§0. Introduction

We assume AD⁺ throughout.

Def 0.1 Hod pair capturing (HPC) is the statement: For every Suslin -co-Suslin set A, there is a least branch hod pair (P, Σ) such that A ≤w Code(Σ).

Here Code(Σ) is the set of all x ∈ R coding stacks x ∈ HC that are by Σ. Since Σ normalizes well, we could restrict to stacks of length one. The structure P is an 1pm, so all extenders on its sequence are short. If our AD⁺ world has iteration strategies for mice with long extenders, HPC should fail. Below that point, it should hold.
Let $\mathcal{J}$ and $\mathcal{J}'$ show that

$\text{HPC} \Rightarrow \text{HOD} = \theta$ is an $\text{Ipm}$, and hence $\text{HPC} \Rightarrow \text{HOD} = \text{GCH}$.

It is natural to try to prove HPC by an induction on pointclasses with the scale property. One support for this idea comes from [27], which shows that HPC implies its local versions.

**Def. 0.2** Let $\Delta$ be a pair boldface pointclass. We say $\Delta \models \text{HPC}$ "iff" whenever $A \in \Delta$, and both $A$ and $\neg A$ have scales $\psi$ and $\psi'$ such that

$\langle \leq \psi \rangle \in \Delta$ and $\langle \leq \psi' \rangle \in \Delta$, then there is an $\text{Ibr}$ hod pair $(\rho, \mathcal{Z})$ such that $A \equiv \text{w} \text{Code}(\mathcal{Z})$, and $\text{code}(\mathcal{Z}) \in \Delta$.\]
The following is implicit in [22J]

**Theorem 3 (22J)** Assume HPC, and let $\mathcal{A}$ be a boldface pointclass such that $\mathcal{A} = \mathcal{P}(\mathbb{R}) \cap L(\mathcal{A}, \mathbb{R})$; then $\mathcal{A} \models \text{HPC}$.

This is proved in [22J] under the assumption $L(\mathcal{A}, \mathbb{R}) = \text{AD}_\mathbb{R}$, but we believe the proof given theorem 3. In fact, with more work it should show that under $\text{AD}^+ + \text{HPC}$, we have $\mathcal{A} \models \text{HPC}$ for all $\mathcal{A}$ closed under $\mathcal{F}$ and $\mathcal{F}_\mathbb{R}$.

Of course, we want to drop the HPC assumption of theorem 3, and then we must replace it by a "no long extender" assumption.
**Def. 0.4** "No long extenders" (NLE) is the statement: there is no countable plus one premouse $P$ such that $F^P$ is long, and $P$ has an $\omega_1$ iteration strategy.

Here $F^P$ is the last extender of $P$.

Plus-one premises are defined in [33].

What we want to prove is

**Conjecture** $\vdash AD^+ + \text{NLE} \Rightarrow \text{HPC}$.

In this note, we take care of what might be one case in an inductive proof that under $AD^+ + \text{NLE}$, $\Delta \vdash \text{HPC}$ for all suitable $\Delta$ is closed under $\rightarrow$ and $\exists^R$. We show
Theorem 0.5 Assume $AD^+ + NLE$.

Let $\Gamma$ be inductive-like and have the scale property, and $\Delta = \Gamma \cap \Gamma$.

Suppose $\Delta \in HPC$, and suppose that all sets in $\Gamma$ are Suslin, then there is an 1br hod pair $(P, \Sigma)$ such that $\text{Code}(\Sigma) \not\in \Delta$.

The proof of theorem 5 is relatively easy, modulo [117] and [127]. It proceeds in two steps.

Step 1 We show there is a pair $(Q, \Psi)$ such that $Q$ is an 1pm, and $\Psi$ is a partial strategy for $Q$ such that $\text{Code}(\Psi) \not\in \Gamma - \Delta$. 

The motivation here is as follows. Suppose there is an HOD pair \((P, E)\) such that Code \((E) \notin A\) and let \((P, E)\) be least in the mouse order with this property. It is shown in [27] that the short-tree component of \(E\), or \(E^{\text{stc}}\), is \(K\)-Suslin where \(K = 6\). Indeed, \(E^{\text{stc}}\) can be recovered by looking for embeddings into \(U\), where \(U\) is the short, normal tree on \(P\) whereby \(P\) iterates past HOD. We'd like to see, without using the existence of \((P, E)\), that there is a \(1\)-path \(Q\) and a normal tree \(W\) of length \(K\) on \(Q\) such that \(W\) induces a partial strategy for \(Q\) that can serve as \(P\), via its method of looking for embeddings into \(W\).

But note that the desired \((Q, W)\) is essentially a subset of \(K\). So we can use the Coding lemma.
Let $A$ be a universal $\Gamma$ set, and $\phi$ a $\Gamma$-norm on $A$. If the desired $(Q, W)$ exists, it will be of the form $(Q_x, W_x)$, where $Q_x \in HC$ is coded by $(x)_0$, and $W_x \in K$ is coded by $(x)_1$, via $W_x = \exists y ((x)_1, y) \mid A((x), y) ^3$.

It turns out that the statement "for all $x$, $(Q_x, W_x)$ fails to produce a $(Q, y)$ that resembles $(P, \exists x \theta)$ sufficiently" can be expressed as

$$M_\Gamma \models \forall x \in R \exists \theta \exists x \theta,$$

where $M_\Gamma$ is the spectral companion of $\Gamma$ and $\theta$ is $\Sigma_1$. But then by admissibility, if $M_\Gamma \models \forall x \in R \exists x \theta$, we have some $\alpha < K$ such that $M_\Gamma \models \exists \alpha \in R \forall \exists x \theta$. But this is impossible: since
Δ = HPC, we get an ind hod past (P, Σ) ≤ Δ that iterates past HOD \[ \mathcal{H} \] via a normal tree \( W \). Letting \( x \) be such that \( Q_x = P \) and \( W_x = W \), we get \( M_{\mathcal{H}} \models \neg \Theta 2x \).

The main thing we need to do to fill out this sketch is to explain what we mean by "resembles \((P, \Sigma)\) sufficiently." We shall do that in the next section. (See p. 8a.)

Notice that in this step we use only the reflection property of \( \Gamma \) not the existence of Suslin cardinals \( > \kappa \).

So the argument works if \( \Gamma = \text{Sob} \).

In that case, the reflection is easier to do, and the \((Q, \psi)\) we get
Remark The sketch just given works if in $\Delta$ there are $\omega_1$-many $1$-br hod pairs $(P, Z)$ such that $Z$ is $\Delta$-fullness preserving. This follows directly from $\Delta \in HPC$ in various cases, e.g., if $\Delta = PC(R) \cap L(\Delta, R)$, or if $\Gamma = S_{\infty}$.

In the general case, we must modify the sketch a bit. See section 4.
resembles a \((P, Z^{\text{src}})\) very strongly with respect to all properties that can be seen in \(L(R)\). In the last section of this paper, we shall record various of these properties in a general definition of "short tree strategy pair".

**Step 2**

Letting \((Q, \mathcal{Y})\) be as in step 1, we show that there is a pseudo-\(Y\)-iterate \((R, \mathcal{F})\) of \((Q, \mathcal{Y})\) such that \(\mathcal{F} \not\in \Delta\) and such that for some \(Z \supseteq \mathcal{F}\), \((R, Z)\) is an lbwr hod pair.

We carry our step 2 by iterating \((Q, \mathcal{Y})\) into the maximal lbwr hod pair construction of some \(\Gamma^*\) Woodin model \(N^*\).
whose $\Gamma^*$ is a good pointclass with
the scale property such that $\Gamma \cup \Gamma \subseteq \Gamma^*$.
We need that all $\Gamma$ sets are Suslin
in order to see that there is such a $\Gamma^*$.
However, we do not need that $\Gamma$ is inductive
like in order to carry out step 2.

Remark. In step 2, the tree $T$ from
$Q$ to $R$ is normal, and it may be
"$\Psi$-maximal", in that it has limit length,
and $\Psi(T)$ is undefined. In that case,
$R = M(T)$. In that case, step 1
guarantees that $(Q, \Psi)$ resembles a
short tree strategy pair sufficiently that
$\Psi \in M(\beta)R$
makes sense, and will do for $\Gamma$.

We would guess that the argument
in step 2 is part of a general proof.
of HPC, under the assumptions AD^+ + NLE, that goes by induction on pointclasses with the scale property. First you get a sufficiently rich fragment (Q, \mathcal{Y}) of the next short tree strategy pair \tau, then you extend some tail of (Q, \mathcal{Y}) to an Ibr hod pair via the argument of step 2. Constructing (Q, \mathcal{Y}) is the problem. We do not see how to make the "construction" described above work if \Gamma is not inductive like, and suspect that a different approach is needed.
Step one executed

We assume that $\Gamma$ is a non-selfdual boldface pointclass closed under $JR$ and $V^R$, and having the scale property. Our definability calculations can be done using a "Spector companion" of $\Gamma$, as shown by Moschovakis ([3], §9.) We fix such a structure $\mathcal{M} = M_{\Gamma}$. Its properties are

(i) $\mathcal{M} = (\bar{M}, e, \overline{\bar{R}}, R_0, R_1, R_2)$, where $\bar{M}$ is transitive and rigid closed, $R \in \mathcal{M}$, and $\mathcal{M}$ satisfies the $\Delta_0$-separation and $\Sigma_0$-collection schema (i.e., $\mathcal{M}$ is admissible) in the language $\bar{L}$.

(ii) (Projectability) The predicate $R_0$ is in $\bar{L}$, as is the graph of a partial function from $R$ onto $\bar{M}$.

(iii) (Resolvability) The predicate $R_1$ is a sequence $\langle M_\alpha | \alpha < \text{o}(M) \rangle$ such that for $\alpha < \text{o}(M)$,

$$M_\alpha = (M_\alpha, e, R_0 \cap M_\alpha, R_1 \cap M_\alpha, A_{\bar{L}})$$
where \( M_\alpha \) is transitive and rudimentary, and every \( R \in M_\alpha \), and \( M_\alpha \) is a \( \Sigma_1 \)-structure of \( M_\alpha \), and \( M_\beta \in M_\alpha \), and

\[(a) \bigcup_{\alpha < \omega(M)} M_\alpha = M, \]

(b) for \( \beta < \alpha \), \( M_\beta \) is a substructure of \( M_\alpha \), and \( M_\beta \in M_\alpha \), and

\[(c) \bigcup_{\alpha < \omega(M)} A_\alpha \) is a complete \( \Gamma \) set, \]

\[(iv) \sum_{i=1}^m \in P(\mathbb{R}) = \Gamma.\]

Remark. This is a slight strengthening of "good companion" in the terminology of Wilson [44]. We have that each \( M_\alpha \) is projectable and \( \Sigma_1 \)-resolvable via \( R^{M_\alpha} \) and \( R_1^{M_\alpha} \). The norm \( \gamma(x) = \text{least } \alpha \) such that \( x \in A_\alpha \) is a \( \Gamma \) norm on \( A \). By [33], \( M \)
consists of all sets whose transitive closure is coded by some well-founded relation in \( A \),

\[0(M) = \omega_\Gamma \] is the prewellordering ordinal of \( \Gamma \).

Our condition guarantees each \((M_\alpha, e)\) is OD from \( P(\mathbb{R}) \cap M_\alpha \), hence OD by Wadge.
We are also assuming $\Delta = HPC$. From this we get

**Lemma 1.1** The mouse order, restricted to the pairs $(P, \Sigma) \subseteq M_{t-}$, has order type $o(M_{t-})$.

**Proof** Let $\leq^+$ be the mouse order. Note that if $(Q, \Psi) \leq^+ (P, \Sigma)$, then $\text{Code}(\Psi)$ is projective in $\text{Code}(\Sigma)$. If there is a tail $(R, \Phi)$ of $(P, \Sigma)$ such that $(Q, \Psi)$ iterates without dropping to $(R, \Phi)$. Let $\pi : Q \to R$ be the iterates map. Then $\Psi = \Phi^\pi$ because $\Psi$ is pullback consistent, so $\text{Code}(\Psi)$ is projective in $\text{Code}(\Phi)$, and hence projective in $\text{Code}(\Sigma)$. Similarly, the mouse order on $(Q, \Psi) \leq^+ (P, \Sigma)$ is projective in $\text{Code}(\Sigma)$. So $\leq^+ \cap M_{t-}$ has order type $\leq o(M_{t-})$. 

For $\alpha < 1 \leq^* n M_{T_1}$, let

$$\Gamma_\alpha = \text{common value of } \text{Proj}(\alpha)(\text{Code}(2))$$

for all $(P, Y)$ of mouse rank $\alpha$.

The proof just given shows that $\Gamma_\alpha$ is well-defined, and

$$\alpha \leq \beta \implies \Gamma_\alpha \subseteq \Gamma_\beta.$$

Each $\Gamma_\alpha \in M_{T_1}$, so $\Gamma_\alpha \in \Delta$. By HPC,

$$\Delta = \bigcup_{\alpha \leq 1 \leq^* n M_{T_1}} \Gamma_\alpha.$$

Since $o(M_{T_1})$ is regular in $V$, we must have $1 \leq^* n M_{T_1} = o(M_{T_1})$.

For $\alpha < o(M_{T_1})$, let

$$H_\alpha = \text{common value of } \text{Moo}(P, \Sigma), \text{ for all } (P, \Sigma) \text{ of mouse rank } \alpha.$$
$H_\alpha$ is a cardinal cutpoint initial segment of $H_\beta \gamma$ for all $\beta > \alpha$. If there are cotinually many $\alpha$ such that $H_\alpha$ has this form, then $\text{HOD}^M$ is just their union.

Remark: By $\text{HOD}^M$ we mean the union of all transitive sets $x$ such that every $y \in x \exists \alpha \in \text{Ord} \in x$ is definable over some $(M_x, \in)$ from ordinals, $\alpha < \zeta(M_x)$. Note here that $M_x$ is definable from $P(\mathbb{R}) \cap M_x$ (as all sets coded by wff relations in $\mathbb{P}(\mathbb{R}) \cap M_x$), and $P(\mathbb{R}) \cap M_x$ is OD over all $M$ by Wadge.

In any case, we have that each $H_\alpha \in M_x$ and in fact the relation $R(\alpha, x)$ iff $x = H_\alpha$ is $\Delta_1$. No parameters are needed, and only $\varepsilon_x$ in the definition.

We modify the sketch in §0 as follows: we find a countable $1_{\text{pm}} \mathbb{Q}$.
and a sequence \( \langle W_\alpha | \alpha < \theta(M) \rangle \)
such that each \( W_\alpha \) is a normal tree on \( Q \) iterating it past \( H_\alpha \) and collectively the \( W_\alpha \)'s determine a sufficiently strong partial iteration strategy \( \gamma \) for \( Q \).

Def. 1.2 Let \( P \) be an \( \mathcal{I} \)-path. A partial \( P \)-stack is a sequence \( S = \langle \langle x_i, k_i, \xi_i \rangle \rangle_{i \in \mathbb{N}} \)
such that setting \( P_0 = P \), there are \( P_i \) for \( 1 \leq i \leq n \) such that for \( 0 \leq i \leq n 

(i) \( \xi_i \) is a normal tree on \( P_i \downarrow \langle 0, x_i, k_i \rangle \),
and for \( 1 \leq i \leq n 

(ii) \( \varepsilon(x_i) \) is the last model of \( \xi_{i-1} \)
or \( \xi_{i-1} \) has limit length, and

\[ P_i = M(\xi_i). \]

We write \( \xi_i(s) \) for \( \xi_i \), etc. We set \( U(s) = \)
\[ \xi_{\text{dom}(s)-1}(s) \in \text{last tree in } S. \quad M_0(s) \text{ is the} \]
Last model if $U(s)$ if it has one.

For $\lambda \leq n$, $M(\lambda) = P_\lambda$. This is consistent with [LI], in that a $P$-stack in the sense of [LI], 5,6, is just a partial $P$-stack such that each $T_\lambda(s)$ has a last model.

If $s$ is a $P$-stack, then we can define the embedding normalization $W(s)$ and the full normalization $X(s)$ as it was done in [LI] and [L2] in the case each $T_\lambda(s)$ for $\lambda \leq \text{dom}(s)$ has a last model. If any $T_\lambda(s)$ for $\lambda \leq \text{dom}(s)$ has no last model, then $W(s)$ and $X(s)$ have no last model. For example, suppose $T$ is a normal tree on $P$ of limit length, and $U$ is a normal tree on $M(T)$ with last model $M(U)$. Then $W = W(T, U)$ is a normal tree on $P$ of limit length. Each branch extender $S_\eta$ comes from a mod-$\eta$ branch extender $S_\eta^T$ for $\eta < \text{lh}(T)$, by inserting
the extenders from some \( s_8^H \), \( s_8 \leq \gamma \).

We have an embedding \( \phi : lhJ \rightarrow lh W \),
where \( \phi(\eta) = \xi \) if \( s_8^x \) is generated by \( s_7^\xi \) and \( s_8^\xi \) for \( \gamma \leq \eta \gamma \), and \( \gamma \) as large as possible. \( \text{ran}(\phi) \) is cofinal in \( lh W \). If \( b \) is a cofinal branch of \( \tilde{\mathcal{I}} \),
then \( W(\tilde{\mathcal{I}}^b, U) \), as defined in [41],
is the extension of \( W(\tilde{\mathcal{I}}, U) \) via \( \phi'' b \).

We derive partial iteration strategies from normal trees as follows.

**Def 1.3** Let \( P \) be an \( 1 pm \), and let \( U \) be a normal tree on \( P \). The partial strategy for \( P \) derived from \( U \) is \( \Sigma \), where

(i) for \( \tilde{\mathcal{I}} \) normal on \( P \), \( \tilde{\mathcal{I}} \) is by \( \Sigma \) iff there is a weak hull embedding of \( \tilde{\mathcal{I}} \) into \( U \),
and (ii) for \( s \) a partial \( P \)-tree, \( s \) is by \( \Sigma \) iff \( X(s) \) is by \( \Sigma \).

We write \( \Sigma = \Sigma(P, U) \). In the abstract,
There is no reason why $\Sigma$ would look remotely like an iteration strategy. But if $(P, Y)$ is an $\omega$-born pair, and $\Psi$ is by $Y$, then $\Sigma(P, U) \leq Y$. If $\Psi$ is a normal iteration of $P$ past some $H_\alpha$, and all its countable hulls are by $Y$, then $\Sigma(P, U)$ is a reasonably large fragment of $\Psi$; indeed, some major relevant part of the $\omega$-relevance part of the tail of $\Sigma(P, U)$ is a complete strategy. Let $\Psi$ be an $\ellpm$. A partial strategy for $G^+(P, \Theta)$ is a winning strategy for $\Sigma$ for $\Pi$ in the variant of $G^+(P, \Theta)$ in which at any $\omega$ limit ordinal, $\Pi$ is allowed to quit and be declared the victor. (Of course, we have to talk about strategies that elect to win the hard way in some cases!) If $\Sigma$ is a tree of $\omega$ limit length by $\Sigma$, then $\check{\Delta}_1^0$ by
convention, the "instruction to quit" is given if \( s \notin \text{dom}(\xi) \). We write \( \xi(s) \uparrow \), and say \( \xi \) is \( \Sigma \)-maximal, in this case.

Given a partial strategy \( \xi \) for \( G^+(P,\theta) \), we extend \( \xi \) to a partial strategy \( \hat{\xi} \) acting on partial \( P \)-stacks as follows. Let \( s \) be a partial \( P \)-stack by \( \xi \) and \( c \) be a cofinal branch of \( U(s) \), where \( U(s) \) has limit length. Then

\[ s \upharpoonright c \text{ is by } \hat{\xi} \text{ iff } X(s \upharpoonright c) \text{ is by } \xi. \]

If \( \xi \) is the restriction to normal trees of a strategy \( \eta \) that normalizes well, then \( \hat{\xi} \in P \), so there is at most one such \( c \). We say \( s \) is \( \hat{\xi} \)-maximal iff \( \hat{\xi}(s) \uparrow \). If \( s \) is by \( \xi \), and either \( N \) is the last model of \( U(s) \), or \( N = M(U(s)) \) and there is no last model, then \( \hat{\xi}_{s,N} \) is the tail of \( \hat{\xi} \):

\[ \hat{\xi}_{s,N}(t) = \xi_{s,N}(s \upharpoonright t) \uparrow. \]
Def. 1.4 Let $P$ be an lbr hod and let $\Sigma$ be a partial strategy for $G^+(P, \theta)$; then $\Sigma$ is reasonable iff for any $s$ by $\Sigma$

(a) If $\Sigma$ is normal and there is a weak hull embedding of $\mathcal{U}$ into $\mathcal{W}$, then $\mathcal{U}$ is by $\Sigma_{s, \mathcal{W}}$, and

(b) $\Sigma$ is well-defined, i.e., it $s \sim c$ and $s \sim d$ are by $\Sigma$, then $c = d$, and

(c) if $2 \in \text{dom}(\Sigma_{s, \mathcal{W}})$ and $2 \in \text{dom}(\Sigma_{s, \mathcal{W}})$, then $\Sigma_{s, \mathcal{W}}(2) = \Sigma_{s, \mathcal{W}}(2)$.

(d) See p. 22a.

Prop 1.5 Suppose $(P, \mathcal{Y})$ is an lbr hod pair, and $\mathcal{U}$ is a normal tree on $P$ all of whose weak hulls are by $\mathcal{Y}$, then $\Sigma(P, \mathcal{U})$ is a reasonable partial strategy for $G^+(P, \mathcal{W})$. 
(d) There is no infinite \( P \)-stack \( s \) such that each \( S_R \) is by \( \hat{\mathcal{E}} \), and for infinitely many \( i \),
\[
\langle v_i(s), R_i(s) \rangle \prec \text{lex} \langle o(M_i(s)), R(M_i(s)) \rangle,
\]
(That is, there is no infinite stack \( s \) by \( \hat{\mathcal{E}} \) that has a proper drop at the beginning of infinitely many trees.)
Proof: $Y$ fully normalizes well and has very strong hull condensation, by [2.7]. It is easy to see (a), (c) follow.

Remark: If $Z$ is a strategy for $G^+(P, \Theta)$, then $\Sigma$ normalizes well, simply by definition. However, one cannot bypass Theorem 2.8 of [1] easily stating that background-induced strategies normalize well, by such a definitional trick. If $Z$ is background-induced, and we define $Z$ as above, then we will have to show (a) and (c) of reasonability, and that seems to lead back to $Y/\Sigma$ being background-induced. So we are led to 2.8 of [1] again. (This remark is due to Hugh Woodin.)
Lemma 1.6 Let $(P, \mathcal{E}) \in M_\beta$ be an $\mathbb{L}$br had pair with mouse rank $\alpha$; then for any $\beta \leq \alpha$ there is a unique normal tree $W$ on $P$ such that every countable weak hull of $W$ is by $\mathcal{E}_\beta$ and $H_\beta$ is the last model of $W$.

Proof A simple Shelah hull argument shows that there is at most one such normal tree $W$. For existence, pick any $(Q, \Psi)$ such that $H_\beta = M_{\omega_0}(Q, \Psi)$. We compare $(P, \mathcal{E})$ with $(Q, \Psi)$. Since $\beta \leq \alpha$, we have a countable normal tree $I$ on $P$ with lost model $R$, and such that every $(R, \mathcal{E}_{\alpha}, R)$ is a non-dropping instance of $(Q, \Psi)$. Hence

$$M_{\omega}(R, \mathcal{E}_{\beta}, R) = H_\beta.$$
But then in \( V_{\mathrm{coll}}(w, R) \) there is a non-dropping stack \( s \) on \( (R, \Sigma_{\exists}, R) \) of length \( w \) having direct limit \( H_{\beta} \). But \( \exists \) fully normalizes well for inifinity stacks, so we have a normal tree \( U \) on \( R \) with last model \( H_{\beta} \), and such that \( R \to H_{\beta} \) does not drop, and all weak hulls of \( U \) are by \( \Sigma_{\exists}, R \). We then can take

\[
W = X(\Sigma, \mathcal{U}),
\]

and all weak hulls of \( W \) are by \( \exists \) because \( \exists \) now fully normalizes well \((\Sigma^2 I)\) and has very strong hull condensations \((\Sigma^2 I)\).

Remark: The proof of 1.6 was a little sloppy. One should replace \( V_{\mathrm{coll}}(w, R) \) by \( \Lambda[I, \mathcal{U}] \) for \( \mathcal{U} \subseteq \mathcal{P}_w(\Lambda) \) and \( I \) if \( I \) is a complete \( \Gamma \) set at some point.
Lemma 1.7  Let \((P, \Sigma)\) be an lbr hod pair of mouse rank \(\alpha\), and for each \(\beta \leq \alpha\), let \(W_\beta\) be the normal tree on \(P\) with last model \(H_\beta\) given by 106. Then \(U \Sigma(P, W_\beta)\) is the restriction of \(\Sigma\) to the normal trees.

Proof  Let \(U = U \Sigma(P, W_\beta)\), so that \(U \subseteq \Sigma\). Let \(T\) be a normal tree by \(U\) of limit length, and let \(\Sigma(T) = \mathfrak{b}\). We must see \(U(T) = \mathfrak{b}\).

But let

\[ H_\beta = M_{\mathfrak{b}}(\mathcal{M}_{\mathfrak{b}}, \Sigma_{\mathfrak{b}}) \]

where clearly \(\beta \leq \alpha\). By full normalization there is a normal \(U\) on \(M_{\mathfrak{b}}\) with last model \(H_\beta\) s.t. all countable weak hulls of \(U\) are by \(\Sigma_{\mathfrak{b}}\). This implies

\[ W_\beta = X(\beta^{\mathfrak{b}}, U) \]

so \(\beta^{\mathfrak{b}}\) is a weak hull of \(W_\beta\), so \(U(P, W_\beta)(T) = \mathfrak{b}\). \(\blacksquare\)
There is actually a one-one correspondence between the $H^k$'s and mouse-equivalence classes:

**Lemma 1.8** Let $(P, \mathcal{E})$ and $(Q, \psi)$ be

1) Mor. pairs such that $\mathcal{M}_{\alpha}(P, \mathcal{E}) = \mathcal{M}_{\alpha}(Q, \psi)$;

2) $(P, \mathcal{E}) \equiv^* (Q, \psi)$.

**Proof** Let $\mathcal{H} = \mathcal{M}_{\alpha}(P, \mathcal{E}) = \mathcal{M}_{\alpha}(Q, \psi)$.

Let $\mathcal{W}_0$ on $P$ and $\mathcal{W}_1$ on $Q$ be

normal with last model $\mathcal{H}$ be normal

and all countable hulls by $\mathcal{E}$ and $\psi$ resp.

We compare $(P, \mathcal{E})$ with $(Q, \psi)$; the worry is that one side comes out strictly shorter. So suppose we have $(R, \lambda)$

that is a normal, non-dropping iterate

of $(Q, \psi)$, and we have a normal tree $T$ on $P$ with last model $S$

such that
R is a cutpoint initial segment of S
\[ \Lambda = \sum_{\alpha \in R} \]
and either \( R \neq S \), or \( P \text{-} to \text{-} S \) drops in \( \mathcal{I} \).

If \( P \text{-} to \text{-} S \) does not drop, then we are done, because \( \text{Mao}(R, \Lambda) = H \) since \( Q \text{-} to \text{-} R \) did not drop, while \( \text{Mao}(S, \sum_{\alpha \in S}) = H \) because \( P \text{-} to \text{-} S \) did not drop, so \( (R, \Lambda) \) cannot be a proper initial segment of \( (S, \sum_{\alpha \in S}) \). So suppose \( P \text{-} to \text{-} S \) drops.

Let \( U \) be normal on \( R \) with last model \( H \), and all its countably built by \( \Lambda \). We can think of \( U \) as a tree \( U^* \) param on \( S \), with last model \( S^* \Delta H \).

(In \( U^* \), there may be dropping strictly above the image of \( R \), not just but on branches.)
of \( U \) that do not drop. \( U^+ \) never drops to the \( U^- \) image of \( R \). But then \( X(T, U^*) \) is a normal tree on \( P \) with last model \( S^* \) and such that all its countable hulls are by \( \Xi \). Since \( H \leq S^* \), \( Wo \) is an initial segment of \( X(T, U^*) \). But \( H \) is the last model of \( Wo \) so it cannot be a proper initial segment of any normal \( X \) extending \( Wo \).

A pleasing corollary of 1.8 is that given \( P \), \( \Xi \) is determined by \( \mathcal{M}_\infty^{\omega}(P, \Xi) \) and \( \Pi P, \Xi \). In particular, \( \Xi \) is OD(s), where \( s \) is a countable subset of \( \mathcal{C}(\mathcal{M}_\infty^{\omega}(P, \Xi)) \).
Corollary 1.9. Let $(P, Σ)$ and $(P', Ψ)$ be bounded pairs such that $M_0(P, Σ) = M_0(P', Ψ)$ and $\Pi_0^* Σ = \Pi_0^* Ψ$. Then $Σ = Ψ$.

Proof. By 1.8, $(P, Σ) ≅ (P', Ψ)$, so we have $(R, Λ)$ which is a common, non-dropping iterate. Let $i$ and $j$ be the iteration maps. We have

![Diagram](image)

where $i$ is the map by $Σ$; $j$ is the map by $Ψ$; and $\Pi = \Pi^* Λ / R_{j,0}$ is by the tail common tail of $Λ$ of $Σ$ and $Ψ$. By assumption, $\Pi \circ i = \Pi \circ j$. But then $i = j$, so $Σ = Λ^{i} = Λ^{j} = Ψ$ by pullback consistency.
Of more immediate relevance is

**Cor. 1.10**

(a) \( \sup \{ 0(\mathcal{M}_0(P, \mathcal{E})) | (P, \mathcal{E}) \text{ is an lbr hod pair coded in } \Delta^3 \} = S_T \).

(b) For any \( \mathcal{A} \in \Delta \), there is an lbr hod pair \( (P, \mathcal{E}) \) such that \( \text{Code}(\mathcal{E}) \in \mathcal{A} \), and \( \mathcal{A} \preceq \text{Code}(\mathcal{E}_{rel}) \).

**Proof** For (a), suppose not; then we have a fixed \( \alpha < S_T \) such that for all \( \beta < S_T \), \( 0(\mathcal{H}_\beta) < \alpha \). But \( \beta \neq \gamma \Rightarrow H_\beta \neq H_\gamma \) by 1.8, so we get a \( S_T \)-sequence of distinct subposets of \( \alpha \), contrary to "boldface GCH" and just the existence of measurable \( \text{for } (\mathcal{E}_{rel} \in \mathcal{A}) \).

For (b), note that \( \mathcal{M}_0(P, \mathcal{E}) \) can be computed from \( P \) and just \( \mathcal{E}_{rel} \) in a projective way. So (a) gives us (b).
Remark: For $\Delta \subseteq (P, \subseteq)$ be a lbh hod pair, let $\Sigma^\Delta$ be the restriction of $\Sigma$ to normal trees that do not drop anywhere (not just along a main branch). Assuming $\Delta \models \text{HPC}$, we can show that the Code($\Sigma^\Delta$) are wedge cofinal in $\Delta$.

This immediately implies 1.10 (b), and then since $\Sigma^\Delta \subseteq \text{o}(\text{Mo}o(P, \subseteq))$ - Suslin,

we get 1.10 (a).

To see the $\Sigma^\Delta$ are wedge cofinal,

Take any lbh pair $(P, \subseteq)$ with Code($\Sigma$) in $\Delta$, and let $(N^*, \mathcal{Z}^*)$ be a coarse $\Gamma_0$-Woodin that captures Code($\Sigma$),

where $\Gamma_0 \subseteq A$. Let $(M, \mathcal{Z})$

be the output of the lbh-pair construction of $(N^*, \mathcal{Z}^*)$ up to its Woodin $S^*$. 
We claim that Code ($\mathcal{Z}$) is projective in Code ($\mathcal{Z}^\mathcal{M}$), which is enough. For that, one shows that ($P, \Sigma$) iterates to some level $(\mathcal{Q}, \mathcal{Y})$ of the Ibr hod pair construction of $(\mathcal{M}, L)$. Iteration trees on $(\mathcal{Q}, \mathcal{Y})$ then get lifted to trees on $\mathcal{M}$ that drop nowhere.

**Lemma 1.11** Let ($P, \Sigma$) be an Ibr hod pair of mouse rank $\alpha < \mathcal{S}$. For $\xi \leq \alpha$, let $W_\xi$ be the unique normal tree on $P$ with last model $H_\xi$, all of whose countable hulls are by $\Sigma$. For $\beta \leq \alpha$, let

$$\Sigma_\beta = \bigcup_{\xi \leq \beta} \Sigma(P, W_\xi).$$

Let $\beta < \alpha$ and let $J$ be a countable normal tree of limit length that is by $\Sigma_\beta$, and $\Sigma_\beta$ is maximal, i.e.,

$$\Sigma_\beta(\mathcal{J}) \uparrow.$$

Then there is a tail $(\mathcal{Q}, \mathcal{Y})$ of $(\Sigma_\beta)_{\beta, \mathcal{M}(\alpha)}$ such that

$$H_\beta = M_{\mathcal{M}(\alpha)}(\mathcal{Q}, \mathcal{Y}).$$
Proof Let \( b = \Xi(I) \). Let \((R, \Phi) \in M_I\) and \( \text{Maa} (R, \Phi) = H_3 \).

Claim \((R, \Phi) \leq^* (M(I), \Xi, \Sigma, M(\alpha))\), where \( \leq^* \) is the mouse order.

Proof Suppose not; then we have countable normal trees \( U \) on \( M(I) \) and \( V \) on \( R \) by \( \Xi, \Sigma, M(\alpha) \) and \( \Phi \) respectively such that

(a) \( U \) and \( V \) have common last level pair \((S, \Delta)\), and

(b) if, branch \( M(I) \to S \) of \( U \) does not drop, then

\( R \to S \) in \( \Delta \) may drop. (The last model of \( \Delta \) can be taken to literally be \( S \) by including one final drop at the end if necessary.)
We have $(S, \Delta) \leq^* (R, \Phi)$, and hence we have $\xi \leq \beta$ such that

$$M_\omega(S, \Delta) = H_\xi.$$

Let $W$ be the normal tree on $S$ whose last model is $H_\xi$, and all countable hulls are by $\Delta$. Let

$$X = X(S, U, W).$$

We have the diagram

```
\begin{align*}
P &\xrightarrow{a} M(\delta) &\xrightarrow{\iota_M} S \\
&\downarrow &\downarrow \\
& \downarrow &\downarrow \\
& H_\xi
\end{align*}
```

and $i^W$ exists, and $i^P \circ i^W$ is continuous at $s(\delta)$, with $\sigma(H_\xi) = \text{sup} i^P \circ i^W \cap s(\delta)$.\]
The properties of full normalization then tell us that $\mathcal{A} \uparrow b$ is a weak hull of $X$. But $X = X_{\mathcal{A}}$, so $Z_{\mathcal{A}}(\mathcal{A}) = b$, so $Z_{\mathcal{B}}(\mathcal{B}) = b$, contradiction.

By the claim we have $\mathcal{U}$ on $M(\mathcal{A})$ and $\mathcal{V}$ on $\mathcal{R}$ normal, by $\mathcal{E}_{\mathcal{A}}, \mathcal{M}(\mathcal{A})$ and $\mathcal{F}$ respectively, with common last model $(S, \Lambda)$, and such that $R \rightarrow S$ does not drop. Thus $M_{\mathcal{U}}(S, \Lambda) = H_{\mathcal{B}}$, and we have the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\mathcal{I}} & M(\mathcal{A}) & \xrightarrow{\mathcal{U}} & S \\
& & & \uparrow \mathcal{W} & \\
& & & & R \\
& & & & \downarrow \mathcal{H}_\mathcal{B} \\
W_{\mathcal{B}} = X & \rightarrow & \\
& & \\
\end{array}
\]
where

\[ X = X(\mathcal{E}, U, W) = W_\beta. \]

(Notice here that although \( \mathcal{E} \) has no last model, \( U \) has last model \( S \); and the branch of \( U \) from \( M(\mathcal{E}) \) to \( S \) drops.

We have that \( W_\beta = X(\hat{\mathcal{E}}, U, W) \) as well, but because \( U \) dropped, there is no weak hull embedding from \( \hat{\mathcal{E}} \) to \( W_\beta \) coming out of that. Indeed, \( Z_\beta(\mathcal{E}) \) by hypothesis.) Letting

\[ Y = X(\mathcal{E}, U) \], we have that \( Y \) is a weak hull of \( W_\beta \), so \( Y \) is by \( Z_\beta \), so \( \langle \hat{\mathcal{E}}, U \rangle \) is a partial P-stack by \( Z_\beta \), with last model \( S \).
To finish the proof, it is enough to show that $(\mathcal{E}_B < s)$ is a finite stack from $S$ to $N$ by $\Lambda$ such that $S$ to $N$ does not drop, then $Y = X(\mathcal{E}, \mathcal{U}, Y)$ weakly embeds into $W_B$. But the diagram below makes this clear.

We have $W_B = X(Y, \mathcal{E})$, so $Y$ weakly embeds into $W_B$. 

\[ H_B = M_{\alpha}(N, \Delta_{3, N}) \]
Remark: If \( \beta \) from \( S \) to \( N \) drops, then the diagram above becomes

\[
P \xrightarrow{\xi} M(S) \xrightarrow{U} S \xrightarrow{\beta} N
\]

\[
W_\beta \xleftarrow{W_\xi} H_\xi = M_0 \left( N, \Lambda_{\beta, N} \right)
\]

where \( \xi \leq \beta \). We then get that \( Y \) is a weak hull of \( W_\beta \)

hence \( Y \) is by \( \xi \). This gives that \( \tilde{\mathcal{E}}^{1} \frac{\mathcal{L}}{\mathcal{U}}, \frac{\mathcal{V}}{\mathcal{W}} \) is by \( \tilde{\mathcal{E}}^{\beta} \), where \( \tilde{\mathcal{E}}^{\beta} \) consists of all weak partial \( P \)-stacks \( S \)
such that each \( \chi(s, \mathcal{E}) \) is by \( \xi \) is weakly embedded into a \( W_\beta \) with \( \xi \leq \beta \). That would have been a better definition of \( \tilde{\mathcal{E}}^{\beta} \).

Then we get \( \left( \tilde{\mathcal{E}}^{\beta} \right)_{\mathcal{L}} = \Lambda \) in the above.
Definition 1.12

Let $\alpha \leq \delta r$. A pair $(P, \langle W_\beta : \beta \leq \alpha \rangle)$ is $\alpha$-good iff

1. $P$ is an $Lpm$, and $W_\beta$ is a normal tree on $P$ with last model $H_\beta$, and

2. For $\beta < \alpha$, let $E_\beta$ be the partial strategy for $P$: if $c$ is a partial strategy for $P$, if $e$ is a partial strategy for $P$ - stack by $E_\beta$ with last tree $U = U(s)$ of limit length, and $c$ is cotinal in $U$, then $E_\beta(s) = c$ iff $X(s^c)$ is weakly embedded into some $W_\gamma$, for $\gamma \leq \beta$;

(a) each $E_\beta$ is reasonable

(b) there is some partial stack $s$ by $E_\beta$ with last model $N$ such that $(N, (E_\beta)_s)$ is an $Lbr$ hod pair with mouse rank $\beta$, and

(c) for any partial stack $s$ by $E_\beta$, such that $U = U(s)$ has limit length and $E_\beta(s) \uparrow$, there is a normal $U$ on $M(\beta)$ by $(E_\beta)_s, M(\beta)$ such that $U$ has last model $Q$, and $(Q, (E_\beta)_s, M(\beta), U, Q)$ is an $Lbr$ hod pair of mouse rank $\beta$. 
We have shown

Lemma 1.13 For any \( \alpha < \delta \), there is an \( \alpha \)-good pair.

Corollary 1.14 There is a \( \delta \)-good pair.

Proof Let \( A = \mathbb{R} \times \mathbb{R} \) be the universal \( \Gamma \) set \( \Gamma \) we fixed above, and if our \( \Gamma \) norm on \( A \). For \( \alpha < \delta \), say that a real \( \varepsilon \) is \( \alpha \)-strong if letting

\[
\mu^\alpha_\varepsilon = \sup \left\{ 0 \left( \frac{1}{n} \right) \right\}_{\varepsilon} \left( \varepsilon \leq \frac{\varepsilon}{n} \right),
\]

and

\[
B_\varepsilon = \left\{ (x, y) \mid \varphi(x) \leq \varphi(y) < \mu_\varepsilon \right\}
\]

for each \( \varepsilon < \alpha \), and

\[
S^2_\varepsilon = \text{the } \frac{\varepsilon}{n} \text{ subset of } \mu_\varepsilon \text{ coded by } \varepsilon,
\]

for \( \varepsilon < \alpha \), then for all \( \delta < \alpha \).
(a) $B^*_x$ codes an $x$-good pair
\[ \langle P^*_x, \langle W^*_x\mid \nu<x\rangle \rangle, \] and

(b) if $x \leq y$, then $P^*_x = P^*_y$ and
\[ W^*_x = W^*_y \] for all $\nu < y$.

The functions $\xi \mapsto \mu^*_x$ and $\xi \mapsto B^*_x$
are $\aleph_1$-dense over $M^*_x$. So the predicate

\[ S(z, y) \text{ iff } z \text{ is } x\text{-strong} \]
is $\Delta_3$ over $M^*_x$. By the Coding Lemma
and $\text{LoL} \Rightarrow \forall x \exists y (z \text{ is } x\text{-strong})$.
This implies $\exists z \forall x (z \text{ is } x\text{-strong})$;
otherwise
\[ M^*_x = \forall z \exists x (z \text{ is not } x\text{-strong}) \]
and by admissibility, there is a $\beta < \delta^*_x$
such that $\forall z \exists x < \beta (z \text{ is not } x\text{-strong})$,
so $\forall z (z \text{ is not } \beta\text{-strong})$. But
letting \( Z \) be \( d \)-strong for all \( d < s \), we see that \( Z \) codes a \( d \)-good pair.
§2. Step two

We complete the proof of theorem 0.5. We have $\Gamma$ inductive like with the scale property, and $\Delta \in HPC$, we wish to construct a bad pair $<\delta, \phi>$ such that $\text{Code}(\delta) \not\in \Delta$, provided that there are

Sulcin $\delta$ cardinals $> \delta$. By 1.14, we may fix a $\delta$-good pair

$(P, <\omega \alpha < \delta>)$.

Let $\Sigma$ be the partial strategy for $P$ determined by the $\omega \alpha$'s; that is, for $\delta$ a partial $P$-move by $\Sigma$ whose last tree has limit length,

$$\Sigma(\delta) = c \iff \exists \xi (\chi(\delta^+ c) \text{ is a weak hull of } \omega \xi).$$

$\text{Code}(\Sigma)$ is a $\Gamma$ set because $(P, \omega \xi)$ is coded in $\Gamma$ by $\xi$, coding lemma.
Code $\varnothing \not\in \Delta$ by (2)(b) of def. 1.12.

In fact, if $A$ is $\Sigma$-maximal, then
\[
\text{Code } \bigcup_{\varnothing \in \Delta} A
\]
by (2)(c) of 1.12.

---

Remark: We can have $\Sigma$ being $\Sigma^*_\omega$, but $\varnothing \notin M^*$ because $\Sigma$ is a partial strategy, not a complete strategy. This shows already that there are $\Sigma$-maximal trees, and in fact, that if $A$ is $\Sigma$-maximal, then there are $\Sigma^*_\omega \subseteq M^*$ maximal trees on $M^*$.

---

Now let $\Gamma^*$ be a good pointclass with the scale property such that $\varnothing \notin \Delta^*$.

Let $(N^*, \varnothing^*, \sigma^*)$ be a coarse $\Gamma^*$-Woodin model that Suslin captures Code $(\varnothing)$ and its complement, with $\varnothing$ in HC$N^*$. Let $\Gamma$ be the maximal Ipm construction of $N^*$. 

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We shall compare \((\emptyset, \Sigma)\) with \((M_e^{\Sigma_e}, \Sigma_{\nu, k})\). For that to work, we need that if \(s\) is by \(\Sigma\), with last model \(N\) such that \(P \to N\) drops, then \(\Sigma_{s, N}\) is a complete strategy.

(This ensures that \(\Sigma_P \subseteq \Sigma\) by reasonability. For \(\Sigma_P\) contains only information about how to iterate the \(N \lor P\), and by reasonability, this information is consistent with \(\Sigma\). So if \(\Sigma_{N}\) is complete, then the information is contained in \(\Sigma_{N}\), and so \(\Sigma_P \subseteq \Sigma\). Similarly, \(\Sigma_Q \subseteq \Sigma_{s, Q}\) for any stack \(s\) by \(\Sigma_0\).

But we have

Claim: There is a stack \(s\) by \(\Sigma\) with last model \(\emptyset\) such that \(\text{Code}(\Sigma_{s, Q}) \notin A\) and whenever \(t\) is by \(\Sigma_{s, Q}\) with last model \(N\) and \(Q \to N\) drops in \(t\), then \(\Sigma_{s, Q, t, N}\) is a complete strategy.)
Proof: If \( s = \emptyset \) does not work, then there is to by \( Z \) with last model \( \mathcal{N}_0 \) such that \( \mathcal{P} \rightarrow \mathcal{N}_0 \) drops, and \( Z_{t_0, \mathcal{N}_0} \) is not complete. But then means we have a normal \( \mathcal{T}_0 \) on \( \mathcal{N}_0 \) by \( Z_{t_0, \mathcal{N}_0} \) such that \( Z_{t_0, \mathcal{N}_0} (\mathcal{T}_0) \) a.

By \( \mathcal{S}_r \)-goodness of \( \mathcal{W} \), we get that \( \text{Code}(Z_{t_0, \mathcal{N}_0}) \not\in \mathcal{W} \). So we can replace \( (\mathcal{P}, Z) \) by \( (\mathcal{N}_0, Z_{t_0, \mathcal{N}_0}) \).

If \( (t_0, \mathcal{N}_0) \) does not work for the claim, we get \( t_1 \) on \( \mathcal{N}_0 \) with last model \( \mathcal{N}_1 \) s.t. \( \mathcal{N}_0 \rightarrow \mathcal{N}_1 \) drops, etc.

So if the claim fails, to \( t_1 \) is an infinite stack by \( Z \) with infinitely many drops, contrary to 1.4 (d). \( \checkmark \)
Now let us fix $s, Q$ as in the claim, and let $\Lambda = \Xi/s, Q$. The arguments of $L$ show that for each $(s, k) \leq (s^*, 0)$, the comparison of $(Q, \Lambda)$ with $(M^e_s, L^e_k)$ is such that only the $Q$-side moves, and no strategy disagreements show up. Here $Q$ is being iterated by $\Lambda$, and we stop the comparison with $(M^e_s, L^e_k)$ if the current tree $T$ on the $Q$ side is $\Lambda$-maximal.

$(Q, \Lambda)$ cannot traverse past $(M^e_s, L^e_k)$ because $\text{Code}(\Lambda)$ is in $\Delta^e_{s^*}$, so the tree $T$ of length $s^* + 1$ on $Q$ by $\Lambda$ that is responsible would be in $\mathcal{N}^*$, and this yields $\mathcal{N}^* \models s^*$ is not Woodin by the universality
argument, contradiction.

If \((Q, \Lambda)\) iterates to some \((M_{\tau,k} \leq \lambda^k, L_{\tau,k})\) via a normal tree \(T\) by \(\Lambda\) of successor length whose main branch does not drop, then letting \(\Lambda = \Lambda_{\tau,b}\) and \(M = M_{\tau,\lambda}^\dagger\) and \(\pi : Q \to M\) be the iteration map of \(T\), we have that \(\Lambda \subseteq L_{\tau,k}\) because \(\Lambda\) is pullback consistent (as a partial strategy). So \(L_{\tau,k}\) is a complete strategy for \(Q\), and \(L_{\tau,k} \in \Delta\) because \(\Lambda \subseteq \Delta\). (More precisely, because there are complete strategies \(\Lambda\) at arbitrarily high in \(\Delta\).) So we have the conclusion of Theorem 0.5.

The final possibility is that for some \(\langle \tau, k \rangle \leq \lambda^\kappa \langle \kappa, 0 \rangle\), setting \((M, L) = (M_{\tau,k} \leq \lambda^k, L_{\tau,k})\), we have a normal
A maximal tree \( T \) on \( Q \) with last model \( M \) and such that \( \Lambda \models \mathcal{L} \). But then, by (2)(c) of \( \mathcal{L} \)-goodness, there are complete strategies \( (\Lambda \models \mathcal{L}) \) of arbitrarily high Wedge degree in \( \mathcal{L} \). Thus \( \text{Code}(\mathcal{L}) \neq \mathcal{L} \) and \( (M,T) \) witnesses the conclusion of Theorem 0.5.

Thm. 0.5 \( \checkmark \)
References


