HOD pair capturing and short tree strategies

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§0. Introduction

We assume AD^+ throughout.

<u>Def 0.1</u> Hod pair capturing (HPC) is the statement: for every Suslin-co-Suslin set A, there is a least branch hod pair (P, Σ) such that $A \leq_w \mathrm{Code}(\Sigma)$.

Here $\mathrm{Code}(\Sigma)$ is the set of all $x \in \mathbb{R}$ coding stacks $s_x \in HC$ that are by Σ. Since Σ normalizes well, we could restrict to stacks of length one. The structure P is an lpm, so all extenders on its sequence are short. If our AD^+ world has iteration strategies for mice with long extenders, HPC should fail. Below that point, it should hold.

[1] and [2] show that

HPC $\Rightarrow$ HOD$|\theta$ is an lpm, and

hence HPC $\Rightarrow$ HOD $\models$ GCH.

It is natural to try to prove HPC by an induction on pointclasses with the scale property. One support for this idea comes from [2], which shows that HPC implies its local versions.

<u>Def. 0.2</u> Let $\underset{\sim}{\Delta}$ be a boldface pointclass. We say $\underset{\sim}{\Delta} \models$ HPC iff whenever $A \in \underset{\sim}{\Delta}$, and A and $\neg A$ have scales $\vec{\varphi}$ and $\vec{\psi}$ such that $\langle \le_{\varphi_i} \mid i \in \omega \rangle \in \underset{\sim}{\Delta}$ and $\langle \le_{\varphi_i} \mid i \in \omega \rangle \in \underset{\sim}{\Delta}$, then there is an lbr hod pair (P, Σ) such that $A \le_w \text{Code}(\Sigma)$, and $\text{code}(\Sigma) \in \underset{\sim}{\Delta}$.

The following is implicit in [2].

<u>Theorem 3.3</u> ([2]) Assume HPC, and let $\underset{\sim}{\Delta}$ be a boldface pointclass such that $\underset{\sim}{\Delta} = P(\mathbb{R}) \cap L(\underset{\sim}{\Delta}, \mathbb{R})$; then $\underset{\sim}{\Delta} \models \mathrm{HPC}$.

This is proved in [2] under the assumption $L(\underset{\sim}{\Delta}, \mathbb{R}) \models AD_{\mathbb{R}}$, but we believe the proof gives theorem 3. In fact, with more work it should show that under $AD^+ + \mathrm{HPC}$, we have $\underset{\sim}{\Delta} \models \mathrm{HPC}$ for all $\underset{\sim}{\Delta}$ closed under $\neg$ and $\exists^{\mathbb{R}}$.

Of course, we want to drop the HPC assumption of theorem 3, and then we must replace it by a "no long extender" assumption.

Def. 0.4 "No long extenders" (NLE) is the statement: there is no countable plus one premouse P such that $\dot{F}^P$ is long, and P has an ω_1 iteration strategy.

Here $\dot{F}^P$ is the last extender of P. Plus-one premice are defined in [3]. What we want to prove is

Conjecture $AD^+ + NLE \Rightarrow HPC$.

In this note, we take care of what might be one case in an inductive proof that under $AD^+ + NLE$, $\underset{\sim}{\Delta} \models HPC$ for all $\underset{\sim}{\Delta}$ closed under $\neg$ and $\exists^{\mathbb{R}}$. We show

Theorem 0.5 Assume $AD^+ + NLE$.

Let $\underset{\sim}{\Gamma}$ be inductive-like and have the scale property, and $\underset{\sim}{\Delta} = \underset{\sim}{\Gamma} \cap \check{\underset{\sim}{\Gamma}}$. Suppose $\underset{\sim}{\Delta} \models HPC$, and suppose that all sets in $\check{\underset{\sim}{\Gamma}}$ are Suslin; then there is an lbr hod pair (P, Σ) such that $\mathrm{Code}(\Sigma) \notin \underset{\sim}{\Delta}$.

The proof of theorem 5 is relatively easy, modulo [1] and [2]. It proceeds in two steps.

Step 1 We show there is a pair (Q, Ψ) such that Q is an lpm, and Ψ is a ~~positional short tree~~ partial strategy for Q such that $\mathrm{Code}(\Psi) \in \underset{\sim}{\Gamma} - \underset{\sim}{\Delta}$.

The motivation here is as follows. Suppose there is an lbr hod pair (P, Σ) such that $\mathrm{Code}(\Sigma) \notin \underset{\sim}{\Delta}$, and let (P, Σ) be least in the mouse order with this property. It is shown in [2] that the short-tree component of Σ, or Σ^{stc}, is κ-Suslin, where $\kappa = \delta_\Gamma$. Indeed, Σ^{stc} can be recovered by looking for embeddings into $\mathcal{U}$, where $\mathcal{U}$ is the short, normal tree on P whereby P iterates past $\mathrm{HOD}^{\underset{\sim}{\Delta}}$. We'd like to see, without using the existence of (P, Σ), that there is a lpm Q and a normal tree $\mathcal{W}$ of length κ on Q such that $\mathcal{W}$ induces a partial strategy for Q that can serve as ψ, via the method of looking for embeddings into $\mathcal{W}$.

But note that the desired $(Q, \mathcal{W})$ is essentially a subset of κ. So we can use the Coding Lemma.

Let A be a universal Γ set, and φ a Γ-norm on A. If the desired (Q, W) exists, it will be of the form (Q_x, W_x), where $Q_x \in HC$ is coded by $(x)_0$ and $W_x \subseteq \kappa$ is coded by $(x)_1$ via $W_x = \{\varphi((x)_1, y) \mid A((x)_1, y)\}$.

It turns out that the statement "for all x, (Q_x, W_x) fails to produce a (Q, Ψ) that resembles $(\mathbb{P}, \Sigma^{STC})$ sufficiently" can be expressed as

$$M_\Gamma \models \forall x \in \mathbb{R}\ \Theta[x],$$

where M_Γ is the Spector companion of Γ and Θ is Σ_1. But then by admissibility, if $M_\Gamma \models \forall x \in \mathbb{R}\ \Theta[x]$, we have some $\alpha < \kappa$ such that $M_\Gamma \restriction \alpha \models \forall x \in \mathbb{R}\ \Theta[x]$.

But this is impossible: since

$\underset{\sim}{\Delta} \models HPC$, we get an lbr hod pair $(P, \Sigma) \in \underset{\sim}{\Delta}$ that iterates past $HOD^{\underset{\sim}{\Delta}} | \alpha$ via a normal tree $\mathcal{W}$. Letting x be such that $Q_x = P$ and $\mathcal{W}_x = \mathcal{W}$, we get $M_\Gamma | \alpha \models \neg\theta[x]$.

The main thing we need to do to fill out this sketch is to explain what we mean by "resembles (P, Σ^{stc}) sufficiently". We shall do that in the next section. (See p. 8a.)

Notice that in this step we use only the reflection property of Γ, not the existence of Suslin cardinals $> \kappa$. So the argument works if $\Gamma = S_\infty^{L(\Gamma, \mathbb{R})}$. ~~and~~ In that case, the reflection is easier to do, and the (Q, Ψ) we get

Remark The sketch just given works if in $\underset{\sim}{\Delta}$ there are Wadge-cofinally many lbr hod pairs (P, Σ) such that Σ is $\underset{\sim}{\Delta}$-fullness preserving. This follows directly from $\underset{\sim}{\Delta} \models HPC$ in various cases, e.g. if $\underset{\sim}{\Delta} = P(\mathbb{R}) \cap L(\underset{\sim}{\Delta}, \mathbb{R})$, or if $\Gamma = S_\infty$.

In the general case, we must modify the sketch a bit. See section I.

resembles a (P, Σ^{stc}) very strongly: with respect to all properties that can be seen in $L(\Gamma, \mathbb{R})$. In the last section of this paper, we shall record various of these properties in a general definition of "short tree strategy pair".

<u>Step 2</u> Letting (Q, Ψ) be as in step I, we show that there is a pseudo-Ψ-iterate (R, Φ) of (Q, Ψ) such that $\Phi \notin \underset{\sim}{\Delta}$, and such that for some $\Sigma \supseteq \Phi$, (R, Σ) is an lbr hod pair.

We carry out step 2 by iterating (Q, Ψ) into the maximal lbr hod pair construction of some Γ^* Woodin model N^*, where Γ^* is a good pointclass with the scale property such that $\Gamma \cup \check{\Gamma} \subseteq \Gamma^*$. We need that all $\check{\Gamma}$ sets are Suslin in order to see that there is such a Γ^*. However, we do not need that Γ is inductive like in order to carry out step 2.

<u>Remark</u> In step 2, the tree $\tilde{\mathcal{T}}$ from Q to R is normal, and it may be "Ψ-maximal", in that it has limit length, and $\Psi(\mathcal{T})$ is undefined. In that case, $R = M(\mathcal{T})$. In that case, step 1 guarantees that (Q, Ψ) resembles a short tree strategy pair sufficiently that $\Psi_{M(\mathcal{T}), R}$ makes sense, and will do for Φ.

We would guess that the argument in step 2 is part of a general proof of HPC, under the assumptions $AD^+ + NLE$, that goes by induction on pointclasses with the scale property. First you get a sufficiently rich fragment (Q, Ψ) of the next short tree strategy pair, then you extend some tail of (Q, Ψ) to an lbr hod pair via the argument of step 2. Constructing (Q, Ψ) is the problem. We do not see how to make the "construction" described above work if Γ is not inductive like, and suspect that a different approach is needed.

§1. Step one executed

We assume that Γ is a nonselfdual boldface pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, and having the scale property. Our definability calculations can be done using a "Spector companion" of Γ, as shown by Moschovakis ([3], §9.) We fix such a structure $\mathcal{M} = \mathcal{M}_\Gamma$. Its properties are

(i) $\mathcal{M} = (M, \in, \vec{R})$, where $\vec{R} = R_0, R_1$, M is transitive and rud closed, $\mathbb{R} \in M$, and $\mathcal{M}$ satisfies the Δ_0-separation and Σ_0-collection shema (i.e. $\mathcal{M}$ is admissible) in the language with $\in, R_0, R_1$.

(ii) (Projectability) The predicate R_0 is the graph of a partial function from $\mathbb{R}$ onto M

(iii) (Resolvability) The predicate R_1 is a sequence $\langle \mathcal{M}_\alpha \mid \alpha < o(M) \rangle$ such that for $\alpha < o(M)$,

$$\mathcal{M}_\alpha = \langle M_\alpha, \in, R_0 \cap M_\alpha, R_1 \cap M_\alpha, A_\alpha \rangle$$

where $\mathcal{M}_\alpha$ is transitive, amenable, and rud closed, $\mathbb{R} \in M_\alpha$, and $M_\alpha \models$ Every set is coded by a wf relation on $\mathbb{R}$, and every wf relation codes a set

and (a) $\bigcup_{\alpha < o(M)} M_\alpha = M$,

(b) for $\beta < \alpha$, $\mathcal{M}_\beta$ is a substructure of $\mathcal{M}_\alpha$, and $\mathcal{M}_\beta \in M_\alpha$, and

(c) $\bigcup_{\alpha < o(M)} A_\alpha$ is a complete Γ set.

(iv) $\underset{\sim}{\Sigma}_1^{\mathcal{M}} \cap P(\mathbb{R}) = \Gamma$.

Remark This is a slight strengthening of "good companion", in the terminology of Wilson [4]. We have that each $\mathcal{M}_\alpha$ is projectable and ~~wel~~ resolvable via $R_0^{\mathcal{M}_\alpha}$ and $R_1^{\mathcal{M}_\alpha}$. The norm $\varphi(x)$ = least α such that $x \in A_\alpha$ is a Γ norm on A. By [3], M consists of all sets whose transitive closure is coded by some wellfounded relation in $\underset{\sim}{\Delta}$. $o(M) = \underset{\sim}{\delta}_\Gamma$ is the prewellordering ordinal of Γ. Our conditions guarantee each $(M_\alpha, \in)$ is OD from $P(\mathbb{R}) \cap M_\alpha$, hence OD by Wadge.

We are also assuming $\underset{\sim}{\Delta} \models$ HPC. From this we get

<u>Lemma 1.1</u> The mouse order, restricted to lbr pairs $(P, \Sigma) \in \mathcal{M}_\Gamma$, has order type $o(\mathcal{M}_\Gamma)$.

<u>Proof</u> Let $\leq^*$ be the mouse-order. Note that if $(Q, \Psi) \leq^* (P, \Sigma)$, then Code$(\Psi)$ is projective in Code(Σ). [There is a tail (R, Φ) of (P, Σ) such that (Q, Ψ) iterates without dropping to (R, Φ). Let $\pi: Q \to R$ be the iteration map. Then $\Psi = \Phi^\pi$ because Ψ is pullback consistent, so Code(Ψ) is projective in Code(Φ), and hence projective in Code(Σ).] Similarly, the mouse order on $(Q, \Psi) \leq^* (P, \Sigma)$ is projective in Code(Σ). So $\leq^* \cap M_\Gamma$ has order type $\leq o(M_\Gamma)$.

For $\alpha < |\leq^* \cap M_\Gamma|$, let

Γ_α = common value of Projective(Code(Σ)), for all (P, Σ) of mouse rank α.

The proof just given shows that Γ_α is well-defined, and

$$\alpha \leq \beta \Rightarrow \Gamma_\alpha \subseteq \Gamma_\beta .$$

Each $\Gamma_\alpha \in M_\Gamma$, so $\Gamma_\alpha \subsetneq \underset{\sim}{\Delta}$. By HPC,

$$\underset{\sim}{\Delta} = \bigcup_{\alpha < |\leq^* \cap M_\Gamma|} \Gamma_\alpha .$$

Since $o(M_\Gamma)$ is regular in V, we must have $|\leq^* \cap M_\Gamma| = o(M_\Gamma)$.

$\boxtimes$

For $\alpha < o(M_\Gamma)$, let

H_α = common value of $M_\infty(P, \Sigma)$, for all (P, Σ) of mouse rank α.

If we have that $H_\alpha = M_\infty(P, \Sigma)$ where Σ is $\underset{\sim}{\Delta}$-fullness preserving, then

H_α is a cardinal cutpoint initial segment of H_β, for all $\beta > \alpha$. If there are cofinally many α such that H_α has this form, then $HOD^{\underset{\sim}{\Delta}}$ is just their union.

<u>Remark</u> By $HOD^{\underset{\sim}{\Delta}}$ we mean the union of all transitive sets x such that every $y \in x \cup \{x\}$ is definable over some $(M_\alpha, \in)$ from ordinals, $\alpha < o(M_\Gamma)$. Note here that M_α is definable from $P(\mathbb{R}) \cap M_\alpha$ (as all sets coded by wf relations in $P(\mathbb{R}) \cap M_\alpha$), and $P(\mathbb{R}) \cap M_\alpha$ is OD over ~~any~~ M by Wadge.

In any case, we have that each $H_\alpha \in M_\Gamma$, and in fact the relation $R(\alpha, x)$ iff $x = H_\alpha$ is $\Delta_1^{(M_\Gamma, \in)}$. No parameters are needed, and only $\in$, in the definition.

We modify the sketch in §0 as follows: we find a countable lpm Q, and a sequence $\langle \mathcal{W}_\alpha \mid \alpha < o(M_T) \rangle$ such that each $\mathcal{W}_\alpha$ is a normal tree on Q iterating it past H_α, and collectively the $\mathcal{W}_\alpha$'s determine a sufficiently strong partial iteration strategy Ψ for Q.

<u>Def. 1.2</u> Let P be an lpm. A <u>partial P-stack</u> is a sequence $s = \langle\langle \nu_i, k_i, \mathcal{T}_i \rangle \mid i \leq n \rangle$ such that setting $P_0 = P$, there are P_i for $1 \leq i \leq n$ such that for $0 \leq i \leq n$,

(i) $\mathcal{T}_i$ is a normal tree on $P_i | \langle \nu_i, k_i \rangle$,

and for $1 \leq i \leq n$

(ii) either P_i is the last model of $\mathcal{T}_{i-1}$, or $\mathcal{T}_{i-1}$ has limit length, and $P_i = M(\mathcal{T}_i)$.

We write $\mathcal{T}_i(s)$ for $\mathcal{T}_i$, etc. We set $\mathcal{U}(s) = \mathcal{T}_{\mathrm{dom}(s)-1}(s)$ = last tree in s. $M_\infty(s)$ is the last model of $\mathcal{U}(s)$ if it has one.

For $i \le n$, $M_i(s) = P_i$. This is consistent with [1], in that a P-stack in the sense of [1], 5.6, is just a partial P-stack such that each $\mathcal{T}_i(s)$ has a last model.

If s is a partial P-stack, then we can define the embedding normalization $W(s)$ and the full normalization $X(s)$ as it was done in [1] and [2] in the case each $\mathcal{T}_i(s)$ for $i+1 < \mathrm{dom}(s)$ has a last model. If any $\mathcal{T}_i(s)$ for $i < \mathrm{dom}(s)$ has no last model, then except in dropping cases $W(s)$ and $X(s)$ have no last model. For example, suppose $\mathcal{T}$ is a normal tree on P of limit length, and $\mathcal{U}$ is a normal tree on $M(\mathcal{T})$ with last model $M^{\mathcal{U}}_\gamma$. Then $W = W(\mathcal{T}, \mathcal{U})$ is a normal tree on P of limit length. Each branch extender s^W_ξ comes from a branch extender $s^{\mathcal{T}}_\eta$ for $\eta < \mathrm{lh}(\mathcal{T})$, by "inserting the extenders from some $S^{\mathcal{U}}_\delta$, $\delta \leq \gamma$.

We have an embedding $\varphi : \mathrm{lh}\,\mathcal{T} \to \mathrm{lh}\,\mathcal{W}$, where $\varphi(\eta) = \xi$ if $S^{\mathcal{W}}_\xi$ is generated by $S^{\mathcal{T}}_\eta$ and $S^{\mathcal{U}}_\delta$ for $\delta \leq_{\mathcal{U}} \gamma$, and δ as large as possible. $\mathrm{ran}(\varphi)$ is cofinal in $\mathrm{lh}\,\mathcal{W}$. If b is a cofinal branch of $\mathcal{T}$, then $\mathcal{W}(\mathcal{T}^\frown b, \mathcal{U})$, as defined in [1], is the extension of $\mathcal{W}(\mathcal{T}, \mathcal{U})$ via $\varphi``b$.

We derive partial iteration strategies from normal trees as follows.

<u>Def 1.3</u> Let P be an lpm, and let $\mathcal{U}$ be a normal tree on P. The <u>partial strategy for P derived from $\mathcal{U}$</u> is Σ, where

(i) for $\mathcal{T}$ normal on P, $\mathcal{T}$ is by Σ iff there is a weak hull embedding of $\mathcal{T}$ into $\mathcal{U}$,

and (ii) for s a partial P-stack, s is by Σ iff $X(s)$ is by Σ.

We write $\Sigma = \Sigma(P, \mathcal{U})$. In the abstract, There is no reason why Σ would look remotely like an iteration strategy. But if (P, Ψ) is an lbr hod pair, and $\mathcal{U}$ is by ~~Ψ~~ Ψ, then $\Sigma(P, \mathcal{U}) \subseteq \Psi$. If $\mathcal{U}$ is a normal iteration of P past some H_α, and all its countable hulls are by Ψ, then $\Sigma(P, \mathcal{U})$ is a reasonably large frament of Ψ; indeed, some tail of $\Sigma(P, \mathcal{U})$ is the M_∞-relevant part of a complete strategy iterating some lpm to H_α. (See [2], def. 8, for "M_∞-relevant".)

Let P be an lpm. A *partial strategy for $G^+(P, \Theta)$* is a winning strategy ~~for~~ Σ for II in the variant of $G^+(P, \Theta)$ in which II is allowed to quit at any limit ordinal, and be declared the victor. (Of course, we care about strategies that elect to win the hard way in some cases!) If $\mathcal{T}$ is a tree of limit length by Σ, then ~~by~~ by convention, the "instruction to quit" is given iff $\vec{\mathcal{T}} \notin \mathrm{dom}(\Sigma)$. We write $\Sigma(\vec{\mathcal{T}})\uparrow$, and say <u>$\vec{\mathcal{T}}$ is Σ-maximal</u>, in this case.

Given a partial strategy Σ for $G^+(P,\theta)$, we extend Σ to a partial strategy $\hat{\Sigma}$ acting on partial P-stacks as follows. Let s be a partial P-stack by $\hat{\Sigma}$ and c be a cofinal branch of $\mathcal{U}(s)$, where $\mathcal{U}(s)$ has limit length. Then

$$s^\frown c \text{ is by } \hat{\Sigma} \text{ iff } X(s^\frown c) \text{ is by } \Sigma.$$

If Σ is the restriction to normal trees of a strategy Ψ that normalizes well, then $\hat{\Sigma} \subseteq \Psi$, so there is at most one such c. We say s is $\hat{\Sigma}$ maximal iff $\hat{\Sigma}(s)\uparrow$. If s is by Σ, and either N is the last model of $\mathcal{U}(s)$, or $N = M(\mathcal{U}(s))$ and there is no last model, then $\hat{\Sigma}_{s,N}$ is the tail of $\hat{\Sigma}$:

$$\hat{\Sigma}_{s,N}(t) = \hat{\Sigma}(s^\frown t).$$

<u>Def. 1.4</u> Let P be an lpm and let Σ be a partial strategy for $G^+(P,\theta)$; then Σ is <u>reasonable</u> iff for any s by $\hat{\Sigma}$

(a) If $\mathcal{T}$ is normal and by $\hat{\Sigma}_{s,N}$ and there is a weak hull embedding of $\mathcal{U}$ into $\mathcal{T}$, then $\mathcal{U}$ is by $\hat{\Sigma}_{s,N}$, and

(b) $\hat{\Sigma}$ is well-defined, i.e. if $s^\frown c$ and $s^\frown d$ are by $\hat{\Sigma}$, then $c=d$, and

(c) if $\mathcal{T} \in \mathrm{dom}(\hat{\Sigma}_{s,N}) \cap \mathrm{dom}(\dot{\Sigma}^N)$, then $\hat{\Sigma}_{s,N}(\mathcal{T}) = \dot{\Sigma}^N(\mathcal{T})$.

(d) See p. 22a.

<u>Prop 1.5</u> Suppose (P,Ψ) is an lbr hod pair, and $\mathcal{U}$ is a normal tree on P all of whose countable weak hulls are by Ψ; then $\Sigma(P,\mathcal{U})$ is a reasonable partial strategy for $G^+(P,\omega_1)$.

(d) There is no infinite P-stack s such that each stk is by $\hat{\Sigma}$, and for infinitely many i,

$$\langle \nu_i(s), k_i(s) \rangle <_{lex} \langle o(M_i(s)), k(M_i(s)) \rangle .$$

(That is, there is no infinite stack s by $\hat{\Sigma}$ that has a proper drop at the beginning of infinitely many trees.)

Proof Ψ fully normalizes well and has very strong hull condensation, by [2]. It is easy to see (a)-(c) follow. $\boxtimes$

Remark If Σ is a strategy for $G^+(P, \theta)$, then $\hat{\Sigma}$ normalizes well, simply by definition. However, one cannot bypass ~~the~~ theorem 2.8 of [ ], stating that background-induced strategies normalize well, by such a definitional trick. If Σ is background-induced, and we define $\hat{\Sigma}$ as above, then we will have to show (a) and (c) of reasonability, and that seems to lead back to $\hat{\Sigma}$ being background-induced. So we are led to 2.8 of [1] again. (This remark is due to Hugh Woodin.)

<u>Lemma 1.6</u> Let $(P, \Sigma) \in \mathcal{M}_r$ be an lbr hod pair with mouse rank α; then for any $\beta \leq \alpha$ there is a unique normal tree $\mathcal{W}$ on P such that every countable weak hull of $\mathcal{W}$ is by Σ, and H_β is the last model of $\mathcal{W}$.

<u>Proof</u> A simple Skolem hull argument shows that there is at most one such normal tree $\mathcal{W}$. For existence, pick any (Q, Ψ) such that $H_\beta = M_\infty(Q, \Psi)$. We compare (P, Σ) with (Q, Ψ). Since $\beta \leq \alpha$, we have a countable normal tree $\mathcal{T}$ on P with last model R, and such that ~~$M_{\mathcal{T}}$~~ $(R, \Sigma_{\mathcal{T}, R})$ is a non-dropping iterate of (Q, Ψ). Hence

$$M_\infty(R, \Sigma_{\mathcal{T}, R}) = H_\beta .$$

But then in $V^{Col(\omega,\mathbb{R})}$ there is a non-dropping stack s on $(R, \Sigma_{\mathcal{I},R})$ of length ω having direct limit H_β. But Σ fully normalizes well for infinite stacks, so we have a normal tree $\mathcal{U}$ on R with last model H_β, and such that R-to-H_β does not drop, and all weak hulls of $\mathcal{U}$ are by $\Sigma_{\mathcal{I},R}$. We then can take

$$W = X(\mathcal{I}, \mathcal{U}),$$

and all weak hulls of W are by Σ because Σ ~~norma~~ fully normalizes well ([2]) and has very strong hull condensation ([2]).

$\boxtimes$

<u>Remark</u> The proof of 1.6 was a little sloppy. One should replace $V^{[illegible]}$ by $L[T,\sigma]$ for $\sigma \in P_{\omega_1}(\mathbb{R})$ and T the tree for a complete Γ set at some point.

<u>Lemma 1.7</u> Let (P, Σ) be an lbr hod pair of mouse rank α, and for each $\beta \leq \alpha$, let W_β be the normal tree on P with last model H_β given by 1.6. Then

$$\bigcup_{\beta \leq \alpha} \Sigma(P, W_\beta)$$ is the restriction of Σ to ~~the~~ normal trees.

<u>Proof</u> Let $\Psi = \bigcup_{\beta \leq \alpha} \Sigma(P, W_\beta)$, so that $\Psi \subseteq \Sigma$. Let $\mathcal{T}$ be a normal tree by Ψ of limit length, and let $\Sigma(\mathcal{T}) = b$. We must see $\Psi(\mathcal{T}) = b$.

But let

$$H_\beta = M_\infty(M_b^{\mathcal{T}}, \Sigma_{\mathcal{T}^\frown b}),$$

where clearly $\beta \leq \alpha$. By full normalization, there is a normal $\mathcal{U}$ on $M_b^{\mathcal{T}}$ with last model H_β s.t. all countable weak hulls of $\mathcal{U}$ are by $\Sigma_{\mathcal{T}^\frown b}$. This implies

$$W_\beta = X(\mathcal{T}^\frown b, \mathcal{U}),$$

so $\mathcal{T}^\frown b$ is a weak hull of W_β, so $\Sigma(P, W_\beta)(\mathcal{T}) = b$. $\boxtimes$

There is actually a one-one correspondence between the H_β's and mouse-equivalence classes:

<u>Lemma 1.8</u> Let (P, Σ) and (Q, Ψ) be lbr hod pairs such that $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$; then $(P, \Sigma) \equiv^* (Q, \Psi)$.

<u>Proof</u> Let $H = M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$. Let $\mathcal{W}_0$ on P and $\mathcal{W}_1$ on Q be normal with last model H be normal and all countable hulls by Σ and Ψ resp. We compare (P, Σ) with (Q, Ψ); the worry is that one side comes out strictly shorter. So suppose we have (R, Λ) that is a normal, non-dropping iterate of (Q, Ψ), and we have a normal tree $\mathcal{T}$ on P with last model S such that

- R is a cutpoint initial segment of S
- $\Lambda = \Sigma_{\mathcal{T},R}$

and

- either $R \neq S$, or P-to-S drops in $\mathcal{T}$.

If P-to-S does not drop, then we are done, because $M_\infty(R,\Lambda) = H$ since Q-to-R did not drop, while $M_\infty(S, \Sigma_{\mathcal{T},S}) = H$ because P-to-S did not drop, so (R,Λ) cannot be a proper initial segment of $(S, \Sigma_{\mathcal{T},S})$. So suppose P-to-S drops.

Let $\mathcal{U}$ be normal on R with last model H, and all its countable hulls by Λ. We can think of $\mathcal{U}$ as a tree $\mathcal{U}^*$ ~~tree~~ on S, with last model $S^* \trianglerighteq H$, (In $\mathcal{U}^*$, there may be dropping strictly above the image of R, ~~that is not~~ but on branches of $\mathcal{U}$ that do not drop, $\mathcal{U}^*$ never drops to the $\mathcal{U}$-image of R.) But then $X(\mathcal{T}, \mathcal{U}^*)$ is a normal tree on P with last model S^*, and such that all its countable hulls are by Σ. Since $H \trianglelefteq S^*$, $\mathcal{W}_0$ is an initial segment of $X(\mathcal{T}, \mathcal{U}^*)$. But H is the last model of $\mathcal{W}_0$, so it cannot be a proper initial segment of any normal X extending $\mathcal{W}_0$.

$\boxtimes$

A pleasing corollary of 1.8 is that given P, Σ is determined by $M_\infty(P, \Sigma)$ and $\pi^{\Sigma}_{P,\infty}$. In particular, Σ is $OD(s)$, where s is a countable subset of $o(M_\infty(P, \Sigma))$.

Corollary 1.9 Let (P, Σ) and (P, Ψ) be lbr hod pairs such that $M_\infty(P, \Sigma) = M_\infty(P, \Psi)$, and $\pi^{\Sigma}_{P,\infty} = \pi^{\Psi}_{P,\infty}$; then $\Sigma = \Psi$.

Proof By 1.8, $(P, \Sigma) \equiv^* (P, \Psi)$, so we have (R, Λ) which is a common, non-dropping iterate. Let i and j be the iteration maps. We have

$$\begin{tikzcd} P \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] & R \arrow[r, "\pi"] & M_\infty \end{tikzcd}$$

where i is the map by Σ, j is the map by Ψ, and $\pi = \pi^{\Lambda}_{R,\infty}$ is by the tail common tail ~~of~~ Λ of Σ and Ψ. By assumption, $\pi \circ i = \pi \circ j$. But then $i = j$, so $\Sigma = \Lambda^i = \Lambda^j = \Psi$ by pullback consistency. $\boxtimes$

Of more immediate relevance is

Cor. 1.10

(a) $\sup\{o(M_\infty(P,\Sigma)) \mid (P,\Sigma)$ is an lbr hod pair coded in $\underset{\sim}{\Delta}\} = \delta_\Gamma$.

(b) For any $A \in \underset{\sim}{\Delta}$, there is an lbr hod pair (P,Σ) such that $\mathrm{Code}(\Sigma) \in \underset{\sim}{\Delta}$, and $A \leq_w \mathrm{Code}(\Sigma^{rel})$.

Proof For (a), suppose not; then we have ~~[illegible]~~ a fixed $\alpha < \delta_\Gamma$ such that for all $\beta < \delta_\Gamma$, $o(H_\beta) < \alpha$. But $\beta \neq \gamma \Rightarrow H_\beta \neq H_\gamma$ by 1.8, so we get a δ_Γ-sequence of distinct subsets of α, contrary to "boldface GCH". ~~(or just the existence of measurables~~ ~~in~~ (Or just measurables in (α, δ_Γ).)

For (b), note that $M_\infty(P,\Sigma)$ can be computed from P and just Σ^{rel} in a projective way. So (a) gives us (b).

$\boxtimes$

Remark For (P, Σ) be a lbr hod pair, let Σ^{nd} be the restriction of Σ to normal trees that do not drop anywhere (not just along a main branch). Assuming $\underset{\sim}{\Delta} \models$ HPC, we can show that the Code(Σ^{nd}) are Wedge cofinal in $\underset{\sim}{\Delta}$. This immediately implies 1.10 (b), and then since Σ^{rel} is $\theta(M_\infty(P, \Sigma))$-Suslin, we get 1.10 (a).

To see the Σ^{nd} are Wadge cofinal: Take any lbr pair (P, Σ) with Code(Σ) in $\underset{\sim}{\Delta}$, and let (N^*, Σ^*) be a coarse Γ_0-Woodin that captures Code(Σ), where $\Gamma_0 \subsetneq \underset{\sim}{\Delta}$. Let (M, Ω) be the output of the lbr-pair construction of (N^*, Σ^*) up to its Woodin δ^*.

We claim that $\mathrm{Code}(\Sigma)$ is projective in $\mathrm{Code}(\Sigma^{\mathrm{nd}})$, which is enough. For that, one shows that (P,Σ) iterates to some level (Q,Ψ) of the lbr hod pair construction of (M,Ω). Iteration trees on (Q,Ψ) then get lifted to trees on M that drop nowhere.

<u>Lemma 1.11</u> Let (P,Σ) be an lbr hod pair of mouse rank $\alpha < \delta_\Gamma$. For $\xi \leq \alpha$, let W_ξ be the unique normal tree on P with last model H_ξ all of whose countable hulls are by Σ. For $\beta \leq \alpha$, let

$$\Sigma_\beta = \bigcup_{\xi \leq \beta} \Sigma(P, W_\xi).$$

Let $\beta < \alpha$, and let $\mathcal{T}$ be a countable normal tree of limit length that is by Σ_β, and Σ_β-maximal, i.e.

$$\Sigma_\beta(\mathcal{T})\uparrow.$$

Then there is a tail (Q,Ψ) of $\left(\hat{\Sigma}_\beta\right)_{\mathcal{T},M(\mathcal{T})}$ such that

$$H_\beta = M_\infty(Q,\Psi),$$

Proof Let $b = \Sigma(\mathcal{J})$. Let $(R, \Phi) \in M_r$ and $M_\infty(R, \Phi) = H_\beta$.

Claim $(R, \Phi) \leq^* (M(\mathcal{J}), \Sigma_{\mathcal{J}, M(\mathcal{J})})$, where $\leq^*$ is the mouse order.

Proof Suppose not; then we have countable normal trees $\mathcal{U}$ on $M(\mathcal{J})$ and $\mathcal{V}$ on R by $\Sigma_{\mathcal{J}, M(\mathcal{J})}$ and Φ respectively such that

(a) $\mathcal{U}$ and $\mathcal{V}$ have common last lbr pair (S, Λ), and

(b) the branch $M(\mathcal{J})$-to-S of $\mathcal{U}$ does not drop.

R-to-S in $\mathcal{V}$ may drop. (The last model of $\mathcal{V}$ can be taken to literally be S by including one final drop at the end if necessary.)

We have $(S, \Lambda) \le^* (R, \Phi)$, and hence we have $\xi \le \beta$ such that

$$M_\infty(S, \Lambda) = H_\xi .$$

Let W be the normal tree on S whose last model is H_ξ, and all countable hulls are by Λ. Let

$$X = X(\mathcal{T}, \mathcal{U}, W) .$$

We have the diagram

$$\begin{tikzcd}
 & & R \arrow[d, dash, "v"] \\
P \arrow[r, dash, "\tilde{\mathcal{T}}"] \arrow[rrd, dash, "X"'] & M(\tilde{\mathcal{T}}) \arrow[r, "i^{\mathcal{U}}"] & S \arrow[d, "i^{W}"] \\
 & & H_\xi
\end{tikzcd}$$

$i^{\mathcal{U}}$ and i^{W} exist, and $i^{W} \circ i^{\mathcal{U}}$ is continuous at $\delta(\tilde{\mathcal{T}})$, with $o(H_\xi) = \sup i^{W} \circ i^{\mathcal{U}} \text{ " } \delta(\tilde{\mathcal{T}})$.

The properties of full normalization then tell us that $\vec{d}^\frown b$ is a weak hull of X. But $X = \mathcal{W}_\xi$, so $\Sigma_\xi(\vec{d}) = b$, so $\Sigma_\beta(\vec{d}) = b$, contradiction.

$\boxtimes$

By the claim we have $\mathcal{U}$ on $M(\vec{d})$ and $\mathcal{V}$ on R normal, by $\Sigma_{\vec{d},M(\vec{d})}$ and Φ respectively, with common last model (S,Λ), and such that R-to-S does not drop. Thus $M_\infty(S,\Lambda) = H_\beta$, and we have the diagram

$$\begin{tikzcd}
P \arrow[r, dash, "\vec{d}"] \arrow[drr, dash, "\mathcal{W}_\beta = X"'] & M(\vec{d}) \arrow[r, dash, "\mathcal{U}"] & S \arrow[d, "i^{\mathcal{W}}"] & R \arrow[l, "i^{\mathcal{V}}"'] \\
 & & H_\beta &
\end{tikzcd}$$

where

$$\mathcal{X} = \mathcal{X}(\mathcal{T}, \mathcal{U}, \mathcal{W})$$
$$= \mathcal{W}_\beta .$$

(Notice here that although $\mathcal{T}$ has no last model, $\mathcal{U}$ has last model S, and the branch of $\mathcal{U}$ from $M(\mathcal{T})$ to S drops. We have that $\mathcal{W}_\beta = \mathcal{X}(\mathcal{T}^\frown b, \mathcal{U}, \mathcal{W})$ as well, but because $\mathcal{U}$ dropped, there is no weak hull embedding from $\mathcal{T}^\frown b$ to $\mathcal{W}_\beta$ coming out of that. Indeed, $\Sigma_\beta(\mathcal{T})\uparrow$ by hypothesis.) Letting $\mathcal{Y} = \mathcal{X}(\mathcal{T}, \mathcal{U})$, we have that $\mathcal{Y}$ is a weak hull of $\mathcal{W}_\beta$, so $\mathcal{Y}$ is by Σ_β, so $\langle \mathcal{T}, \mathcal{U} \rangle$ is a partial $\mathbb{P}$-stack by $\hat{\Sigma}_\beta$, with last model S.

To finish the proof, it is enough to show $\left(\hat{\Sigma}_\beta\right)_{\langle \gamma, \mathcal{U}\rangle, S} = \Lambda^{rel}$, where Λ^{rel} is the M_∞-relevant part of Λ.

For that, it is enough to show that whenever $\vec{\mathcal{V}}$ is a finite stack from S to N by Λ that and S-to-N does not drop, then

$$Y = X(\gamma, \mathcal{U}, \vec{\mathcal{V}})$$

weakly embeds into W_β. But the diagram below makes this clear.

$$\begin{tikzcd}
P \arrow[r, "\gamma"] \arrow[rrr, bend right, "Y"'] \arrow[drrr, "W_\beta"'] & M(\gamma) \arrow[r, "\mathcal{U}"] & S \arrow[r, "\vec{\mathcal{V}}"] & N \arrow[d, "Z"] \\
 & & & H_\beta = M_\infty(N, \Lambda_{\vec{\mathcal{V}}, N})
\end{tikzcd}$$

We have $W_\beta = X(Y, Z)$, so Y weakly embeds into W_β. $\boxtimes$

__Remark__ If $\vec{v}$ from S to N drops, then the diagram above becomes

$$\begin{tikzcd}
P \arrow[r, "\sigma"] \arrow[rrr, bend right, "Y"] \arrow[drrr, bend right, "W_\xi"'] & M(\sigma) \arrow[r, "u"] & S \arrow[r, "\vec{v}"] & N \arrow[d, "Z"] \\
 & & & H_\xi = M_\infty(N, \Lambda_{\vec{v},N})
\end{tikzcd}$$

where $\xi \leq \beta$. We then get that Y is a weak hull of W_ξ, hence Y is by Σ_ξ. This gives that $\langle \sigma, u, \vec{v} \rangle$ is by $\hat{\Sigma}'_\beta$, where $\hat{\Sigma}'_\beta$ consists of all ~~stacks~~ partial P-stacks s such that each $X(s{\restriction}k)$ is ~~by~~ is weakly embedded into a W_ξ with $\xi \leq \beta$. That would have been a better definition of $\hat{\Sigma}_\beta$.

Then we get $(\hat{\Sigma}'_\beta)_{\langle \sigma, u \rangle, s} = \Lambda$ in the above.

<u>Definition 1.12</u> Let $\alpha \le \delta_\Gamma$. A pair $(P, \langle W_\beta \mid \beta < \alpha \rangle)$ is *α-good* iff

(1) P is an lpm, and W_β is a normal tree on P with last model H_β, and

(2) For $\beta < \alpha$, let Σ_β be the partial strategy for P: if s is a partial P-stack by Σ_β with last tree $\mathcal{U} = \mathcal{U}(s)$ of limit length, and c is cofinal in $\mathcal{U}$, then $\Sigma_\beta(s) = c$ iff $X(s^\frown c)$ is weakly embedded into some W_ξ, for $\xi \le \beta$; then

(a) each Σ_β is reasonable

(b) there is some partial stack s by Σ_β with last model N such that $(N, (\Sigma_\beta)_s)$ is an lbr hod pair with mouse rank β, and

(c) for any partial stack s by Σ_β such that $\mathcal{T} = \mathcal{U}(s)$ has limit length and $\Sigma_\beta(s)\uparrow$, there is a normal $\mathcal{U}$ on $M(\mathcal{T})$ by $(\Sigma_\beta)_{s, M(\mathcal{T})}$ such that $\mathcal{U}$ has last model Q, and $(Q, (\Sigma_\beta)_{s, M(\mathcal{T}), \mathcal{U}, Q})$ is an lbr hod pair of mouse rank β.

We have shown

Lemma 1.13 For any $\alpha < \delta_\Gamma$, there is an α-good pair.

Corollary 1.14 There is a δ_Γ-good pair.

Proof Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be the universal Γ set we fixed above, and φ our Γ norm on A. For $\alpha < \delta_\Gamma$, say that a real z is α-strong iff letting

$$\mu_\xi = \sup\{o(H_\eta) \mid \eta \leq \xi\},$$

and

$$B_\xi = \{(x,y) \mid \varphi(x) \leq \varphi(y) < \mu_\xi\}$$

for each $\xi \leq \alpha$, and

$$S^z_\xi = \text{the } \Sigma^1_1(B_\xi) \text{ subset of } \mu_\xi \text{ coded by } z,$$

for $\xi < \alpha$, then for all $\xi < \alpha$:

(a) B_ξ codes an ξ-good pair
$\langle P^\xi, \langle W^\xi_\nu \mid \nu < \xi \rangle \rangle$, and

(b) if $\gamma < \xi$, then $P^\gamma = P^\xi$, and
$W^\gamma_\nu = W^\xi_\nu$ for all $\nu < \gamma$.

The functions $\xi \longmapsto \mu_\xi$ and $\xi \longmapsto B_\xi$ are Σ_1 over M_Γ. So the predicate

$$S(z, \alpha) \text{ iff } z \text{ is } \alpha\text{-strong}$$

is Δ_1 over M_Γ. By the Coding Lemma and 1.14, $\forall \alpha\, \exists z\, (z \text{ is } \alpha\text{-strong})$.
This implies $\exists z\, \forall \alpha\, (z \text{ is } \alpha\text{-strong})$:
otherwise
$M_\Gamma \models \forall z\, \nexists \alpha\, (z \text{ is not } \alpha\text{-strong})$,
and by admissibility, there is a $\beta < \delta_\Gamma$ such that $\forall z\, \nexists \alpha < \beta\, (z \text{ is not } \alpha\text{-strong})$, so $\forall z\, (z \text{ is not } \beta\text{-strong})$. But

letting z be α-strong for all $\alpha < \delta_r$, we see that z codes a δ_r-good pair.

$\boxtimes$

§2. Step two

We complete the proof of theorem 0.5. We have Γ inductive like with the scale property, and $\underset{\sim}{\Delta} \models$ HPC. We wish to construct a hod pair (Q, Ψ) such that $\mathrm{Code}(\Psi) \notin \underset{\sim}{\Delta}$, granted that there are Suslin cardinals $> \delta_\Gamma$. By 1.14, we may fix a δ_Γ-good pair

$$(P, \langle W_\alpha \mid \alpha < \delta_\Gamma \rangle).$$

Let Σ be the partial strategy for P determined by the W_α's; that is, for s a partial P-stack by Σ whose last tree has limit length,

$$\Sigma(s) = c \text{ iff } \exists \alpha \,(\mathcal{Q}(s^\frown c) \text{ is a weak hull of } W_\alpha).$$

$\mathrm{Code}(\Sigma)$ is a Γ set because $(P, \vec{W})$ is coded in Γ by the coding lemma. ~~$\mathrm{Code}(\Sigma) \notin \underset{\sim}{\Delta}$~~

Code$(\Sigma) \notin \underset{\sim}{\Delta}$ by (2)(b) of def. 1.12. In fact, if $\mathcal{T}$ is Σ-maximal, then Code$(\Sigma_{\mathcal{T}, M(\mathcal{T})}) \notin \underset{\sim}{\Delta}$, by (2)(c) of 1.12.

<u>Remark</u> We can have Σ being $\underset{\sim}{\Sigma}_1^{M_\Gamma}$ but $\Sigma \notin M_\Gamma$ because Σ is a partial strategy, not a complete strategy. This shows already that there are Σ-maximal trees, and in fact, that if $\mathcal{T}$ is Σ-maximal, then there are $\Sigma_{\mathcal{T}, M(\mathcal{T})}$-maximal trees on $M(\mathcal{T})$.

Now let Γ^* be a good pointclass with the scale property such that $\underset{\sim}{\Gamma} \subsetneq \underset{\sim}{\Delta}^*$. Let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ^*-Woodin model that Suslin captures Code(Σ) and its complement, with P in HC^{N^*}. Let $\mathbb{C}$ be the maximal lpm construction of N^*.

We shall compare (P, Σ) with the $(M^{c}_{\nu,k}, \Omega^{c}_{\nu,k})$. For that to work, we need that if s is by Σ, with last model N such that P-to-N drops, then $\Sigma_{s,N}$ is a complete strategy.

(This insures that ~~[illegible]~~

$\dot{\Sigma}^P \subseteq \Sigma$ by reasonability. For $\dot{\Sigma}^P$ contains only information about how to iterate the $N \lhd P$, and by reasonability, this information is consistent with Σ. So if Σ_N is complete, then the information is contained in Σ_N, so $\dot{\Sigma}^P \subseteq \Sigma$. Similarly,

$\dot{\Sigma}^Q \subseteq \Sigma_{s,Q}$ for any stack s by Σ.)

But we have

<u>Claim</u> There is a stack s by Σ with last model Q such that $\mathrm{Code}(\Sigma_{s,Q}) \notin \underset{\nu}{\Delta}$, and whenever t is by $\Sigma_{s,Q}$ with last model N, and Q-to-N drops in t, then $\Sigma_{s,Q,t,N}$ is a complete strategy.

<u>Proof</u> If $s = \emptyset$ does not work, then there is t_0 by Σ with last model N_0 such that P-to-N_0 drops, and Σ_{t_0,N_0} is not complete. But that means we have a normal $\mathcal{T}_0$ on N_0 by Σ_{t_0,N_0} such that $\Sigma_{t_0,N_0}(\mathcal{T}_0)\uparrow$.

By δ_r-goodness of $\vec{W}$, we get that $\mathrm{Code}(\Sigma_{t_0,N_0}) \notin \underset{\sim}{\Delta}$. So we can replace (P,Σ) by (N_0, Σ_{t_0,N_0}). If (t_0, N_0) does not work for the claim, we get t_1 on N_0 with last model N_1 s.t. N_0-to-N_1 drops, etc.

So if the claim fails, $t_0{}^\frown t_1{}^\frown \cdots$ is an infinite stack by Σ with infinitely many drops, contrary to 1.4(d). $\boxtimes$

Now let us fix s, Q as in the claim, and let $\Lambda = \Sigma_{s,Q}$. The arguments of [ ] show that for each $\langle \nu, k \rangle \leq \langle \delta^*, 0 \rangle$, the comparison of (Q, Λ) with $(M^c_{\nu,k}, \Omega^c_{\nu,k})$ is such that only the Q-side moves, and no strategy disagreements show up. Here Q is being iterated by Λ, and we stop the comparison with $(M^c_{\nu,k}, \Omega^c_{\nu,k})$ if the current tree $\mathcal{T}$ on the Q side is Λ-maximal.

(Q, Λ) cannot iterate ~~past~~ past $(M^c_{\delta^*,0}, \Omega^c_{\delta^*,0})$ because $Code(\Lambda)$ is in $\underset{\sim}{\Delta}^*$, so the tree $\mathcal{T}$ of length δ^*+1 on Q by Λ that is responsible would be in N^*, and this yields $N^* \models \delta^*$ is not Woodin by the universality argument, contradiction.

If (Q, Λ) iterates to some $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ via a normal tree $\mathcal{T}$ by Λ of successor length whose main branch does not drop, then letting $\Omega = \Omega^{\mathbb{C}}_{\nu,k}$ and $M = M^{\mathbb{C}}_{\nu,k}$ and $\pi : Q \to M$ be the iteration map of $\mathcal{T}$, we have that $\Lambda \subseteq \Omega^{\pi}$, because Λ is pullback consistent (as a partial strategy), so Ω^{π} is a complete strategy for Q, and $\Omega^{\pi} \notin \underset{\sim}{\Delta}$ because $\Lambda \notin \underset{\sim}{\Delta}$. (More precisely, because there are complete strategies Λ_t arbitrarily high in $\underset{\sim}{\Delta}$.) So we have the conclusion of Theorem 0.5.

The final possibility is that for some $\langle \nu, k \rangle \leq_{lex} \langle \delta^*, 0 \rangle$, setting $(M, \Omega) = (M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, we have a normal,

Λ-maximal tree $\mathcal{T}$ on Q with last model M and such that $\Lambda_{\mathcal{T},M} \subseteq \Omega$. But then, by (2)(c) of δ_Γ-goodness, there are complete strategies $(\Lambda_{\mathcal{T},M})_t$ of arbitrarily high Wadge degree in $\underset{\sim}{\Delta}$. Thus $\mathrm{Code}(\Omega) \notin \underset{\sim}{\Delta}$, and (M, Ω) witnesses the conclusion of theorem 0.5.

Thm. 0.5 $\boxtimes$

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