The Comparison Lemma

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Abstract

The standard comparison lemma of inner model theory is deficient, in that it does not in general produce a comparison of all the relevant inputs. How two mice compare can depend upon which iteration strategies are used to compare them. We shall outline here a method for comparing iteration strategies that removes this defect.

0 Introduction

Inner model theory begins with Gödel's 1937 work on L, but its true scope only came into view in 1966-68, with the pioneering work of Jack Silver and Kenneth Kunen on $L[U]$, the canonical minimal inner model with one measurable cardinal. Underlying the Kunen-Silver theory of $L[U]$ is a method for comparing structures that resemble some level of $L[U]$, and a *comparison lemma* stating that this method succeeds.

Kunen's version of the comparison method has proved to be the right tool in more general contexts. In the years since his paper [17], inner model theory has grown more or less continuously in several directions, and Kunen's method of comparing via iterated ultrapowers has been central to this growth. It now applies to models satisfying large cardinal hypotheses much stronger than the existence of one measurable cardinal. At the same time, we have come to realize that the large cardinal hypotheses themselves do not always provide the best way to identify targets and measure progress. The logical complexity of the predicate "resembles a level of M", which is coupled to the extent to which the levels of M can be correct about truth in the full universe, may be a better guide. Complexity and correctness are naturally measured using concepts from descriptive set theory.

It is customary to call the structures that resemble some level of a canonical inner model *mice*.[1] Certain first order features of the levels are recorded in the notion of *premouse*, and the remaining second order resemblance condition is called *iterability*. That is, a mouse is an iterable premouse. In general, M is iterable iff there is an *iteration strategy* Σ for M. In the most important case, M is countable, and Σ is coded by a Universally Baire (uB) set of reals. Descriptive set theoretic complexity attaches to the pairs (M, Σ), rather than to M itself; it

[1]The term is due to Ronald Jensen.

amounts to the Wadge order on the sets of reals that code such pairs. Similarly, we shall see that the right general statement of the comparison lemma involves the comparison of *mouse pairs* like (M, Σ), rather than simply the comparison of mice like M. That is, we must compare not just the mice, but also the iteration strategies witnessing that they are mice.

Our goal in this paper is to state and motivate a comparison lemma for mouse pairs. The lemma[2] is stated as Theorem 6.2 below. It is proved in [35]. In §1 we review the work of Kunen and Mitchell on comparison by linear iteration and discuss its limitations. In §2 we review some basic definitions and results from [21] and [38] related to premice and iterability, and introduce a few modifications to them that seem necessary for strategy comparison. In §3 we prove the general comparison lemma of [21] on comparison by means of iteration trees, and discuss some of its immediate corollaries. We shall see at this point that the heart of the comparison problem concerns the comparison of countable mice in models of the Axiom of Determinacy (AD). We shall be working in models of AD for much of the rest of the paper. In §5 we discuss various regularity properties of iteration strategies. This leads us to a formal definition of pure extender pairs, as pairs (M, Σ) such that M is a premouse, and Σ is an iteration strategy for M having certain of these properties. In §6 we state the general comparison lemma for pure extender pairs, and in §8 we outline its proof. §7 concerns the construction of such pairs in models of AD. In §9 we state some theorems on the structure of HOD in models of AD that can be proved using the strategy comparison process. §10 discusses some directions for future work.

The reader who has a passing familiarity with inner model theory at the level of [21] or [38] should be able to follow most of the paper. [35], [33], and [36] are full, technical descriptions of the proof of Theorem 6.2, and of some of its applications.

1 Comparison by linear iteration

There is a comparison lemma underlying Gödel's work too. The premice are the models M that satisfy $\mathsf{V} = \mathsf{L}$ together with some rudimentary fragment of ZFC. The iterability condition is wellfoundedness. Thus the mice are simply the premice that are isomorphic to transitive structures. Henceforth we shall always assume that wellfounded, extensional structures have been transitivized. With this convention, all of Gödel's mice are L_α's, and his comparison lemma just states that for any two mice M and N, one is an initial segment of the L-hierarchy of the other. This leads to

Theorem 1.1. *(Gödel 1937, [10]) Assume* ZF*; then*

(1) $L \models \mathsf{ZFC} + \mathsf{V} = \mathsf{L}$,

(2) $L \models \mathsf{GCH}$,

(3) $L \models$ "$\mathbb{R}$ *has a* Δ^1_2 *wellorder", and*

[2]One might call it a theorem, but somehow "lemma" is more common. The Dodd-Jensen Lemma is a closely related result with a similar non-theorem status.

(4) every real in L is Δ^1_2 in a countable ordinal.

Item (4) places an upper bound on the complexity of the reals in L; in particular, they are all ordinal definable in an absolute way. Gödel's upper bound is best possible:

Theorem 1.2. *Assume* ZF*; then*

(1) (Shoenfield [29])L is Σ^1_2 correct,

(2) (Solovay [14]) $\mathbb{R} \cap L = \{x \mid x$ is Δ^1_2 is a countable ordinal$\}$.

In Kunen's work, the premice are models $(M, \in U)$ such that M that satisfy "U is a normal ultrafilter witnessing that $\mathrm{crit}(U)$ is measurable and $V = L[U]$", together with some fragment of ZFC.[3] We say that $(M, \in, U)$ is an intial segment of $(N, \in, W)$, and write $(M, \in, U) \trianglelefteq (N, \in, W)$, iff $o(M) \leq o(N)$, $W \cap M = U$, and $M = L_{o(M)}[W]$.[4]

Given a premouse $\mathcal{M}_0 = (M_0, \in, U_0)$, we let

$$\begin{aligned} \mathcal{M}_{\alpha+1} &= \mathrm{Ult}(\mathcal{M}_\alpha, U_\alpha), \\ i_{\alpha,\alpha+1} &= \text{ canonical embedding from } \mathcal{M}_\alpha \text{ to } \mathcal{M}_{\alpha+1}, \\ U_{\alpha+1} &= i_{\alpha,\alpha+1}(U_\alpha), \text{ and} \\ \mathcal{M}_\lambda &= \lim_{\alpha<\lambda} \mathcal{M}_\alpha \end{aligned}$$

for λ a limit ordinal, where the direct limit is taken with respect to the iteration maps $i_{\alpha,\beta}$. $\mathcal{M}_0$ is *iterable* iff all $\mathcal{M}_\alpha$ are wellfounded. Iterability is Π_1 in the language of set theory. As applied to countable $\mathcal{M}_0$, it is Π^1_2. Kunen's comparison lemma is:

Theorem 1.3. *(Kunen Comparison Lemma, [17, Theorem 5.8]) Let $(M, \in, U)$ be an iterable premouse and $\lambda > \mathrm{crit}(U)^{+,M}$ be a regular cardinal. Let $(N, \in, W)$ be the λ-th iterate of $(M, \in, U)$; then $(N, \in, W)$ is an initial segment of $(L[F], \in, F)$, where F is the club filter on λ.*

Comparison leads to a complexity order on mice.

Definition 1.4. Let $\mathcal{M}$ and $\mathcal{N}$ be mice; then $\mathcal{M} \leq^* \mathcal{N}$ iff there is an elementary embedding from $\mathcal{M}$ into an initial segment of an iterate of $\mathcal{N}$. We call $\leq^*$ the *mouse order*.

The mouse order does not appear explicitly in Kunen's work, but it is not too hard to use his comparison lemma to show

Corollary 1.5. *On mice of the form $(M, \in, U)$, the mouse order is a prewellorder. Moreover $\mathcal{M} <^* \mathcal{N}$ iff there is an elementary embedding from $\mathcal{M}$ into a proper initial segment of an iterate of $\mathcal{N}$.*

[3] [17] requires that $M \models \mathsf{ZFC}$, but does not require that $U \in M$. These differences are not relevant to the discussion here.

[4] Here $o(M) = \mathrm{OR} \cap M$.

The fact that there is only one way to iterate a Kunen mouse figures heavily in the proof of this corollary.

Iteration maps do not move reals, so we get at once that for $x, y \in \mathbb{R}$,

$$x <_{L[U]} y \text{ iff } \exists \mathcal{M}(\mathcal{M} \text{ is a mouse and } \mathcal{M} \models x <_{L[U]} y),$$

where $<_{L[U]}$ is the order of construction. Since iterability is Π^1_2, $\mathbb{R} \cap L[U]$ is a Σ^1_3 set. It is independent of U, as is $<_{L[U]}$, which is a Σ^1_3 wellorder of it.

This led to a better proof of:

Theorem 1.6. *(Solovay, Silver [30], [31]) Let U be a normal ultrafilter witnessing that κ is measurable; then*

(1) $L[U] \models \mathsf{ZFC} +$ "$U \cap L[U]$ is a normal measure on κ + $V = L[U \cap L[U]]$",

(2) $L[U] \models \mathsf{GCH}$,

(3) $L[U] \models$ "$\mathbb{R}$ has a Δ^1_3 wellorder", and

(4) every real in $L[U]$ is Δ^1_3 in a countable ordinal.

The idea of studying $L[U]$ and part (1) of 1.6 are due to Solovay. The rest is due to Silver. The proof of the theorem shows that $\mathbb{R} \cap L[U]$ is independent of U, so at least in this respect, $L[U]$ is "canonical". The iterability condition and comparison lemma behind (2)-(4) in Silver's work involve stretching premice whose measures are generated by indiscernibles. This is good enough if we are working with mice at the level of one measurable cardinal and slightly beyond, but it soon becomes impossibly complicated.[5] Kunen found the right way forward, by bringing in Gaifman's method of iterated ultrapowers.

Kunen's methods lead to a full canonicity theorem for $L[U]$:

Theorem 1.7. *(Kunen [17]) Let $\mathcal{M} = (M, \in, U)$ and $\mathcal{N} = (N, \in, W)$ be mice such that $o(M) = o(N) = OR$; then either $\mathcal{M}$ is an iterate of $\mathcal{N}$, or $\mathcal{M} = \mathcal{N}$, or $\mathcal{N}$ is an iterate of $\mathcal{M}$. In particular, if $\mathrm{crit}(U) = \mathrm{crit}(W)$, then $U = W$.*

Thus U is definable over $L[U]$ as the unique non-principal normal ultrafilter. It follows that $L[U] \models V = \mathrm{HOD}$.

With $L[U]$ established as independent of U and unique up to iteration, it is natural to look for ways to construct it beyond the naive one that involves starting with a normal measure over V. It was shown very quickly that this can be done. Solovay and Kunen showed that $L[U]$ exists if there are ideals with certain saturation properties ([34] and [17, 11.12,11.13]). Martin, building on work of Solovay, showed that a small fragment of Δ^1_2 determinacy implies

[5] In Silver's work, the role of $\mathcal{M}_0$ as a premouse is played by the type of the iteration points $\langle \mathrm{crit}(U_n) \mid n < \omega \rangle$ inside the ω-th iterate $\mathcal{M}_\omega$. That is, premice are not just approximations to levels of the model, they include information about one way of iterating it, and the iterability condition states that continuing this particular iteration leads to wellfounded models. This only works if there are not many ways to iterate the model.

that $L[U]$ exists. By 1976, Jensen had proved the Covering Lemma, and Dodd and Jensen had developed *core model theory* at the level of one measurable cardinal. (Cf. [3], [5],[6],[7].) This is a powerful and systematic tool for constructing $L[U]$ under a wide variety of hypotheses.

It is also natural to look at the first order theory of $L[U]$. It satisfies GCH, but can we go beyond that? Indeed we can; the methods for unlocking the first order theory of L apply also to $L[U]$. In unpublished work from the mid-1970s, Solovay extended Jensen's fine structural analysis of L to $L[U]$, making the most powerful of these methods applicable.

Most importantly for our story, the work of Kunen and Silver suggests that there are canonical inner models beyond $L[U]$ for large cardinal hypotheses stronger than one measurable cardinal. Kunen and Silver themselves extended their theory to models having many measurable cardinals. (See [17] and [31].) Moreover, even below $L[U]$ there are many mice, and in this context, mice like L and $L[U]$ that contain all the ordinals have no special status. Thus even at this early stage, an optimist might have envisioned a hierarchy of canonical inner models (mice), understood in fine structural detail, wellordered by complexity (the mouse order) the way the large cardinal hypotheses are ordered by consistency strength, and going as far as the large cardinals do. This hierarchy would be the model-theoretic counterpart of the proof-theoretic hierarchy of consistency strengths.

In 1974 and 1978, William Mitchell took an important step in this direction. His papers [19] and [20] isolate what seem to be central features of the first order form of canonical inner models, all the way up to inner models with superstrong cardinals. Mitchell's notion of models constructed from *coherent sequences of extenders* has figured in all further work in inner model theory.[6][7] Mitchell himself used it to construct inner models satisfying "there is a κ that is $\kappa + 2$-strong", a large cardinal hypothesis significantly beyond the existence of measurables, and seemingly well on the way to the existence of superstrongs, and even supercompacts.[8]

Theorem 1.8 (Mitchell 1978)**.** *Suppose there is a cardinal κ such that κ is $\kappa + 3$-strong; then there is a model M constructed from a coherent sequence of extenders such that*

(1) $M \models \mathsf{ZFC} + \exists\kappa($ *κ is $\kappa + 2$-strong*$)$*,*

(2) $M \models \mathsf{GCH} +$ *"$\mathbb{R}$ has a Δ^1_3 wellorder", and*

(3) $\mathbb{R} \cap M$ is enumerated by a Δ^1_3 real.

[6]See §2.1 for precise definitions. Extenders are a simplification, due to Jensen, of what Mitchell called *hypermeasures*. Roughly speaking, an extender E over M is a system of M-ultrafilters $\langle E_a \mid a \in [\mathrm{lh}(E)]^{<\omega}\rangle$ coding an elementary embedding $i_E \colon M \to \mathrm{Ult}(M, E)$. The ultrapower $\mathrm{Ult}(M, E)$ is the direct limit of all the $\mathrm{Ult}(M, E_a)$ for $a \in [\mathrm{lh}(E)]^{<\omega}$. $\mathrm{lh}(E)$ is the *length* or *support* of E, and $\mathrm{crit}(E)$ is the critical point of i_E. When we construct from E, we identify it with the predicate $\{(a, x) \mid x \in E_a\}$.

[7]Roughly, coherence means that $\alpha < \beta \Rightarrow \mathrm{lh}(E_\alpha) < \mathrm{lh}(E_\beta)$, and for all α, the initial segment of $i_{E_\alpha}(\vec{E})$ consisting of extenders with length $\leq \mathrm{lh}(E)$ is $\vec{E} \restriction \alpha$. In a phrase, the extenders are listed in order of increasing strength, without leaving gaps.

[8]κ is β-strong iff there is an elementary $j \colon V \to M$ such that $\kappa = \mathrm{crit}(j)$ and $V_\beta \subseteq M$. Note that if $\beta \geq \kappa+2$, then we cannot have $j = i_D$ for some ultrafilter D on κ, because $D \in V_{\kappa+2}$ and $D \notin \mathrm{Ult}(V, D)$.

There is a comparison lemma behind this theorem. In [20], a premouse is a structure $\mathcal{M} = (M, \in, \vec{E})$ such that M satisfies "$\vec{E}$ is a coherent sequence of extenders and $V = L[\vec{E}]$", together with some reasonable fragment of ZFC. We call $\vec{E}$ the *$\mathcal{M}$-sequence.* A *linear iteration* of $\mathcal{M}$ is a sequence $\langle\langle \mathcal{M}_\alpha, E_\alpha\rangle \mid \alpha < \theta\rangle$, equipped with iteration maps $i_{\alpha,\beta} \colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, such that for all $\alpha + 1 < \theta$

$$E_\alpha \text{ is on the } \mathcal{M}_\alpha\text{-sequence},$$

and

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\alpha, E_\alpha).$$

At limit steps $\lambda < \theta$ we take direct limits:

$$\mathcal{M}_\lambda = \lim_{\alpha<\lambda} \mathcal{M}_\alpha,$$

where the limit is under the iteration maps. $\mathcal{M}$ is *linearly iterable* iff all models in all linear iterations of $\mathcal{M}$ are wellfounded. For countable premice, linear iterability is a Π^1_2 property. Mitchell showed it is enough for comparison:

Theorem 1.9. *(Comparison by linear iteration, Mitchell 1978) Let $\mathcal{M}$ and $\mathcal{N}$ be linearly iterable premice, and suppose that neither has an initial segment satisfying "there is a κ such that κ is $\kappa + 3$-strong"; then there are linear iterates $\mathcal{P}$ of $\mathcal{M}$ and $\mathcal{Q}$ of $\mathcal{N}$ such that $\mathcal{P}$ is an initial segment of $\mathcal{Q}$, or vice-versa.*

Mitchell's proof involves comparing $\mathcal{M}$ and $\mathcal{N}$ directly, rather than iterating them into some standard structure fixed in advance, as Kunen did. The idea is to iterate away least extender disagreements, and show, using a reflection argument, that eventually all disagreements have been removed. That linear iteration eventually removes all disagreements relies on our assumption that no initial segment of $\mathcal{M}$ or $\mathcal{N}$ has a cardinal κ that is $\kappa + 3$-strong.

The mouse order on such "$\kappa+3$-small" mice is defined just as in 1.4: $\mathcal{M} \leq^* \mathcal{N}$ iff $\mathcal{M}$ can be elementarily embedded into an initial segment of a linear iterate of $\mathcal{N}$. Theorem 1.9 shows that $\leq^*$ is pre-linear. Mitchell also showed that it is wellfounded.[9] On countable $\kappa + 3$-small mice, the mouse order is Δ^1_3. This leads to the Δ^1_3 wellorder of $\mathbb{R}$ in Theorem 1.8. That Mitchell's $L[\vec{E}]$ satisfies GCH is a good deal more difficult to prove than it was in the case of $L[U]$, but eventually comparison by linear iteration carries the day.

This was a big step forward, but as Theorem 1.8 shows, the models whose theory is fully developed by Mitchell [20] did not go very far descriptive-set-theoretically. None of them is Σ^1_3 correct, and the set of all reals in any such model are enumerated by a single Δ^1_3 real. All of them satisfy "there is a Δ^1_3 wellorder of the reals", and therefore Δ^1_2 determinacy fails to hold in any of them. S. Baldwin and A. Dodd ([1],[2], [4]) extended Mitchell's work to inner models satisfying "$\exists\kappa\forall\beta(\kappa$ is β-strong)", and slightly stronger large cardinal hypotheses. But their comparison process involved only linear iterations, so again the set of reals in any of the Baldwin-Dodd models can be enumerated by a single Δ^1_3 real.

[9] This is a special case of the Dodd-Jensen Lemma, a stronger result discovered independently of Mitchell by Dodd and Jensen. See Theorem 6.5.

The weakness of linear iteration as a comparison method shows up as follows. Once we reach premice satisfying large cardinal hypotheses just past those dealt with by Baldwin and Dodd, it can happen that the least extender disagreement between $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ involves an extender E_α on the $\mathcal{M}_\alpha$ sequence such that $\mathrm{crit}(E_\alpha) < \mathrm{lh}(E_\beta)$ for some $\beta < \alpha$. Applying E_α to $\mathcal{M}_\alpha$ can then bring back to life a disagreement that we removed by using E_β. So comparison by linear iteration can enter a loop.

The right thing to do in this situation is to set

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}\,\mathcal{M}_\beta, E_\alpha),$$

where β is least such that $\mathrm{crit}(E_\alpha) < \mathrm{lh}(E_\beta)$.[10] This leads to nonlinear iterations with a tree structure, and embeddings along each branch from the earlier to the later models on that branch. That is, to *iteration trees*. In 1985-86 Martin and the author developed the theory of iteration trees, and used it to prove a rudimentary comparison lemma for mice that contain all the reals that are Δ^1_3 in a countable ordinal. (See [18].) Iterability with respect to iteration trees can be very complicated in descriptive set theoretic terms, so it is not limited as a guarantor of comparability the way that linear iterability is.

But there remains another limitation. We have not given Mitchell's definition of "premouse" in detail, nor those of the other authors, because the hierarchies they use make fine structural analysis something between difficult and impossible. The basic problem is the demand that all extenders E on the coherent sequence must be total over $L[\vec{E}]$. This means the simple condensation properties of the L hierarchy do not go over to $L[\vec{E}]$. For example, it is not true that every real in $L[\vec{E}]$ is constructed at a stage that is countable in $L[\vec{E}]$, or that the Σ_ω-hull of one of its levels collapses to a countable level. The standard $L_\alpha[U]$ hierarchy has these features too, and one can work around them. Solovay did produce a fine structural analysis of the $L[U]$ hierarchy, and Dodd and Jensen used it in their work on core model theory. Dodd produced a manuscript of many hundreds of pages on the fine structure of his $L[\vec{E}]$ models. But this old fine structure is just too complicated to go much further.

In order to form iteration trees properly, we do need a fine structural analysis, because in the situation described above, it could be that $\mathrm{crit}(E_\alpha) < \mathrm{lh}(E_\beta)$ but E_α does not measure all sets in $\mathcal{M}_\beta$. In that case, the right thing to do in a comparison process is to apply E_α to the first initial segment $\mathcal{P}$ of $\mathcal{M}_\beta$ that defines a subset of $\mathrm{crit}(E_\alpha)$ not measured by E_α, taking an ultrapower of $\mathcal{P}$ that preserves the definition of some such set. The fine structure of $\mathcal{P}$ enters crucially at this point. Thus at the level of iteration trees, even the comparison of mice satisfying ZFC can involve their fine structure.[11]

In 1988, Baldwin and Mitchell found the right solution, namely, we should allow partial extenders on the coherent sequence. When we add a new extender E to the level $L_\alpha[\vec{E}]$ that we have just reached, the measures in E are total over $L_\alpha[\vec{E}]$. But we may later construct more

[10] See footnote 24. The expert reader will notice that in more modern indexing schemes, $\mathrm{lh}(E_\beta)$ should be replaced by $\nu(E_\beta)$ or $\lambda(E_\beta)$.

[11] [18] uses iteration trees to prove a fine-structure-free, partial comparison lemma at the level of one Woodin cardinal. It is of limited value in determining the properties of the mice being compared, however. For example, it is open whether those mice satisfy the GCH.

subsets of $\text{crit}(E)$, and these will not be measured by E. [21] works out a general fine structure theory for hierarchies of this kind. The key fact is that all levels in the hierarchy satisfy a local version of the GCH known as *soundness*. In this respect they resemble the levels of L, and not the levels in the standard stratifications of $L[U]$ or $L[\vec{E}]$. This makes their fine structure much more transparent.

2 Fine structure and iteration trees

Let us give some formal definitions. Most of the material here can be found in [21], [38], and [48]. We do need to vary and extend the standard definitions a little, in order to lay a foundation for Theorem 6.2.

2.1 Extenders and ultrapowers

Our notation for extenders is standard.

Definition 2.1. Let M be transitive and rudimentarily closed; then $E = \langle E_a \mid a \in [\theta]^{<\omega}\rangle$ is a (κ, θ)-extender over M with spaces $\langle \mu_a \mid a \in [\theta]^{<\omega}\rangle$ if and only if

(1) Each E_a is an (M, κ)-complete ultrafilter over $P([\mu_a]^{|a|}) \cap M$, with μ_a being the least μ such that $[\mu]^{|a|} \in E_a$.

(2) (Compatibility) For $a \subseteq b$ and $X \in M$, $X \in E_a \iff X^{ab} \in E_b$.

(3) (Uniformity) $\mu_{\{\kappa\}} = \kappa$.

(4) (Normality) If $f \in M$ and $f(u) < \max(u)$ for E_a a.e. u, then there is a $\beta < \max(a)$ such that for $E_{a\cup\{\beta\}}$ a.e. u, $f^{a,a\cup\{\beta\}}(u) = u^{\{\beta\},a\cup\{\beta\}}$.

The unexplained notation here can be found in [38]. We shall often identify E with the binary relation $(a, X) \in E$ iff $X \in E_a$. We call θ the *length* of E, and write $\theta = \text{lh}(E)$. The *space of* E is

$$\text{sp}(E) = \sup\{\mu_a \mid a \in [\text{lh}(E)]^{<\omega}\}.$$

The *domain of* E is the family of sets it measures, that is, $\text{dom}(E) = \{Y \mid \exists (a, X) \in E(Y = X \vee Y = [\mu_a]^{|a|} - X)\}$. If M is a premouse of some kind, we also write $M|\eta = \text{dom}(E)$, where η is least such that $\forall (a, X) \in E(X \in M|\eta)$. In premice, $\eta = \sup(\{\mu_a^{+,M} \mid a \in [\theta]^{<\omega}\})$.

Given an extender E over M, we form the Σ_0 ultrapower

$$\text{Ult}_0(M, E) = \{[a, f]^M_E \mid a \in [\text{lh}(E)]^{<\omega} \text{ and } f \in M\},$$

as in [38]. Our M will always be rudimentarily closed and satisfy the Axiom of Choice, so we have Los' theorem for Σ_0 formulae, and the canonical embedding

$$i^M_E \colon M \to \text{Ult}_0(M, E)$$

is cofinal and Σ_0 elementary, and hence Σ_1 elementary. By (1) and (3), $\mathrm{crit}(i_E) = \kappa$. We write $\mathrm{crit}(E)$ or κ_E for κ. By normality, $a = [a, \mathrm{id}]^M_E$, so $\mathrm{lh}(E)$ is included in the (always transitivized) wellfounded part of $\mathrm{Ult}_0(M, E)$. More generally,

$$[a, f]^M_E = i^M_E(f)(a).$$

If $X \subseteq \mathrm{lh}(E)$, then $E \restriction X = \{(a, Y) \in E \mid a \subseteq X\}$. $E \restriction X$ has the properties of an extender, except possibly normality, so we can form $\mathrm{Ult}_0(M, E \restriction X)$, and there is a natural factor embedding $\tau\colon \mathrm{Ult}_0(M, E \restriction X) \to \mathrm{Ult}_0(M, E)$ given by

$$\tau([a, f]^M_{E\restriction X}) = [a, f]^M_E.$$

In the case that $X = \nu > \kappa_E$ is an ordinal, $E \restriction \nu$ is an extender, and $\tau \restriction \nu$ is the identity. We say ν is a *generator of* E iff ν is the critical point of τ, that is, $\nu \neq [a, f]^M_E$ whenever $f \in M$ and $a \subseteq \nu$. Let

$$\nu(E) = \sup(\{\nu + 1 \mid \nu \text{ is a generator of } E\}).$$

So $\nu(E) \leq \mathrm{lh}(E)$, and E is equivalent to $E \restriction \nu(E)$, in that the two produce the same ultrapower.

let

$$\lambda(E) = \lambda_E = i^M_E(\kappa_E).$$

Note that although E may be an extender over more than one M, $\mathrm{sp}(E)$, κ_E, $\mathrm{lh}(E)$, $\mathrm{dom}(E)$, $\nu(E)$, and $\lambda(E)$ depend only on E itself. If N is another transitive, rudimentarily closed set, and $P(\mu_a) \cap N = P(\mu_a) \cap M$ for all $a \in [\mathrm{lh}(E)]^{<\omega}$, then E is also an extender over N; moreover i^M_E agrees with i^N_E on $\mathrm{dom}(E)$. However, i^M_E and i^N_E may disagree beyond that. We say E is *short* iff $\nu(E) \leq \lambda(E)$. It is easy to see that E is short if $\mathrm{lh}(E) \leq \sup(i^M_E``(\kappa_E^{+,M}))$. If E is short, then all its interesting measures concentrate on the critical point. When E is short, i^M_E is continuous at $\kappa^{+,M}$, and if M is a premouse, then $\mathrm{dom}(E) = M|\kappa_E^{+,M}$. In this paper, we shall deal almost exclusively with short extenders.

If we start with $j\colon M \to N$ with critical point κ, and an ordinal ν such that $\kappa < \nu \leq o(N)$, then for $a \in [\nu]^{<\omega}$ we let μ_a be the least μ such that $a \subseteq j(\mu)$, and for $X \subseteq [\mu_a]^{|a|}$ in M, we put

$$(a, X) \in E_j \iff a \in j(X).$$

E_j is an extender over M, called the (κ, ν)-*extender derived from* j. We have the diagram

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "i"'] & N \\ & \mathrm{Ult}_0(M, E_j) \arrow[u, "k"'] \end{tikzcd}$$

where $i = i^M_{E_j}$, and

$$k(i(f)(a)) = j(f)(a).$$

$k \upharpoonright \nu$ is the identity. If E is an extender over M, then E is derived from i^M_E.

The *Jensen completion* of a short extender E over some M is the $(\kappa_E, i^M_E(\kappa_E^{+,M}))$ extender derived from i^M_E. E and its Jensen completion E^* are equivalent, in that $\nu(E) = \nu(E^*)$, and $E = E^* \upharpoonright \mathrm{lh}(E)$.

2.2 Pure extender premice

Inner model theory deals with canonical objects, but inner model theorists have presented them in various ways. Clearly we could vary the $L_\alpha[U]$ hierarchy trivially, say by only allowing U to be used as a predicate at limit ordinals, and we would get the same model. On the other hand, if the stages at which U is used to define new sets code some random set, we get a random model, not a canonical one.

When constructing a model of the form $L[\vec{E}]$ for $\vec{E}$ a sequence of extenders, we need a rule that specifies at what stages the information in the next extender can be added. The rule itself should not contribute random information. There are two principal such indexing schemes, ms-indexing (as in [21]) and λ-indexing (as in [12]). The essential equivalence of these two schemes has been carefully demonstrated by Fuchs in [8] and [9]. We shall use λ-indexing here.

The possibility of different indexing schemes may cast doubt on the claim that these models are canonical. The comparison lemma is what answers such doubts in general. One only directly compares premice of some fixed type, it is true. But as we shall see, one corollary of comparison is that all reals in such mice are absolutely ordinal definable in some way. For example, if M is the minimal iterable proper class premouse with one Woodin cardinal, in whatever sense of premouse we have developed, then $\mathbb{R} \cap M$ will be the set of reals that are Δ^1_3 in a countable ordinal. This can be seen on very general, abstract grounds from the prewellordering of inner model operators.[12] More generally, so long as the witnesses to iterability for countable mice are uB sets of reals, as they are anywhere we have a theory now, the reals of M will be an initial segment of the reals that are $(\Sigma^2_1)^{uB}$ in a countable ordinal. This is a reason to believe that any two notions of mice that reach equally far must lead to intertranslatable hierarchies, at least insofar as mice defining new reals are concerned.

It is of course possible for a premouse notion (indexing scheme) to be restrictive. In fact, at present they all are. The restriction in the notion we are about to define is that all extenders on the coherent sequence of M must be short. As a consequence, such M cannot satisfy "there is a κ such that κ is κ^+ supercompact". They can get close to that; for example, they can have subcompact cardinals.

The reader should see [48] for further details on the following definition. A *potential pure extender premouse* is an acceptable J-structure

$$M = \langle J^{\vec{E}}_\alpha, \in, \vec{E}, \gamma, F \rangle$$

with various properties. $o(M) = \mathrm{OR} \cap M = \omega\alpha$. The language $\mathcal{L}_0$ of M has $\in$, predicate symbols $\dot{E}$ and $\dot{F}$, and a constant symbol $\dot{\gamma}$. We call $\mathcal{L}_0$ the *language of (pure extender) premice.*

[12] See [39].

If M is a potential pure extender premouse, then $\dot{E}^M$ is a sequence of extenders, and either $\dot{F}^M$ is empty (i.e. M is *passive*), or $\dot{F}^M$ codes a new extender being added to our model by M. The main requirements are

(1) (λ-indexing) If $F = \dot{F}^M$ is nonempty (i.e., M is *active*), then $M \models \mathrm{crit}(F)^+$ exists, and for $\mu = \mathrm{crit}(F)^{+,M}$, $o(M) = i_F^M(\mu) = \mathrm{lh}(F)$. $\dot{F}^M$ is just the graph of $i_F^M \restriction (M|\mu)$.

(2) (Coherence) $i_F^M(\dot{E}^M) \restriction o(M) + 1 = (\dot{E}^M)^\frown\langle\emptyset\rangle$.

(3) (Initial segment condition, $\mathsf{J\text{-}ISC}$) If G is a whole proper initial segment of F, then the Jensen completion of G must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is the index of its Jensen completion in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$.

(4) (Weak $\mathsf{ms\text{-}ISC}$) Whenever E is an extender on the M-sequence with critical point κ and F is the Jensen completion of $E_{\{\kappa\}}$, then F is on the sequence of $M|\,\mathrm{lh}(E)$.

(5) If N is a proper initial segment of M, then N is a potential premouse.

Here an initial segment $G = F \restriction \eta$ of F is *whole* iff $\eta = \lambda_G$.

Since potential premice are acceptable J-structures, the basic fine structural notions apply to them. We recall some of them in the next section. We then define a *premouse* as a potential premouse all of whose proper initial segments are sound.

Figure 1 illustrates a common situation, one that occurs at successor steps in an iteration tree, for example.

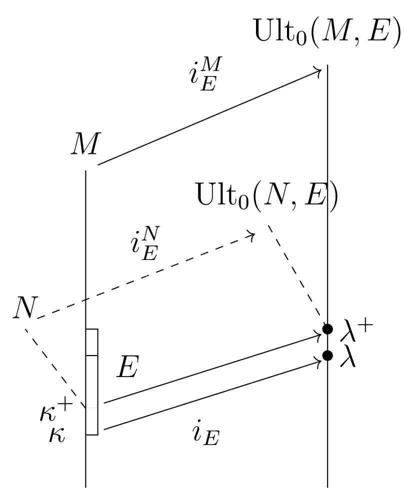

Figure 1: E is on the coherent sequence of M, $\kappa = \mathrm{crit}(E)$, and $\lambda = \lambda(E)$. $P(\kappa)^M = P(\kappa)^N = \mathrm{dom}(E)$, so $\mathrm{Ult}_0(M,E)$ and $\mathrm{Ult}_0(N,E)$ make sense. The ultrapowers agree with M below $\mathrm{lh}(E)$, and with each other below $\mathrm{lh}(E)+1$.

2.3 Projecta and cores

Fine structure theory relies on a careful analysis of the condensation properties of mice; that is, of the extent to which Skolem hulls of a mouse M collapse to initial segments of M. Jensen's theory of projecta, standard parameters, and cores is the foundation for this analysis. The fact that we are allowing partial extenders on our coherent sequence leads to models whose levels are all *sound*; roughly speaking, any level M is the Σ_k hull of its k-th projectum and k-th standard parameter.[13] This leads to a fine structure much closer to that of L; for example, every subset of κ in M is constructed at a level of M-cardinality κ, so the GCH is immediate.

The comparison of iteration strategies is much easier to describe if we depart slightly from the usual notions of cores and soundness. We call this mildly new fine structure the *projectum-free spaces* fine structure, because one of its main features is that no projectum of a premouse M is the critical point of an extender on the M-sequence. We call the corresponding premice *pfs premice*.

This change to the standard fine structure is something between a necessity and a great convenience when it comes to the construction of iteration strategies that are nice enough to be compared with one another. Nevertheless, the new premice are intertranslatable with the standard ones, and the main new ideas in the strategy comparison proof lie elsewhere, so we shall go quickly. The reader can find a full account of the projectum-free spaces fine structure in [35].

Some terminology from [11]: for any acceptable J-structure (N, B)[14]

$$\begin{aligned}
\rho_1(N,B) = &\text{ least } \alpha \text{ s.t. } \quad \exists A \subset \alpha(A \in \boldsymbol{\Sigma}_1^{(N,B)} \wedge A \notin N),\\
p_1(N,B) = &\text{ first standard parameter of } (N,B)\\
= &\text{ lex-least descending sequence of ordinals } r \text{ such that}\\
&\quad \exists A \subseteq \rho_1(N,B)(A \notin N \wedge A \text{ is } \Sigma_1^{(N,B)} \text{ definable from } r.).
\end{aligned}$$

We allow $\rho_1(N,B) = o(N)$ and $p_1(N,B) = \emptyset$.

Premice are acceptable J-structures, and the key to their fine structure is that if they are sufficiently iterable, then their standard parameters are solid and universal.

Definition 2.2. Let $M = (N, B)$ be an acceptable J-structure, and $r \in [o(N)]^{<\omega}$; then

$$W_M^{\alpha,r} = \mathrm{cHull}_1^M(\alpha \cup r \setminus (\alpha+1)).$$

We call $W_M^{\alpha,r}$ the *standard solidity witness* for r at α. We say r is *solid over M* iff all its standard solidity witnesses belong to M.[15]

[13] If $\kappa = \mathrm{crit}(U)$, then $0^\sharp$ is Σ_1 definable over $L_{\kappa+1}[U]$ and not in $L_{\kappa+1}[U] = L_{\kappa+1}$. So the first projectum of $L_{\kappa+1}[U]$ is ω, and hence $L_{\kappa+1}[U]$ is not sound. Its core is essentially equivalent to $0^\sharp$.

[14] It is enough for us to consider the case that N is a potential premouse and B is amenable to N.

[15] $\mathrm{Hull}_1^M(X) = \{a \in M \mid \{a\} \text{ is } \Sigma_1^M \text{ in parameters from } X\}$. "cHull" stands for the transitive collapse of the hull in question.

Definition 2.3. Let $M = (N, B)$ be an acceptable J-structure, and $r \in [o(N)]^{<\omega}$. We say that r is *universal* over M if for $\rho = \rho_1(M)$ and $W = W_M^{\rho,r}$,

(a) $M|\rho^{+,M} = W|\rho^{+,W}$, and

(b) for any $A \subseteq \rho$, A is boldface $\mathbf{\Sigma}_1^M$ iff A is boldface $\mathbf{\Sigma}_1^W$.

It is easy to see that there is at most one parameter $r \in [o(M)]^{<\omega}$ that is both solid and universal over M.[16]

Now let M be a potential premouse. We set $\rho_0(M) = o(M)$, $p_0(M) = \emptyset$, $\mathfrak{C}_0(M) = M$, and say that M is 0-sound. Moving on to level 1, let

$$\begin{aligned}
&\rho_1 = \rho_1(M), \\
&p_1 = p_1(M), \\
&\bar{\mathfrak{C}}_1 = \mathrm{cHull}_1^M(\rho_1 \cup \{p_1\}),
\end{aligned}$$

and

$$\mathfrak{C}_1 = \mathrm{cHull}_1^M(\rho_1 \cup \{p_1, \rho_1\}).$$

$\mathfrak{C}_1$ is the 1-core of M. $\bar{\mathfrak{C}}_1$ is the hull that is taken in the usual theory; we call it the strong 1-core of M.

Let $\sigma\colon \bar{\mathfrak{C}}_1 \to M$ and $\pi\colon \mathfrak{C}_1 \to M$ be the anticollapse maps, and $\bar{p}_1 = \sigma^{-1}(p_1)$. We call π the *anticore map*. We say that M is *parameter solid* if p_1 is solid and universal over M and $\bar{p}_1$ is solid and universal over $\bar{\mathfrak{C}}_1$. We say that M is *projectum solid* iff ρ_1 is not measurable by the M-sequence, and either

$$\mathfrak{C}_1 = \bar{\mathfrak{C}}_1,$$

or

$$\mathfrak{C}_1 = \mathrm{Ult}_0(\bar{\mathfrak{C}}_1, D)$$

where D is the order zero measure of $\bar{\mathfrak{C}}_1$ on ρ_1, and

$$\sigma = \pi \circ i_D.$$

We say that M is *1-solid* iff M is parameter solid and projectum solid. We say that M is 1-sound iff M is 1-solid and $M = \mathfrak{C}_1(M)$.

Let $\tau = \pi^{-1} \circ \sigma$, so that either τ is the identity, or $\tau = i_D$ for D the order zero measure of $\bar{\mathfrak{C}}_1$ on ρ_1. Using the elementarity of τ, we see that $\tau(\bar{p}_1) = p_1(\mathfrak{C}_1)$, and hence $p_1(\mathfrak{C}_1)$ is solid

[16] Suppose r and s were distinct such parameters, and let α be largest in $r \triangle s$. Suppose $\alpha \in r$; then for $\rho = \rho_1(M)$, one can compute $W^{\rho,s}$ from $W^{\alpha,r}$, so $W^{\rho,s} \in M$, contrary to the universality of s.

and universal over $\mathfrak{C}_1$.[17] Since $\pi(\rho_1) = \rho_1$, ρ_1 is not measurable by the $\mathfrak{C}_1$-sequence. Thus if M is 1-solid, then $\mathfrak{C}_1(M)$ is 1-sound.

If M is 1-sound, then its is coded by $(M||\rho_1, A^1)$, where $M||\rho_1$ is the initial segment of M of height ρ_1 and

$$A^1 = \{\langle \varphi, b\rangle \mid \varphi \text{ is } \Sigma_1 \wedge b \in M||\rho_1 \wedge M \models \varphi[b, \eta_1, \rho_1, p_1]\},$$

where η_1 is the Σ_1^M cofinality of ρ_1.[18] We then set

$$\begin{aligned} M^1 &= (M||\rho_1, A^1),\\ \rho_2 &= \rho_1(M^1),\\ p_2 &= p_1(M^1), \end{aligned}$$

and go on to define the 2-core $\mathfrak{C}_2$ and strong 2-core $\bar{\mathfrak{C}}_2$ of M. M is 2-solid if these behave well in a fashion analogous to 1-solidity, with the additional requirement that if $\rho_2 \leq \eta_1$, then η_1 is not measurable by the M-sequence.[19] M is 2-sound iff M is 2-solid and $M = \mathfrak{C}_2(M)$.

If M is 2-sound, then we go on to define $\rho_3(M)$, $p_3(M)$, and the 3-cores $\mathfrak{C}_3$ and $\bar{\mathfrak{C}}_3$. And so on. In general, we have

$$\begin{aligned} M^k &= (M||\rho_k, A^k),\\ \mathfrak{C}_k(M) &= \text{ decoding of } M^k,\\ \rho_{k+1} &= \rho_1(M^k), \text{ and}\\ p_{k+1} &= p_1(M^k), \end{aligned}$$

where A^k is the $\Sigma_1^{M^{k-1}}$ theory of parameters in $\rho_k \cup \{\eta_k, \rho_k, p_k\}$, for η_k the $\Sigma_1^{M^{k-1}}$ cofinality of ρ_k.[20] We call M^k the *k-th reduct* of M.

Definition 2.4. A *pfs premouse* is a pair $M = (\hat{M}, k)$ such that $\hat{M}$ is a potential pfs premouse, and

(a) $\hat{M}$ is k-sound,

(b) whenever P is an initial segment of $\hat{M}$ such that $o(P) < o(\hat{M})$, then P is an ω-sound potential pfs premouse.

We write $k(M) = k$, say that $\hat{M}$ is the *bare premouse* associated to N, and identify $\hat{M}$ with M when context permits.

[17] If b is a solidity witness for $\bar{p}_1$, then $\tau(b)$ is a generalized solidity witness for p_1. See [48].

[18] A^1 has distinguished names for η_1, ρ_1, and p_1. Since $\eta_1 \leq \rho_1$, we don't need this name to decode M, but by including it, we guarantee that η_1 is in the hull that collapses to $\mathfrak{C}_2$.

[19] This additional requirement helps insure that iteration maps preserve ρ_1.

[20] Again, A^k has distinguished names for η_k, ρ_k, p_k, so they are automatically put into the hull that collapses to $\mathfrak{C}_{k+1}(M)$.

The convention that each premouse has a distinguished degree of soundness is due to Itay Neeman. It is useful for simplifying statements about premice, while retaining precision. In this vein:

Definition 2.5. Let M be a pfs premouse and $k = k(M)$; then

$$\rho^-(M) = \rho_k(M),$$
$$\rho(M) = \rho_{k+1}(M),$$

and

$$\mathfrak{C}(M) = \mathfrak{C}_{k+1}(M).$$

We call $\rho(M)$ and $\mathfrak{C}(M)$ *the* projectum and core of M. We say that M is *solid* iff M is $k+1$-solid, M is *sound* iff M is $k+1$-sound. We let $M \downarrow n = (\hat{M}, n)$, and $M^- = M \downarrow k-1$.

Definition 2.6. Let M and N be pfs premice, $\pi\colon M \to N$, and $k = k(M) = k(N)$, and $\sigma = \pi \restriction M^k$.

(1) π is *nearly elementary* iff σ is Σ_0 elementary and cardinal preserving as a map from M^k to N^k, and π is the decoding of σ.[21]

(2) π is *elementary* iff in addition, σ is Σ_1 elementary as a map from M^k to N^k.

(3) π is *exact* iff $\pi(\eta_i(M)) = \eta_i(N)$ and $\pi(\rho_i(M) = \rho_i(N)$ for all $i \leq k$.[22]

The ultrapower and iteration maps we consider are all elementary and exact. Nearly elementary maps come up when we consider factor embeddings from one ultrapower to another that uses a larger class of functions.

When the fine structure of M becomes relevant, the special case that $k(M) = 0$ is very often representative of the general one. In this special case, a map $\pi\colon M \to N$ is elementary iff it is Σ_1 elementary, and nearly elementary iff it is Σ_0 elementary and cardinal preserving. Exactness follows from our convention that $\pi(o(M)) = o(N)$.

We often identify M with $\hat{M}$. Abusing notation this way, if M is a premouse, then we set $o(M) = \mathrm{OR} \cap M$, so that $o(M) = \omega\alpha$ for $M = (J^A_\alpha, ...)$. We write $\hat{o}(M)$ for α itself. The index of M is

$$l(M) = \langle \hat{o}(M), k(M)\rangle.$$

If $\langle \nu, l\rangle \leq_{\mathrm{lex}} l(M)$, then $M|\langle \nu, l\rangle$ is the initial segment N of M with index $l(N) = \langle \nu, l\rangle$. (So $\dot{E}^N = \dot{E}^M \cap N$, and when $\nu < \hat{o}(M)$, $\dot{F}^N = \dot{E}^M_{\omega\nu}$.) If $\nu \leq \hat{o}(M)$, then we write $M|\nu$ for $M|\langle \nu, 0\rangle$. We write $M||\nu$ for the structure that agrees with $M|\nu$ except possibly on the interpretation of $\dot{F}$, and satisfies $\dot{F}^{M||\nu} = \emptyset$. By convention, $k(M||\nu) = 0$.[23]

[21] Meaning that $\pi \circ h = g \circ \sigma$ for $h\colon M^k \to M$ and $g\colon N^k \to N$ the natural surjections given by k-soundness.

[22] We adopt the convention that $\pi(o(M)) = o(N)$ here and in what follows.

[23] Many authors, for example [48], reverse the meanings of $M|\nu$ and $M||\nu$. We find it more logical to let $M||\nu$ stand for cutting M twice, first to $M|\nu$, and then again by throwing away the top extender.

Definition 2.7. If P and Q are pfs premice, then

(i) $P \trianglelefteq Q$ iff there are μ and l such that $P = Q|\langle \mu, l\rangle$.

(ii) $P \triangleleft Q$ iff $P \trianglelefteq Q$ and $P \neq Q$.

If $P \trianglelefteq Q$ we say that P is an *initial segment* of Q, and if $P \triangleleft Q$ we say it is a proper initial segment.

If $M = (\hat{M}, k)$ is a premouse, then its extender sequence is $\dot{E}^M = \dot{E}^{\hat{M}}$ together with a last (or *top*) extender $\dot{F}^M = \dot{F}^{\hat{M}}$.

2.4 Iteration trees

Strategy comparison also leads us to consider iteration strategies defined on a slightly larger class of iteration trees than is usual. We call these iteration trees *plus trees*.

Definition 2.8. Let M be a pfs premouse, and E be an extender on the M-sequence; then

(1) E^+ is the extender with generators $\lambda_E \cup \{\lambda_E\}$ that represents $i_F^{\mathrm{Ult}(M,E)} \circ i_E^M$, where F is the order zero total measure on λ_E in $\mathrm{Ult}(M, E)$,

(2) $\hat{\lambda}(E^+) = \lambda_E$, and

(3) $\mathrm{lh}(E^+) = \mathrm{lh}(E)$.

Definition 2.9. G is *of plus type* iff $G = E^+$, for some extender E that is on the sequence of a pfs premouse M. In this case, we let $G^- = E$. The *extended M-sequence* consists of all extenders E such that either E or E^- is on the M-sequence.

We wish to consider iteration trees that are allowed to use extenders of the form E^+, where E is on the coherent sequence of the current model. To unify notation, if E is an extender on the sequence of some premouse, let us set $\hat{\lambda}(E) = \lambda(E) = \hat{\lambda}(E^+)$ and $E^- = E$.

Definition 2.10. Let M be a pfs premouse; then a *plus tree* on M is a system $\mathcal{T} = \langle T, \langle E_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T})\rangle\rangle$ such that there are M_α and $i_{\alpha,\beta}$ and D satisfying:

(1) $M_0 = M$, and T is a tree order;

(2) if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then E_α is on the extended M-sequence, and

  (a) if $\xi < \alpha$, then $\hat{\lambda}(E_\xi) \leq \hat{\lambda}(E_\alpha)$, and

  (b) if $\xi < \alpha$ and E_ξ is of plus type, then $\mathrm{lh}(E_\xi) < \hat{\lambda}(E_\alpha)$;

(3) if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then letting β be least such that either $\beta = \alpha$, or $\mathrm{crit}(E_\alpha) < \hat{\lambda}(E_\beta)$,

  (a) T-pred$(\alpha + 1) = \beta$,

(b) $M_{\alpha+1} = \mathrm{Ult}(M^*_{\alpha+1}, E_\alpha)$, for $M^*_{\alpha+1}$ the shortest initial segment N of M_β such that $\rho(N) \leq \mathrm{crit}(E_\alpha)$, if one exists, and $M^*_{\alpha+1} = M_\beta$ otherwise,

(c) $\alpha + 1 \in D$ iff $M^*_{\alpha+1} \neq M_\beta$

(d) $\hat{\imath}_{\beta,\alpha+1} = i^*_{\alpha+1} \colon M^*_{\alpha+1} \to M_{\alpha+1}$ is the canonical embedding, and

(e) if $\xi \leq_T \beta$, then $\hat{\imath}_{\xi,\alpha+1} = \hat{\imath}_{\beta,\alpha+1} \circ \hat{\imath}_{\xi,\beta}$.

(4) if $\lambda < \mathrm{lh}(\mathcal{T})$ is a limit ordinal, then $D \cap [0, \lambda)_T$ is finite, and M_λ is the direct limit of the M_α for $\alpha <_T \lambda$ under the $\hat{\imath}^{\mathcal{T}}_{\alpha,\eta}$; moreover $\lambda \notin D$.

Here we have extended the standard notation slightly, in that $\mathrm{dom}(\hat{\imath}_{\alpha,\beta}) \lhd M_\alpha$ is possible. This happens if $D \cap (\alpha, \beta]_T \neq \emptyset$.

It may seem that clause (3) of 2.10 allows generators to move along branches of $\mathcal{T}$. The whole point of iteration trees is that we want to avoid that, so that comparisons making use of them will terminate. The worry would be the case that $\beta = \xi + 1$, where $E_\xi = F^+$ for some F, so that $\hat{\lambda}(E_\xi) = \lambda_F$. But in this case, the only important generators of E_ξ are in $\lambda_F \cup \{\lambda_F\}$. Clause (3) requires that generators below $\lambda_F = \hat{\lambda}(E_\xi)$ are not moved. λ_F itself has no total measures in M_β, and hence in M_α. There are no partial extenders on the sequence of M_α with critical point λ_F because the proper initial segments of M_α are projectum solid. Thus E_α is not moving any important generators of E_ξ. It is quite possible that $\mathrm{crit}(E_\alpha) < \lambda(E_\xi)$, however.

The ultrapower referred to in (3)(b) is formed as follows. Given P a pfs premouse, $k = k(P)$, and E an extender over P with $\mathrm{crit}(E) < \rho_k(P)$, then we can form the Σ_0 ultrapower $N = \mathrm{Ult}_0(P^k, E)$, then decode it to some Q. We say that $Q = \mathrm{Ult}(P, E)$.

If Q is wellfounded and the canonical embedding i^P_E is continuous at $\rho_k(P)$, then Q is a pfs premouse, and i^P_E is elementary. No first order condition will guarantee wellfoundedness, but the continuity of i^P_E does follow from a first order condition:

Definition 2.11. Let M be a pfs premouse and $k = k(M)$; then M is *projectum stable* iff $k = 0$, or the $(\Sigma_1)^{M^{k-1}}$ cofinality of $\rho_k(M)$ is not the critical point of an M-total extender on the M-sequence.

When the base model of a plus tree is projectum stable, then its models are all pfs premice, and its branch embeddings are elementary and exact.

Lemma 2.12. *Let M be a projectum stable pfs premouse, and let $\mathcal{T}$ be a plus tree on M; then*

(i) all $\mathcal{M}^{\mathcal{T}}_\alpha$ are pfs premice,

(ii) all branch embeddings are elementary and exact,

(iii) whenever $\alpha + 1 \in D^{\mathcal{T}}$, then $M^*_{\alpha+1}$ is sound, and

(iv) if $\alpha+1 \in D^{\mathcal{T}}$, $\alpha+1 \leq_T \beta$, and $D^{\mathcal{T}} \cap (\alpha+1, \beta]_T = \emptyset$, then $M^*_{\alpha+1} = \mathfrak{C}(M_\beta)^-$ and $i_{\alpha+1,\beta} \circ i^{*,\mathcal{T}}_{\alpha+1}$ is the anticore map.

We have defined projectum stability in order to be able to state our comparison theorem correctly, but it is the sort of fine structural complication that is best ignored on a first reading. If $k(M) = 0$, then M is projectum stable, and if $\rho_{k(M)}(M) = \omega$, then M is projectum stable.

Plus trees are not necessarily length increasing. We say *the plus case occurs at* α iff E_α is of plus type.

Definition 2.13. Let $\mathcal{T}$ be a plus tree on M; then

(a) $\mathcal{T}$ is *normal* (or *length-increasing*) iff whenever $\alpha < \beta < \mathrm{lh}(\mathcal{T})-1$, then $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$,

(b) $\mathcal{T}$ is λ-*tight* iff for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $E^{\mathcal{T}}_\alpha$ is not of plus type, and

(c) $\mathcal{T}$ is λ-*separated* iff for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $E^{\mathcal{T}}_\alpha$ is of plus type.

One can re-organize any plus tree $\mathcal{T}$ as a λ-tight tree $\mathcal{U}$ in a fairly straightforward way, but this reduction is not useful until we have have shown that the iteration strategies we define treat $\mathcal{T}$ and $\mathcal{U}$ the same way. That is a consequence of strategy comparison, so the reduction is useless in the comparison proof itself. In fact, our initial results on the good behavior of iteration strategies apply at the other extreme, to their restrictions to λ-separated trees. Notice that λ-separated trees are normal, by (2)(b) of 2.10. Every extender E used in a λ-separated tree has a largest generator, and this helps in the comparison proof.

The agreement between models in a normal plus tree is given by

Lemma 2.14. *Let $\mathcal{U}$ be a normal plus tree, $M_\alpha = \mathcal{M}^{\mathcal{U}}_\alpha$, and $E_\alpha = E^{\mathcal{U}}_\alpha$; then for $\alpha < \beta < \mathrm{lh}(\mathcal{U})$,*

(1) $M_\alpha || \mathrm{lh}(E_\alpha)) = M_\beta | \mathrm{lh}(E_\alpha)$,

(2) $\mathrm{lh}(E_\alpha)$ *is a cardinal of* M_β, *so* $M_\alpha | \mathrm{lh}(E_\alpha) \neq M_\beta | \mathrm{lh}(E_\alpha)$, *and*

(3) if $\alpha + 1 \leq_T \beta$, *then* $\mathrm{lh}(E_\alpha) \leq \rho^-(M_\beta)$.

Part (3) is easy to prove by induction. It comes down to the fact that if $\mathrm{Ult}(M, E)$ exists, then $\rho^-(\mathrm{Ult}(M, E)) = \sup i_E``\rho^-(M)$.

Figure 2 shows how the agreement of models in a normal iteration tree is propagated when the tree is augmented by one new extender. (Figures like this were first drawn by Itay Neeman.)

The agreement of models in an arbitrary plus tree is a bit awkward to state. It is easy to see that any plus tree $\mathcal{T}$ breaks up into disjoint maximal finite intervals in which the exit extenders have strictly decreasing length. That is, $\mathrm{lh}(\mathcal{T})$ can be partitioned into intervals $[\alpha, \alpha+n]$, where $0 \leq n < \omega$, such that

(i) for all $\beta < \alpha$, $\mathrm{lh}(E_\beta) < \mathrm{lh}(E_\alpha)$,

(ii) for all $i < n$, $E_{\alpha+i}$ is not of plus type, and $\hat{\lambda}(E_{\alpha+i}) \leq \hat{\lambda}(E_{\alpha+i+1}) < \mathrm{lh}(E_{\alpha+i+1}) < \mathrm{lh}(E_{\alpha+i})$, and

(iii) $\mathrm{lh}(E_{\alpha+n}) < \hat{\lambda}(E_{\alpha+n+1})$, or $\alpha + n + 1 = \mathrm{lh}(\mathcal{T})$.

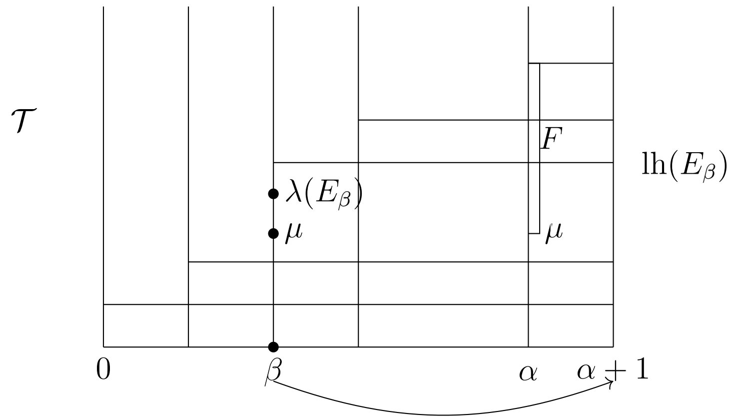

Figure 2: A normal tree $\mathcal{T}$, extended normally by F. The vertical lines represent the models, and the horizontal ones represent their levels of agreement. $\mathrm{crit}(F) = \mu$, and β is least such that $\mu < \lambda(E^{\mathcal{T}}_\beta)$. The arrow at the bottom represents the ultrapower embedding generated by F.

Of course $n = 0$ is possible. Part (iii) implies $\mathrm{lh}(E_{\alpha+n}) < \hat{\lambda}(E_\beta)$ for all $\beta > \alpha + n$. Part (ii) is justified by clause (2)(b) in Definition 2.10. We call $[\alpha, \alpha + n]$ a *maximal delay interval*, and we say that $\alpha + n$ *ends a delay interval*.

It may seem pointless to allow decreasing lengths, because given a maximal delay interval $[\alpha, \alpha + n]$, we could have just skipped using $E_\alpha, ..., E_{\alpha+n-1}$, and taken $E_{\alpha+n}$ out of $\mathcal{M}^{\mathcal{T}}_\alpha$ to continue the iteration. Doing this everywhere would produce a normal iteration tree $\mathcal{S}$ with the same last model as $\mathcal{T}$, differing only in that the nontrivial delay intervals in $\mathcal{T}$ are eliminated. We call $\mathcal{S}$ the *normal companion* of $\mathcal{T}$.

So why bother with $\mathcal{T}$, why not just use its normal companion? The answer is that we shall be considering trees by some iteration strategy Σ. It may happen that $\mathcal{T}$ is by Σ, but its normal companion is not. In the strategy-comparison proof, we have to live with the possibility that this happens when Σ is a background-induced strategy, as in §7. One can show that background-induced strategies are not pathological in this way, but the proof involves a strategy comparison.

Nevertheless, the reader will lose little by restricting his attention to normal plus trees.

3 Iteration strategies and comparison

What qualifies a premouse as a mouse, comparable with others of its kind, is an iteration strategy.

Let M be a premouse. $G^+(M, \theta)$ is the game of length θ in which I and II cooperate to produce a plus tree $\mathcal{T}$ on M. Given $\mathcal{T} \upharpoonright \alpha + 1$ with last model $\mathcal{M}^{\mathcal{T}}_\alpha$, player I chooses E from the extended sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ such that E meets the requirements (2) of Definition 2.10, and sets

$E^{\mathcal{T}}_\alpha = E$. The rules for plus trees then determine $\mathcal{T} \restriction \alpha + 2$, and I wins if its last model $\mathcal{M}^{\mathcal{T}}_{\alpha+1}$ is illfounded. At limit steps λ II must pick a branch b that is cofinal in λ such that the direct limit $\mathcal{M}^{\mathcal{T}}_b$ along b is wellfounded. If he does so, then $\mathcal{M}^{\mathcal{T}}_\lambda = \mathcal{M}^{\mathcal{T}}_b$ and $\mathcal{T} \restriction \lambda + 1 = \mathcal{T}^\frown b$. If he fails to do so, then I wins. If II manages to stay in the category of wellfounded models for θ rounds, then he wins. See [38], where the corresponding game is called $G_k(M, \theta)$, for $k = k(M)$. A *θ-iteration strategy* for M is a winning strategy for II in $G^+(M, \theta)$.

The following comparison lemma is essentially Theorem 3.11 of [38].

Theorem 3.1. *Let P and Q be projectum stable premice of size $\leq \theta$, and suppose Σ and Ψ are $\theta^+ + 1$-iteration strategies for P and Q respectively; then there are normal, λ-tight plus trees $\mathcal{T}$ by Σ and $\mathcal{U}$ by Ψ of size θ, with last models R and S, such that either*

(a) $R \trianglelefteq S$, and P-to-R does not drop, or

(b) $S \trianglelefteq R$, and Q-to-S does not drop.

Proof. (Sketch.) We build $\mathcal{T}$ and $\mathcal{U}$ inductively, by "iterating away the least disagreement" at successor steps, and using our iteration strategies at limit steps. At step α we have $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$ with last models P_α and Q_α respectively. We begin with $P_0 = P$, $Q_0 = Q$, and $\mathcal{T}_0 = \mathcal{U}_0$ being the empty tree. At step $\alpha + 1$, let

$$\gamma = \text{ least } \beta \text{ such that } P_\alpha | \beta \neq Q_\alpha | \beta.$$

If there is no such β, the comparison is complete. Otherwise, let

$$\begin{aligned} \mathcal{T}_{\alpha+1} &= \mathcal{T}_\alpha^\frown \langle \dot{E}^{P_\alpha}_\gamma \rangle, \text{ and} \\ \mathcal{U}_{\alpha+1} &= \mathcal{U}_\alpha^\frown \langle \dot{E}^{Q_\alpha}_\gamma \rangle. \end{aligned}$$

Here $\mathcal{S}^\frown \langle E \rangle$ stands for the unique normal extension of $\mathcal{S}$ whose last extender used is E, with the understanding that $\mathcal{S}^\frown \langle E \rangle = \mathcal{S}$ if $E = \emptyset$. At limit steps we let $\mathcal{T}_\lambda$ be $\bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$, extended by the branch $\Sigma(\bigcup_{\alpha<\lambda} \mathcal{T}_\alpha)$ if this tree has limit length. Similarly on the $\mathcal{U}$ side.

We claim that the comparison is complete at some stage $\alpha < \theta$. For suppose not, and let $\mathcal{T} = \mathcal{T}_{\theta^+ + 1}$ and $\mathcal{U} = \mathcal{U}_{\theta^+ + 1}$ be the normal trees of length $\theta^+ + 1$ that result. Let $\pi \colon H \to V_\xi$ be elementary, where ξ is large, everything relevant is in $\text{ran}(\pi)$, H is transitive, and $\theta < \text{crit}(\pi) < \theta^+$. Let $\alpha = \text{crit}(\pi)$. We have $\pi(\langle P, Q \rangle) = \langle P, Q \rangle$ and $\pi(\alpha) = \theta^+$, and it is not hard to see that

$$\begin{aligned} \pi(\mathcal{T} \restriction \alpha + 1) &= \mathcal{T}, \\ \pi(\mathcal{U} \restriction \alpha + 1) &= \mathcal{U}, \end{aligned}$$

and

$$\begin{aligned} \pi \restriction \mathcal{M}^{\mathcal{T}}_\alpha &= i^{\mathcal{T}}_{\alpha,\theta^+}, \\ \pi \restriction \mathcal{M}^{\mathcal{U}}_\alpha &= i^{\mathcal{U}}_{\alpha,\theta^+}. \end{aligned}$$

Also,
$$P(\alpha)^{\mathcal{M}_\alpha^\mathcal{T}} = P(\alpha)^{\mathcal{M}_{\theta+}^\mathcal{T}} = P(\alpha)^{\mathcal{M}_{\theta+}^\mathcal{U}} = P(\alpha)^{\mathcal{M}_\alpha^\mathcal{U}}.$$

Thus π, $i^\mathcal{T}_{\alpha,\theta^+}$, and $i^\mathcal{U}_{\alpha,\theta^+}$ all generate the same (α,θ^+)-extender; call it G. Let E be the first extender used in $\mathcal{T}$ along the branch $[\alpha,\theta^+]_T$, and F the first extender used in $\mathcal{U}$ along $[\alpha,\theta^+]_U$.

Because generators are not moved, E is an initial segment of G. That is, for $x \subseteq \mathrm{crit}(E)$ in M_α and $a \subseteq \lambda_E$ finite, and letting $E = E^\mathcal{T}_\gamma$,

$$\begin{aligned} x \in E_a &\Leftrightarrow a \in i_E(x) \\ &\Leftrightarrow a \in i_{\gamma+1,\theta^+} \circ i_E(x) \\ &\Leftrightarrow a \in i_{\alpha,\theta^+}(x) \\ &\Leftrightarrow x \in G_a. \end{aligned}$$

Line 2 follows from line 1 because $a \in [\lambda_E]^{<\omega}$ and $\mathrm{crit}(i^\mathcal{T}_{\gamma+1,\theta^+}) \geq \lambda_E$.[24] By the Jensen initial segment condition, E is then the first whole initial segment of G that is not on the sequence of the common lined up part $N = \mathcal{M}^\mathcal{T}_{\theta^+}|\theta^+ = \mathcal{M}^\mathcal{U}_{\theta^+}|\theta^+$. For the same reasons, F is the first whole initial segment of G that is not on the N-sequence. Thus $E = F$. Since $\mathrm{lh}(E) = \mathrm{lh}(F)$, they were used at the same stage in the comparison. But we were iterating away disagreements, so $E \neq F$, contradiction.

This gives us $\mathcal{T} = \mathcal{T}_\alpha$ and $\mathcal{U} = \mathcal{U}_\alpha$ with last models R and S such that $R \trianglelefteq S$ or $S \trianglelefteq R$. If $R \triangleleft S$, then R is sound, and therefore by 2.12(iv)(a) the branch P-to-R did not drop, so we have conclusion (a). Similarly, if $S \triangleleft R$ we get conclusion (b). Thus we may assume $R = S$. It is now enough to show that one of the two branches P-to-R and Q-to-S did not drop. Assume otherwise, and let
$$C = \mathfrak{C}(R) = \mathfrak{C}(S)$$
be the core, and π the anticore map. By 2.12(iv)(a), C occurs on both branches, and π is the iteration map of both the branch C-to-R of $\mathcal{T}$, and the the branch C-to-S of $\mathcal{U}$. But as in the termination proof, this means the first extenders used in these two branches are the same, a contradiction. □

Notice that although the successful comparison only involves trees of size θ, we really did need $\theta^+ + 1$-iterability to show that it exists. In particular, to compare countable mice, we need $\omega_1 + 1$-iterability.

Corollary 3.2. *Let M and N be countably iterable premice such that $\rho^-(M) = \rho^-(N) = \omega$; then either $M \trianglelefteq N$ or $N \trianglelefteq M$.*

Proof. M and N are projectum stable, so 3.1 applies. Let $\mathcal{T}$ on M with last model R and $\mathcal{U}$ on N with last model S be as in 3.1, and suppose without loss of generality that $R \trianglelefteq S$ and M-to-R does not drop. Since $\rho^-(M) = \omega$, it is impossible to take an ultrapower of M without dropping, so $\mathcal{T}$ is empty and $M = R$. It is enough to show that $\mathcal{U}$ is also empty. But otherwise,

[24] This is why it is important that the generators of an extender used in $\mathcal{T}$ are not moved by extenders used later on the same branch.

N-to-S must drop, and letting $C = \mathfrak{C}(S)$, the last drop is to C, and the anticore map $\pi\colon C \to S$ is the same as the branch embedding of $\mathcal{U}$.[25] We have

$$\rho^-(M) = \omega < \mathrm{crit}(\pi) < \rho^-(C) \leq \rho^-(S),$$

so if $\hat{M} = \hat{S}$, then $k(S) < k(M)$, contrary to $M \trianglelefteq S$. Thus $\hat{M} \neq \hat{S}$, which implies that $M \in S$. But this means $M \triangleleft S|\omega_1^S$. Since $S|\omega_1^S = C|\omega_1^C$, we get that $M \triangleleft C$, so $M \triangleleft \mathcal{M}_\gamma^{\mathcal{U}}$ for $\gamma = U\text{-pred}(\alpha+1)$, so the comparison was over before we reached S, contradiction. □

Corollary 3.3. *Let M and N be countably iterable premice such that $\rho^-(M) = \rho^-(N) = \omega$ and $o(M) = o(N)$; then $M = N$. Thus M is ordinal definable from $o(M)$.*

Corollary 3.4. *Let M be a countably iterable premouse; and $x \in P(\omega) \cap M$; then x is ordinal definable.*

Proof. Let α be least such that x is definable over $M|\alpha$; then $\rho_k(M|\alpha) = \omega$ for some k, so $M|\alpha$ is ordinal definable, so x is ordinal definable. □

We have called Theorem 3.1 a comparison lemma, but it has a clear defect in that regard. We have not compared all the data. How M and N compare could depend on which iteration strategies for them are used in making the comparison.[26] Because of this, 3.1 does not lead to a mouse order on the mice to which it applies.

A full comparison process would compare all the data; not just the mice, but also their iteration strategies. The field of the mouse order would then consist of pairs (M, Σ) such that Σ is an iteration strategy for M. This is where we are headed, but first we need to record some regularity properties of iteration strategies that make comparing them possible. These properties are byproducts of the known proofs of iterabilty; moreover, assuming AD^+, every countable mouse has an iteration strategy with these properties. We shall then define a *pure extender pair* as a pair (M, Σ) such that Σ is an iteration strategy for M having the regularity properties we have isolated. Theorem 6.2 is a comparison lemma for pure extender pairs.

By Corollary 3.2, how sound mice projecting to ω compare *is* independent of their iteration strategies, and there is a well defined mouse order on them, namely, $M \leq^* N$ iff $M \trianglelefteq N$. In fact, if M is a sound mouse projecting to ω, then there is exactly one iteration strategy Σ for M[27]; moreover Σ has the properties that make (M, Σ) a pure extender pair. If $M \trianglelefteq N$ and Ψ is an iteration strategy for N, then Σ is just the restriction of Ψ to iteration trees that drop to an initial segment of M along every branch. So $(M, \Sigma) \trianglelefteq (N, \Psi)$ in a natural sense. Thus in the case of mice projecting to ω, the comparison lemma for pure extender pairs has already been proved. It follows from the uniqueness of their iteration strategies, and no iteration is needed in order to compare them.

[25] In other words, $\pi = i^{\mathcal{U}}_{\alpha+1,\beta} \circ i^{*,\mathcal{U}}_{\alpha+1}$, where $C = \mathcal{M}^{*,\mathcal{U}}_{\alpha+1}$ and $S = \mathcal{M}^{\mathcal{U}}_\beta$.

[26] There are examples of this in [35].

[27] This follows from Theorem 4.11 of [38], and the fact that any iteration strategy for a pointwise definable M has the Weak Dodd-Jensen property with respect to all enumerations of M.

4 Universally Baire iteration strategies

The reals belonging to mice are not just ordinal definable, they are ordinal definable in a generically absolute way. This shows up in the way that the iteration strategies that witness their ordinal definability are constructed. The direct proofs of iterability only produce branches for countable iteration trees, even in the realm of linear iterations. Yet $\omega_1 + 1$-iterability is the minimal useful kind of iterability; for example, it is the kind needed to compare countable premice. All known proofs of $\omega_1 + 1$-iterability involve at some point producing an ω_1-strategy Σ, and showing that Σ is sufficiently absolutely definable that one can extend it to an $\omega_1 + 1$ strategy. Here is a simple proposition in this vein.

Proposition 4.1. *Assume* AD*, and let* Σ *be an* ω_1*-iteration strategy for a countable premouse* M*; then* Σ *can be extended to an* $\omega_1 + 1$ *strategy for* M*.*

Proof. Let $\mathcal{T}$ be a normal tree of length ω_1 on M that is played by Σ. It will suffice to show $\mathcal{T}$ has a cofinal, wellfounded branch. But let $j : V \to N$ with $\mathrm{crit}(j) = \omega_1$ witness the measurability of ω_1. The pair $\langle \mathcal{T}, M \rangle$ can be coded by a set of ordinals A, and Los's Theorem holds for ultrapowers of wellordered structures, so $j : L[A] \to L[j(A)]$ is elementary. It follows that $j(\mathcal{T})$ is an iteration tree on M, $\mathcal{T} = j(\mathcal{T}) \restriction \omega_1$, and $\omega_1 < \mathrm{lh}(j(\mathcal{T}))$. But this implies that $[0, \omega_1)_{j(T)}$ is a cofinal, wellfounded branch of $\mathcal{T}$. □

In 4.1 the absolute definability of Σ is manifested in its membership in a model of AD. In some contexts, absolute definability has to be more finely calibrated, and a model of some fragment of AD that contains Σ constructed along with Σ. This leads into the *core model induction method*, our most all-purpose method for constructing mice and their iteration strategies.[28] In this paper, we shall avoid such difficulties by simply working in models of AD, and considering the comparison and mouse existence problems there. In fact, we shall usually work in the theory $\mathsf{ZF} + \mathsf{AD}^+$.[29] In that context it is natural to focus on countable premice, and ω_1-iteration strategies for them. That is what we shall do.

Proposition 4.1 and its refinements stand at a key junction in inner model theory. They constitute one of the main reasons inner model theory and descriptive set theory have become so entangled in the years since we discovered iteration trees.

Corollary 3.4 and the proposition imply

Corollary 4.2. *Assume* AD*, and let* M *be an* ω_1*-iterable premouse; then every real in* M *is* Σ^2_1 *in a countable ordinal.*

The converse of Corollary 4.2 holds in $L(\mathbb{R})$, that is, every ordinal definable real belongs to a mouse. (In $L(\mathbb{R})$, a real is OD iff it is Σ^2_1 in a countable ordinal.) In fact, we can say something about how the large cardinal properties of the mice match up with the levels of ordinal definability they capture.

[28]See [27].

[29]AD^+ is a technical strengthening of AD that was isolated by Woodin. If A is uB and there are infinitely many Woodin cardinals with a measurable above them all, then $L(A, \mathbb{R}) \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$. See [40].

Definition 4.3. Let $n \leq \omega$; then $M_n^\sharp$ is the minimal countably iterable, sound, active premouse satisfying "there are n Woodin cardinals". M_n is the result of iterating the last extender of M_n through the ordinals.

Thus M_n is the canonical minimal proper class extender model with n Woodins, and $M_n^\sharp$ is its sharp. $M_0 = L$. The basic theory of these and somewhat larger extender models with many Woodin cardinals was developed by Martin, Mitchell, and Steel in [18] and [21]. See also [38]. The optimal correctness results for these models were established by Woodin, using genericity iterations and the extender algebra. This led to the following *mouse capturing* theorem for $L(\mathbb{R})$.

Theorem 4.4 (Martin, Mitchell, Steel, Woodin 1985-1990)**.** *Suppose there are ω Woodin cardinal, plus a measurable cardinal above them all; then*

(1) for any $n < \omega$,

(a) $\mathbb{R} \cap M_n = \{x \mid x$ is Δ^1_{n+2} in a countable ordinal$\}$, and

(b) $M_n \models$ "$\mathbb{R}$ has a Δ^1_{n+2} wellorder.

(2) (a) $\mathbb{R} \cap M_\omega = \{x \in \mathbb{R} \mid x$ is $OD^{L(\mathbb{R})}\}$, and

(b) $M_\omega \models$ "$\mathbb{R}$ has an $OD^{L(\mathbb{R})}$ wellorder".

The upper bounds on the definability of the reals in mice in this theorem are refinements of Corollary 3.4 obtained by putting an upper bound on the complexity of the relevant iteration strateges. For example, let $x \in \mathbb{R} \cap M_\omega$, and let $N \lhd M_\omega$ be such that x is definable over n but $x \notin N$. Because N projects to ω, its iteration strategy Σ is *unique*. Because $N \lhd M_\omega$, $\Sigma \cap \mathrm{HC} \in L(\mathbb{R})$, so N is ω_1-iterable in $L(\mathbb{R})$. But ω_1 is measurable in $L(\mathbb{R})$, so N is ω_1+1-iterable in $L(\mathbb{R})$. Applying Corollary 3.4 inside $L(\mathbb{R})$, we see that x is $\mathrm{OD}^{L(\mathbb{R})}$.

The reader should see [44] and [38, §7,§8] for a detailed account of the proof of Theorem 4.4.

What about pointclasses beyond $L(\mathbb{R})$? Does AD, or better AD^+, imply that every real that is Σ^2_1 in a countable ordinal belongs to a mouse? If by "mouse" we mean simply the structures we defined in §2, this is quite unlikely, because it is very likely that there is a minimal long extender mouse, that is, a structure (M, F) such that M is a short extender premouse of the sort we defined in §2, F is a long extender, and (M, F) has an ω_1 iteration strategy in some model of AD^+. The first order theory of (M, F) will be Δ^2_1 in that model of AD^+, but every short extender mouse projecting to ω will be a proper initial segment of M.

There is as yet no general theory of mice with long extenders, but there is a theory for this minimal one, and somewhat stronger mice. See [22] or [47] for more detail.

Definition 4.5. ($\mathsf{ZF} + \mathsf{AD}^+$) Let $\Gamma \subseteq P(\mathbb{R})$.

(a) $\Gamma - \mathsf{NLE}$ is the assertion that there is no ω_1-iteration strategy Σ for a premouse (in the sense of [22]) with a long extender on its sequence such that $\mathrm{Code}(\Sigma) \in \Gamma$.

(b) For $x, y \in \mathbb{R}$, x is $(\Sigma_1^2)^\Gamma$ in y and α iff there ia a formula φ such that x is the unique z such that for some $A \subseteq \mathrm{HC}$ such that $\mathrm{Code}(A) \in \Gamma$, $(\mathrm{HC}, \in, A) \models \varphi[z, y, \alpha]$.

(c) Γ-*mouse capturing* ($\Gamma - \mathsf{MC}$) is the statement: whenever x is $(\Sigma_1^2)^\Gamma$ in y and a countable ordinal, then there is a mouse M over y such that $x \in M$, and M has an ω_1-iteration strategy with code in Γ.

We write NLE for $P(\mathbb{R}) - \mathsf{NLE}$.

The appropriate converse to Corollary 4.2 is then the following small variation on the well-known *Mouse Set Conjecture.*[30]

Mouse Set Conjecture: (MSC) Assume $\mathsf{ZF}+\mathsf{AD}^+$, and let Γ be a strongly closed pointclass[31]; then $\Gamma - \mathsf{NLE}$ implies $\Gamma - \mathsf{MC}$.

The main open problems in inner model theory have to do with the existence of iteration strategies. MSC and its counterpart HPC for strategy mice are good candidates for the most important of them. HPC follows from MSC; we shall state it precisely in §9. It seems quite unlikely that one could obtain inner models with superstrong cardinals from strong hypotheses that do not directly imply the existence of large cardinals without proving HPC along the way. Even granted consistency strength that is close to the surface, say the assumption that there are supercompacts, it is plausible that the construction of iteration strategies for mice with superstrongs will involve a proof of HPC, and perhaps also MSC.

MSC was identified as an important target in the late 1990s or early 2000s. Our comparison lemma for iteration strategies has some consequences that are relevant to it. The simplest is

Theorem 4.6. *([36, 3.6]) Assume $\mathsf{ZF}+\mathsf{AD}^+$, and that there is an ω_1-iterable premouse with a long extender on its sequence; then* MSC *holds.*

This is pretty good evidence that MSC is true.

Mouse capturing localizes, in that if Γ_1 and Γ_2 are strongly closed and $\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_2 - \mathsf{MC}$ implies $\Gamma_1 - \mathsf{MC}$. For this and other reasons, it is natural to try to prove MSC by induction on the Wadge hierarchy. This has been done for Γ contained in the minimal model of $\mathsf{AD}_\mathbb{R}$ + "θ is regular", and somewhat beyond.[32]

Although the large cardinal pattern in M sometimes matches nicely its correctness and the definabilty of its iteration strategies, this is not generally true. If MSC is to be proved by an induction on the Wadge hierarchy, then the descriptive set theory associated to levels at which new iteration strategies appear is more important than the first order properties of the mice. One important consequence of our strategy comparison theorem is that these levels correspond precisely to the Suslin cardinals in our model of AD^+. This is shown in [36] and [?]..

[30]See [41, 0.1.2].

[31]That is, closed under complements, real quantification, and Wadge reducibility.

[32]See [23], [24], and [25].

5 Regularity properties of iteration strategies

For most of the rest of this paper, we shall be working in a model of $\mathsf{ZF} + \mathsf{AD}^+$ and considering iteration strategies for countable premice. In this section we introduce three regularity properties of such iteration strategies: normalizing well, strong hull condensation, and internal lift consistency. Strategies with these properties can be compared with one another, and as a consequence are well behaved in many other ways.

Strategy comparison applies to iteration strategies that act on stacks of plus trees. Let us introduce the relevant terminology.

If λ is a limit ordinal, then $G^+(M, \lambda, \theta)$ is the game in which the players play λ rounds, the α-th round being a play of $G^+(N, \theta)$, where N is an initial segment, chosen by I, of the direct limit along the branch produced by the prior rounds. I moves at successor stages, by playing an extender or starting a new round if he wishes.[33] If the current round lasts θ moves, then there are no further rounds, and the game is over.[34]

II picks branches at limit stages, and his obligation is just to insure all models are wellfounded, including the direct limit of the base models in the final stack of length λ. We say that *s is an M-stack* of length α whose component plus trees have length $< \theta$ iff s is a position in $G^+(M, \lambda, \theta)$ that represents α completed rounds and is not yet a loss for II. A *(λ, θ)-iteration strategy* for M is a winning strategy for II in $G^+(M, \lambda, \theta)$, and M is (λ, θ)-iterable iff there is such a strategy. See [38]. Clearly $G^+(M, 1, \theta) = G^+(M, \theta)$.[35]

Definition 5.1. Let M be a premouse; then M is *countably iterable* iff every countable elementary submodel of M is $(\omega_1, \omega_1 + 1)$-iterable.

Countable iterability is what one needs to prove that M is well-behaved in a fine structural sense; for example, that its standard parameter is solid and universal.

5.1 Tail strategies

Iterates of an iterable structure are iterable, via a *tail strategy*.

Definition 5.2. Let Ω be a winning strategy for II in $G^+(M, \lambda, \theta)$, let s be an M-stack according to Ω with $\mathrm{lh}(s) < \lambda$, and let $N = M_\infty(s)|\langle \nu, k \rangle$ for some ν, k; then $\Omega_{s,N}$ is the strategy for $G^+(N, \lambda - \mathrm{lh}(s), \theta)$ given by:

$$\Omega_{s,N}(t) = \Omega(s^\frown \langle N \rangle^\frown t),$$

for all N-stacks t. We set $\Omega_s = \Omega_{s,M_\infty(s)}$. For $N \trianglelefteq M$, we let $\Omega_N = \Omega_{\langle\emptyset\rangle,N}$.

We are assuming here that the position $s^\frown\langle N \rangle$ includes the information that N is the base model for a new round. There are other tails of Ω one might consider.

[33]For notational reasons, we allow I to move immediately from round α to round $\alpha + 1$, without playing any extenders.

[34]Thus if M is countable, a position in $G^+(M, \omega_1, \omega_1)$ is a member of HC, and a strategy for it is a subset of HC.

[35]Up to minor details in how they are presented.

Our definitions so far allow the tails of an iteration strategy to be inconsistent with the strategy itself; for example, one could have a strategy Ω for $G^+(M, \lambda, \theta)$ such that $\Omega \neq \Omega_M$. One could have more subtle inconsistencies, for example, $N \trianglelefteq M$ and some normal $\mathcal{T}$ by both Ω_M and Ω_N such that $\Omega_M(\mathcal{T}) \neq \Omega_N(\mathcal{T})$. The iteration strategies that we shall construct in §7 do not have such internal inconsistencies, and one basic task is to spell that out precisely and prove it. For example,

Definition 5.3. Let Ω be a winning strategy for II in $G^+(M, \lambda, \theta)$; then Ω is *positional* iff whenever s and t are M-stacks by Ω of length $< \lambda$, and $N \trianglelefteq M_\infty(s)$ and $N \trianglelefteq M_\infty(t)$, then $\Omega_{s,N} = \Omega_{t,N}$.

The background induced iteration strategies of §7 are positional. This is not at all clear from their construction, but it is a consequence of strategy comparison, proved in [33].

5.2 Pullback strategies

Given an elementary $\pi \colon M \to N$ and a plus tree $\mathcal{T}$ on M, we can lift $\mathcal{T}$ to a copied tree $\pi\mathcal{T}$ on N with the same tree order as $\mathcal{T}$. The construction produces elementary copy maps

$$\pi_\alpha \colon M_\alpha \to N_\alpha,$$

where $M_\alpha = \mathcal{M}_\alpha^{\mathcal{T}}$ and $N_\alpha = \mathcal{M}_\alpha^{\pi\mathcal{T}}$. Let $E_\alpha = E_\alpha^{\mathcal{T}}$ and $F_\alpha = E_\alpha^{\pi\mathcal{T}}$. For any extender G, let $\varepsilon(G) = \mathrm{lh}(G)$ if G has plus type, and $\varepsilon(G) = \lambda_G$ otherwise. We show by induction that the copy maps commute with the branch embeddings of $\mathcal{T}$ and $\pi\mathcal{T}$, and agree with one another, in that

(1) if $\beta \leq \alpha$, then $\pi_\alpha \restriction \varepsilon(E_\beta) = \pi_\beta \restriction \varepsilon(E_\beta)$ and $N_\alpha | \varepsilon(F_\beta) = N_\beta | \varepsilon(F_\beta)$, and

(2) if $\beta \leq_T \alpha$, then $\pi_\alpha \circ \hat{\imath}^{\mathcal{T}}_{\beta,\alpha} = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha} \circ \pi_\beta$.

Set $\pi_0 = \pi$. The successor step is as follows: let $E = E_\alpha$, $\beta = T\text{-pred}(\alpha + 1)$, and

$$\begin{aligned} F &= \pi_\alpha(E), \\ P &= \mathcal{M}^{*,\mathcal{T}}_{\alpha+1}, \\ Q &= \pi_\beta(P). \end{aligned}$$

Here if $E = G^+$, where G is on the M_α sequence, then $\pi_\alpha(E) = \pi_\alpha(G)^+$.[36] The agreement between π_α and π_β implies that β is least such that $\mathrm{crit}(F) < \hat{\lambda}(F_\beta)$, and $\hat{\lambda}(F_\xi) \leq \hat{\lambda}(F)$ for all $\xi < \alpha$, with $\mathrm{lh}(F_\xi) < \hat{\lambda}(F)$ if F_ξ has plus type. We set $F_\alpha = F$, and for $k = k(P)$, let

$$\pi_{\alpha+1} \colon \mathrm{Ult}_k(P, E) \to \mathrm{Ult}_k(Q, F)$$

[36] If $G = \dot{F}^{M_\alpha}$ then $\pi_\alpha(G) = \dot{F}^{N_\alpha}$.

be the completion of the map

$$\pi_{\alpha+1}([a,f]_E^{P^k}) = [\pi_\alpha(a), \pi_\beta(f)]_F^{Q^k},$$

for $a \in [\varepsilon(E)]^{<\omega}$. One can show that $\pi\mathcal{T}$ is still a plus tree, that $\pi_{\alpha+1}$ is well defined and elementary, and that the induction hypotheses (1) and (2) still hold.[37]

Since the copy maps commute with the branch embeddings, at limit steps λ we have a unique elementary $\pi_\lambda\colon M_\lambda \to N_\lambda$ that commutes with the branch embeddings of $\mathcal{T}$ and $\pi\mathcal{T}$ along $[0,\lambda)_T$. It is easy to check (1) and (2).

If $\pi\mathcal{T}$ ever reaches an illfounded model, we stop the construction.

We can copy stacks of plus trees by successively copying the individual plus trees in the stack. For example, if $s = \langle \mathcal{T}, \mathcal{U}\rangle$, then $\pi s = \langle \pi\mathcal{T}, \sigma\mathcal{U}\rangle$, where σ is the last copy map in the $\pi\mathcal{T}$ system.

Definition 5.4. If Ω is an iteration strategy for N, and $\pi\colon M \to N$ is elementary, then Ω^π is the pullback strategy for M, given by

$$\Omega^\pi(s) = \Omega(\pi s),$$

for all s such that $\pi s \in \mathrm{dom}(\Omega)$.

If Ω is a (λ,θ)-iteration strategy for N, then Ω^π is a (λ,θ) iteration strategy for M. Thus every elementary submodel of a mouse is also a mouse.

5.3 Internal lift consistency

If $Q \trianglelefteq N$, then iteration trees on Q can be lifted to trees on N. More generally, given

$$\pi\colon M \to Q \trianglelefteq N$$

such that π is nearly elementary as a map from M to Q, and a plus tree $\mathcal{T}$ on M, we can lift $\mathcal{T}$ to a plus tree $\mathcal{U} = \pi\mathcal{T}^+$ on N in a natural way. $\mathcal{U}$ will have the same tree order as $\mathcal{T}$, so long as it is defined. Let M_α and N_α be the α-th models of $\mathcal{T}$ and $\mathcal{U}$, and E_α and F_α the α-th extenders. We shall have a nearly elementary

$$\pi_\alpha\colon M_\alpha \to Q_\alpha \trianglelefteq N_\alpha.$$

Here $\pi_0 = \pi$ and $Q_0 = Q$. We have agreement and commutativity conditions like those above. Drops in $\mathcal{T}$ of more than one degree will cause corresponding drops in $\mathcal{U}$. Drops of one degree may not. $\mathcal{U}$ may drop where $\mathcal{T}$ does not.

[37] See [35]. The fine structural parts take more work than one might expect. It is possible, and sometimes necessary, to copy under maps π that are only nearly elementary. In that case $\pi\mathcal{T}$ may not literally be a plus tree, because it may sometimes take a k-ultrapower when it could have taken a $k+1$-ultrapower.

The successor step is the following. We are given E_α on M_α; set $F_\alpha = \pi_\alpha(E_\alpha)$. Let $\beta = T\text{-pred}(\alpha+1)$; then one can show that $\beta = U\text{-pred}(\alpha+1)$ according to the rules of plus trees for $\mathcal{U}$. Let

$$M_{\alpha+1} = \mathrm{Ult}(M^*_{\alpha+1}, E_\alpha),$$

and

$$N_{\alpha+1} = \mathrm{Ult}(N^*_{\alpha+1}, F_\alpha),$$

where $M^*_{\alpha+1}$ and $N^*_{\alpha+1}$ are determined by rules of plus trees. Let

$$S = \pi_\beta(M^*_{\alpha+1}),$$

where if $M^*_{\alpha+1} = M_\beta \downarrow n$ then $S = Q_\beta \downarrow n$. Clearly $\pi_\beta \restriction M^*_{\alpha+1}$ is nearly elementary as a map into S, so $\mathrm{crit}(F_\alpha)$ is a cardinal of S and $\mathrm{crit}(F_\alpha) < \rho^-(S)$. It follows that

$$S \trianglelefteq N^*_{\alpha+1}.$$

Let $i^*\colon N^*_{\alpha+1} \to N_{\alpha+1}$ be the canonical embedding, and

$$Q_{\alpha+1} = i^*(S),$$

with $Q_{\alpha+1} = N_{\alpha+1} \downarrow n$ if $S = N^*_{\alpha+1} \downarrow n$. We obtain $\pi_{\alpha+1}$ by a variant of the Shift Lemma: let $R = M^*_{\alpha+1}$ and $k = k(R) = k(S)$. We obtain $\sigma\colon\ \mathrm{Ult}_0(R^k, E_\alpha) \to Q^k_{\alpha+1}$ by setting

$$\sigma([a,f]^{R^k}_{E_\alpha}) = [\pi_\alpha(a), \pi_\beta(f)]^{N^*_{\alpha+1}}_{F_\alpha},$$

where the equivalence class on the right is formed using functions appropriate to $\mathrm{Ult}(N^*_{\alpha+1}, F_\alpha)$. One can show that σ is Σ_0 elementary and cardinal preserving map from $M^k_{\alpha+1}$ to $Q^k_{\alpha+1}$. We let $\pi_{\alpha+1}$ be its completion.

The rest is the same as in the copying construction. The difference here is that $Q_{\alpha+1}$ is an ultrapower of S formed using functions from $N^*_{\alpha+1}$, not just those from S. As a consequence, $\pi_{\alpha+1}$ may be only nearly elementary, even if π_0 is elementary.

We can make sense of πs^+ for s an M-stack by repeatedly lifting the plus trees in s, as before.

Definition 5.5. Let Ω be a (λ,θ)-iteration strategy for a pfs premouse M. We say Ω is *internally lift consistent* iff whenever s is a stack by Ω and $N \trianglelefteq M_\infty(s)$, then for any N-stack t, letting π be the identity, t is by $\Omega_{s,N}$ iff πt^+ is by Ω_s.

5.4 Strong Hull Condensation

This is probably the most important regularity property for iteration strategies. It asserts that if $\mathcal{U}$ is by Σ, and there is a *tree embedding* from $\mathcal{T}$ to $\mathcal{U}$, then $\mathcal{T}$ is by Σ.

Strong Hull Condensation is inspired by the Hull Condensation property of [23]. There one is given map $\sigma : \mathrm{lh}(\mathcal{T}) \to \mathrm{lh}(\mathcal{U})$ and and embeddings $\tau_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{\sigma(\alpha)}$ for $\alpha < \mathrm{lh}(\mathcal{T})$. σ

preserves tree order and tree predecessor. The τ_α's have the agreement one would get from a copying construction, and they commute with the branch embeddings of $\mathcal{T}$ and $\mathcal{U}$. Moreover, $\tau_\alpha(E_\alpha^\mathcal{T}) = E_{\sigma(\alpha)}^\mathcal{U}$. A simple example is the way $\mathcal{T} = \pi\mathcal{W}$ sits inside $\mathcal{U} = \pi(\mathcal{W})$, in the case $\pi\colon H \to V$ is elementary and $\pi \restriction (M \cup \{M\}) = \text{ id }$.

A tree embedding from $\mathcal{T}$ into $\mathcal{U}$ is a tuple with most of the properties of σ, $\vec{\tau}$, ψ above.[38] The pair $(\sigma, \vec{\tau})$ is resolved into two pairs: the pair $(v, \vec{s})$, which embeds the models of $\mathcal{T}$ into models of $\mathcal{U}$ in a minimal way, and the pair $(u, \vec{t})$, which connects the exit extenders of $\mathcal{T}$ to exit extenders in $\mathcal{U}$. The requirement that σ preserves tree predecessors is relaxed to the requirement that if $\beta = T\text{-pred}(\gamma+1)$, then $U\text{-pred}(u(\gamma)+1) \in [v(\beta), u(\beta)]_U$. We shall also allow the t_α's to be partial, in a controlled way. Recall here the partial branch embeddings $\hat{\imath}^\mathcal{U}_{\alpha,\beta}$. Recall also that $\varepsilon(E) = \text{lh}(E)$ if E has plus type, and $\varepsilon(E) = \lambda(E)$ otherwise.

Definition 5.6. Let $\mathcal{T}$ and $\mathcal{U}$ be plus trees on a premouse M, with $\text{lh}(\mathcal{T}) > 1$. A *tree embedding* of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, v, \langle s_\beta \mid \beta < \text{lh}(\mathcal{T})\rangle, \langle t_\beta \mid \beta + 1 < \text{lh}(\mathcal{T})\rangle\rangle$$

such that

(a) $u : \{\alpha \mid \alpha + 1 < \text{lh}(\mathcal{T})\} \to \{\alpha \mid \alpha + 1 < \text{lh}(\mathcal{U})\}$, and $\alpha < \beta \to u(\alpha) < u(\beta)$.

(b) $v\colon \text{lh}(\mathcal{T}) \to \text{lh}(\mathcal{U})$, v preserves tree order and is continuous at limit ordinals, $v(0) = 0$, and $v(\alpha+1) = u(\alpha)+1$.

(c) $s_\beta : \mathcal{M}_\beta^\mathcal{T} \to \mathcal{M}^\mathcal{U}_{v(\beta)}$ is elementary, and $s_0 = \text{ id }$; moreover for $\alpha <_T \beta$,

$$s_\beta \circ \hat{\imath}^\mathcal{T}_{\alpha,\beta} = \hat{\imath}^\mathcal{U}_{v(\alpha),v(\beta)} \circ s_\alpha.$$

In particular, the two sides have the same domain.

(d) For $\alpha + 1 < \text{lh}(\mathcal{T})$, $v(\alpha) \leq_U u(\alpha)$, and

$$t_\alpha = \hat{\imath}^\mathcal{U}_{v(\alpha),u(\alpha)} \circ s_\alpha.$$

Moreover, if $E_\alpha^\mathcal{T}$ is of plus type, then

$$E^\mathcal{U}_{u(\alpha)} = t_\alpha(E_\alpha^\mathcal{T}),$$

and if $E_\alpha^\mathcal{T}$ is not of plus type, then

$$E^\mathcal{U}_{u(\alpha)} \in \{t_\alpha(E_\alpha^\mathcal{T}), t_\alpha(E_\alpha^\mathcal{T})^+\}.$$

(e) For $\alpha < \beta < \text{lh}(\mathcal{T})$,

$$s_\beta \restriction \varepsilon(E_\alpha^\mathcal{T}) = t_\alpha \restriction \varepsilon(E_\alpha^\mathcal{T}).$$

[38] The notion was isolated independently by Farmer Schlutzenberg. See [28]. Schlutzenberg's term is *inflationary map*.

(f) If $\beta = T\text{-pred}(\alpha+1)$, then $U\text{-pred}(u(\alpha)+1) \in [v(\beta), u(\beta)]_U$, and setting $\beta^* = U\text{-pred}(u(\alpha)+1)$, $P = \mathcal{M}^{*,\mathcal{T}}_{\alpha+1}$, and $Q = \mathcal{M}^{*,\mathcal{U}}_{u(\alpha)+1}$

$$s_{\alpha+1}([a, f]^P_{E^{\mathcal{T}}_\alpha}) = [t_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta(f)]^Q_{E^{\mathcal{U}}_{u(\alpha)}}.$$

The map $s_{\alpha+1}$ in clause (f) is essentially the copy map associated to $(t_\alpha, \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta, E^{\mathcal{T}}_\alpha)$. (It is not literally that if $E^{\mathcal{U}}_\alpha$ is of plus type but $E^{\mathcal{T}}_\alpha$ is not.) One can show that there is always enough agreement between t_α and $\hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta$ that copying is possible, and produces an elementary map.

The appropriate diagram to go with (f) of Definition 5.6 (for the non-dropping case is)

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[rr, "s_{\alpha+1}"] & & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} \\
& \mathcal{M}^{\mathcal{U}}_{u(\beta)} & \\
& & \mathcal{M}^{\mathcal{U}}_{\beta^*} \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \arrow[ul] \\
\mathcal{M}^{\mathcal{T}}_\beta \arrow[uuu, "E^{\mathcal{T}}_\alpha"] \arrow[uur, "t_\beta"] \arrow[urr, "\rho"] \arrow[rr, "s_\beta"] & & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u] \\
\mathcal{M}^{\mathcal{T}}_\alpha \arrow[rr, "t_\alpha"] & & \mathcal{M}^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

Definition 5.7. For plus trees $\mathcal{T}$ and $\mathcal{U}$,

(a) $\Phi\colon \mathcal{T} \to \mathcal{U}$ iff Φ is a tree embedding of $\mathcal{T}$ into $\mathcal{U}$,

(b) $\mathcal{T}$ is a *pseudo-hull* of $\mathcal{U}$ iff there is a tree embedding of $\mathcal{T}$ into $\mathcal{U}$.

Definition 5.8. Let Σ be a complete (λ, θ) iteration strategy for a pfs premouse M; then Σ has *strong hull condensation* iff whenever s is a stack of plus trees by Σ and $N \trianglelefteq \mathcal{M}_\infty(s)$, and $\mathcal{U}$ is a plus tree on N by $\Sigma_{s,N}$, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is a tree embedding, then

(a) $\mathcal{T}$ is by $\Sigma_{s,N}$, and

(b) if $\alpha < \text{lh}(\mathcal{T})$ and $v(\alpha) \leq_U \beta$ and $\pi = \hat{\imath}_{v(\alpha),\beta} \circ s^\Phi_\alpha$, then for any $Q \trianglelefteq \text{dom}(\pi)$, $\Sigma_{s^\frown\langle\mathcal{T}\restriction\alpha+1\rangle,Q} = (\Sigma_{s^\frown\langle\mathcal{U}\rangle,\pi(Q)})^\pi$.

Because less is required of a tree embedding than is required of a hull embedding in [23], the property is stronger than the property called Hull Condensation in [23]. Hence its name.

Clause (b) of 5.8 implies a very useful property of iteration strategies, *pullback consistency*. Roughly, a strategy is pullback consistent iff it pulls back to itself under its own iteration maps.

Definition 5.9. Let Ω be a complete (λ, θ) iteration strategy for a premouse M. We say that Ω is *pullback consistent* iff whenever $s^\frown\langle P, \mathcal{T}\rangle$ is an M-stack by Ω, $\alpha <_T \beta$, $K \trianglelefteq \mathcal{M}^{\mathcal{T}}_\alpha$, and $L = \hat{\imath}^{\mathcal{T}}_{\alpha,\beta}(K)$, then

$$\Omega_{s^\frown\langle P,\mathcal{T}\restriction\alpha+1\rangle,K} = (\Omega_{s^\frown\langle P,\mathcal{T}\restriction\beta+1\rangle,L})^{\hat{\imath}^{\mathcal{T}}_{\alpha,\beta}}.$$

The definition applies even if there are drops along the branch of $\mathcal{T}$ from α to β, so long as K is in the domain of the partial iteration map $\hat{\imath} = \hat{\imath}^{\mathcal{T}}_{\alpha,\beta}$. Indeed $K = \mathrm{dom}(\hat{\imath})$ is possible, in which case $L = \mathcal{M}^{\mathcal{T}}_{\beta}$.

We have stated pullback consistency for pullbacks within a single normal tree $\mathcal{T}$, but this implies we can pull back consistently from one normal tree in a stack into any previous one, step by step. This is simply because $\Omega^{i \circ j} = (\Omega^i)^j$.

Lemma 5.10. *Let Σ be a (λ, θ) iteration strategy for M that has strong hull condensation; then Σ is pullback consistent.*

Proof. (Sketch.) For example, let $\mathcal{U}$ be a plus tree on M, $N = \mathcal{M}^{\mathcal{U}}_{\beta}$, and $i = i^{\mathcal{U}}_{0,\beta}$. Let $\mathcal{T}$ be the empty tree on M. There is a trivial tree embedding Φ of $\mathcal{T}$ into $\mathcal{U}$: $v^{\Phi}(0) = 0$ and $s^{\Phi}_0 =$ id. Applying 5.8(b), we get that $\Sigma = (\Sigma_{\mathcal{U}\restriction\beta+1,N})^i$, as desired. The general case just involves more notation. □

5.5 Normalizing well

Given an M-stack $s = \langle \mathcal{T}, \mathcal{U} \rangle$ with last model N such that $\mathcal{T}$ and $\mathcal{U}$ are normal, shuffling the extenders of $\mathcal{U}$ into $\mathcal{T}$ in a minimal way produces a normal tree $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ with last model R, and a nearly elementary map $\pi \colon N \to R$. We call $W(\mathcal{T}, \mathcal{U})$ the *embedding normalization* of $\langle \mathcal{T}, \mathcal{U} \rangle$. The idea is simple, but there are many technical details, so we refer the reader to [35] for a formal definition.[39]

It proves useful to consider a slightly less minimal shuffling $V(\mathcal{T}, \mathcal{U})$ that we call the *quasi-normalization* of $\langle \mathcal{T}, \mathcal{U} \rangle$. $V(\mathcal{T}, \mathcal{U})$ is a plus tree, but it may not be length-increasing. If $\mathcal{T}$ is λ-separated and $\mathcal{U}$ is normal, then $W(\mathcal{T}, \mathcal{U}) = V(\mathcal{T}, \mathcal{U})$. If they are merely normal, then $W(\mathcal{T}, \mathcal{U})$ is the normal companion of $V(\mathcal{T}, \mathcal{U})$, but they may be different. We refer the reader to [35] for a full definition.

Definition 5.11. Let Σ be a (λ, θ)-iteration strategy for a pfs premouse M, where $\lambda > 1$. We say that Σ *quasi-normalizes well* iff whenever s is an M-stack by Σ, and $\langle \mathcal{T}, \mathcal{U} \rangle$ is a maximal 2-stack by Σ_s such that $\mathcal{U}$ is normal, then

(a) $V(\mathcal{T}, \mathcal{U})$ is by Σ_s, and

(b) letting $\mathcal{V} = V(\mathcal{T}, \mathcal{U})$ and $\pi \colon \mathcal{M}^{\mathcal{U}}_{\infty} \to \mathcal{M}^{\mathcal{V}}_{\infty}$ be the map generated by quasi-normalization, we have that $\Sigma_{s^\frown\langle \mathcal{T}, \mathcal{U} \rangle} = (\Sigma_{s^\frown \mathcal{V}})^{\pi}$.

In clause (b), the map π may be only nearly elementary, but that is sufficient to pull back an iteration strategy.

Definition 5.12. Let Σ be a (λ, θ)-iteration strategy for M, where $\lambda > 1$; then Σ *normalizes well* iff

[39] Much of the general theory of normalization was developed independently by Schlutzenberg. See [28]. See also [13] and [32].

(a) Σ quasi-normalizes well, and

(b) whenever s is an M-stack by Σ, and $\mathcal{T}$ is a plus tree by Σ_s, and $\mathcal{U}$ is the normal companion of $\mathcal{T}$, then $\mathcal{U}$ is by Σ_s.

Clearly, if Σ normalizes or quasi-normalizes well, then so do all its tail strategies. One can make sense of $W(s)$ and $V(s)$ for arbitrary stacks s, and show that a strategy that (quasi-) normalizes well for stacks of length 2 must (quasi-) normalize well for all finite stacks.[40]

6 A comparison lemma for pure extender pairs

We collect some of these regularity properties in a definition.

Definition 6.1. (P, Σ) is a *pure extender pair* with scope H_δ iff

(1) P is a pfs premouse, and $P \in H_\delta$,

(2) Σ is a (ω, δ) iteration strategy for P, and

(3) Σ quasi-normalizes well, has strong hull condensation, and is internally lift consistent.

(P, Σ) is *projectum stable* iff P is projectum stable.

We are only interested in the case that Σ is absolutely definable. In the most important context, P is countable, Σ has scope H_{ω_1}, and its absolute definability is witnessed by membership in a model of AD^+. One can show that in any case, Σ is determined by its action on countable, λ-separated plus trees.([35].)

It would be more natural to require that an iteration strategy with scope H_δ be a (δ, δ)-strategy, but then our comparison proof for pure extender pairs would need to go into quasi-normalizing infinite stacks. There seems to be no obstacle to doing this.

If (M, Σ) is a pure extender pair, and s is a stack by Σ with last model N, then we call (N, Σ_s) an *iterate of* (M, Σ). If the branch M-to-N of s does not drop, we call it a non-dropping iterate. In that case, we have an iteration map $i_s \colon M \to N$.

Theorem 6.2 (Comparison for pure extender pairs)**.** *Assume* AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be projectum stable pure extender pairs with scope HC; then they have a common iterate* (R, Ω) *such that on at least one of the two sides, the iteration does not drop.*

The comparison process in the proof produces single λ-separated trees $\mathcal{T}$ and $\mathcal{U}$ leading from P and Q to R. However, it does not proceed by iterating away least disagreements. It is not even clear what that would mean in the case that the current tail strategies have a disagreement. Instead, we go back toward the original Kunen method of fixing a standard

[40]The routine proof is in [35]. There are deeper results about $W(s)$ when s is infinite that are due to Schlutzenberg. See [28]. Benjamin Siskind proved in [32] that embedding normalization is associative.

structure in advance, although now we start with an array of candidates for (R, Ω), not just one. §8 has a brief outline of the proof.[41]

Working in the category of mouse pairs enables us to state a general Dodd-Jensen lemma. Let us say $\pi\colon (P, \Sigma) \to (Q, \Psi)$ is elementary iff π is elementary from P to Q, and $\Sigma = \Psi^\pi$.

Lemma 6.3. *If (Q, Ψ) is a pure extender pair with scope H_δ and $\pi\colon (P, \Sigma) \to (Q, \Psi)$ is elementary, then (P, Σ) is a pure extender pair with scope H_δ.*

Proof. (Sketch.) One must show that strong hull condensation, normalizing well, and internal lift consistency pass from Ψ to its π-pullback Σ. This involves a lot of diagram-chasing, all of it routine. □

The iteration maps associated to a pure extender pair are elementary in the category of mouse pairs. For example

Lemma 6.4. *Let (P, Σ) be a projectum stable pure extender pair, and (R, Ω) a non-dropping iterate of (P, Σ) with iteration map $i\colon P \to R$; then i is elementary as a map from (P, Σ) to (R, Ω).*

This is just a re-statement of pullback consistency.

Theorem 6.5 (Dodd-Jensen lemma)**.** *Let (P, Σ) be a projectum stable pure extender pair, and (Q, Ψ) be an iterate of (P, Σ) via the stack s. Suppose $\pi\colon (P, \Sigma) \to (Q, \Psi)$ is elementary; then s does not drop, and for all ordinals $\eta \in P$, $i_s(\eta) \leq \pi(\eta)$.*

The proof is just the usual Dodd-Jensen proof; the point is just that the language of mouse pairs enables us to formulate the theorem in its proper generality. There is no need to restrict to mice with unique iteration strategies, as is usually done.

Similarly, we can define the mouse order in its proper generality, without restricting to mice with unique iteration strategies.

Definition 6.6. If (P, Σ) and (Q, Ψ) are projectum stable pure extender pairs with scope H_δ, then $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) can be elementarily embedded into an iterate of (Q, Ψ).

Theorems 6.2 and 6.5 yield

Corollary 6.7. *Assume AD^+; then $\leq^*$ is a prewellorder on the projectum stable pure extender pairs with scope HC. Moreover, $(P, \Sigma) <^* (Q, \Psi)$ iff (P, Σ) can be elementarily embedded into a dropping iterate of (Q, Ψ).*

We have made quasi-normalizing well part of the definition of pure extender pair because this is what one gets directly from the construction in §7, and it suffices for strategy comparison. Using comparison arguments, one can show

Theorem 6.8. *([33]) Assume AD^+, and let (P, Σ) be a projectum stable pure extender pair with scope HC; then Σ normalizes well and is positional.*

[41] It is possible to compare pairs that are not projectum stable. See [35].

7 Background-induced iteration strategies

We construct a mouse M by adding extenders to its coherent sequence, one by one. If we add E, then $M|\,\mathrm{lh}(E)$ must be a premouse, and this imposes a fairly severe restriction on E. Nevertheless, no first-order requirement like premousehood can guarantee that we are building a standard structure, one that can be compared with others of its kind. We need to be building an *iterable* premouse. Moreover, it is not enough that $M|\,\mathrm{lh}(E)$ be iterable, for we need the full M to be iterable, and when we add E, we don't know what M will be.

The standard way to solve these difficulties is to demand a *background certificate* E^* for E. What exactly one demands of E^* depends on the context. In this paper we shall ask that E^* be a nice extender over V such that $E \restriction \lambda_E \subseteq E^*$. In contexts where one is trying to construct mice without assuming there are large cardinals at all, much more care is needed at this point, and the iterability proofs become more difficult.

In any of its forms, the background certificate demand conflicts with the demand that our mice be sound. The standard way to solve that difficulty is to "core down" at every step, replacing the current approximation to M by its core. There are highly nontrivial comparison arguments involved in showing that this core exists, and agrees sufficiently with M that the process of adding certified extenders and coring down converges to anything. These arguments rely on the iterability of M.

The existence of full background extender certificates means that we can lift iteration trees on M to iteration trees on V, and thus use an iteration strategy Σ^* for V to induce an iteration strategy Σ for M. This of course does not solve the iterability problem for M, it just reduces it to the problem for V. But some such reduction, ideally using weaker background certificates, seems inevitable in any construction of iteration strategies for premice. M cannot see the iteration trees with respect to which it must be iterable, but V can see their lifts. Moreover, those lifts can be taken to be simple (for example, use only nice extenders) in ways that the trees on M being lifted are not.

Of course, one cannot prove that there are any nontrivial mice without making assumptions that go beyond ZF. Determinacy assumptions are particularly useful in this regard. Under AD^+, every Suslin-co-Suslin set is Wadge reducible to an iteration strategy for a countable *coarse premouse*. This is a result of Hugh Woodin from the late 1980s. We shall use these coarse premice as background universes in which to construct the fine ones.

Definition 7.1. Let E be an extender over V; then E is *nice* iff

(a) E is strictly short, that is, $\mathrm{lh}(E) < \lambda(E)$,

(b) for some ν, $\mathrm{lh}(E)$ is the least strongly inaccessible η such that $\nu < \eta$,

(c) $V_{\mathrm{lh}(E)} \subseteq \mathrm{Ult}(V, E)$.

Definition 7.2. A sequence $\vec{F} = \langle F_\alpha \mid \alpha < \mu \rangle$ is *coarsely coherent* iff each F_α is a nice extender over V, and

(1) if G is a nice initial segment of F_α, then $G = F_\beta$ for some $\beta < \alpha$,

(2) if $\beta < \alpha$, then $\mathrm{lh}(F_\beta) \le \mathrm{lh}(F_\alpha)$, and

(3) $i\colon V \to \mathrm{Ult}(V, F_\alpha)$ is the canonical embedding, and $\vec{E} = i(\vec{F})$, then $\langle E_\xi \mid \mathrm{lh}(E_\xi) \le \mathrm{lh}(F_\alpha)\rangle = \langle F_\xi \mid \xi < \alpha\rangle$.

An $\vec{F}$-*tree* is an iteration tree $\mathcal{T}$ such that for all α, $E_\alpha^{\mathcal{T}} \in i_{0,\alpha}^{\mathcal{T}}(\vec{F})$.

Definition 7.3. We say that $((M, \vec{F}, \delta), \Sigma)$ is a *coarse extender pair* iff

(a) M is countable and transitive, and $M \models \mathsf{ZFC}$ + "δ is Woodin via $\vec{F}$", and

(b) Σ is an (ω_1, ω_1)-iteration strategy acting on $\vec{F}$-trees that normalizes well and has strong hull condensation.

Definition 7.4. Let $((M, \vec{F}, \delta), \Sigma)$ be a coarse extender pair, and $A \subset \mathbb{R}$; then $((M, \vec{F}, \delta), \Sigma)$ *captures* A iff there is a $\tau \in M$ such that

(a) τ is a $\mathrm{Col}(\omega, \delta)$-term for a set of reals, and

(b) whenever $i\colon M \to N$ is an iteration map by Σ and g is $\mathrm{Col}(\omega, i(\delta))$-generic over N, then $i(\tau)_g = A \cap N[g]$.

Notice here that $(M, \vec{F}, \delta, \tau, \Sigma)$ determines A, because for every real x there is a genericity iteration leading to N and g as in (d) such that $x \in N[g]$.

The following is a direct consequence of Woodin's work in the late 1980s on large cardinals in HOD under determinacy hypotheses. See [16] and [42].

Theorem 7.5. *Assume* AD*; then for any Suslin and co-Suslin set A, there is a coarse extender pair $((M, \vec{F}, \delta), \Sigma)$ that captures A.*

The reason that the iteration strategy Σ produced in [42] normalizes well and has strong hull condensation is that there is some set T of ordinals such that $V_\delta^M = V_\delta^{L[T,M]}$, and Σ chooses unique wellfounded branches when thought of as a strategy for $\vec{F}$-trees with base model $L[T, M]$.[42] Normalizing well and strong hull condensation are properties of the way Σ acts on trees on the countable model M that follow easily from the fact that it chooses unique wellfounded branches for trees on the uncountable model $L[T, M]$.

Now let $(\mathcal{N}, \Sigma)$ be a coarse extender pair, with $\mathcal{N} = (N, \vec{F}, \delta)$. In $\mathcal{N}$, we do a maximal full background extender construction, where the background extenders are taken from $\vec{F}$. This produces a sequence

$$\mathbb{C} = \langle (M_{\nu,k}, \Omega_{\nu,k}) \mid \langle \nu, k\rangle \le_{\mathrm{lex}} \langle \delta, 0\rangle\rangle$$

such that for each ν, k, $M_{\nu,k}$ is a pfs premouse of soundness degree k and $\Omega_{\nu,k}$ is the (ω_1, ω_1)-iteration strategy induced by converting stacks s on $M_{\nu,k}$ to stacks $\mathrm{lift}(M_{\nu,k}, s, \mathbb{C})$ of $\vec{F}$-trees on $\mathcal{N}$, and defining

$$s \text{ is by } \Omega_{\nu,k}(\mathcal{T}) \text{ iff } \mathrm{lift}(M_{\nu,k}, s, \mathbb{C}) \text{ is by } \Sigma.$$

[42] T codes Suslin representations for A and its complement, thereby generating a term that captures A. There is some work beyond [42] involved in producing a coarsely coherent $\vec{F}$.

We start with $M_{0,0}$ equal to the passive premouse with universe V_ω. Given $M_{\nu,k}$, we show using the existence of $\Omega_{\nu,k}$ that $M_{\nu,k}$ is solid. Here the pfs fine structure complicates the standard comparison arguments, but not fatally.[43] We then core down, setting

$$M_{\nu,k+1} = \mathfrak{C}(M_{\nu,k}),$$

and letting $\Omega_{\nu,k+1}$ be the strategy induced by lifting to $\mathcal{N}$. Extenders get added to the sequence of our evolving model as follows: at any limit ordinal ν we have produced a passive premouse $M^{<\nu}$ from the previous stages. Suppose there is an F such that

(i) $(M^{<\nu}, F)$ is a pfs premouse of degree 0, and

[(ii) for some F^* in $\vec{F}$, $F \restriction \lambda_F \subseteq F^*$.

One can show using iterability that there is at most one such F.[44] We then set

$$M_{\nu,0} = (M^{<\nu}, F).$$

We can use the existence of F^* to define $\text{lift}(M_{\nu,0}, s, \mathbb{C})$, and thereby $\Omega_{\nu,0}$.[45] If there are no such F and F^*, then we set $M_{\nu,0} = (M^{<\nu}, \emptyset)$.

$\text{lift}(M_{\nu,k}, s, \mathbb{C})$ is defined for all $M_{\nu,k}$-stacks s in V, not just those in $\mathcal{N}$. Since Σ is an (ω_1, ω_1)-strategy in V, so is $\Omega_{\nu,k}$. Moreover,

Lemma 7.6. *Let Ψ be the restriction of $\Omega_{\nu,k}$ to finite stacks; then $(M_{\nu,k}, \Psi)$ is a pure extender pair.*

It is the proof of this lemma that makes the complications of pfs fine structure, plus trees, and quasi-normalization necessary. The connection between Σ and its induced strategy $\Omega_{\nu,k}$ is not sufficiently tight that one can prove directly that $\Omega_{\nu,k}$ inherits the property of normalizing well. One gets only that it quasi-normalizes well. Moreover, even the proof that $\Omega_{\nu,k}$ quasi-normalizes well would fall apart if we took cores in the standard way.

8 The comparison proof

Here is a very brief sketch of the comparison process behind Theorem 6.2. Our goal is just to indicate where the regularity properties of pure extender pairs enter.

Definition 8.1. Let (M, Σ) and (N, Ω) be pure extender; then

[43] One complication is that $M_{\nu,k}$ will sometimes fail to be projectum stable. This is why we did not make projectum stability part of the definition of *pfs premouse*.

[44] This is the *Bicephalus Lemma*.

[45] It is important here to prove that $F^+ \restriction (\lambda_F + 1) \subseteq F^*$, so that we can lift plus trees. The proof is a comparison argument like the proof of [21, Theorem 10.1]. It is not good enough to simply make $F^+ \restriction (\lambda_F + 1) \subseteq F^*$ an additional requirement for adding F, because then the premice we are constructing might not have enough extenders on their sequences to be useful.

(a) (M, Σ) *iterates past* (N, Ω) iff there is a λ-separated tree $\mathcal{T}$ by Σ on M whose last pair is (N, Ω).

(b) (M, Σ) *iterates to* (N, Ω) iff there is a normal $\mathcal{T}$ as in (a) such that the branch M-to-N of $\mathcal{T}$ does not drop.

(c) (M, Σ) *iterates strictly past* (N, Ω) iff it iterates past (N, Ω), but not to (N, Ω).

The main lemma is

Lemma 8.2. *Assume* AD^+, *let* (P, Σ) *be a pure extender pair, and let* $(\mathcal{N}, \Psi)$ *be a coarse extender pair such that* $P \in HC^{\mathcal{N}}$ *and* $(\mathcal{N}, \Psi)$ *captures* Code(Σ). *Let* $\mathbb{C}$ *be the maximal full background construction of* $\mathcal{N}$*; then there is a level* (M, Ω) *of* $\mathbb{C}$ *such that*

(a) (P, Σ) *iterates to* (M, Ω)*, and*

(b) (P, Σ) *iterates strictly past all levels of* $\mathbb{C}$ *that are strictly earlier than* (M, Ω).

This is enough to compare two pure extender pairs (P, Σ) and (Q, Λ). We simply find a coarse extender pair $(\mathcal{N}, \Psi)$ that captures both of them, let $\mathbb{C}$ be its construction, and then look for the least level (M, Ω) of $\mathbb{C}$ that one of the two pairs iterates to it. If that pair is (P, Σ), then $(P, \Sigma) \leq^* (Q, \Lambda)$. Otherwise $(Q, \Lambda) <^* (P, \Sigma)$.

Let us sketch the proof of Lemma 8.2. Suppose (M, Ω) is a level of $\mathbb{C}$ such that (P, Σ) iterates strictly past all earlier levels. The main new thing is to show that no strategy disagreements show up when we compare (P, Σ) with (M, Ω).[46] Suppose we have produced a λ-separated tree $\mathcal{T}$ on P with last model Q, and that $Q|\alpha = M|\alpha$, and that $\mathcal{U}$ is a normal tree on $R = Q|\alpha = M|\alpha$ played by both $\Sigma_{\mathcal{T}, Q|\alpha}$ and Ω. Let $\mathcal{U}$ have limit length, and let $b = \Omega_R(\mathcal{U})$. We must see $b = \Sigma_{\mathcal{T}, R}(\mathcal{U})$. .

The internal lift consistency of Σ lets us reduce to the case that $R = Q$, so let us assume that. We now look we look at the embedding normalization $W(\mathcal{T}, \mathcal{U})$ of $\langle \mathcal{T}, \mathcal{U} \rangle$, which also has limit length. Since $\mathcal{T}$ is λ-separated, $W(\mathcal{T}, \mathcal{U}) = V(\mathcal{T}, \mathcal{U})$. For any cofinal branch c of $\mathcal{U}$, let $\mathcal{W}_c = W(\mathcal{T}, \mathcal{U}^\frown c)$, and let $\lambda = \text{lh}(W(\mathcal{T}, \mathcal{U})$. One can show

(1) For any cofinal branch c of $\mathcal{U}$, $\mathcal{W}_c \restriction \lambda = W(\mathcal{T}, \mathcal{U})$; moreover, c is determined by $[0, \lambda]_{W_c}$.

(2) Since Σ normalizes well, $\Sigma(\langle \mathcal{T}, \mathcal{U} \rangle) = c$ iff $\mathcal{W}_c \restriction \lambda + 1$ is by Σ.

(3) Letting $i_b^* : \mathcal{N} \to \mathcal{N}_b$ come from lifting $i_b^{\mathcal{U}}$ to $\mathcal{N}$, we have that $\mathcal{W}_b$ is a pseudo-hull of $i_b^*(\mathcal{T})$. This is the key step in the proof.

(4) $i_b^*(\Sigma) \subseteq \Sigma$ because Code(Σ) was captured by $\mathcal{N}$, so $i_b^*(\mathcal{T})$ is by Σ.

(5) Since Σ has strong hull condensation, $\mathcal{W}_b$ is by Σ.

(6) By (2), $\Sigma(\langle \mathcal{T}, \mathcal{U} \rangle) = b$.

$$
\begin{tikzcd}
M_b^{\mathcal{U}} \arrow[r] & M_\infty^{\mathcal{W}_b} \arrow[r] & i_b^*(R) & N_b^* \\
R \arrow[u, "\mathcal{U}"] & & R \arrow[u, "i_b^*"'] & N^* \arrow[u, "i_b^*"'] \\
P \arrow[u, "\mathcal{T}"] \arrow[uur, "\mathcal{W}_b"] \arrow[uurr, "i_b^*(\mathcal{T})"'] & & &
\end{tikzcd}
$$

Figure 3: Proof of Lemma 8.2. $\mathcal{W}_b$ is a psuedo-hull of $i_b^*(\mathcal{T})$.

Figure 3 is a diagram of the situation.

To finish the proof of Lemma 8.2, we must show that (P, Σ) does not iterate past $(M_{\delta,0}, \Omega_{\delta,0})$. This is a consequence of the fact that δ is Woodin in $\mathcal{N}$.

9 Strategy mice and HOD

The study of HOD in models of AD has a long history.[47] HOD was studied by purely descriptive set theoretic methods in the late 1970s and 1980s, and partial results on basic questions such as whether HOD $\models \mathsf{GCH}$ were obtained then. It was known then that inner model theory, if only one could develop it in sufficient generality, would be relevant to characterizing the reals in HOD. Theorem 4.4 bears that out, as do the instances of MSC for models of AD beyond $L(\mathbb{R})$ that have been proved since then.

Just how relevant inner model theory is to the study of HOD in models of AD became clear in 1994, when the author showed that if there are ω Woodin cardinals with a measurable above

[46] No extenders on the M-sequence are part of a least disagreement by the proof of Theorem 3.1. This is basically folklore; see for example [26, Theorem 2.5].

[47] See [43] for a survey of some of this history.

them all, then $\mathrm{HOD}^{L(\mathbb{R})}$ up to $\theta^{L(\mathbb{R})}$ is a pure extender mouse.[48] Shortly afterward, this result was improved by Hugh Woodin, who reduced its hypothesis to $\mathsf{AD}^{L(\mathbb{R})}$, and identified the full $\mathrm{HOD}^{L(\mathbb{R})}$ as a model of the form $L[M, \Sigma]$, where M is a pure extender premouse, and Σ is a partial iteration strategy for M. $\mathrm{HOD}^{L(\mathbb{R})}$ is thus a new type of mouse, sometimes called a *strategy mouse*, sometimes called a *hod mouse*. See [46] for an account of this work.

Since the mid-1990s, there has been a great deal of work devoted to extending these results to models of determinacy beyond $L(\mathbb{R})$. Woodin analyzed HOD in models of AD^+ below the minimal model of $\mathsf{AD}_{\mathbb{R}}$ fine structurally, and Sargsyan extended the analysis further, first to determinacy models below $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular", and more recently, to models of still stronger forms of determinacy.[49] Part of the motivation for this work is that it seems to be essential in the core model induction: in general, the next iteration strategy seems to be a strategy for a hod mouse, not for a pure extender mouse. This idea comes from work of Woodin and Ketchersid around 2000.[50]

This work has been limited by very complicated notions of strategy premouse, made necessary by the lack of a general method for comparing iteration strategies. The comparison process behind Theorem 6.2 fills that gap, at least in the short extender realm, and makes possible a much simpler and more natural premouse notion. The resulting premice are called *least branch premice* (lpm's), and the pairs (M, Σ) are called *least branch hod pairs* (lbr hod pairs).

A least branch premouse M is like a pure extender premouse, but it has an additional predicate $\dot{\Sigma}^M$ that is used to describe an iteration strategy for M. The lpm rules require that the least missing piece of strategy information be addded at essentially every stage.[51] A least branch hod pair (M, Σ) consists of a countable lpm M together with an (ω, ω_1)-iteration strategy Σ for M that is internally lift consistent, normalizes well, and has strong hull condensation. In addition we demand that $\dot{\Sigma}^M \subseteq \Sigma$, and more generally, that whenever N is a Σ-iterate of M via the stack s, then

$$\dot{\Sigma}^N \subseteq \Sigma_{s,N}.$$

This property of (M, Σ) is called *pushforward consistency.*

Least branch hod pairs can be used to analyze HOD fine structurally, provided there are enough of them.

Definition 9.1 (AD^+). (a) *Hod Pair Capturing* (HPC) is the assertion: for every Suslin-co-Suslin set A, there is a least branch hod pair (P, Σ) such that A is definable from parameters over $(\mathrm{HC}, \in, \Sigma)$.

(b) $L[E]$ *capturing* (LEC) is the assertion: for every Suslin-co-Suslin set A, there is a pure extender pair (P, Σ) such that A is definable from parameters over $(\mathrm{HC}, \in, \Sigma)$.

---

[48] See [45]. In a determinacy context, θ denotes the least ordinal that is not the surjective image of the reals.

[49] (See [23], [24]), and [25]. The determinacy principles dealt with here are all weaker than a Woodin limit of Woodin cardinals.

[50] See [15] and [27].

[51] Adding the strategy this way was originally suggested by Woodin. There are some fine-structural problems with the precise method for inserting strategy information he proposed. One method for strategy insertion that is correct in detail is due to Schlutzenberg and Trang. Cf. [35].

An equivalent (under AD^+) formulation would be that the sets of reals coding strategies of the type in question, under some natural map of the reals onto HC, are Wadge cofinal in the Suslin-co-Suslin sets of reals. The restriction to Suslin-co-Suslin sets A is necessary, for AD^+ implies that if (P, Σ) is a pair of one of the two types, then the codeset of Σ is Suslin and co-Suslin. This is the main result of [36], where it is also shown that the Suslin representation constructed is of optimal logical complexity.

Remark. HPC is a cousin of Sargsyan's *Generation of Full Pointclasses*. See [23] and [24], §6.1.

Assuming AD^+, LEC is equivalent to MSC, as shown in [41, 16.6]. [35] shows that under AD^+, LEC implies HPC. We do not know whether HPC implies LEC.

Granted $\mathsf{AD}_{\mathbb{R}}$ and HPC, we have enough hod pairs to analyze HOD.

Theorem 9.2 ([36]). *Assume $\mathsf{AD}_{\mathbb{R}}$ and HPC; then $V_\theta \cap HOD$ is the universe of a least branch premouse, and thus $HOD \models \mathsf{GCH}$.*

If we assume $\mathsf{AD}^+ +$ "there is an (ω_1, ω_1) iteration strategy for a pure extender premouse with a long extender on its sequence"; then LEC and hence HPC hold in all initial segments of the Wadge hierarchy below the Wadge-least such strategy. This leads to

Theorem 9.3 ([36]). *Assume $\mathsf{AD}^+ +$ "there is an (ω_1, ω_1) iteration strategy for a pure extender premouse with a long extender on its sequence".*

(1) For any $\Gamma \subseteq P(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} + \mathsf{NLE}$, $HOD^{L(\Gamma,\mathbb{R})}$ is a least branch premouse.

(2) There is a $\Gamma \subseteq P(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} + \mathsf{NLE}$ and $HOD^{L(\Gamma,\mathbb{R})} \models$ "there is a subcompact cardinal".

Of course, one would like to remove the mouse existence hypothesis of 9.3, and prove its conclusion under AD^+ alone. Finding a way to do this is one manifestation of the long standing iterability problem of inner model theory. Although we do not yet know how to do this, the theorem does make it highly likely that in models of $\mathsf{AD}_{\mathbb{R}}$ that have not reached an iteration strategy for a pure extender premouse with a long extender, HOD is a least branch premouse. It also makes it very likely that there are such HOD's with subcompact cardinals. Subcompactness is one of the strongest large cardinal properties that can be represented with short extenders.[52]

10 Some conjectures

The natural conjecture is that LEC and HPC hold in all models of AD^+ that have not reached an iteration strategy for a premouse with a long extender. Because our capturing mice have only short extenders on their sequences, LEC and HPC cannot hold in larger models of AD^+.

Conjecture 10.0.1. Assume AD^+ and NLE; then LEC.

[52]Theorem 9.3 is the first strong evidence that the HOD of a determinacy model can satisfy that there are cardinals that are strong past a Woodin cardinal.

Conjecture 10.0.2. Assume AD^+ and NLE; then HPC.

As we remarked above, 10.0.1 implies 10.0.2. It is not clear how far we are from a proof of these conjectures. There are intermediate levels that could be important. Our progress is closely related to

Conjecture 10.0.3. If there is a strongly compact cardinal, then there is an inner model of $\mathsf{ZFC}+$ " there is a subcompact cardinal".

Kunen's landmark paper [17] obtained the first nontrivial results in this direction, by constructing models with many measurable cardinals. We can obtain stronger conclusions now using the core model induction method, the strongest present being those of [25].[53] Progress along this line involves an inductive construction of lbr hod pairs, so that one is simultaneously proving approximations to HPC.

One might make the hypothesis of 10.0.3 PFA, or "$\Box_\kappa$ fails at a singular strong limit κ", or any of many other strong propositions. It seems unlikely that we will be able to prove such nontrivial consistency strength lower bounds without proving Conjecture 10.0.2. Typically, Conjecture 10.0.3 is stated as an equiconsistency, with "supercompact" replacing "subcompact". This form is likely true, but reaching supercompacts involves a general comparison lemma for mice with long extenders. At present we have only a very weak approximation to Theorem 3.1 in this realm (see [22]), and no strategy comparison theorem at all for the mice it covers.

Proofs of the conjectures above would be major steps forward. They may not be close at hand. But let us conclude with a conjecture that is well and truly beyond the reach of current inner model theory.

Conjecture 10.0.4. Assume $\mathsf{ZF}+\mathsf{AD}+V=L(P(\mathbb{R}))$; then $\mathrm{HOD}\models\mathsf{GCH}$.

In a sense this conjecture originates with the work of the Cabal in descriptive set theory in the late 1970s and early 1980s. Since then, we have seen that it belongs to inner model theory, and likely involves a general notion of mouse pair, a comparison lemma for these pairs, and a proof that their codesets are Wadge cofinal in the Suslin-co-Suslin sets of reals.[54] A ZFC version of 10.0.4 is: Assume $\mathsf{ZFC}+$ " there are arbitrarily large Woodin cardinals", and let Γ be a proper Wadge initial segment of the Universally Baire sets; then $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}\models\mathsf{GCH}$. The conclusion here is $(\Pi^2_1)^{uB}$, and hence set generically absolute granted arbitrarily large Woodin cardinals.[55] This makes it very likely that the ZFC version is either provable in ZFC, or refutable in ZFC augmented by some large cardinal hypothesis beyond the reach of current inner model theory.

What we know from descriptive set theory, and from inner model theory where we have it, suggests that Conjecture 10.0.4 is true, and that its proof involves a general theory of mouse pairs. The mice must be capable of having long extenders, and probably supercompact cardinals

[53] [25] show that if there is a strongly compact cardinal, then there is a strategy mouse with an extender overlapping a Woodin cardinal.

[54] One might make the hypothesis AD^+ rather than AD without destroying the intent. We are guessing that its proof would show that AD implies AD^+.

[55] This is a result of Woodin. See [40].

and beyond. Although 10.0.1, 10.0.2, 10.0.3, and their approximations at still lower levels are much better targets at present, Conjecture 10.0.4 points to a longer term future. The intricately structured world of inner model theory extends well beyond the part that we have discovered so far.

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