.2. A MAXIMAL MODEL OF AD$^+$ $\theta_0 = \theta$

AD. Since $\mathcal{P}$ is countably iterable via strategies in $\gamma_0$, $L(\mathcal{P}) \models \theta_0 = \theta$, so
$\mathcal{P} \in \Gamma_0$, contradiction.)

(2) follows at once from (1). Part (3) holds because if $A$ is OD($\mathbb{R}$) in
some model of AD $+$ $V = L(P(\mathbb{R}))$, then $A$ is OD($\mathbb{R}$) in any bigger model of
AD.

Lemma 6.2.3  (a) Let $\gamma$ be the Wadge ordinal of $\Gamma_0$; then $j(\Gamma_0)$ and $j(M_0)$
are definable in $V[G]$ from $j(\gamma)$, uniformly in $G$. Thus if $G, H$ are $\mathbb{P}$-generics such that $V[G] = V[H]$, then $j_G(\Gamma_0) = j_H(\Gamma_0)$ and $j_G(M_0) = j_H(M_0)$.

(b) If $x$ is HOD$^{M_0}$, then $j(x) \in V$, and $j(x)$ is independent of $G$.

(c) HOD$^{j(M_0)}$ is contained in $V$, and is independent of $G$.

Proof. For (a), we claim that in $V[G]$, $j(M_0)$ is the unique model of AD
containing all reals and ordinals whose $\theta$ is $j(\gamma)$. For if not, there is a
Wadge initial segment of $j(M_0)$ satisfying AD$^+_\mathbb{R}$ $+$ DC. This implies that
there is a Wadge initial segment of $M_0$ satisfying AD$^+_\mathbb{R}$ $+$ DC, contrary to our
hypotheses.

We leave (b) and (c) as exercises. \qed

Lemma 6.2.4  (a) if $A \in \Gamma_0$, then there is an $\mathbb{R}$-premouse $\mathcal{P}$ such that
$A \in \mathcal{P}$, and $\mathcal{P}$ is countably iterable, both in $V$ and in $M_0$.

(b) $P(\mathbb{R}) \cap M_0 = P(\mathbb{R}) \cap HOD^j(M_0)$.

Proof. (a) MSC implies every OD($\mathbb{R}$) set of reals belongs to a countable
iterable $\mathbb{R}$-mouse. In $M_0$, MSC holds, and every set of reals is OD($\mathbb{R}$), and
so we get an $\mathbb{R}$-mouse $\mathcal{P} \in M_0$ such that $A \in \mathcal{P}$, and $\mathcal{P}$ is countably iterable
in $M_0$. We may assume $\mathcal{P}$ projects to $\mathbb{R}$. Letting then $\mathcal{S}$ be a countable
elementary submodel of $\mathcal{P}$, we have $j(M_0) \models \mathcal{S}$ is $\omega_1$-iterable, via a unique
strategy $\Sigma$. But then $\Sigma \models V \in V$, and witnesses $\mathcal{S}$ is $j(\omega_1^V)$-iterable in $V$.

For (b), let $A \in \Gamma_0 = (P(\mathbb{R}) \cap M_0)$. Let $\mathcal{P}$ be as in part (a). Then by an
easy absoluteness argument, $j(M_0)$ believes that $\mathcal{P}$ embeds into $j(\mathcal{P})$, and
thus believes that $\mathcal{P}$ is $\omega_1 + 1$ iterable, so that $\mathcal{P} \in HOD^j(M_0)$.

Conversely, let $A \subseteq \mathbb{R}^V$ be in $HOD^j(M_0)$. By MSC in $j(M_0)$, we get an
$\mathbb{R}^V$-mouse $\mathcal{P}$ projecting to $\mathbb{R}^V$, having $A$ in it, and being $\omega_1 + 1$-iterable in
$j(M_0)$. $\mathcal{P} \in V$ by 6.2.3(c). We leave it as an exercise to show that every
countable elementary submodel of $\mathcal{P}$ has an $\omega_1$-iteration strategy in $M_0$. This gives $\mathcal{P} \in M_0$ by 6.2.2(1).

\section{The Plan}

Our goal is to show that there is a model of $\text{AD}^+ + V = L(P(\mathbb{R}))$ properly including $M_0$. Suppose $\mathcal{N}$ were such a model. We have $\theta^{M_0} = \theta^\mathcal{N}_0 < \theta^\mathcal{N}$ because $M_0$ was maximal. This also gives $(\Sigma^2_1)^{M_0} = (\Sigma^2_1)^\mathcal{N}$. By results of Woodin, there is in $\mathcal{N}$ a sjs $\vec{A}$ containing the universal $\Sigma^2_1$-Woodin $\mathcal{N}$, with each $A_i \in M_0$. (The sequence of $A_i$ cannot be in $M_0$ since the universal $\Pi^2_1$ set is not Suslin in $M_0$.)

The sequence $\vec{A}$ is a Wadge minimal set of reals not in $M_0$, so it seems clear that our first step towards $\mathcal{N}$ should be to construct an sjs $\vec{A}$ containing the universal $\Sigma^2_1$-Woodin set, with each $A_i$ in $M_0$. This is actually all we need to do, since then a core model induction like our $L(\mathcal{R})$ one will give $L(\vec{A}, \mathbb{R}) \models \text{AD}$, so that $\mathcal{N} = L(\vec{A}, \mathbb{R})$ will do.

As we have seen, such an sjs $\vec{A}$ determines fullness-preserving, $\vec{A}$-guided strategy $\Sigma$ for a suitable $\Sigma^2_1$-Woodin $\mathcal{N}$, and $\vec{A}$ is in turn determined by $\mathcal{N}$ and $\Sigma$. (See Theorem 4.5.13 and Lemma 4.7.4.) We shall obtain $\vec{A}$ by making our way to $\mathcal{N}$ and $\Sigma$. Letting $j: V \to M \subseteq V[G]$ be as in HI, we proceed as follows:

1. We use $j$ to show that $\text{cof}(\theta^{M_0}) = \omega$. Thus $\text{cof}(\theta^{j(M_0)}) = \omega$ in $M$, and we can pick a countable family $\mathcal{B}$ of sets of reals Wadge cofinal in $j(M_0)$.

2. Let $H_0 = \text{HOD}^{M_0}[\theta^{M_0}]$. We show that in $M$, $H_0$ is $(\Sigma^2_1)^{j(M_0)}$-full, and has a fullness-preserving iteration strategy $\Sigma$ guided by $\mathcal{B}$.

3. We show that, when restricted to some $\Sigma$-iterate of $H_0$, the strategy $\Sigma$ condenses well.

4. Pulling back to $V$, we have a $(\Sigma^2_1)^{M_0}$-suitable $\mathcal{N}$ with an iteration strategy guided by some countable $\mathcal{A}$ which condenses well. We use this to get our sjs $\vec{A}$.

Woodin’s theory of approximations to sjs-guided iteration strategies plays a heavy role in steps (2) and (3).
6.4 HOD$^M_0$ as viewed in $j(M_0)$

We shall show that HOD$^M_0$ yields a mouse which is suitable from the point of view of $j(M_0)$.

Let us define suitability again. (See Chapter 4, §5,6.) Assume AD$^+$ and MSC for a bit. Let $\Gamma = \Sigma_1^2$. By MSC, we have that for any countable transitive $a$ and $b \subseteq a$, $b \in \text{OD}(a)$ iff $b \in C_\Gamma(a)$ iff $b$ is in some $\omega_1$-iterable $a$-mouse. Let $\text{Lp}(a)$ be the union of all $\omega_1$-iterable $a$-mice projecting to $a$.

We let $a^+$ be the $a$-mouse given by

**Definition 6.4.1** For any countable, transitive $a$, let $a^+ = \bigcup_i (M_i)$, where $M_0 = a$, and $M_{i+1} = \text{Lp}(M_i)$.

As before, a premouse $N$ is suitable (with respect to $\Gamma$) just in case $N$ is countable, and

(a) $N \models$ there is exactly one Woodin cardinal, called $\delta^N$,

(b) $N = (N|\delta^N)^+$, and

(c) If $\xi < \delta^N$ is a cardinal of $N$, then $\text{Lp}(N|\xi) \models \xi$ is not Woodin.

**Definition 6.4.2** $H_0 = \text{HOD}^M_0|\theta^M_0$.

The maximality of $M_0$, and in particular the fact that $P(\mathbb{R}) \cap \text{HOD}^j(M_0) \subseteq M_0$, leads to

**Theorem 6.4.3 ([4])** In $j(M_0)$, we have

(a) $H_0$ is $\Sigma_1^2$-full, that is, every OD($H_0$) bounded subset of $\theta^M_0$ belongs to $H_0$, and

(b) $H_0^+ \models \theta^M_0$ is Woodin.

Thus $H_0^+$ is suitable in $j(M_0)$.

**Proof.** See Chapter 5. \hfill $\square$

This gives at once

**Corollary 6.4.4** Let $Q \in \text{HOD}^j(M_0)$; then

(a) $L[Q, H_0]\|\theta^M_0 = H_0$, 


(b) $L[Q, H_0] = \theta^{M_0}$ is Woodin.

And from this, we get some important confirmation that our plan of constructing an sjs containing the universal $(\Sigma^2_1)^{M_0}$ set and consisting only of sets in $M_0$ is plausible.

**Theorem 6.4.5** Assume $\text{HI}$, and let $M_0$, etc., be as above; then

(a) $\theta^{M_0} < j(\omega_1^V)$,

(b) $j^*\theta^{M_0}$ is cofinal in $j(\theta^{M_0})$, and

(c) In $V$, $\text{cof}(\theta^{M_0}) = \omega$.

**Proof.** See Chapter 5. □

### 6.5 HOD below $\theta_0$

We have already used

**Theorem 6.5.1 (Woodin)** Assume $\text{AD}^+$ and $\text{MSC}$; then $\text{HOD}|\theta_0$ is a pre-mouse.

We shall need the proof of 6.5.1, as well as its statement. The main ideas are exposited in [17] and [15][§8], but the full proof has never been written up, and so we shall do that here. Let us assume $\text{AD}^+$ plus $\text{MSC}$ for the remainder of this section.

#### 6.5.1 Quasi-iterability

We need some material from earlier lectures. See Chapter 4, §5.6.

If $\mathcal{N}$ is suitable and $A$ is an OD set of reals, then for any cardinal $\mu$ of $\mathcal{N}$, $\tau^{\mathcal{N}}_{A,\mu}$ is the unique standard $\text{Col}(\omega, \mu)$-term which captures $A$ over $\mathcal{N}$. Notice that if $\mu < \nu$, then $\tau^{\mathcal{N}}_{A,\mu}$ is easily definable over $\mathcal{N}$ from $\tau^{\mathcal{N}}_{A,\nu}$.

Let $A = (A_0, \ldots, A_{k-1})$ be a sequence of OD sets of reals, and let $\nu_k$ be the $k$-th cardinal of $\mathcal{N}$ which is $\geq \delta^V$; then

$$\gamma^N_A = \sup(\{\xi \mid \xi \text{ is definable over } (\mathcal{N}|\nu_k+2, \tau^{\mathcal{N}}_{A_0,\nu_k}, \ldots, \tau^{\mathcal{N}}_{A_k,\nu_k})\}).$$

Let also

$$H^N_A = \text{Hull}^{\mathcal{N}|\nu_k+2}(\gamma^N_A, \tau^{\mathcal{N}}_{A_0,\nu_k}, \ldots, \tau^{\mathcal{N}}_{A_k,\nu_k})).$$
where we take the full elementary hull, but without transitively collapsing. Using the regularity of \( \delta^N \) in \( N \), we have that

\[
H^N_A \cap \delta^N = \gamma^N_A,
\]

so that \( \gamma^N_A \) is the image of \( \delta^N \) in the collapse of \( H^N_A \).

From the proof of 4.5.13, we have

**Lemma 6.5.2** Let \( T \) be a maximal tree on a suitable \( N \), with cofinal branches \( b \) and \( c \) such that \( i_b(\delta^N) = i_c(\delta^N) = \delta(T) \). Let \( \bar{A} = \langle A_0, ..., A_k \rangle \) be a sequence of OD sets of reals, and suppose

\[
i_b(\tau^{N}_{A_i,\nu}) = i_c(\tau^{N}_{A_i,\nu})
\]

for all \( i \leq k \), where \( \nu \) is the \( k \)-th cardinal of \( N \) above \( \delta^N \). Then \( i_b \upharpoonright H^N_A = i_c \upharpoonright H^N_A \).

**Definition 6.5.3** Let \( i : H^N_A \to M \), where \( N \) and \( M \) are suitable, and \( A \) is an OD set of reals. We say that \( i \) is \( A \)-correct iff \( i(\tau^N_{A_i,\nu}) = \tau^M_{A_i,\nu} \). In the case that \( M = M^T_b = M(T)^+ \) and \( i = i^T_b \), we say also that \( b \) is \( A \)-correct. Finally, correctness with respect to a set or sequence of OD sets of reals means correctness with respect to each member of the set or sequence.

**Corollary 6.5.4** Let \( \bar{A} \) be a sequence of OD sets of reals, and let \( T \) be a maximal tree on a suitable \( N \). Suppose \( b \) and \( c \) are \( \bar{A} \)-correct branches of \( T \); then \( i_b \upharpoonright H^N_A = i_c \upharpoonright H^N_A \).

**Definition 6.5.5** An iteration tree \( T \) on premouse \( N \) is \( \Sigma^2_1 \)-guided (or \( \Sigma_1 \)-guided, or OD-guided) just in case for all limit \( \lambda < \text{lh}(T) \), \( Q([0, \lambda]) \text{ exists, and } Q([0, \lambda]) \subseteq \text{Lp}(M(T)) \).

So an \( \text{Lp} \)-guided tree on \( N \) is according to all iteration strategies for \( N \). We have seen that if \( \Sigma \) is an \( \omega_1 \)-strategy for a suitable \( N \), and \( T \) is a normal, \( \text{Lp} \)-guided tree on \( N \) of limit length, and \( b = \Sigma(T) \), then either

(a) \( T \) is short, and \( T \downarrow b \) is the unique \( \text{Lp} \)-guided extension of \( T \), or

(b) \( T \) is maximal, and \( M_b^T = M(T)^+ \).

In case (a) we can define \( b \) from \( T \), without referring to \( \Sigma \), but in case (b) we may not be able to do that. However, in both cases we can define \( M_b^T \) from \( T \), without referring to \( \Sigma \). This leads to
Definition 6.5.6 A countable sequence \( \langle N_\alpha \mid \alpha < \beta \rangle \) is pre-suitable iff whenever \( \alpha + 1 < \beta \), then

1. \( N_\alpha \) and \( N_{\alpha + 1} \) are suitable, and
2. there is a normal, \( \mathcal{L}_p \)-guided tree \( T \) on \( N_\alpha \) such that either
   
   (a) \( T \) is short, and \( N_{\alpha + 1} \) is the last model of \( T \), or
   
   (b) \( T \) is maximal, and \( N_{\alpha + 1} = M(T) \).

Notice that \( N_\alpha \) and \( N_{\alpha + 1} \) uniquely determine \( T \). There is nothing in this definition, however, which connects \( N_\lambda \), for \( \lambda \) a limit, to the earlier \( N_\alpha \). We would like some way defining the direct limit of \( \langle N_\alpha \mid \alpha < \lambda \rangle \), given that \( \langle N_\alpha \mid \alpha < \lambda \rangle \) is played according to some iteration strategy for \( N_0 \), without fully knowing the branches of maximal trees. For this, we use the approximation lemma 6.5.4.

Let \( \langle N_\alpha \mid \alpha < \beta \rangle \) be pre-suitable. We shall define by induction on \( \gamma \leq \beta \):

1. \( \langle N_\alpha \mid \alpha < \gamma \rangle \) is suitable,
2. for \( \xi < \gamma \) and \( \tilde{A} \) a finite sequence of OD sets of reals: \( [\xi, \gamma) \) is \( \tilde{A} \)-good,
3. if \( [\xi, \gamma) \) is \( \tilde{A} \)-good, and \( \xi \leq \eta < \gamma \), the \( \tilde{A} \)-guided embedding \( \pi_{\tilde{A}}^{\xi, \eta} : H_{\tilde{A}}^{N_\xi} \to H_{\tilde{A}}^{N_\eta} \),
4. for \( \gamma \) a limit: the quasi-limit \( \text{qlim}_{\alpha < \gamma} N_\alpha \).

Let \( T_\alpha \) be the unique normal tree leading from \( N_\alpha \) to \( N_{\alpha + 1} \), as in presuitability.

To begin with, we say that \( \langle N_\alpha \mid \alpha < \gamma \rangle \) is suitable just in case for all limit \( \lambda < \gamma \), \( N_\lambda = \text{qlim}_{\alpha < \lambda} N_\alpha \). Concepts (2)-(4) will only be defined when \( \langle N_\alpha \mid \alpha < \gamma \rangle \) is suitable, so we assume that. It will follow from suitability that our embeddings commute appropriately, in that whenever \( \xi < \nu < \mu < \gamma \) and \( [\xi, \mu + 1) \) is \( \tilde{A} \)-good, then \( \pi_{\tilde{A}}^{\xi} = \pi_{\tilde{A}}^{\nu} \circ \pi_{\tilde{A}}^{\xi, \nu} \). So we assume that too.

Now let \( \gamma \) be a limit ordinal. We say \( [\xi, \gamma) \) is \( \tilde{A} \)-good just in case \( [\xi, \eta) \) is \( \tilde{A} \)-good for all \( \eta < \gamma \). We say that the quasi-limit \( \text{qlim}_{\alpha < \gamma} N_\alpha \) exists iff for every finite sequence \( \tilde{A} \) of OD sets of reals, there is a \( \xi < \gamma \) such that \( [\xi, \gamma) \) is \( \tilde{A} \)-good. In this case, we have a direct limit system \( \mathcal{F} \) whose indices are pairs \( (\xi, \tilde{A}) \) such that \( [\xi, \gamma) \) is \( \tilde{A} \)-good, with the directed ordering \( (\xi, \tilde{A}) \leq (\nu, \tilde{B}) \) iff \( \xi \leq \nu \) and \( \tilde{A} \) is a subsequence of \( \tilde{B} \), with \( H_{\tilde{A}}^{N_\xi} \) being the
structure indexed in $\mathcal{F}$ by $(\xi, \vec{A})$, and the map from $H^N_{\vec{A}}$ to $H^N_{\vec{B}}$ being $\pi^{\vec{A}}_{\xi,\nu}$. (Note here $H^N_{\vec{A}} \subseteq H^N_{\vec{B}}$.) Writing $\mathcal{F} = \mathcal{F}(\langle N_\alpha \mid \alpha < \gamma \rangle)$, we set

$$q\lim_{\alpha<\gamma} N_\alpha = \operatorname{dirlim}(\mathcal{F}(\langle N_\alpha \mid \alpha < \gamma \rangle)).$$

This completes the definitions of (2)-(4) in the case that $\gamma$ is a limit ordinal.

Next, suppose $\gamma = \eta + 1$, where $\eta$ is a limit ordinal. We declare that $[\xi, \gamma)$ is $\vec{A}$-good just in case $\xi = \eta$, or $[\xi, \eta)$ is $\vec{A}$-good, and letting $\sigma: H^N_{\vec{A}} \to N_\eta$ be the map given by $\mathcal{F}(\langle N_\alpha \mid \alpha < \eta \rangle)$ (whose limit is $N_\eta$ by suitability), we have that $\sigma$ is $\vec{A}$-correct. In the latter case, we set $\pi^{\vec{A}}_{\xi,\eta} = \sigma$. This completes the definitions (2)-(4) in the case that $\gamma$ is the successor of a limit ordinal.

Finally, suppose $\gamma = \eta + 1$, where $\eta = \nu + 1$. We say $[\xi, \gamma)$ is $\vec{A}$-good just in case $\xi = \eta$, or $[\xi, \eta)$ is $\vec{A}$-good, and either

(a) $T_\nu$ is short, and its canonical embedding $i: N_\nu \to N_\eta$ is $\vec{A}$-correct, or

(b) $T_\nu$ is maximal, and has a cofinal, $\vec{A}$-correct branch.

In both cases, we have a canonical embedding $\pi^{\vec{A}}_{\nu,\eta}: H^N_{\vec{A}} \to H^N_{\vec{A}}$. If $\xi < \nu$, we define $\pi^{\vec{A}}_{\xi,\eta} = \pi^{\vec{A}}_{\nu,\eta} \circ \pi^{\vec{A}}_{\xi,\nu}$. This is all we need to do in the current case. This completes our inductive definitions of (1)-(4).

**Definition 6.5.7** $N$ is quasi-iterable just in case $N$ is suitable, and every suitable $\langle N_\alpha \mid \alpha < \gamma \rangle$ of limit length, with $N = N_0$, has a suitable quasi-limit. We call the models on any suitable sequence beginning with $N$ quasi-iterates of $N$.

**Definition 6.5.8** Let $\vec{A}$ be a sequence of OD sets of reals. We say $N$ is $\vec{A}$-quasi-iterable iff $N$ is quasi-iterable, and whenever $\langle N_\alpha \mid \alpha < \gamma \rangle$ is a suitable sequence with $N_0 = N$, then $[0, \gamma)$ is $\vec{A}$-good. We call the models on any such suitable sequence $\vec{A}$-quasi-iterates of $N$.

**Remark 6.5.9** It is easy to see that any tail end of a suitable sequence is suitable, and any tail end of an $\vec{A}$-good sequence is $\vec{A}$-good. Thus any $\vec{A}$-quasi-iterate of an $\vec{A}$-good iterate is itself $\vec{A}$-quasi-iterable.

We need to strengthen $\vec{A}$-quasi-iterability by adding a Dodd-Jensen property.
Definition 6.5.10 Let $\vec{A}$ be a sequence of OD sets of reals. We say $\mathcal{N}$ is strongly $\vec{A}$-quasi-iterable iff $\mathcal{N}$ is $\vec{A}$-quasi-iterable, and whenever $\mathcal{P}$ is a quasi-iterate of $\mathcal{N}$, and $\langle Q_\alpha \mid \alpha \leq \beta \rangle$ and $\langle S_\alpha \mid \alpha \leq \xi \rangle$ are suitable with $Q_0 = S_0 = \mathcal{P}$ and $Q_\beta = S_\xi = \mathcal{R}$, then the $\vec{A}$-iteration maps from $H^P_\vec{A}$ to $\mathcal{R}$ of the two systems are the same.

The following is central:

Theorem 6.5.11 (Woodin) Assume AD$^+$ plus MSC, and let $\vec{A}$ be a sequence of OD sets of reals; then there is a strongly $\vec{A}$-quasi-iterable mouse.

Remark 6.5.12 Although we do not need it in this section, notice that from 6.5.4 we immediately get: if $\mathcal{A}$ is a collection of OD sets of reals, and $T$ is a maximal tree on a suitable $\mathcal{N}$ such that $\delta(\mathcal{N}) = \sup(\{\gamma_\mathcal{N} \mid A \in \mathcal{A}\})$, then $T$ has at most one cofinal, $\mathcal{A}$-correct branch. This was part of the proof of Theorem 4.5.13. If $b$ is an $\mathcal{A}$-correct branch of $T$, and $\delta(T) = \sup(\{\gamma_\mathcal{N} \mid A \in \mathcal{A}\})$, then we can apply 6.5.4 to maximal trees on $\mathcal{M}^T_b$, and continue iterating.

We shall use this to show that if $\mathcal{B}$ is a Wadge-cofinal countable collection of OD$^0_\alpha$ sets of reals, and $\mathcal{A} = j^*\mathcal{B}$, then $\mathcal{A}$ guides a fullness preserving strategy on $H_0^+\|\theta_0$ in this way.

6.5.2 HOD$|\theta_0$ as a direct limit of mice

We now define a direct limit system which will give us HOD$|\theta_0$. It is just the same system we used in the definition of $\text{qlim}_{\alpha<\gamma} N_\alpha$, but with $\{N_\alpha \mid \alpha < \gamma\}$ replaced by the family of all quasi-iterable mice. More precisely, let

$$I = \{ (\mathcal{N}, \vec{A}) \mid \mathcal{N} \text{ is strongly } \vec{A}\text{-quasi-iterable} \},$$

and order $I$ by

$$(\mathcal{N}, \vec{A}) \leq_I (\mathcal{M}, \vec{B}) \iff \vec{A} \subseteq \vec{B} \text{ and } \mathcal{M} \text{ is a quasi-iterate of } \mathcal{N}.$$
is enough to carry out that argument.) In either case, we get a common quasi-iterate $\mathcal{R}$ of $M, N, P$. Because $\mathcal{R}$ is a quasi-iterate of $\mathcal{P}$, we have that $\mathcal{R}$ is strongly $\vec{A} \prec \vec{B}$ quasi-iterable. This implies that $(\mathcal{R}, \vec{A} \prec \vec{B})$ is the desired upper bound in $I$ for $(M, \vec{A})$ and $(N, \vec{B})$.

We regard $(N, \vec{A}) \in I$ as an index of the structure $H^N_{\vec{A}}$, and the maps

$$\pi_{(N, \vec{A}), (M, \vec{B})}: H^N_{\vec{A}} \rightarrow H^M_{\vec{B}}$$

of our system are those given by strong $\vec{A}$-quasi-iterability. (“Strong” implies the maps are unique, that they commute appropriately, and that the system is OD.)

We use $\mathcal{F}$ to denote this direct limit system. Let

$$M_\infty = \text{dirlim} \mathcal{F}.$$ 

The system is not quite countably directed, but we do have

**Lemma 6.5.14** Let $(M_i, \vec{A}_i) \in I$ for all $i < \omega$; then there is a suitable $\mathcal{N}$ such that for all $i$, $(M_i, \vec{A}_i) \leq_I (\mathcal{N}, \vec{A})$.

The proof uses a simultaneous comparison of the $M_i$, like the proof of directedness. It follows at once that $M_\infty$ is well-founded.

We use $\delta_\infty$ for the common image of the Woodin cardinal $\delta^N$ of the various $(N, \vec{A}) \in I$.

**Lemma 6.5.15** $\delta_\infty = \theta_0$.

**Proof.** That $\delta_\infty \leq \theta_0$ is easy to see: given $(N, \vec{A})$ in $I$, one can show that there is a prewellorder of $\mathbb{R}$ of order type $\pi_{(N, \vec{A}), \infty}(\gamma^N_{\vec{A}})$ which is ordinal definable from $(N, \vec{A})$, and hence from $N$. This prewellorder is just the natural one given by $\mathcal{F}$, which is definable because the comparison process (including the choice of “sufficiently much” of the last branches of the comparison trees) is appropriately definable.

So suppose $\delta_\infty < \theta_0$. We reflect this to some Wadge level where we have a sjs, and then use that to get a contradiction. This sort of argument is used often in Woodin’s work.

Let $(\alpha, \beta)$ be lexicographically least such that $L_\beta(P_\alpha(\mathbb{R})) \models \text{ZF}^- + \delta_\infty < \theta_0$. Let $S = L_\beta(P_\alpha(\mathbb{R}))$. By MSC, we have that $S$ is an initial segment of $K(\mathbb{R})$, and it is easy to see from our definition of it that it ends a weak gap in $K(\mathbb{R})$. We therefore have an sjs $\vec{A}$ containing the universal $(\Sigma^2_1)^S$ set, and such that each $A_i$ is $\text{OD}^S$. 

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6.5. HOD BELOW $\theta_0$  83
Remark 6.5.16 [14] only gives a real $x$ such that each $A_i$ is $OD^S(x)$. Some additional work is needed to avoid the real parameter.

We showed earlier in the seminar (see Chapter 4) that there is a $(\Sigma^2_1)^S$-suitable $\mathcal{N}$ with a fullness-preserving iteration strategy $\Sigma$ which is guided by $\mathbf{A}$. $\Sigma$ has the Dodd-Jensen property, by the usual argument, and thus we can let $P_\infty$ be the direct limit of all the non-dropping $\Sigma$-iterates of $\mathcal{N}$, under the maps given by comparison. It is easy to see that $P_\infty$ embeds into the direct limit of $\mathcal{F}^S$. (In fact, it’s not hard to show they are equal.) Letting $\pi: \mathcal{N} \to P_\infty$ be the natural map, and $\tau_i = \pi(\tau_{A_i,\delta_{\mathcal{N}}})$, we have that

$$H_i = \text{Hull}_{P_\infty} \delta_\infty \cup \{\tau_k \mid k < i\},$$

we have that $H_i$ is coded by a subset $G_i$ of $\delta_\infty$. By the Coding Lemma, we have then that $\langle G_i \mid i < \omega \rangle$, and the sequence of natural embeddings between the $G_i$, are in $S$. Thus $o(P_\infty) < \theta_0^S$, and $P_\infty \in S$. We have also that $\langle \tau_i \mid i \in \omega \rangle \in S$. But $\mathbf{A}$ was an sjs, so from $\langle \tau_i \mid i \in \omega \rangle$ we can define a tree $T$ (in the sense of descriptive set theory) whose projection is the universal $(\Pi^2_1)^S$ set of reals. Since $T$ is on some ordinal $< \theta_0^S$, we have a contradiction.  

\[ \square \]

Lemma 6.5.17 $M_\infty|\delta_\infty = \text{HOD}|\theta_0$.

Proof. Clearly, $M_\infty|\delta_\infty \subseteq \text{HOD}|\theta_0$.

Let $A$ be a bounded, ordinal definable subset of $\theta_0$. We must show $A \in M_\infty$. Let $(\mathcal{N}, \mathcal{B}) \in I$ and $\alpha < \gamma^\mathcal{N}_\mathcal{B}$ be such that $A \subseteq \pi(\mathcal{N}, \mathcal{B}),\infty(\alpha)$. We define

$$C(x) \iff x \text{ is a real coding some } (P, \xi) \text{ such that } \exists (\mathcal{N}, \mathcal{B}) \in I(P = \mathcal{N}(\delta^\mathcal{N}) \land \xi < \gamma^\mathcal{N}_{\mathcal{B}} \land \pi(\mathcal{N}, \mathcal{B}),\infty(\xi) \in A).$$

(Note that $\mathcal{N}$ is determined by $P$, if it exists, by $\mathcal{N} = P^+$.) Clearly, $C$ is an OD set of reals. Let

$$\tau_\infty = \text{common value of } \pi(\mathcal{Q},\infty)(\tau^\mathcal{Q}_C,\delta_{\mathcal{Q}}),$$

for all strongly $C$-quasi-iterable $\mathcal{Q}$.

We claim that for all $\beta$,

$$\beta \in A \iff M_\infty \models (\text{Col}(\omega, \delta_\infty) \land \hat{x}_{M_\infty} \in \tau_\infty),$$
where for any appropriate $R$, $\dot{x}_R$ is the canonical collapse term for a real coding $R|\delta_R$. This gives $A \in M_\infty$, as desired. To see this claim, fix $\beta < \alpha$, and fix $(Q, \vec{B}) \in I$ and $\xi$ such that $\pi_{(Q, \vec{B}), \infty}(\xi) = \beta$. We may assume that $(Q, \vec{B}^\frown C) \in I$. Letting $\pi = p_i_{(Q, \vec{B}^\frown C), \infty}$, we then have

$$\beta = \pi(\xi) \in A \iff H^Q_{\vec{B}^\frown C} \models (\Col(\omega, \delta^Q) \models \dot{x}_Q \in \tau^Q_{C, \delta^Q}) \iff M_\infty \models (\Col(\omega, \delta_\infty) \models \dot{x}_{M_\infty} \in \tau_\infty),$$

by the elementarity of $\pi$.

This finishes the proof of Lemma 6.5.17, and hence of Theorem 6.5.1.

In fact, we do not need to consider the full system $F$ in order to obtain $\text{HOD}|_{\theta_0}$. It is enough to take any collection $A$ of OD sets of reals which is Wadge cofinal (i.e. $\forall B \in P(\mathbb{R}) \cap \text{OD} \exists A \in \mathcal{A}(B \leq_w A)$), and consider only the subsystem of $F$ corresponding to indices $(N, \vec{A}) \in I$ with $\vec{A} \in \mathcal{A}^{<\omega}$. This follows from

Lemma 6.5.18 Assume $\text{AD}^+$ and MSC. Let $A$ be a Wadge-cofinal collection of OD sets of reals, and let $x \in \text{HOD}|_{\theta_0}$. Then there is a suitable $\mathcal{N}$ and an $\vec{A} \in \mathcal{A}^{<\omega}$ such that $x \in \text{ran}(\pi_{(\mathcal{N}, \vec{A}), \infty})$.

Proof. We may assume $x$ is an ordinal. Let $(\mathcal{N}, B) \in I$ and $\xi < \gamma^N_B$ be such that $\pi_{(\mathcal{N}, B), \infty}(\xi) = x$. Let $A \in \mathcal{A}$ be such that $B \leq_w A$ via the Wadge reduction $\vec{z}$. Using genericity iteration, we can find $(\mathcal{M}, (B, A)) \in I$ such that $(\mathcal{N}, B) \leq_{I} (\mathcal{M}, (B, A))$, and $\vec{z}$ is generic over $\mathcal{M}$ for the extender algebra at $\delta^\mathcal{M}$. It is an easy exercise to show, using the $\delta^\mathcal{M}$-chain condition for the extender algebra, that then $\gamma^\mathcal{M}_B \leq \gamma^\mathcal{M}_A$. But then $\pi_{(\mathcal{N}, B), (\mathcal{M}, (B, A))}(\xi) < \gamma^\mathcal{M}_A$, so $x \in \text{ran}(\pi_{(\mathcal{M}, A), \infty})$, as desired. □

6.6 A fullness preserving strategy for $H^+_0$

We return to our particular situation. By the results of the last section, $H_0$ is the direct limit of $\mathcal{F}^{M_0}$, restricted to $\theta^{M_0}$. We have shown already that $H_0$ is $\Sigma^3_1$-full in $j(M_0)$, and that in fact $H^+_0$ is suitable there. We have not yet shown that $H^+_0$ is the full direct limit of $\mathcal{F}^{M_0}$, but that will follow from what we are about to do.

Definition 6.6.1 \(\mathcal{P}_\infty\) is the direct limit of the system $\mathcal{F}^{M_0}$.
(b) \( \mathcal{O} \) is the collection of finite sequences of ordinal definable sets of reals.

(c) Working in \( j(M_0) \), let \( ((N_i, \bar{A}_i) \mid i < \omega) \) be a cofinal, linearly ordered sequence from \( (I, \leq I)^{M_0} \). Then

\[
P^*_\infty = \text{qlim}_{i<\omega} N_i,
\]

where the quasi-limit is computed in \( j(M_0) \).

Concerning part (c), notice that \( N_i = j(N_i) \) is suitable, and in fact strongly \( j(\bar{A}_i) \)-quasi-iterable, in \( j(M_0) \). It follows that \( \mathcal{P}^*_\infty \) exists, and that in \( j(M_0) \), \( \mathcal{P}^*_\infty \) is suitable, and in fact strongly \( j(\bar{A}) \)-quasi-iterable, for all \( \bar{A} \in \mathcal{O}^{M_0} \).

There is a natural embedding \( \sigma \) from \( \mathcal{P}_\infty \) into \( \mathcal{P}^*_\infty \): given \( x \in \mathcal{P}_\infty \), we can find an \( i \) such that \( x = \pi_{(N_i, \bar{A}_i), \infty}(\bar{x}) \) for some \( \bar{x} \), and we then set

\[
\sigma(x) = \pi_{j(\bar{A}_i)}(\bar{x}).
\]

The range of \( \sigma \) is just the direct limit of the subsystem of \( \mathcal{F}(\langle N_i \mid i < \omega \rangle)^{j(M_0)} \) corresponding to sets in \( j^w\mathcal{O}^{M_0} \).

**Definition 6.6.2** Let \( \mathcal{W} \) be suitable, and \( \mathcal{A} \) a collection of OD sets of reals; then

(a) \( H^W_{\mathcal{A}} = \bigcup_{\bar{B} \in \mathcal{A}^{<\omega}} H^W_{\bar{B}} \),

(b) \( S \) is an \( \mathcal{A} \)-quasi-iterate of \( \mathcal{W} \) iff for all \( \bar{B} \in \mathcal{A}^{<\omega} \), \( S \) is a \( \bar{B} \)-quasi-iterate of \( \mathcal{W} \),

(c) if \( S \) is an \( \mathcal{A} \)-quasi-iterate of \( \mathcal{W} \), then \( \pi^A_{\mathcal{W}, S} : H^W_{\mathcal{A}} \rightarrow H^S_{\mathcal{A}} \) is the union of the quasi-iteration maps \( \pi^B_{\mathcal{W}, S} : H^W_{\bar{B}} \rightarrow H^S_{\bar{B}} \),

(d) \( \mathcal{W} \) is \( \mathcal{A} \)-quasi-iterable iff \( \mathcal{W} \) is \( \bar{B} \)-quasi-iterable, for all \( \bar{B} \in \mathcal{A}^{<\omega} \),

(e) if \( \mathcal{W} \) is \( \mathcal{A} \)-quasi-iterable, then \( \pi^A_{\mathcal{W}, \infty} : H^W_{\mathcal{A}} \rightarrow M_\infty \) is the natural map into the direct limit of the system \( \mathcal{F} \).

We want to use these notions in \( j(M_0) \), with \( \mathcal{A} = j^w\mathcal{O}^{M_0} \). Although \( \mathcal{A} \notin j(M_0) \) in this case, \( \mathcal{A} \subseteq j(M_0) \), and this is enough to make sense of \( \mathcal{A} \)-quasi-iterability “in” \( j(M_0) \).
Lemma 6.6.3 In \( j(M_0) \), \( P_\infty^* \) and all of its quasi-iterates are \( j^-\mathcal{O}^M_0 \)-quasi-iterable.

Proof. We observed this for \( P_\infty^* \) above. It passes to quasi-iterates trivially. \( \square \)

Lemma 6.6.4 Let \( \mathcal{A} \) be any Wadge cofinal collection of \( OD^M_0 \) sets of reals; then \( \text{ran}(\sigma) = H_{P_\infty^*}^A \).

Proof. Clear. \( \square \)

We shall show that \( \sigma \) is onto, so that \( P_\infty = P_\infty^* \). To begin with,

Lemma 6.6.5 \( \sigma \upharpoonright (\delta_\infty + 1) \) is the identity.

Proof. We get that \( \sigma \upharpoonright \delta_\infty \) is the identity from the elementary properties of the direct limit systems in question. We leave the easy proof to the reader.

Suppose \( \delta_\infty < \sigma(\delta_\infty) \). It follows that \( H_0 = P_\infty^* | \eta \) where \( \eta = \delta_\infty < \delta_\infty^P = \sigma(\delta_\infty) \). But \( P_\infty^* \) is suitable in \( j(M_0) \), so \( \eta \) is not Woodin in \( P_\infty^* \), so \( (H_0^+)^{j(M_0)} \models \delta_\infty \) is not Woodin. This contradicts 6.4.3. \( \square \)

The following extends our fullness and suitability results for \( H_0 \).

Lemma 6.6.6 Let \( k: H_0 \to S \) and \( i: S \to j(H_0) \) be such that \( j = i \circ k \), with \( S \) countable in \( V[G] \). Then in \( j(M_0) \):

(a) \( S \) is \( \Sigma^2_1 \)-full, and

(b) \( S^+ \) is suitable.

Proof. Working in \( j(M_0) \), let \( T \) be the tree of a \( \Sigma^2_1 \) scale on the set of reals coding \( \omega_1 \)-iterable relativised mice. We have that \( T \in \text{HOD}^{j(M_0)} \). As in the proof that \( \theta^M_0 \) has cofinality \( \omega \) (see Chapter 5), we can extend \( k \) and \( i \) so as to obtain

\[ k^*: L[T, H_0] \to \text{Ult}(L[T, H_0], E_k) \]

and

\[ i^*: \text{Ult}(L[T, H_0], E_k) \to L[j(T), j(H_0)] \]

such that \( j = i^* \circ k^* \). \( k^* \) is just the canonical embedding into the ultrapower, and \( i^* \) is given by: \( i^*(k^*(f)(a)) = j(f)(i(a)) \), for all \( a \in |o(S)|^{< \omega} \). Note that since \( j \) maps \( o(H_0) \) cofinally into \( o(j(H_0)) \), \( k \) maps \( o(H_0) \) cofinally into...
88 CHAPTER 6. THE CORE MODEL INDUCTION BEYOND $L(\mathbb{R})$

We have that $S = k^*(H_0)$. Also, $H_0$ is a rank initial segment of $L[T, H_0]$, and $o(H_0) = \theta^{M_0}$ is Woodin in $L[T, H_0]$, so $S$ has these properties in $Ult(L[T, H_0], E_k)$.

Now an absoluteness argument like that we had before shows that indeed $S$ is $\Sigma^2_1$-full, and $S^+$ is suitable, in $j(M_0)$. For if not, there is a countable minimal mouse $Q \in j(M_0)$ witnessing that, and there is a tree in $Ult(L[T, H_0], E_k)$ whose only branch is $Q$. The key is that we can use the tree $k^*(T)$ to certify the iterability of the mice our tree produces. Since $T$ embeds into $k^*(T)$, $Q \in p[k^*(T)]$. On the other hand, $k^*(T)$ embeds into $j(T)$, and so any $S \in p[k^*(T)]$ is $\omega_1$-iterable in $j(M_0)$. Pulling back, it is enough to show that in $V$, every $S \in p[T]$ is $\omega_1$-iterable via a strategy in $M_0$. But fix $S \in p[T]$ with $S \in V$; then $j(M_0) \models S \in p[T]$, so $j(M_0) \models S$ is $\omega_1$-iterable. But $j(S) = S$, so $M_0 \models S$ is $\omega_1$-iterable, as desired.

It follows that $Q \in Ult(L[T, H_0], E_k)$, a contradiction. 

Lemma 6.6.7 Let $A$ be Wadge cofinal in the $OD^{M_0}$ sets of reals, and let $W$ be a quasi-iterate of $P^*_\infty$ in the sense of $j(M_0)$; then $\delta^W = \sup \{ \gamma^W_{\tilde{A}, o(W)} \mid \tilde{A} \in (j^{+} A)^{\leq \omega} \}$. That is, $(\delta^W + 1) \subseteq H^W_{j^+ A}$.

Proof. The key is

Claim. $j \upharpoonright P_\infty = \pi_{(P^*_\infty, \infty)} \circ \sigma$.

Proof. Let $x \in P_\infty$. By 6.5.18, we have $x = \pi^{M_0}_{(\mathcal{N}, \tilde{A}), \infty}(\bar{x})$, where $\tilde{A} \in A^{\leq \omega}$.

We have then that

$$j(x) = j(\pi^{M_0}_{(\mathcal{N}, \tilde{A}), \infty}(\bar{x}))$$

$$= \pi^{j(M_0)}_{(\mathcal{N}, j(\tilde{A})), \infty}(\bar{x})$$

$$= \pi^{j(M_0)}_{(P^{\mathcal{N}, j(\tilde{A})}, \infty)}((\pi^{j(M_0)}_{(\mathcal{N}, \tilde{A}), (P^{\mathcal{N}, j(\tilde{A})}, \infty)}(\bar{x})))$$

$$= \pi^{j(M_0)}_{(P^{\mathcal{N}, j(\tilde{A})}, \infty)}(\sigma(x)).$$

This proves our claim. 

Now let

$$S = \text{collapse of } H^W_{j^+ A},$$
6.6. A FULLNESS PRESERVING STRATEGY FOR $H_0^+$

and $l: S \to \mathcal{W}$ be the collapse map, and $k: \mathcal{P}_\infty \to S$ the collapse of $\pi_{\mathcal{P}_\infty, \mathcal{W}}^{j(A)}: H^{\mathcal{P}_\infty}_{j^*(A)} \to H^{\mathcal{W}}_{j^*(A)}$. By our claim

$$j \upharpoonright \mathcal{P}_\infty = \pi_{\mathcal{W}, \infty}^{j(A)} \circ l \circ k.$$  

So $k$ factors into $j$, and thus by 6.6.6, $S$ is suitable. From the minimality of $\mathcal{W}$, we then have that $l \upharpoonright (\delta^S + 1) = \text{identity}$, which is what we want. \hfill \Box

We can now show that the quasi-iterates of $\mathcal{P}_\infty^*$ are “sound”, in a certain sense. This allows us to improve 6.6.7.

**Lemma 6.6.8** Let $A$ be Wadge cofinal in the $OD^{M_0}$ sets of reals, and working in $j(M_0)$, let $\mathcal{W}$ be a quasi-iterate of $\mathcal{P}_\infty^*$; then

1. $\mathcal{W} = H^{\mathcal{W}}_{j^*(A)}$.
2. $\sigma$ is the identity, so $\mathcal{P}_\infty = \mathcal{P}_\infty^*$.
3. for any quasi-iterate $Q$ of $\mathcal{W}$, $\pi_{\mathcal{W}, Q}^{j(A)}: \mathcal{W} \to Q$, and $\pi_{\mathcal{W}, Q}^{j(A)}(\tau^\mathcal{W}_{j(B), \nu}) = \tau^Q_{j(B), \mu}$, where $\mu$ is the image of $\nu$, for all $OD^{M_0}$ sets of reals $B$.
4. $j \upharpoonright \mathcal{P}_\infty = \pi_{\mathcal{P}_\infty^*, \mathcal{P}_\infty}^{j(A)}$.

**Proof.** (i): To come.

(ii): This is immediate from the special case of 6.6.8(i) in which $\mathcal{W} = \mathcal{P}_\infty^*$.

(iii): This is immediate from (i) and the definitions.

(iv): This is just the key claim in the proof of 6.6.7, using that $\sigma$ is the identity. \hfill \Box

We can now show that the embeddings $\pi_{\mathcal{W}, Q}^{j(A)}$ of part (iii) of 6.6.8 comes from an iteration strategy for $\mathcal{P}_\infty$.

**Theorem 6.6.9** Let $A$ be countable, and Wadge cofinal in the $OD^{M_0}$ sets of reals; then in $M$ (where $j: V \to M$) there is a unique fullness-preserving, $j(A)$-guided $\omega_1$-iteration strategy $\Sigma$ for $\mathcal{P}_\infty$. Moreover, $\Sigma$ moves the term relations associated to $j(B)$ correctly, for any $OD^{M_0}$ set of reals $B$. 
**Proof.** We shall follow the proof of Theorem 4.5.13. There our counterpart to \( j(\mathcal{A}) \) was a self-justifying system, but 6.6.7 and 6.6.8 give us the consequences of being a self-justifying system which we actually used.

We shall just define \( \Sigma \) on stacks of maximal trees. (Maximal in \( j(M_0) \), that is.) On short trees, \( \Sigma \) is \( L^p \)-guided.

So let \( \mathcal{R} \) be a non-dropping \( \Sigma \)-iterate of \( P_\infty \), via the linear stack of maximal trees \( T \). Since \( \Sigma \) is fullness-preserving, we have that \( \mathcal{R} \) is a quasi-iterate of \( P_\infty \). Now let \( \mathcal{U} \) be a maximal tree on \( \mathcal{R} \), and \( W = M(\mathcal{U})^+ \). We have then that for all \( \vec{A} \in \mathcal{A}^{<\omega} \), \( \delta^W = \sup(\{ \gamma^W_{j(\vec{A})} | \vec{A} \in \mathcal{A}^{<\omega} \}) \), by 6.6.7(b). Now for each \( \vec{A} \in \mathcal{A}^{<\omega} \) we can pick a \( j(\vec{A}) \)-correct branch \( b_\vec{A} \) of \( \mathcal{U} \). Since the \( \gamma^W_{j(\vec{A})} \) are cofinal in \( \delta(\mathcal{U}) \), these branches converge to a cofinal branch \( b \) of \( \mathcal{U} \), and as in Theorem 4.5.13, we have

(i) \( i_b(\delta^\mathcal{R}) = \delta(\mathcal{U}) \),

(ii) letting \( T^S_B \) be the theory of parameters in \( \delta^S \cup \{ \tau^S_{\vec{B},\nu} | i < \text{lh}(\vec{B}) \} \) in \( S|(\nu^{++})^S \), where \( \nu \) is the \( \text{lh}(\vec{B}) \)-th cardinal of \( S \) above \( \delta^S \), we have

\[
i_b(T^\mathcal{R}_{j(\vec{A})}) = T^W_{j(\vec{A})},
\]

for all \( \vec{A} \in \mathcal{A}^{<\omega} \).

By 6.6.8, \( \mathcal{R} \) is coded by the \( T^\mathcal{R}_{j(\vec{A})} \), so \( M^\mathcal{U}_b \) is coded by the \( i_b(T^\mathcal{R}_{j(\vec{A})}) \). Thus by (ii) above, there is an embedding \( \sigma : M^\mathcal{U}_b \rightarrow W \), with \( \text{ran}(\sigma) = \bigcup_{\vec{A} \in \mathcal{A}^{<\omega}} H^W_{j(\vec{A})} \). By 6.6.8 then, \( \text{ran}(\sigma) = W \), so \( \sigma \) is the identity. This implies that \( i_b(\sigma^\mathcal{R}_{j(\vec{A}),\nu}) = \tau^W_{j(\vec{A}),i_b(\nu)} \) for all appropriate \( \vec{A} \) and \( \nu \). Thus \( b \) is \( j(\vec{A}) \)-correct for all \( \vec{A} \in \mathcal{A}^{<\omega} \), and we can set

\[
\Sigma(T^\mathcal{R}_\vec{A}) = b.
\]

We leave the easy proof that \( b \) is the unique fullness-preserving, \( j^\mathcal{A} \)-correct branch to the reader.

Finally, we must deal with linear stacks \( \langle \mathcal{T}_\alpha | \alpha < \lambda \rangle \) of maximal trees of limit length. For this, it is enough to show that the direct limit \( \mathcal{S} \) along the branches chosen by \( \Sigma \), under those branch embeddings, is the same as the quasi-limit \( W \) of the \( M(\mathcal{T}_\alpha)^+ \). This is done by induction on \( \lambda \). The key is
that, as in the successor case, there is a natural embedding \( \sigma : S \to \mathcal{W} \) such that \( \sigma | \delta^S \) is the identity, and such that all \( \tau^W_{j(A), \nu} \) are in the range of \( \sigma \). We leave the details to the reader.

We want a fullness-preserving strategy which condenses well. As a first step in that direction, we have

**Theorem 6.6.10 (Weak condensation)** Let \( A \) be Wadge cofinal in the \( \text{OD}^{M_0} \) sets of reals, and \( \Sigma \) be the \( j(A) \)-guided \( \omega_1 \) iteration strategy for \( P_\infty \) given by 6.6.9. Let \( k : P_\infty \to \mathcal{W} \) be an iteration map by \( \Sigma \), and suppose there are \( l : P_\infty \to S \) and \( t : S \to \mathcal{W} \) be such that \( k = t \circ l \). Then \( S \) is suitable.

**Proof.** We have that \( k = \pi^{j_{A}^*}_{P_\infty, \mathcal{W}} \), so

\[
j \upharpoonright P_\infty = \pi^{j_{A}^*}_{P_\infty, \mathcal{W}} \circ \pi^{j_{A}^*}_{P_\infty} \circ \pi^{j_{A}^*}_{\mathcal{W}} = \pi^{j_{A}^*}_{P_\infty} \circ t \circ l.
\]

Thus \( k \) factors into \( j \), and we can apply 6.6.6.

Finally, we can pull back the existence of a fullness-preserving strategy with weak condensation to \( V \). We get

**Theorem 6.6.11** Let \( A \) be countable, and Wadge cofinal in the \( \text{OD}^{M_0} \) sets of reals; then there is in \( V \) a suitable \( N \) and a fullness-preserving (with respect to the pointclass \( (\Sigma_2^{1})^{M_0}_0 \)), \( A \)-guided iteration strategy \( \Sigma \) for \( N \) with the Dodd-Jensen property. Moreover

(a) for any \( \mathcal{W} \), \( \mathcal{W} \) is a non-dropping \( \Sigma \)-iterate of \( N \) iff \( \mathcal{W} \) is a quasi-iterate (in the sense of \( M_0 \)) of \( N \) iff \( \mathcal{W} \) is an \( A \)-quasi-iterate of \( N \),

(b) for \( \mathcal{W} \) a non-dropping \( \Sigma \)-iterate of \( N \), \( H_\mathcal{W}^{N} = \mathcal{W} \),

(c) the \( \Sigma \)-iteration maps are \( A \)-quasi-iteration maps, and thus \( P_\infty \) is the direct limit of all \( \Sigma \)-iterates of \( N \), and

(d) if \( t : N \to \mathcal{W} \) is an iteration map by \( \Sigma \), and \( t = l \circ k \) where \( k : N \to S \), then \( S \) is suitable.

**Proof.** \( j(A) = j_{A}^{*} \) is in \( M \), and in \( M \) there is such a \( j(A) \)-guided strategy for \( P_\infty \).
6.7 Branch condensation

In this section, we prove

**Theorem 6.7.1 (Branch condensation)** Assume $\mathcal{H}_1$, and let $A$ be countable and Wadge cofinal in the $\mathcal{O}_1^{\mathcal{M}_0}$ sets of reals; then there is a suitable $\mathcal{N}$ and $A$-guided strategy $\Sigma$ for $\mathcal{N}$ having all the properties of 6.6.11, and such that in addition

(e) Let $R$ be a $\Sigma$-iterate of $\mathcal{N}$, and let $i: R \to W$ be an iteration map by $\Sigma$. Let $T$ be a maximal tree on $R$, and $b$ a cofinal, non-dropping branch of $T$, and suppose there is $t: M^T_b \to W$ such that $i = t \circ i^T_b$; then $b = \Sigma(T)$.

**Corollary 6.7.2** Let $cN, \Sigma$ be as in 6.7.1; then $\Sigma$ condenses well.

Proof. We show that the restriction of $\Sigma$ to normal trees on $\mathcal{N}$ condenses well, and leave the case of stacks of normal trees to the reader. Let $\mathcal{S}$ be a normal tree on $\mathcal{N}$, and $U$ be a hull of $\mathcal{S}$, as witnessed by $\sigma: \text{lh}(U) \to \text{lh}(S)$ and $\pi_\gamma: M^U_{\sigma(\gamma)} \to M^S_{\sigma(\gamma)}$ for $\gamma < \text{lh}(U)$. If $U$ is not by $\Sigma$, then let $\lambda < \text{lh}(U)$ be least such that $b = [0, \lambda]_U$ is not by $\Sigma$. Since $M^U_{\lambda}$ is iterable by the pullback of $\Sigma$, and since $\mathcal{N}$ is suitable, we must have that $T = U|\lambda$ is maximal. Letting $t = \pi_\lambda$ and $W = M^S_{\sigma(\lambda)}$, and $i = i^S_{0,\sigma(\lambda)}$, we have $i = t \circ i^T_b$. By rebranchcon, this implies $b = \Sigma(T)$, a contradiction. □

**Proof of Theorem 6.7.1.** Fix $N^*$ and $\Sigma$ which satisfy the conclusion of 6.6.11 with respect to $A$. The desired $\mathcal{N}$ will be a $\Sigma$-iterate of $N^*$. Note that trivially, every $\Sigma$-iterate of $N_0$ satisfies (a)-(d).

Suppose toward contradiction that no $\Sigma$-iterate of $N^*$ satisfies (e). Then for any such iterate $N_\alpha$, we can find a $\Sigma$-iterates $N_{\alpha+1}$ of $N_\alpha$ and $W$ of $N_{\alpha+1}$, a maximal tree $T$ on $N_{\alpha+1}$, a cofinal, non-dropping branch $b$ of $T$ such that $b \neq \Sigma(T)$, and a $t: M^T_b \to W$ such that $i = t \circ i^T_b$. Notice that since $M^T_b$ embeds into $W$, it is iterable, and thus since $T$ is maximal, $M^T_b = M(T)^+ = M_c(T)$, where $c = \Sigma(T)$. We can compare $W$ with $M^T_c$, using $\Sigma$ to iterate each of them, and arriving at a common $\Sigma$-iterate $N_{\alpha+2}$. This gives maps $k, l$ such that $k \circ i_b$ and $l \circ i_c$ map $N_{\alpha+1}$ to $N_{\alpha+2}$. (Here $k$ is $t$ followed by the coiteration map on $W$.) We get $k \circ i_b = l \circ i_c$ from the Dodd-Jensen property of $\Sigma$. 


By our hypothesis, $\mathcal{N}_{\alpha+1}$ and $\Sigma$ satisfy (a)-(d) but not (e), so we can generate $\mathcal{N}_{\alpha+3}$ and $\mathcal{N}_{\alpha+4}$, etc. Taking direct limits at limit ordinals, we generate this way what we shall call a bad sequence of length $\omega_1$. It is important that in the formal definition of this concept we record only properties which can be seen in $M_0[g]$, where $g$ is a generic object adding the sequence. So we can refer to the models, embeddings, and suitability, but we cannot mention $\Sigma$ or $A$. This leads to

**Definition 6.7.3** A bad sequence is a sequence $\langle (\mathcal{N}_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) | \alpha < \eta \rangle$, where $\eta \leq \omega_1$, such that for all $\alpha$,

(i) $\mathcal{N}_\alpha$ is suitable, and $j_\alpha: \mathcal{N}_\alpha \to \mathcal{N}_{\alpha+1}$ if $\alpha + 1 < \eta$,

(ii) if $\alpha$ is odd, then $T_\alpha$ is a countable, normal tree on $\mathcal{N}_\alpha$, with distinct cofinal non-dropping branches $b_\alpha, c_\alpha$ such that $M_{T_\alpha}^{T_\alpha} = M_{T_\alpha}^{T_\alpha}$ and and $i_{b_\alpha}(\delta^{\mathcal{N}_\alpha}) = i_{c_\alpha}(\delta^{\mathcal{N}_\alpha}) = \delta T_\alpha$,

(iii) if $\alpha$ is odd, then $k_\alpha: M_{T_\alpha}^{T_\alpha} \to \mathcal{N}_{\alpha+1}$ and $l_\alpha: M_{T_\alpha}^{T_\alpha} \to \mathcal{N}_{\alpha+1}$, and $j_\alpha = k_\alpha \circ i_{b_\alpha} = l_\alpha \circ i_{c_\alpha}$, and

(iv) if $\alpha$ is a limit ordinal, then $\mathcal{N}_\alpha$ is the direct limit of the $\mathcal{N}_\beta$ for $\beta < \alpha$,

under the maps $j_\gamma \xi$, generated by the $j_\beta$, for $\beta < \alpha$

(v) if $\alpha$ is even (including $\alpha$ a limit), then $(T_\alpha, b_\alpha, c_\alpha, k_\alpha, l_\alpha) = (0, 0, 0, 0, 0)$.

Our preamble to the definition essentially shows how to construct a bad sequence of length $\omega_1$ such that each $\mathcal{N}_\alpha$ is a $\Sigma$-iterate of $\mathcal{N}^*$. We shall show that there is such a sequence in a certain generic extension of $M_0$. We then reflect this fact to an initial segment of $M_0$ where we have a sjs on the local $\Sigma^2_1$, then use the condensation properties of the associated term relations to reach a contradiction.

The aspect of badness which is not simply a projective property of the sequence, namely the suitability of the $\mathcal{N}_\alpha$, can be certified by embeddings into $P_\infty$.

**Definition 6.7.4** Let $\pi: \mathcal{N} \to \mathcal{P}$, where $\mathcal{N}$ and $\mathcal{P}$ are premice with the first-order properties in suitability, and with $\mathcal{N}$ countable. A $(\pi, \mathcal{N}, \mathcal{P})$-certified bad sequence is a sequence $\langle (\mathcal{N}_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \pi_\alpha) | \alpha < \eta \rangle$, such that for all $\alpha$,

(i) $\mathcal{N}_0 = \mathcal{N}$, and $\pi_0 = \pi$, and $j_\alpha: \mathcal{N}_\alpha \to \mathcal{N}_{\alpha+1}$ if $\alpha + 1 < \eta$, \vspace{10pt}
Lemma 6.7.5 Let \( \langle \pi, T_\alpha \rangle \) be \( N^\ast \)-certified bad sequence; then

\[
\langle \pi, T_\alpha, b_\alpha, c_\alpha, l_\alpha, j_\alpha, k_\alpha, \alpha \rangle \in \text{Col}(\omega, < \omega_1). \]

\[ p_\alpha \colon N_\alpha \rightarrow P_\infty, \text{ and } \pi_\alpha = \pi_\alpha \circ j_\gamma, \alpha, \text{ for all } \gamma < \alpha, \]

(ii) if \( \alpha \) is odd, then \( T_\alpha \) is a countable, normal tree on \( N_\alpha \), with distinct cofinal non-dropping branches \( b_\alpha, c_\alpha \) such that \( M^{T_\alpha}_{b_\alpha} = M^{T_\alpha}_{c_\alpha} \) and and

\[ i_{b_\alpha}(\delta^{N_\alpha}) = i_{c_\alpha}(\delta^{N_\alpha}) = \delta T_\alpha, \]

(iii) if \( \alpha \) is odd, then \( k_\alpha : M_{b_\alpha} \rightarrow N_{\alpha + 1} \) and \( l_\alpha : M_{c_\alpha} \rightarrow N_{\alpha + 1} \), and \( j_\alpha = k_\alpha \circ i_{b_\alpha} = l_\alpha \circ i_{c_\alpha} \), and

(iv) if \( \alpha \) is a limit ordinal, then \( N_\alpha \) is the direct limit of the \( N_\beta \) for \( \beta < \alpha \), under the maps \( j_\gamma, \xi \) generated by the \( j_\beta \), for \( \beta < \alpha \),

(v) \( \pi_\alpha : N_\alpha \rightarrow P \), and \( \pi_\gamma = \pi_\alpha \circ j_\gamma, \alpha \), for all \( \gamma < \alpha \),

(vi) if \( \alpha \) is even (including \( \alpha \) a limit), then \( (T_\alpha, b_\alpha, c_\alpha, k_\alpha, l_\alpha) = (0, 0, 0, 0, 0) \).

Let \( \pi^* : N^* \rightarrow P_\infty \) be the iteration map by \( \Sigma \).

Lemma 6.7.5 Let \( \langle (N_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \pi_\alpha) \mid \alpha < \eta \rangle \) be a \( (\pi^*, N^*, P_\infty) \)-certified bad sequence; then \( \langle (N_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) \mid \alpha < \eta \rangle \) is a bad sequence.

Proof. This follows easily from weak condensation, 6.6.10.

One can construct a \( (\pi^*, N^*, P_\infty) \)-certified bad sequence of length \( \eta \) from \( (\pi^*, N^*, P_\infty) \) and a counting of \( \eta \), via a simple induction. (Here we use our hypothesis that no quasi-iterate of \( N^* \) satisfies (e).) This gives

Lemma 6.7.6 Let \( g \) be \( \text{Col}(\omega, < \omega_1) \)-generic over \( L[\pi^*, N^*, P_\infty] \); then \( L[\pi^*, N^*, P_\infty][g] \models \) “there is a \( (\pi^*, N^*, P_\infty) \)-certified bad sequence of length \( \omega_1 \).”

Proof. We show how to construct a \( (\pi^*, N^*, P_\infty) \)-certified bad sequence of length \( \omega_1 \), and omit the easy absoluteness argument which shows there is such a sequence in \( L[\pi^*, N^*, P_\infty][g] \). (Notice that \( L[\pi^*, N^*, P_\infty][g] \) is correct about which sequences are \( (\pi^*, N^*, P_\infty) \)-certified bad.)

We set \( N_0 = N^* \) and \( \pi_0 = \pi^* \). If \( \alpha > 0 \) is a limit, then we set \( N_\alpha \) to be the direct limit of the \( N_\xi \) for \( \xi < \alpha \) under the \( j_\xi, \gamma \), and let \( \pi_\alpha = \bigcup_{\xi < \alpha} \pi_\xi \).

Now suppose \( \alpha \) is even, and we have defined \( N_\alpha \) and \( \pi_\alpha : N_\alpha \rightarrow P_\infty \) such that \( \pi_0 = \pi_\alpha \circ j_{0, \alpha} \). Since \( P_\infty \) is the direct limit of all \( \Sigma \)-iterates of \( N_0 \), we can cover the range of \( \pi_\alpha \) with the range of some map in the direct limit system of all \( \Sigma \)-iterates of \( N_0 \). This gives

\[ \pi_\alpha = \phi \circ \psi, \]
where \( \psi: N_\alpha \to S \), and \( \phi: S \to P_\infty \), and \( S \) is a \( \Sigma \)-iterate of \( N_0 \), and \( \phi \) is the natural map to the direct limit of the system of \( \Sigma \)-iterates.

By hypothesis, (e) does not hold with \( N = S \), so let \( R = N_{\alpha+1} \) be part of a counterexample, and let \( \rho: S \to N_{\alpha+1} \) be the \( \Sigma \)-iteration map, and put \( j_\alpha = \rho \circ \psi \), and let \( \pi_{\alpha+1}: N_{\alpha+1} \to P_\infty \) be the map of the system of \( \Sigma \)-iterates. For \( x \in N_\alpha \), we have

\[
\pi_{\alpha+1}(j_\alpha(x)) = \pi_{\alpha+1}(\rho(\psi(x))) = \phi(\psi(x)) = \pi_\alpha(x),
\]

because the \( \Sigma \)-iteration maps are unique by Dodd-Jensen.

The remainder of our counterexample to (e) is a \( \Sigma \)-iteration map \( i: N_{\alpha+1} \to W \), a maximal tree \( T = T_{\alpha+1} \) on \( N_{\alpha+1} \), a cofinal, non-dropping branch \( b = b_{\alpha+1} \) of \( T \) such that \( b \neq c \), where \( c = c_{\alpha+1} = \Sigma(T) \), and a \( t: M_b \to W \) such that \( i = t \circ i_b \). Since \( M_b^T \) embeds into \( W \), it is iterable, and thus since \( T \) is maximal, \( M_b^T = M(T)^+ = M_c(T) \). We can compare \( W \) with \( M_c^T \), using \( \Sigma \) to iterate each of them, and arriving at a common \( \Sigma \)-iterate \( N_{\alpha+2} \) with iteration maps \( l = l_{\alpha+1}: M_c \to N_{\alpha+2} \) and \( u: W \to N_{\alpha+2} \). Set \( k = k_{\alpha+1} = u \circ t \). We have that \( k \circ i_b \) and \( l \circ i_c \) map \( N_{\alpha+1} \) to \( N_{\alpha+2} \). Clearly \( l \circ i_c \) is an iteration map by \( \Sigma \), but since \( k \circ i_b = u \circ i \), it is also an iteration map by \( \Sigma \). Thus \( k \circ i_b = l \circ i_c \).

Finally, let \( j_{\alpha+1} = l \circ i_c \), and let \( \pi_{\alpha+2}: N_{\alpha+2} \to P_\infty \) be the map of the \( \Sigma \)-system. It is easy to see that \( \pi_{\alpha+1} = \pi_{\alpha+2} \circ j_{\alpha+1} \).

Although we have used \( \Sigma \) to show the desired extension of a given \( (\pi^*, N^*, P_\infty) \)-certified bad sequence exists, the properties of the extension we wish to have can be verified by a tree obtained from a counting of the given sequence and \( (\pi^*, N^*, P_\infty) \). Thus there is a \( (\pi^*, N^*, P_\infty) \)-certified bad sequence in \( L[\pi^*, N^*, P_\infty][g] \).

Our next goal is to show that if \( g \) is \( \text{Col}(\omega, < \omega_1) \)-generic over \( V \), then \( M_0[g] \models \text{"there is a bad sequence of length } \omega_1\text{"} \). This then gives a first order property of \( M_0 \) which we can reflect to some Wadge level where we have a self-justifying system.

Unfortunately, \( \pi^* \notin M_0 \), so it is not at all clear one can obtain a bad sequence in \( M_0[g] \). This is where we need \( \text{HI}(c) \), which we now spell out more carefully:

\( \text{HI}(c): \) for any countable set \( X \) of ordinals, and any OD of \( M_0 \) set of ordinals \( A \), there is a transitive set \( R \) such that \( X, A \in R \), and \( R \models \text{ZFC}+ \omega_1^Y \text{ is a } \)
measurable cardinal."

It is not actually necessary that the normal measure in $R$ be induced by $j$, although if we obtain $H_1$ via the symmetric collapse up to a superstrong, that will be the case.

Lemma 6.7.7 The following is true in $M_0$: There is a countable structure $(R_0, \in, \mu, \rho, N, Q, t)$ such that

(i) $R_0$ is a transitive model of $\text{ZFC}^- + \"\mu \text{ is a normal measure on a measurable cardinal}\"$,

(ii) if $g$ is any generic over $R_0$ for $\text{Col}(\omega, < \text{crit}(\mu))$, then $t^g$ is a $(\rho, N, Q)$-certified bad sequence, and

(iii) $R_0$ is linearly iterable by $\mu$, and if $i: R_0 \rightarrow S$ is an iteration map from a countable length iteration, and $g$ is $S$-generic over $\text{Col}(\omega, < i(\text{crit}(\mu)))$, then the projection of $i(t)^g$ (dropping out the last coordinate) is truly a bad sequence.

Proof. Let $N = N^*$. Let $R$ witness $H_1(c)$, with respect to $A$ coding $\mathcal{P}_\infty$ and $X$ coding $\pi^*, N^*$. Let $\mu^*$ be a normal measure of $R$ on $\omega_1^V$, and let $t^*$ be the term in $R$ for a $(\pi^*, N^*, \mathcal{P}_\infty)$-certified bad sequence in $R[g]$, for any $g$ is on $\text{Col}(\omega, < \omega_1^V)$. In $R$, let $Y \prec H^R_{\eta}$ with $Y$ countable, where $H^R_{\eta} = \text{ZFC}^-$, and $\psi: R_0 \cong Y$ be the collapse map, suppose $\psi((\mu, \rho, N^*, Q, t)) = (\mu^*, \pi^*, N^*, \mathcal{P}_\infty, t^*)$. We claim that $(R_0, \in, \mu, N^*, Q, t)$ has properties (i)-(iii) in $M_0$.

Parts (i) and (ii) are obvious. For (iii), let $i: R_0 \rightarrow S$ come from iterating by $\mu$ and its images countably many times. Let $\phi: S \rightarrow R$ be some realization map, so that $\psi = \phi \circ i$, and let $g$ be $S$-generic over $\text{Col}(\omega, < i(\text{crit}(\mu)))$. By the elementarity of $\phi$, we have that $i(t)^g$ is a $(i(\rho), N^*, i(Q))$-certified bad sequence. But letting $i(t)^g = ((N_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \phi \circ \pi_\alpha) | \alpha < \eta)$, we can simply compose our realizations into $Q$ with $\phi$ to get $((N_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha, \phi \circ \pi_\alpha) | \alpha < \eta)$. We claim this is a $(\pi^*, N^*, \mathcal{P}_\infty)$-certified bad sequence. Since $i$ is elementary, this is pretty clear, except perhaps for the requirement that $\phi \circ \pi_0 = \pi^*$. There we have

$$\phi(\pi_0(x)) = \phi(i(\rho)(x))$$

$$= \phi(i(\rho(x)))$$

$$= \psi(\rho(x))$$

$$= \pi^*(x),$$
for $x \in \mathcal{N}^*$. The first line comes from $i(\rho) = \pi_0$, which is one of the properties of certified badness of $i(t)^\varphi$ in $S[\varphi]$. The second comes from $i(x) = x$, the third from $\psi = \phi \circ i$, and the last from $\psi(x) = x$.

But the projection of any $(\pi^*, \mathcal{N}^*, \mathcal{P}_\infty)$-certified bad sequence is bad from the point of view of $M_0$. This proves (iii) holds in $M_0$. □

We have just shown that $M_0 \models \varphi_0$, where $\varphi_0$ is the sentence:

$\varphi_0$: There is a suitable $\mathcal{N}$ and a countable structure $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ such that (i)-(iii) of 6.7.7 hold.

Now let $\beta$ be least such that for some $\alpha$,

$$L_\beta(P_\alpha(\mathbb{R})) \models ZFC - \varphi_0.$$ 

Let $W = L_\beta(P_\alpha(\mathbb{R}))$. Using the minimality of $W$, we have a sjs $\langle A_i \mid i < \omega \rangle$ such that the universal $(\Sigma^2_1)^W$ set is $A_0$, and each $A_i$ is OD$_W$. (This is a result of Woodin; we gave a similar argument in the proof of 6.5.15.) We have also that $W, A$ are $\Delta^2_2$ in $M_0$.

Let $\mathcal{N}$ and $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ witness that $W \models \varphi_0$. Working in $M_0$, let $T$ be a tree on some $\omega \times \lambda$ such that $p[T]$ is a universal $\Sigma^2_1$ set. Let $r_0$ be a real coding $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$, and such that $W$ and $A$ are $(\Delta^2_1(r_0))^{M_0}$, and there are Skolem functions for the structure $(HC, \in, W, A)$ in $(\Delta^2_1(r_0))^{M_0}$. The existence of such Skolem functions implies that for any $G \in V$ which is generic over $L[T, r_0]$ for a poset in $L_{\omega_1^V}[T, r_0]$,

$$\langle HC^{L[T, r_0, G]}, \in, W \cap L[T, x, G], A \cap L[T, x, G] \rangle \prec (HC, \in, W, A).$$

Now let $g \in V$ be $L[T, r_0]$-generic for $Col(\omega, < \omega_1^{L[T, r_0]})$. In $L[T, r_0, g]$ we can form $i: R_0 \to S$ by iterating $\omega_1^{L[T, r_0]}$ times. Let $s$ be $i(t)^\varphi$, projected onto its first 6 coordinates; say

$$s = \langle (\mathcal{N}_\alpha, T_\alpha, b_\alpha, c_\alpha, j_\alpha, k_\alpha, l_\alpha) \mid \alpha < \eta \rangle,$$

where $\eta = \omega_1^{L[T, r_0, g]}$. Since $(R_0, \in, \mu, \rho, \mathcal{N}, \mathcal{Q}, t)$ has properties (i)-(iii) in $W$, we have that $W \models s$ is bad. Let $\mathcal{M}$ be the direct limit of the $\mathcal{N}_\alpha$ for $\alpha < \eta$, under the $j_{\alpha, \beta}$. Let $j_{\alpha, \infty}: \mathcal{N}_\alpha \to \mathcal{M}$ be the direct limit map.

Now let $h \in V$ be $L[T, r_0, g]$-generic for $\Pi_{\alpha < \eta} \text{Col}(\omega, \eta)$, where $\eta_1 = \omega_2^{L[T, r_0]}$. Letting $S_1$ be the result of iterating $R_0 \eta_1$ times, we get $i_1: S \to S_1$, and this lifts to $i_1: S[\varphi] \to S_1[\varphi][h]$. It follows that $i_1(i(t))^\varphi$ projects to
a sequence \( u \) of length \( \eta_1 \) which extends \( s \), and is also bad in \( W \). But this means that \( \mathcal{M} = \mathcal{N}_{\eta_1}^a \), from which we conclude

\[ W \models \mathcal{M} \text{ is suitable}. \]

We have that \( \langle \tau_{A_i, \delta M}^M \mid i < \omega \rangle \in L[T, r_0, g] \), because \( \langle A_i \cap L[T, r_0, g] \mid i < \omega \rangle \in L[T, r_0, g] \). Working in \( L[T, r_0, g] \), we then get an odd ordinal \( \alpha < \eta \) such that

\[ \forall i < \omega (\tau_{A_i, \delta M}^M \in \mathrm{ran}(j_{\alpha, \infty})), \]

from which it follows by the condensation properties of the capturing terms for a sjs that

\[ j_{\alpha}(\tau_{A_i, \delta N_\alpha}^N) = \tau_{A_i, \delta N_{\alpha+1}}^N, \]

But \( j_{\alpha} = k_{\alpha} \circ i_{b_{\alpha}} = l_{\alpha} \circ i_{c_{\alpha}} \), so by condensation again, for all \( i \)

\[ i_{b_{\alpha}}(\tau_{A_i, \delta N_\alpha}^N) = i_{b_{\alpha}}(\tau_{A_i, \delta N_{\alpha}}^N) = \tau_{A_i, \delta R}, \]

where \( R = M_{b_{\alpha}} = M_{c_{\alpha}} \). But \( \sup(\{\gamma_{A_i, \delta R}^R \mid i < \omega\}) = \delta R = \delta(\mathcal{T}_\alpha) \) as usual, and this implies \( b_{\alpha} = c_{\alpha} \). This is a contradiction, completing the proof of 6.7.1. \( \square \)

6.8 Conclusion

We can now complete the proof of Theorem 6.1.1.

**Lemma 6.8.1** For any set of reals \( A \) in \( M_0 \), there is a scale on \( A \) all of whose associated prewellorders are in \( M_0 \).

**Proof.** Let \( x \) be a real such that \( A \) is \( \mathrm{OD}(x)^{M_0} \). Relativising 6.7.1 to \( x \), we get a countable, Wadge cofinal collection \( A \) of \( \mathrm{OD}(x)^{M_0} \) sets of reals containing \( A \), and an \( x \)-mouse \( N \) with a fullness-preserving, \( A \)-guided strategy \( \Sigma \) with properties (a)-(e) of 6.7.1. In particular, \( \Sigma \) condenses well and has the Dodd-Jensen property. One can now get a scale on \( A \) by a straightforward adaptation of the construction of [18, §2]. \( \square \)

**Corollary 6.8.2** For set of reals \( A \) in \( M_0 \), there is an sjs \( \vec{B} \) such that \( A = B_0 \), and each \( B_i \in M_0 \).

Now let \( \vec{B} \) be an sjs with each \( B_i \) in \( M_0 \), and with \( B_0 \) being the universal \( (\Sigma_1^2)^{M_0} \) set. Thus \( \vec{B} \notin M_0 \).
Lemma 6.8.3 $L(\vec{B}, \mathbb{R}) \models AD$.

Proof. This is a core model induction like the one for $L(\mathbb{R})$. Things to note are:

(i) The pattern-of-scales results generalize from $L(\mathbb{R})$ to $L(\vec{B}, \mathbb{R})$. For this, it is important that $\vec{B}$ is a sjs. The Friedman games involved in our closed game representations will involve player I proving facts of the form “$y \in B_i$”, and for this, he should use the tree of the scale on $B_i$ coded into $\vec{B}$.

(ii) $W_\alpha^*$ has the same statement as before, with $J_\alpha(\vec{B}, \mathbb{R})$ replacing $J_\alpha(\mathbb{R})$.

(iii) in $W_\alpha$, the lightface capturing mice are now $\vec{B}$-mice. These are obtained by adding extenders and closing under the operation $\mathcal{M} \mapsto \mathcal{M} \oplus \vec{B}$. Here $\mathcal{M} \oplus \vec{B}$ is $(\mathcal{M}^+, T)$, where the “$+$” operation is as above, with respect to the pointclass $(\Sigma^2_1)^{M_0}$, and $T(i, \tau)$ holds iff $\tau = \tau_{\vec{B}, i, \nu}^\mathcal{M}$, for $\nu$ the $i$-th cardinal of $\mathcal{M}^+$ above $\text{o}(\mathcal{M})$.

(iv) Let $S$ code up the trees of the scales on the $B_i$’s which are given by $\vec{B}$. Then $j(S) \in V$. [Exercise, using that each $B_i$ is OD from a real in $M_0$.] This enables us to show that $j(K) \in V$ at the successor steps in our core model induction.

We shall give no further detail here. \qed

Since $\vec{B} \not\in M_0$, we get $L(\vec{B}, \mathbb{R}) \models \theta_0 < \theta$. This completes our proof of Ketchersid’s theorem.
Bibliography


