

CONDENSATION FOR MOUSE PAIRS

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Abstract

This is the first of two papers on the fine structure of HOD in models of the Axiom of Determinacy (AD). Let $M \models \text{AD}^+ + \mathbf{V} = \mathbf{L}(\varphi(\mathbb{R}))$. [13] shows that under a natural hypothesis on the existence of iteration strategies, the basic fine structure theory for pure extender models goes over to HOD^M . In this paper, we prove a fine condensation theorem, quite similar to Theorem 9.3.2 of Zeman's book [16], except that condensation for iteration strategies has been added to the mix. In the second paper, we shall use this theorem to show that in HOD^M , \square_κ holds iff κ is not subcompact.

1. INTRODUCTION

One goal of descriptive inner model theory is to elucidate the structure of HOD (the universe of hereditarily ordinal definable sets) in models M of the Axiom of Determinacy. HOD^M is close to M in various ways; for example, if $M \models \text{AD}^+ + \mathbf{V} = \mathbf{L}(\varphi(\mathbb{R}))$ ¹, then M can be realized as a symmetric forcing extension of HOD^M , so that the first order theory of M is part of the first order theory of its HOD.² For this and many other reasons, the study of HOD in models of AD has a long history. We refer the reader to [12] for a survey of this history.

The study of HOD involves ideas from descriptive set theory (for example, games and definable scales) and ideas from inner model theory (mice, comparison, fine structure). One early result showing that inner model theory is relevant is due to the first author, who showed in 1994 ([11]) that if there are ω Woodin cardinals with a measurable above them all, then in $L(\mathbb{R})$, HOD up to θ is a pure extender mouse. Shortly afterward, this result was improved by W. Hugh Woodin, who reduced its hypothesis to $\text{AD}^{L(\mathbb{R})}$, and identified the full $\text{HOD}^{L(\mathbb{R})}$ as a model of the form $L[M, \Sigma]$, where M is a pure extender premouse, and Σ is a partial iteration strategy for M . $\text{HOD}^{L(\mathbb{R})}$ is thus a new type of mouse, sometimes called a *strategy mouse*, sometimes called a *hod mouse*. See [14] for an account of this work.

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¹ AD^+ is a technical strengthening of AD. It is not known whether $\text{AD} \Rightarrow \text{AD}^+$, though in every model of AD constructed so far, AD^+ also holds. The models of AD that we deal with in this paper satisfy AD^+ .

²This is a theorem of Woodin from the early 1980s. Cf. [15].

Since the mid-1990s, there has been a great deal of work devoted to extending these results to models of determinacy beyond $L(\mathbb{R})$. Woodin analyzed HOD in models of AD^+ below the minimal model of $\text{AD}_{\mathbb{R}}$ fine structurally, and Sargsyan pushed the analysis further, first to determinacy models below $\text{AD}_{\mathbb{R}}$ + “ θ is regular” (see [2]), and more recently, to determinacy models below the minimal model of the theory “ $\text{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_{\alpha}$ is the largest Suslin cardinal” (commonly known as LSA). (See [3].) The hod mice used in this work have the form $M = L[\vec{E}, \Sigma]$, where \vec{E} is a coherent sequence of extenders, and Σ is an iteration strategy for M . The strategy information is fed into the model M slowly, in a way that is dictated in part by the determinacy model whose HOD is being analyzed. One says that the hierarchy of M is *rigidly layered*, or *extender biased*. The object (M, Σ) is called a rigidly layered (extender biased) hod pair.

Putting the strategy information in this way makes comparison easier, but it has serious costs. The definition of “premouse” becomes very complicated, and indeed it is not clear how to extend the definition of rigidly layered hod pairs much past that given in [3]. The definition of “extender biased hod premouse” is not uniform, in that the extent of extender bias depends on the determinacy model whose HOD is being analyzed. Fine structure, and in particular condensation, become more awkward. For example, it is not true in general that the pointwise definable hull of a level of M is a level of M . (The problem is that the hull will not generally be sufficiently extender biased.) Because of this, it is open whether the hod mice of [3] satisfy $\forall \kappa \square_{\kappa}$. (The second author did show that $\forall \kappa \square_{\kappa, 2}$ holds in these hod mice; cf. [3].)

The more naive notion of hod premouse would abandon extender bias, and simply add the least missing piece of strategy information at essentially every stage. This was originally suggested by Woodin. The first author has recently proved a general comparison theorem that makes it possible to use this approach, at least in the realm of short extenders. The resulting premice are called *least branch premice* (lpm’s), and the pairs (M, Σ) are called *least branch hod pairs* (lbr hod pairs).³ Combining results of [13] and [10], one has

Theorem 1.1 ([13],[10]). *Assume AD^{++} “there is an (ω_1, ω_1) iteration strategy for a pure extender premouse with a long extender on its sequence”. Let $\Gamma \subseteq P(\mathbb{R})$ be such that $L(\Gamma, \mathbb{R}) \models \text{AD}_{\mathbb{R}} +$ “there is no (ω_1, ω_1) iteration strategy for a pure extender premouse with a long extender on its sequence”; then $\text{HOD}^{L(\Gamma, \mathbb{R})}$ is a least branch premouse.*

Of course, one would like to remove the iterability hypothesis of 1.1, and prove its conclusion under AD^+ alone. Finding a way to do this is one manifestation of the long standing iterability problem of inner model theory. Although we do not yet know how to do this, the theorem does make it highly likely that in models of $\text{AD}_{\mathbb{R}}$ that have not reached an iteration strategy for a pure extender premouse with a long extender, HOD is an lpm.

Least branch premice have a fine structure much closer to that of pure extender models than that of rigidly layered hod premice. The book [13] develops the basics, the solidity and universality of

³The (pure extender or least branch hod) premice in the paper are called pfs (projectum-free space) premice in [13]. We will occasionally omit the “pfs” for brevity. All premice used in this paper are pfs premice (and their strong cores), see Section 2 for more discussion.

standard parameters, and a coarse form of condensation. The main theorem of this paper, Theorem 4.6, is a stronger condensation theorem. The statement of 4.6 is parallel to that of Theorem 9.3.2 of [16], but it has a strategy-condensation feature that is new even in the pure extender model context. The proof of 4.6 follows the same outline as the proofs of solidity, universality, and condensation given in [13], but there are a number of additional difficulties to be overcome. These stem from the restricted elementarity we have for the ultrapowers of phalanxes that are taken in the course of the proof.

Theorem 4.6 is one of the main ingredients in the proof of the main theorem of our third paper. We say that (M, Σ) is a *mouse pair* iff M is either a pure extender pfs premouse or a least branch pfs premouse, and Σ is an iteration strategy for M that has strong hull condensation, normalizes well, is internally lift consistent, and in the least branch case, is pushforward consistent. See [13, Chapter 9] and Section 2 below for a full definition.⁴

Theorem 1.2 (AD^+). *Let (M, Σ) be a mouse pair. Let κ be a cardinal of M such that $M \models \text{“}\kappa^+$ exists”}; then in M , the following are equivalent.*

1. \square_κ .
2. $\square_{\kappa, < \kappa}$.
3. κ is not subcompact.
4. The set of $\nu < \kappa^+$ such that $M|_\nu$ is extender-active is non-stationary in κ^+ .

The special case of this theorem in which M is a pure extender model is a landmark result of Schimmerling and Zeman. (See [5].) Our proof follows the Schimmerling-Zeman proof quite closely.

Theorem 1.2 has applications to consistency strength lower bound questions that we discuss in the second paper. But our work was also motivated by the desire to put the fine structure theory of [13] to the test, so to speak. Determining the pattern of \square is a good way to go one level deeper into the world of projecta, standard parameters, restricted elementarity, and condensation theorems. We found when we did so that the definition of *hod premouse* given in the first draft of [13] had problems, in that strategy information was being added in a way that might not in general be preserved by Σ_1 hulls.⁵ The better method for strategy insertion comes from [8], and we describe it further below. [13] has been revised so that it now uses this method.

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⁴Theorem 4.6 is also used heavily in the proof in [9] that the iteration strategy component of a mouse pair fully normalizes well, and is therefore positional.

⁵Remark 2.47 of [8] shows that in fact it is preserved by Σ_1 hulls, but the proof involves a phalanx comparison, and so a lot of theory just to prove a property of mice one would like to have available at the beginning.

2. LEAST-BRANCH HOD PREMICE

All premeice used in this paper are in the pfs hierarchy, defined in [13]. We adopt for the most part the fine structure and notation from [13, Chapter 9] concerning least-branch hod premeice (lpm's) and lbr hod pairs. A similar, albeit simpler, fine structure for pure extender premeice is discussed in [13, Chapter 4]. Fine structural condensation is the point of this paper, so we cannot avoid going into the details covered in these chapters. (Our main result is a fine structural refinement of [13, Theorem 4.10.10].) We summarize some main points below. The reader can see [13, Chapters 4,9] for more details.

2.1. Potential least branch premeice

The language for lpm's is \mathcal{L}_1 with symbols $\in, \dot{E}, \dot{F}, \dot{\Sigma}, \dot{B}, \dot{\gamma}$. $\mathcal{L}_0 = \mathcal{L}_1 - \{\dot{B}, \dot{\Sigma}\}$ is the language of pure extender mice. An lpm M is of the form (N, k) where N is an \mathcal{L}_1 amenable structure that is k -sound. We write $k = \text{deg}(M)$. We often identify M with N and suppress k . $o(M)$ denotes the ordinal height of M , and $\hat{o}(M)$ denotes the α such that $o(M) = \omega\alpha$. $l(M) = (\hat{o}(M), \text{deg}(M))$ is the index of M . For $(\nu, l) \leq_{\text{lex}} l(M)$, $M|(\nu, l)$ is the initial segment of M with index (ν, l) . We write $N \trianglelefteq M$ iff $N = M|(\nu, l)$ for some $(\nu, l) \leq_{\text{lex}} l(M)$. If $\nu \leq \hat{o}(M)$, write $M|\nu$ for $M|(\nu, 0)$.⁶

\dot{E}^M codes the sequence of extenders that go into constructing M . \dot{F}^M if non-empty is the amenable code for a new extender being added; in this case, we say that M is *extender-active* (or just *E-active*). If $\dot{F}^M = F$ is nonempty, then $M \models \text{crt}(F)^+$ exists and $o(M) = i_F^M(\mu)$, where $\mu = \text{crt}(F)^+$. Also F must satisfy the Jensen initial segment condition (ISC), that is, whole initial segments of F must be in \dot{E}^M (see [16] for a detailed discussion of ISC). $\dot{\gamma}$ is the index of the largest whole initial segment of F if exists; otherwise, $\dot{\gamma} = 0$. We also demand M is *coherent*, that is $i_F^M(\dot{E}^M) \upharpoonright o(M) + 1 = (\dot{E}^M)^\frown \langle \emptyset \rangle$.

$\dot{\Sigma}^M$ and \dot{B}^M are used to record information about an iteration strategy Ω of M . $\dot{\Sigma}^M$ codes the strategy information added at earlier stages; $\dot{\Sigma}^M$ acts on λ -separated trees.⁷ $\dot{\Sigma}^M(s, b)$ implies that $s = \langle \nu, k, \mathcal{T} \rangle$, where $(\nu, k) \leq l(M)$ and \mathcal{T} is a λ -separated tree on $M|(\nu, k)$ in M of limit length and $\mathcal{T} \frown b$ is according to the strategy. We say that s is an M -tree, and write $s = \langle \nu(s), k(s), \mathcal{T}(s) \rangle$. We write $\dot{\Sigma}_{\nu, k}^M$ for the partial iteration strategy for $M|(\nu, k)$ determined by $\dot{\Sigma}$. We write $\Sigma^M(s) = b$ when $\dot{\Sigma}^M(s, b)$, and we say that s is according to Σ^M if $\mathcal{T}(s)$ is according to $\dot{\Sigma}_{\nu(s), k(s)}^M$.

Now we discuss how to code branch information for a tree $\mathcal{T}(s)$ such that $\Sigma^M(s)$ has not yet been defined into the \dot{B}^M predicate. Here we use the \mathfrak{B} -operator in [8]. We are correcting some errors in the original version of [13]. These corrections have been incorporated in its latest version.

M is *branch-active* (or just *B-active*) iff

- (a) there is a largest $\eta < o(M)$ such that $M|\eta \models \text{KP}$, and letting $N = M|\eta$,

⁶ $M|\nu$ can be active. We write $M||\nu$, or $M|\langle \nu, -1 \rangle$, for $M|\nu$ cut a second time by removing its last extender if it has one.

⁷See [13, Chapter 4] for detailed discussions on λ -separated trees.

- (b) there is a $<_N$ -least N -tree s such that s is by Σ^N , $\mathcal{T}(s)$ has limit length, and $\Sigma^N(s)$ is undefined.
- (c) for N and s as above, $o(M) \leq o(N) + \text{lh}(\mathcal{T}(s))$.

Note that being branch-active can be expressed by a Σ_2 sentence in $\mathcal{L}_1 - \{\dot{B}\}$. This contrasts with being extender-active, which is not a property of the premouse with its top extender removed. In contrast with extenders, we know when branches must be added before we do so.

Definition 2.1. Suppose that M is branch-active. We set

$$\begin{aligned} \eta^M &= \text{the largest } \eta \text{ such that } M|\eta \models \text{KP}, \\ \nu^M &= \text{unique } \nu \text{ such that } \eta^M + \nu = o(M), \\ s^M &= \text{least } M|\eta^M\text{-tree such that } \dot{\Sigma}^{M|\eta^M} \text{ is undefined, and} \\ b^M &= \{\alpha \mid \eta + \alpha \in \dot{B}^M\}. \end{aligned}$$

Moreover,

- (1) M is a *potential lpm* iff b^M is a cofinal branch of $\mathcal{T}(s)|\nu^M$.
- (2) M is *honest* iff $\nu^M = \text{lh}(\mathcal{T}(s))$, or $\nu^M < \text{lh}(\mathcal{T}(s))$ and $b^M = [0, \nu^M)_{\mathcal{T}(s)}$.
- (3) M is an *lpm* iff M is an honest potential lpm.
- (4) M is *strategy active* iff $\nu^M = \text{lh}(\mathcal{T}(s))$.

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Note that η^M is a Σ_0^M singleton, because it is the least ordinal in \dot{B}^M (because 0 is in every branch of every iteration tree), and thus s^M is also a Σ_0^M singleton. We have separated honesty from the other conditions because it is not expressible by a Q -sentence, whereas the rest is. Honesty is expressible by a Boolean combination of Σ_2 sentences. See 2.7 below.

The original version of [13] required that when $o(M) < \eta^M + \text{lh}(\mathcal{T}(s))$, \dot{B}^M is empty, whereas here we require that it code $[0, o(M))_{\mathcal{T}(s)}$, in the same way that \dot{B}^M will have to code a new branch when $o(M) = \eta^M + \text{lh}(\mathcal{T}(s))$. Of course, $[0, \nu^M)_{\mathcal{T}(s)} \in M$ when $o(M) < \eta^M + \text{lh}(\mathcal{T}(s))$ and M is honest, so the current \dot{B}^M seems equivalent to the original $\dot{B}^M = \emptyset$. However, $\dot{B}^M = \emptyset$ leads to Σ_1^M being too weak, with the consequence that a Σ_1 hull of M might collapse to something that is not an lpm.⁸ Our current choice for \dot{B}^M solves that problem.

Remark 2.2. Suppose N is an lpm, and $N \models \text{KP}$. It is very easy to see that $\dot{\Sigma}^N$ is defined on all N -trees s that are by $\dot{\Sigma}^N$ iff there are arbitrarily large $\xi < o(N)$ such that $N|\xi \models \text{KP}$. Thus if M is

⁸The hull could satisfy $o(H) = \eta^H + \text{lh}(\mathcal{T}(s^H))$, even though $o(M) < \eta^M + \text{lh}(\mathcal{T}(s^M))$. But then being an lpm requires $\dot{B}^H \neq \emptyset$. See Remark 2.47 in [8] for a more detailed discussion. Basically, one can show that the $L^\Sigma[\vec{E}]$ constructions doesn't break down because the the models constructed are not Σ -premise; [8, Remark 2.47] outlines an argument that cores of Σ -premise that are constructed in the $L^\Sigma[\vec{E}]$ -constructions are Σ -premise.

branch-active, then η^M is a successor admissible; moreover, we do add branch information, related to exactly one tree, at each successor admissible. Waiting until the next admissible to add branch information is just a convenient way to make sure we are done coding in the branch information for a given tree before we move on to the next one. One could go faster. \dashv

We say that an lpm M is (fully) passive if $\dot{F}^M = \emptyset$ and $\dot{B}^M = \emptyset$. It cannot be the case that M is both E -active and B -active. In the case that M is E -active, using the terminology of [5], the extender \dot{F}^M can be of type A , B , or C .

2.2. Solidity and soundness

We adopt the projectum-free space (pfs) fine structure in [13, Chapter 4]. We write $\rho_n(M)$ for the n -th projectum of M and $p_n(M)$ for the n -th standard parameter of M . We set $\rho(M) = \rho_{\text{deg}(M)+1}(M)$ and $p(M) = p_{\text{deg}(M)+1}(M)$, and call them *the* projectum and parameter of M . We say M is *sound* iff it is $\text{deg}(M) + 1$ -sound. An lpm M must be $\text{deg}(M)$ -sound, but it need not be $\text{deg}(M) + 1$ -sound. There are two types of premice, type 1 and type 2, with the distinction being based on the soundness pattern of the premice. Type 1 are the most important. All proper initial segments of an lpm must be sound type 1 lpms. Type 2 premice can be produced by taking a k -ultrapower that is discontinuous at ρ_k .

If M is type 1 and k -sound for $k \geq 1$, then it is coded by its reduct M^k , where

$$M^k = (M \parallel \rho_k(M), A_M^k),$$

and

$$A_M^k = \{\langle \varphi, b \rangle \mid \varphi \text{ is } \Sigma_1 \wedge b \in M \parallel \rho_k \wedge M^{k-1} \models \varphi[b, w_k]\},$$

where $\rho_k = \rho_k(M)$, $w_k = w_k(M) = \langle \rho_k(M), \eta_k(M), p_k(M) \rangle$ and $\eta_k(M)$ is the Σ_1 -cofinality of $\rho_k(M)$ over M^{k-1} . We also have the decoding function $d^k : M^k \rightarrow M$ and canonical Σ_1 -Skolem function $h_{M^k}^1$ over M^k defined as in [13, Chapter 4]. We have the $k + 1$ -st projectum, parameter, strong core $\bar{\mathfrak{C}}_{k+1}$, and core \mathfrak{C}_{k+1} defined by (we will omit the M from the notation)

$$\begin{aligned} \rho_{k+1} &= \rho_1(M^k), \\ p_{k+1} &= p_1(M^k), \\ \bar{\mathfrak{C}}_{k+1} &= \text{transitive collapse of } d^k \circ h_{M^k}^1[(\rho_{k+1} \cup \{p_{k+1}, w_k\})], \\ \bar{p}_{k+1} &= \sigma^{-1}(p_{k+1}), \\ \mathfrak{C}_{k+1} &= \text{transitive collapse of } d^k \circ h_{M^k}^1[(\rho_{k+1} \cup \{p_{k+1}, \rho_{k+1}, w_k\})]. \end{aligned}$$

Here, we let $\sigma : \bar{\mathfrak{C}}_{k+1} \rightarrow M$ and $\pi : \mathfrak{C}_{k+1} \rightarrow M$ be the uncollapse maps.

For M of type 1, M is $k+1$ -solid iff M^k is parameter solid, projectum solid, stable (see Definition

2.3), and M is weakly ms-solid.⁹ M is $k + 1$ -sound iff M is $k + 1$ -solid and $M = \mathfrak{C}_{k+1}(M)$. M is $k + 1$ -strongly sound if M is $k + 1$ -sound and $M = \bar{\mathfrak{C}}_{k+1}(M)$ ⁻¹⁰.

Let M be a pfs premouse of type 1 and $\text{deg}(M) = k < \omega$. We set

$$\hat{\rho}_{k+1}(M) = \hat{\rho}_1(M^k) = \begin{cases} 0 & \text{if } M^k \text{ is strongly sound,} \\ \text{least } \kappa \text{ s.t. } \kappa \notin \text{Hull}_1^{M^k}(\rho_1(M^k) \cup p_1(M^k)) & \text{otherwise.} \end{cases}$$

$$\hat{\eta}_{k+1}^M = \hat{\eta}_1(M^k) = \text{cof}_1^{M^k}(\hat{\rho}_{k+1}(M)).$$

We say that M is *almost sound* iff

- (a) M is solid;
- (b) $M^k = \text{Hull}_1^{M^k}(\rho_1(M^k) \cup \{p_1(M^k) \cup \hat{\rho}_1(M^k)\})$;
- (c) if $\rho_1(M^k) \leq \hat{\rho}_1(M^k)$, then letting

$$(H, B) = \text{cHull}_1^{M^k}(\rho_1(M^k) \cup p_1(M^k)),$$

with anticollapse map $\pi : (H, B) \rightarrow M^k$, we have

$$M^k = \text{Ult}((H, B), D),$$

where D is the order zero measure of H on $\hat{\rho}_1(M^k)$ and $\pi = i_D$, and

- (d) if $\rho_1(M^k) < \hat{\rho}_1(M^k)$ then $\hat{\eta}_1(M^k) < \rho_1(M^k)$.

Now suppose M is an acceptable J structure, and for $k = \text{deg}(M)$, either $k = 0$ or M^{k-1} is a pfs premouse of type 1. We say that M is a pfs premouse of type 1A iff $\rho_k(M) = \rho_{k-1}(M)$ or $\rho_k(M) \in \text{Hull}_1^{M^{k-1}}(\rho_k(M) \cup p_k(M))$, equivalently iff $\text{deg}(M) = 0$ or $\text{deg}(M) > 0$ and M^- is strongly sound. M has type 1B iff $\text{deg}(M) > 0$ and M^- is sound, but not strongly sound. M has type 2 iff M^- is almost sound, but not sound.¹¹ So for M a pfs premouse of degree k ,

- M has type 1A iff $\hat{\rho}_k(M) = 0$,
- M has type 1B iff $\hat{\rho}_k(M) = \rho_k(M)$, and
- M has type 2 iff $\hat{\rho}_k(M) > \rho_k(M)$.

⁹ M is weakly ms-solid iff either M is passive or the last extender of \mathfrak{C}_1 and $\bar{\mathfrak{C}}_1$ satisfies the weak ms-ISC. M satisfies the weak ms-ISC if letting E be the top extender of M and κ be the critical point of E , then the Jensen completion of E_κ is on the sequence of $M|\text{lh}(E)$. For the other components of solidity, see Definition 2.3 below.

¹⁰For a pfs premouse N , N^- is just N but with soundness degree $\text{deg}(N) - 1$.

¹¹If M is a pfs premouse of type 1B, $k = \text{deg}(M) > 0$, and $N = \text{Ult}_k(M, E)$ where $\eta_k^M = \text{crt}(E)$, then $\sup i_E \rho_k(M) = \rho_k(N) < \hat{\rho}_k(N) = i_E(\rho_k(M))$, so N is a type 2 pfs premouse of degree k . Fine structural hulls of type 1 premouse can have type 2 as well.

Let M be a pfs premouse of either type, with $k = \deg(M) > 0$. We set

$$\hat{w}_k(M) = \langle \hat{\eta}_k(M), \hat{\rho}_k(M), p_k(M) \rangle,$$

where $\hat{\rho}_k(M)$ is the least ρ that is not in $\text{Hull}_1^{M^{k-1}}(\rho_k(M) \cup p_k(M))$ and $\hat{\eta}_k(M) = \text{cof}_1^{M^{k-1}}(\hat{\rho}_k(M))$. When M is clear from the context, we write \hat{w}_k for $\hat{w}_k(M)$ etc. Let

$$\hat{A}_M^k = \{ \langle \varphi, b \rangle \mid \varphi \text{ is } \Sigma_1 \wedge b \in M \mid \rho_k \wedge M^{k-1} \models \varphi[b, \hat{w}_k] \},$$

and

$$\hat{M}^k = (M \mid \mid \rho_k, \hat{A}_M^k).$$

We also have the decoding function $\hat{d}^k : M^k \rightarrow M$ and canonical Σ_1 -Skolem function $h_{\hat{M}^k}^1$ over \hat{M}^k defined as in [13, Chapter 4]. We write $\rho^-(M)$ for $\rho_{\deg(M)-1}(M)$ etc.¹²

We define solidity and soundness for type 2 premeice exactly as we did for type 1 premeice, but with \hat{M}^k replacing M^k . Since $\hat{M}^k = M^k$ when M has type 1, the type 1 and type 2 definitions can be unified. For the record:

Definition 2.3. Let M be a premouse of degree k ; then

$$\begin{aligned} \rho_{k+1}(M) &= \rho_1(\hat{M}^k), \\ p_{k+1}(M) &= p_1(\hat{M}^k), \\ \bar{\mathfrak{C}}_{k+1}(M) &= \text{transitive collapse of } \hat{d}^k \circ h_{\hat{M}^k}^1 \text{ ``}(\rho_{k+1} \cup \{p_{k+1}\}), \text{ and} \\ \mathfrak{C}_{k+1}(M) &= \text{transitive collapse of } \hat{d}^k \circ h_{\hat{M}^k}^1 \text{ ``}(\rho_{k+1} \cup \{p_{k+1}, \rho_{k+1}\}). \end{aligned}$$

Let $\sigma : \bar{\mathfrak{C}}_{k+1} \rightarrow M$ and $\pi : \mathfrak{C}_{k+1} \rightarrow M$ be the anticollapse maps, and $\bar{p}_{k+1} = \sigma^{-1}(p_{k+1})$; then

- (a) \hat{M}^k is *parameter solid* iff p_{k+1} is solid and universal over \hat{M}^k and \bar{p}_{k+1} is solid and universal over the reduct $(\bar{\mathfrak{C}}_{k+1})^k$ of $\bar{\mathfrak{C}}_{k+1}$.
- (b) \hat{M}^k is *projectum solid* iff ρ_{k+1} is not measurable by the M -sequence, and either $\bar{\mathfrak{C}}_{k+1} = \mathfrak{C}_{k+1}$, or $\mathfrak{C}_{k+1} = \text{Ult}(\bar{\mathfrak{C}}_{k+1}, D)$ and $\sigma = \pi \circ i_D$, for i_D the order zero measure of $\bar{\mathfrak{C}}_{k+1}$ on ρ_{k+1} .
- (c) \hat{M}^k is *stable* iff either $\hat{\eta}_k^M < \rho_{k+1}$, or $\hat{\eta}_k^M$ is not measurable by the M -sequence.
- (d) We say that M is *projectum stable* if $\eta_k(M)$ is not measurable by the M -sequence.

¹²It is sometimes useful to have also a coding structure for the strong core. Let M be a pfs premouse and $k = \deg(M) > 0$; then

$$B^k = \{ \langle \varphi, b \rangle \mid \varphi \text{ is } \Sigma_1 \wedge b \in M \mid \rho_k \wedge M^{k-1} \models \varphi[b, p_k] \},$$

and the reduct of $\bar{\mathfrak{C}}_k(M)$ is defined as

$$M_0^k = (M \mid \mid \rho_k, B^k).$$

- (e) We say that M has *stable type 1* iff M has type 1A, or M has type 1B and is projectum stable.

We say that M is *solid* (or $k+1$ -*solid*) iff \hat{M}^k is parameter solid, projectum solid, and stable. We say that M is *sound* (or $k+1$ -*sound*) iff M is solid and $M = \mathfrak{C}_{k+1}(M)$.¹³ \dashv

It is easy to see that M has stable type 1 iff M has type 1, and for any E on the M -sequence, letting $P \trianglelefteq M$ be longest such that $\text{Ult}(P, E)$ is defined, $\text{Ult}(P, E)$ has type 1.

2.3. Elementarity of maps

Suppose that M is an lpm, and $\pi: H \rightarrow M$. What sort of elementarity for π do we need to conclude that H is an lpm? In the proof of square for ordinary mice, we have to deal with embeddings that are only weakly elementary.¹⁴ In the context of proving square in pfs preface, we need a slight strengthening of weak elementarity, called *near elementarity* and defined in [13, Chapters 2,4]. Nearly elementary maps are produced by lifting constructions, and they will occur in the square construction.

Roughly a map $\pi: H \rightarrow M$ with $k = \text{deg}(M) = \text{deg}(H)$ is nearly elementary if it is weakly elementary and maps $\hat{\eta}_k(H)$ to $\hat{\eta}_k(M)$. More precisely, we say that π is nearly elementary if π is the completion of $\pi \upharpoonright \hat{H}^k$ and $\pi \upharpoonright \hat{H}^k$ is a Σ_0 -preserving and cardinal preserving map from \hat{H}^k to \hat{M}^k . π is *elementary* if it is nearly elementary and $\pi \upharpoonright \hat{H}^k$ is Σ_1 -elementary.

We note in the above that

- (a) If H is of type 1A, then H^- is strongly sound and $\hat{w}_k(H) = \langle 0, 0, p_k(H) \rangle$. In this case, M is of type 1A, but π may or may not preserve $\rho_k(H)$.
- (b) If H is of type 1B, then $\rho_k(H) = \hat{\rho}_k(H)$ and hence $\hat{w}_k(H) = w_k(H)$. M may have type 1B or type 2. M is of type 1B if and only if $\hat{w}_k(M) = w_k(M)$ if and only if $\pi(w_k(H)) = w_k(M)$.
- (c) If H is of type 2, then M is of type 1B or 2.

The existence of a nearly elementary $\pi: H \rightarrow M$ does not imply that H is a premouse when $\text{deg}(H) = \text{deg}(M) = 0$. If M is a passive lpm, then so is H , and there is no problem. If M is extender-active, then it could be that H is only a protomouse, in that its last extender predicate is not total. The problem here is solved by the parts of the Schimmerling-Zeman proof related to protomice, which work in our context. Finally, we must consider the case that M is branch-active.

Definition 2.4. A rQ -*formula* of \mathcal{L}_1 is a conjunction of formulae of the form

- (a) $\forall u \exists v (u \subseteq v \wedge \varphi)$, where φ is a Σ_1 formula of \mathcal{L}_1 such that u does not occur free in φ ,

¹³Similarly, if $k = \text{deg}(M)$, then $\rho(M) = \rho_{k+1}(M)$, $\hat{\rho}(M) = \hat{\rho}_{k+1}(M)$, $p(M) = p_{k+1}(M)$, and so on for the other k -free notations.

¹⁴See section 1.4 of [13] for a discussion of the degrees of elementarity. If $\text{deg}(H) = \text{deg}(M) = 0$, then π is weakly elementary iff it is Σ_0 elementary and cardinal-preserving.

or of the form

- (b) “ $\dot{F} \neq \emptyset$, and for $\mu = \text{crt}(\dot{F})^+$, there are cofinally many $\xi < \mu$ such that ψ ”, where ψ is Σ_1 .

⊢

Formulae of type (a) are usually called Q -formulae. Being a passive lpm can be expressed by a Q -sentence, but in order to express being an extender-active lpm, we need type (b) clauses, in order to say that the last extender is total. rQ formulae are Π_2 , and hence preserved downward under Σ_1 -elementary maps. They are preserved upward under Σ_0 maps that are *strongly cofinal*.

Definition 2.5. Let M and N be \mathcal{L}_0 -structures and $\pi: M \rightarrow N$ be Σ_0 and cofinal. We say that π is *strongly cofinal* iff M and N are not extender active, or M and N are extender active, and π “ $(\text{crt}(\dot{F})^+)^M$ is cofinal in $(\text{crt}(\dot{F})^+)^N$ ”.

⊢

It is easy to see that

Lemma 2.6. rQ formulae are preserved downward under Σ_1 -elementary maps, and upward under strongly cofinal Σ_0 -elementary maps.

Lemma 2.7. (a) There is a Q -sentence φ of \mathcal{L}_1 such that for all transitive \mathcal{L}_0 structures M , $M \models \varphi$ iff M is a passive lpm.

(b) There is a rQ -sentence φ of \mathcal{L}_1 such that for all transitive \mathcal{L}_0 structures M , $M \models \varphi$ iff M is an extender-active lpm.

(c) There is a Q -sentence φ of \mathcal{L}_1 such that for all transitive \mathcal{L}_0 structures M , $M \models \varphi$ iff M is a potential branch-active lpm.

Proof. (Sketch.) We omit the proofs of (a) and (b). For (c), note that “ $\dot{B} \neq \emptyset$ ” is Σ_1 . One can go on then to say with a Σ_1 sentence that if η is least in \dot{B} , then $M|_\eta$ is admissible, and s^M exists. One can say with a Π_1 sentence that $\{\alpha \mid \dot{B}(\eta + \alpha)\}$ is a branch of $\mathcal{T}(s)$, perhaps of successor order type. One can say that \dot{B} is cofinal in the ordinals with a Q -sentence. Collectively, these sentences express the conditions on potential lpm-hood related to \dot{B} . That the rest of M constitutes an extender-passive lpm can be expressed by a Π_1 sentence. \square

Corollary 2.8. (a) If M is a passive (resp. extender-active, potential branch-active) lpm, and $\text{Ult}_0(M, E)$ is wellfounded, then $\text{Ult}_0(M, E)$ is a passive (resp. extender-active, potential branch-active) lpm.

(b) Suppose that M is a passive (resp. extender-active, potential branch-active) lpm, and $\pi: H \rightarrow M$ is Σ_1 -elementary; then H is a passive (resp. potential branch-active) lpm.

(c) Let $\text{deg}(M) = \text{deg}(H) = 0$, and $\pi: H \rightarrow M$ be Σ_2 elementary; then H is a branch-active lpm iff M is a branch-active lpm.

Proof. rQ -sentences are preserved upward by strongly cofinal Σ_0 embeddings, so we have (a). They are Π_2 , hence preserved downward by Σ_1 - elementary embeddings, so we have (b).

It is easy to see that honesty is expressible by a Boolean combination of Σ_2 sentences, so we get (c). □

Remark 2.9. It could happen that M is a branch-active lpm, $\pi: H \rightarrow M$ is cofinal and elementary (with $\deg(M) = \deg(H) = 0$), and b^M is not cofinal in $\mathcal{T}(s^M)$, but b^H is cofinal in $\mathcal{T}(s^H)$. If we were using the branch coding in the original version of [13], then $\dot{B}^M = \emptyset$, so $\dot{B}^H = \emptyset$, so H is not an lpm. ⊥

Part (c) of Lemma 2.7 is not particularly useful. In general, our embeddings will preserve honesty of a potential branch active lpm M because $\dot{\Sigma}^M$ and \dot{B}^M are determined by a complete iteration strategy for M that has strong hull condensation. So the more useful preservation theorem in the branch-active case applies to *hod pairs*, rather than to *hod premice*.

2.4. Plus trees

The iteration trees we use below are λ -separated plus trees. These notions are defined in detail in [13, Section 4.4] and we briefly summarize the relevant concepts here. Suppose M is a pfs premouse and E is an extender on the M -sequence, then

- E^+ is the extender with generators $\lambda_E \cup \{\lambda_E\}$ that represents $i_F^{\text{Ult}(M,E)} \circ i_E^M$, where F is the order zero total measure on λ_E in $\text{Ult}(M, E)$.
- $\hat{\lambda}(E^+) = \lambda_E$.
- $\text{lh}(E^+) = \text{lh}(E)$.
- $o(E^+) = (\text{lh}(E)^+)^{\text{Ult}(M,E^+)}$.

We say that an extender G is of *plus type* if $G = E^+$ for some extender E on the sequence of a pfs premouse M ; we let $G^- = E$. In general, if E is an extender (of plus type or not)

- we let $\varepsilon(E) = \text{lh}(E)$ if E is of plus type; otherwise, $\varepsilon(E) = \lambda(E)$.
- if E is on the sequence of some premouse, then
 - (i) $\hat{\lambda}(E) = \lambda(E) = \hat{\lambda}(E^+)$,
 - (ii) $E^- = E$.

The *extended M -sequence* consists of all E such that E^- is on the M -sequence.

A *plus tree* \mathcal{T} on a pfs premouse is like an ordinary normal tree, except that

- (i) We only require that $E_\alpha^\mathcal{T}$ be on the extended $\mathcal{M}_\alpha^\mathcal{T}$ sequence,

- (ii) E_α^T is applied to the longest possible initial segment of \mathcal{M}_β^T , where β is least such that $\text{crt}(E_\alpha^T) < \hat{\lambda}(E_\beta^T)$, and
- (iii) the length-increasing condition is weakened slightly.¹⁵

See [13, Definition 4.4.3] for the complete definition.

A λ -separated tree is a plus tree in which every extender used along the tree is of plus type. The weakening in (iii) above does not affect λ -separated trees; that is, the lengths of the extenders used in a λ -separated tree are strictly increasing. Moreover, quasi-normalization coincides with embedding normalization on stacks of λ -separated plus trees. [13, Section 8.1] shows that λ -separated trees are enough for comparisons. For these and other reasons it is convenient to restrict one's attention to the way an iteration strategy Σ acts on stacks of λ -separated trees. By Lemma 9.3.2 of [13], if (P, Σ) is a mouse pair, then Σ is determined by its action on countable λ -separated trees.

2.5. Mouse pairs and Dodd-Jensen

Two of the main definitions from [13] are

Definition 2.10. (M, Ω) is a *pure extender pair* with scope H_δ iff

- (a) M is a pure extender pfs premouse.
- (b) Ω is a complete (ω, δ) iteration strategy for M ¹⁶, and
 - Ω is internally lift-consistent, quasi-normalizes well, and has strong hull condensation.¹⁷

→

Definition 2.11. (M, Ω) is a *least branch hod pair (lbr hod pair)* with scope H_δ iff

- (a) M is an lpm.
- (b) Ω is a complete (ω, δ) iteration strategy for M ,
- (c) Ω is internally lift-consistent, quasi-normalizes well, and has strong hull condensation, and
- (d) Ω is pushforward consistent, that is if s is by Ω with last model N , then $\dot{\Sigma}^N \subseteq \Omega_s$, where $\Omega_s(t) = \Omega(s \frown t)$.¹⁸

→

Definition 2.12. (M, Ω) is a *mouse pair* iff it is either a pure extender pair or an lbr hod pair. →

¹⁵The length-increasing condition is enough to guarantee that $T\text{-pred}(\alpha + 1)$ is the least β such that $\text{crt}(E_\alpha^T) < \varepsilon(E_\beta^T)$. Thus none of the generators of a plus extender E , including the generator $\hat{\lambda}(E)$, are moved later on a branch in which E has been used.

¹⁶See [13, 4.6.3].

¹⁷See [13, 5.4.4, 7.1.1, 7.1.9].

¹⁸See [13, 9.2.1].

Remark 2.13. [13] required that M have type 1 in order for (M, Ω) to qualify as a mouse pair, because that was sufficient generality for its purposes. Here we are allowing M to have type 2. \dashv

Included in 2.11(b) is the requirement that all Ω -iterates of M be least branch premice. Because of our honesty requirement in the branch-active case, this no longer follows automatically from the elementarity of the iteration maps. That the iterates of M are honest comes out of the construction of Ω , as a consequence of pushforward consistency.

If (M, Ω) is an lbr hod pair and $\pi: H \rightarrow M$ is nearly elementary, then Ω^π is the *pullback strategy*, given by

$$\Omega^\pi(s) = \Omega(\pi s).$$

We show now that, except in the protomouse case, (H, Ω^π) is an lbr hod pair.¹⁹

Lemma 2.14. *Let (M, Ω) be an lbr hod pair with scope H_δ , and let $\pi: H \rightarrow M$ be nearly elementary. Suppose that one of the following holds:*

- (a) M is passive or branch-active, or
- (b) H is an lpm.

Then (H, Ω^π) is an lbr hod pair with scope H_δ .

Proof. We show first that H is an lpm. If (b) holds, this is rather easy. If M is passive, we can apply (a) of 2.7, noting that Q sentences go down under nearly elementary embeddings. So let us assume that M is branch-active.

By (b) of 2.7, H is a potential branch active lpm. So we just need to see that H is honest. Let $\nu = \nu^H$, $b = b^H$, and $\mathcal{T} = \mathcal{T}(s^H)$. If $\nu = lh(\mathcal{T})$, there is nothing to show, so assume $\nu < lh(\mathcal{T})$. We must show that $b = [0, \nu]_{\mathcal{T}}$. We have by induction that for $N = H|\eta^H$, (N, Ω_N^π) is an lbr hod pair. Thus \mathcal{T} is by Ω^π , and so we just need to see that for $\mathcal{U} = \mathcal{T} \upharpoonright \nu$ and $\mathcal{W} = \mathcal{U} \frown b$, \mathcal{W} is by Ω^π , or equivalently, that $\pi\mathcal{W}$ is by Ω . But it is easy to see that $\pi\mathcal{W}$ is a psuedo-hull of $\pi(\mathcal{U}) \frown b^M$, and Ω has strong hull condensation, so we are done.

For the proof that (H, Ω^π) is internally lift-consistent, normalizes well, and has strong hull condensation, the reader should see [13]. We give here the proof that (H, Ω^π) is pushforward consistent, because it extends the honesty proof given above.

Let P be an Ω^π iterate of H via the stack of trees s . Let Q be the corresponding Ω iterate of M via πs , and let $\tau: P \rightarrow Q$ be the nearly elementary copy map. Then for $\mathcal{U} \in P$,

$$\begin{aligned} \mathcal{U} \text{ is by } \dot{\Sigma}^P &\Rightarrow \tau(\mathcal{U}) \text{ is by } \dot{\Sigma}^Q \\ &\Rightarrow \tau\mathcal{U} \text{ is by } \Omega_{\pi s, Q} \\ &\Rightarrow \mathcal{U} \text{ is by } (\Omega^\pi)_{s, P}, \end{aligned}$$

¹⁹This is Lemma 9.2.3 of [13].

as desired. □

The basic results of inner model theory, such as the Comparison Lemma and the Dodd-Jensen Lemma, are better stated and proved as results about mouse pairs than as results about mice, with the notions of elementary submodel and iterate adjusted so that this is possible. For example, if (H, Ψ) and (M, Σ) are mouse pairs, then $\pi: (H, \Psi) \rightarrow (M, \Sigma)$ is elementary (resp. nearly elementary) iff π is elementary (nearly elementary) as a map from H to M , and $\Psi = \Sigma^\pi$. We say that (M, Σ) is an *iterate of* (H, Ψ) iff there is a stack s on H such that s is by Ψ , and $\Sigma = \Psi_s$. It is a *non-dropping iterate* iff the branch H -to- M does not drop. Assuming AD^+ and that our pairs have scope HC , [13] proves the following:

- (1) If (M, Σ) is a mouse pair, H is a premouse, and $\pi: H \rightarrow M$ is nearly elementary, then (H, Σ^π) is a mouse pair.
- (2) If (H, Ψ) is a mouse pair, and (M, Σ) is a non-dropping iterate of (H, Ψ) , then the iteration map $i_s: (H, \Psi) \rightarrow (M, \Sigma)$ is elementary in the category of pairs.
- (3) (Dodd-Jensen) If (H, Ψ) is a mouse pair, (M, Σ) is an iterate of (H, Ψ) via the stack s , and $\pi: (H, \Psi) \rightarrow (M, \Sigma)$ is nearly elementary, then
 - (i) the branch H -to- M of s does not drop, and
 - (ii) for all $\eta < o(H)$, $i_s(\eta) \leq \pi(\eta)$, where i_s is the iteration map.
- (4) (Mouse order) Let $(H, \Psi) \leq^* (M, \Sigma)$ iff there is a nearly elementary embedding of (H, Ψ) into some iterate of (M, Σ) ; then \leq^* is a prewellorder of the mouse pairs with scope HC in each of the two types.

The prelinearity of the mouse pair order is the content of the Comparison Lemma for mouse pairs. For pure extender pairs, it is proved in Theorem 8.4.5 of [13]. The proof for lbr hod pairs is basically the same; it is Theorem 9.5.10 of [13].

3. COMPARING STRATEGIES FOR UNSOUND MICE

Let us assume AD^+ throughout this section.

The comparison theorems of [13] are stated and proved for mouse pairs of stable type 1. (This includes all mouse pairs of degree 0.) Mouse pairs that are not of stable type 1 (that is, those of type 2, or of type 1B and not projectum stable) can arise in our fine condensation results, so we need a comparison process that applies to them as well. In this section we generalize the process of [13] to such pairs.

The problem is that all levels of a background construction are type 1 pairs, so a type 2 pair (P, Σ) cannot literally iterate into a level of a background construction.²⁰ In the end, our solution

²⁰We shall see that a projectum stable pair of type 1B cannot do so either.

to this problem amounts to making small adjustments to the formulation and proof of $(*)(P, \Sigma)$ in [13]. Since there is nothing very new involved, we shall omit some details.²¹

A premouse M has type 2 iff M^- is almost sound, but not sound. The paradigm example is $M = (\text{Ult}_{k+1}(P, E), k+1)$, where P has type 1B and degree $k+1$, and the canonical embedding i_E^P is discontinuous at $\rho_{k+1}(P)$. The point of setting $\text{deg}(M) = k+1$ here, rather than k , is that we want to take $k+1$ -ultrapowers of M , and $\text{deg}(M) = k+1$ fits better with our definitions of elementarity and plus trees in that context.

However, almost soundness records a number of properties of M^- that are peculiar to this particular way of producing unsound mice. If we are going to iterate almost sound type 1 premice N at the $\text{deg}(N) + 1$ level, and compare their iteration strategies, then it is more natural to go all the way, and look at $\text{deg}(N) + 1$ -strategies for arbitrary solid N .²²

So we shall consider arbitrary solid type 1 premice N , and look at $\text{deg}(N) + 1$ -level iterations of them.²³ The solidity hypothesis is useful at several points, and we are building on the theory of [13] rather than trying to re-do it, so there is no value in dropping solidity.

3.1. +1-iteration trees

Definition 3.1. Let N be a solid premouse and $\text{deg}(N) = k$, and let E be an extender over N such that $\text{crt}(E) < \rho_{k+1}(N)$; then

- (a) $\text{Ult}_{k+1}(N, E)$ is the decoding of $\text{Ult}_1(N^k, E)$, where the latter ultrapower is formed using all boldface $\Sigma_1^{N^k}$ functions.
- (b) We set $\text{deg}(\text{Ult}_{k+1}(N, E)) = k$.
- (c) $i_E^{N^k} : N^k \rightarrow \text{Ult}_1(N^k, E)$ is the canonical embedding, and $i_E^N : N \rightarrow \text{Ult}_{k+1}(N, E)$ is its completion.

–

Familiar calculations show

Lemma 3.2. Let N be a solid premouse and $\text{deg}(N) = k$, and let $P = \text{Ult}_{k+1}(N, E)$, where E is an extender over N such that $\text{crt}(E) < \rho_{k+1}(N)$. Let $i = i_E^N$ and $\hat{i} = i \upharpoonright N^k = i_E^{N^k}$. Suppose that P is wellfounded; then

- (1) \hat{i} is Σ_2 -elementary as a map from N^k to P^k .
- (2) P is a type 1 premouse of degree k , and $P^k = \text{Ult}_1(N^k, E)$.

²¹[13, 4.6.12] states a comparison theorem for pure extender pfs mice that are not of stable type 1. There is no discussion of strategy comparison in this case.

²²For one thing, it can happen that (P, Σ) and (Q, Λ) are type 2 pairs, but the result of comparing them is some (R, Ω) that is not type 2, because $\hat{\rho}^-(P)$ and $\hat{\rho}^-(Q)$ are mapped to different points in R . In this case R is two order zero ultrapowers away from its type 1 core, not just one.

²³One could look at $\text{deg}(N) + 2$ -level iterations, etc. This is done in a rudimentary way in [1], and fully and systematically in Jensen's Σ^* theory. [4] proves the basic results about the $\text{deg}(N) + 1$ case.

(3) If $\text{Th}_1^{N^k}(\alpha \cup \{q\}) \in N^k$, then $i(\text{Th}_1^{N^k}(\alpha \cup \{q\})) = \text{Th}_1^{P^k}(i(\alpha) \cup \{i(q)\})$.

(4) $i(\rho_k(N)) = \rho_k(P)$ and $i(\eta_k^N) = \eta_k$.

(5) P is solid, $i(p_1(N^k)) = p_1(P^k)$, and $\rho_1(P^k) = \sup i^{\text{“}\rho_1(N^k)\text{”}}$.

Proof. Of course, Los’s theorem holds for $\text{Ult}_1(N^k, E)$ for all Σ_1 formulae, and the canonical embedding \hat{i} is Σ_2 elementary. So we have (1). Since “ $z = \text{Th}_1(\alpha \cup \{q\})$ ” is Π_2 , we also get (3).

(2) follows from the Upward Extension of Embeddings Lemma.²⁴

For (4): “ $z = \rho_k(N)$ ” is equivalent to $N^{k-1} \models \theta[z]$, where θ is a Boolean combination of Π_3 formulae.²⁵ Since i is Σ_3 elementary as a map from N^{k-1} to P^{k-1} , $i(\rho_k(N)) = \rho_k(P)$. By Lemma 4.3.4 of [13] we get that $i(\eta_k^N) = \eta_k^P$ as well.

For (5): We claim first that $\sup i^{\text{“}\rho_1(N^k)\text{”}} \leq \rho_1^{P^k}$. For if $\text{crt}(E) \leq \alpha < \rho_1(N^k)$ and $r = [a, f]_E^{N^k}$ where f is $\Sigma_1^{N^k}$ in q , then $\text{Th}_1^{P^k}(i(\alpha) \cup \{r\})$ can be easily computed from $\text{Th}_1^{P^k}(i(\alpha) \cup \{i(q)\})$, and the latter is in P^k by (3). We claim next that $\rho_1(P^k) \leq \sup i^{\text{“}\rho_1(N^k)\text{”}}$. For let $q = p_1(N^k)$ and $\rho = \rho_1(N^k)$; then the fact that N is solid implies there is a $\Sigma_1^{N^k}(q)$ map of ρ onto $\rho^{+,N}$, and since $\varepsilon(E) < \sup i^{\text{“}\rho\text{”}}$, this yields a $\Sigma_1^{P^k}(i(q))$ map of $\sup i^{\text{“}\rho\text{”}}$ onto $(\sup i^{\text{“}\rho\text{”}})^{+,P}$, so that $\text{Th}_1^{P^k}(\sup i^{\text{“}\rho\text{”}} \cup \{i(q)\}) \notin P^k$.

The calculations just done show that $p_1(P^k) \leq_{\text{lex}} i(p_1(N^k))$, and $i(p_1(N^k)) \leq_{\text{lex}} p_1(P^k)$ follows from the preservation of solidity witnesses given by (3).

Let us show that P is solid. We have already shown that P is parameter solid. Let $\rho = \rho_1(N^k)$; then since N is projectum solid, ρ is not measurable in N , so $i(\rho)$ is not measurable in P , so if i is continuous at ρ , then P is projectum solid. So assume i is discontinuous at ρ ; then we have a boldface $\Sigma_1^{N^k}$ function $f: \kappa_E \rightarrow \rho$ such that f is order preserving and continuous at limits. This implies ρ is a limit cardinal in N , and $f \upharpoonright \alpha \in N$ for all $\alpha < \kappa_E$, so that $N \models \text{“}f(\alpha) \text{ is singular”}$ for all α . But then $\sup i^{\text{“}\rho = [\{\kappa_E\}, f]_E^{N^k}\text{”}}$ is singular in P , and hence not measurable in P .

Finally, we must show that P is stable. But $\eta_k^P = i(\eta_k^N)$, so if η_k^P is measurable in P , then η_k^N is measurable in N , so $\eta_k^N < \rho_{k+1}(N)$ by stability of N , so $\eta_k^P < \rho_{k+1}(P)$ by (4), as desired. \square

Let us turn the lemma into a definition.

Definition 3.3. Let $i: N \rightarrow P$, where $\text{deg}(N) = \text{deg}(P) = k$; then we say i is *+1-elementary* iff N and P are solid premice of type 1, and conclusions (1), (3), (4), and (5) of Lemma 3.2 are satisfied by i , N , and P . \dashv

Now we look at iteration trees \mathcal{T} in which such “soundness degree +1” ultrapowers are taken. We record the degrees of the ultrapowers that can be taken in the degrees $\text{deg}^{\mathcal{T}}(\alpha)$ of the nodes, so now $\text{deg}^{\mathcal{T}}(\alpha) = \text{deg}(\mathcal{M}_\alpha^{\mathcal{T}}) + 1$ is possible.

Definition 3.4. Let M be a solid premouse of degree k ; then a *1-bounded plus tree on M* is a system $\langle \mathcal{T}, \langle E_\alpha^{\mathcal{T}} \mid \alpha + 1 < \text{lh}(\mathcal{T}) \rangle, \text{deg}^{\mathcal{T}} \rangle$ such that $\text{dom}(\text{deg}^{\mathcal{T}}) = \text{lh}(\mathcal{T})$, and there are $M_\alpha, i_{\alpha,\beta}, D$ satisfying

²⁴See [13, 4.3.7(b)].

²⁵See [13, 2.4.5–2.4.7].

- (1) If $\deg^{\mathcal{T}}(0) = \deg(M)$, then \mathcal{T} is a plus tree on M in the sense of [13, 4.4.3], with models, embeddings, and dropset $M_\alpha, i_{\alpha,\beta}, D$. In this case, we also put $0 \in D$, and say that \mathcal{T} has a *small drop* at 0.
- (2) If $\deg^{\mathcal{T}}(0) = \deg(M) + 1 = k + 1$, then $M_\alpha, i_{\alpha,\beta}, D$ satisfy all the properties listed in [13, 4.4.3], except that clause (3)(b) is modified so that if $T\text{-pred}(\alpha + 1) = \beta$, $[0, \beta]_{\mathcal{T}} \cap D = \emptyset$, and $\text{crt}(E_\alpha^{\mathcal{T}}) < \rho_k(M_\beta)$, then
 - (a) if $\text{crt}(E_\alpha^{\mathcal{T}}) < \rho_{k+1}(M_\beta)$, then $M_{\alpha+1} = \text{Ult}_{k+1}(M_\beta, E_\alpha^{\mathcal{T}})$, and
 - (b) if $\rho_{k+1}(M_\beta) \leq \text{crt}(E_\alpha^{\mathcal{T}})$, then $M_{\alpha+1} = \text{Ult}_k(M_\beta, E_\alpha^{\mathcal{T}})$.
 - (c) In case (b), we put $\alpha + 1 \in D^{\mathcal{T}}$ and say that \mathcal{T} has a *small drop* at $\alpha + 1$.
- (3) β is a *+1-node* of \mathcal{T} iff $[0, \beta]_{\mathcal{T}} \cap D^{\mathcal{T}} = \emptyset$. If β is a +1-node, then we set $\deg^{\mathcal{T}}(\beta) = \deg(M_\beta) + 1$. Otherwise, we set $\deg^{\mathcal{T}}(\beta) = \deg(M_\beta)$.
- (4) \mathcal{T} is *1-maximal* iff 0 is a +1-node of \mathcal{T} . If \mathcal{T} is not 1-maximal, then we say it is *0-bounded*.

+

If \mathcal{T} is 1-bounded, there is at most one small drop along any branch, and if there is a small drop, it must be the first drop along that branch. We call a drop $\alpha + 1 \in D^{\mathcal{T}}$ *large* iff it is not small.

Remark 3.5. It is easy to see that $\mathcal{M}_\alpha^{\mathcal{T}}$ is unsound (i.e. not $\deg(\mathcal{M}_\alpha^{\mathcal{T}})$ sound) iff $[0, \alpha]_{\mathcal{T}}$ has a large drop. +

In a 1-bounded tree, all the drops are forced, including the small ones, except possibly a small drop at 0. If there is such a drop, then \mathcal{T} is 0-bounded, and therefore it is just a plus tree in the sense of [13]. In general, a 1-maximal plus tree may not be a plus tree in the sense of [13], but it seems better to expand the meaning of “plus tree” than to invent a new term. In a 1-bounded tree, after a branch has dropped in any way, the later ultrapowers on the branch are always n -ultrapowers of n -sound premice.²⁶ If M is not just solid, but sound, then a 1-maximal plus tree on M is essentially the same thing as a plus tree on M^+ in the sense of [13].

From Lemma 3.2 we get

Lemma 3.6. *Let \mathcal{T} be a 1-bounded plus tree on the solid premouse M , $\alpha <_{\mathcal{T}} \beta$, and $D^{\mathcal{T}} \cap [\alpha, \beta]_{\mathcal{T}} = \emptyset$; then $i_{\alpha,\beta}^{\mathcal{T}}$ is elementary, and if \mathcal{T} is 1-maximal, then it is exact. If in addition β is a +1-node, then $i_{\alpha,\beta}^{\mathcal{T}}$ is +1-elementary.*

Exactness uses that \mathcal{T} is 1-maximal and M is stable, for otherwise we might take a $\deg(M)$ -ultrapower with critical point η_k^M , producing thereby an inexact canonical embedding.

Along branches that have only a small drop, the embeddings are slightly less elementary.

²⁶We could relax clause (2) by allowing $M_{\alpha+1} = \text{Ult}_k(M_\beta, E_\alpha^{\mathcal{T}})$ when $\text{crt}(E_\alpha^{\mathcal{T}}) < \rho_{k+1}(M_\beta)$. Only a few things become more complicated.

Definition 3.7. Let M and N be solid premice, $k = \deg(M) = \deg(N)$, and $\pi: M \rightarrow N$. We say that π is *nearly +1-elementary* iff

- (i) π is elementary and exact,
- (ii) $\pi^{\ulcorner} \rho_1^{M^k} \subseteq \rho_1^{N^k}$, $\pi(p_1(M^k)) = p_1(N^k)$, and
- (iii) π is $k+1$ -theory preserving, in that for all $\alpha < \rho_1^{M^k}$ and all $q \in M^k$, $\pi(\text{Th}_1^{M^k}(\alpha \cup \{q\})) = \text{Th}_1^{N^k}(\pi(\alpha) \cup \{\pi(q)\})$.

–

Nearly +1-elementary $\pi: M \rightarrow N$ are called $\deg(M)+1$ -apt in [4]. That paper proves a copying lemma and a Dodd-Jensen lemma for such maps.²⁷ We shall use those results below.

Notice that if π is nearly elementary as a map from M to N , then it is nearly (+1)-elementary as a map from M^- to N^- . M and N may have either type, but M^- and N^- have type 1.

Clause (i) of 3.7 implies that $\pi(\eta_k^M) = \eta_k^N$.²⁸

Lemma 3.8. *Let N be a solid premouse and $\deg(N) = k$, and let $P = \text{Ult}_k(N, E)$, where E is an extender over N that is close to N such that $\rho_{k+1}(N) \leq \text{crt}(E) < \rho_k(N)$. Suppose P is wellfounded; then P is solid, and the canonical embedding i_E^N is nearly +1-elementary. Moreover, $\rho_{k+1}(N) = \rho_{k+1}(P)$.*

Proof. This is proved in [13, Lemma 2.4.12], except for the assertions that P is projectum solid and stable, and that i_E^N is exact.

Note that $\text{crt}(E) \notin \{\rho_{k+1}(N), \eta_k^N\}$ because $\text{crt}(E)$ is measurable in N by closeness, while N is projectum solid and stable. Since $\rho_{k+1}(N) < \text{crt}(E)$, P is projectum solid, and since $\eta_k^N \neq \text{crt}(E)$, i_E^N is continuous at $\rho_k(N)$, and therefore exact. Since i_E^N is continuous at $\rho_k(N)$, $i_E^N(\eta_k^N) = \eta_k^P$. But N is stable, so P is stable. \square

Remark 3.9. With more work, one can weaken the hypotheses on E by dropping closeness, and requiring of $\text{crt}(E)$ only that $\text{crt}(E) < \rho_k(N)$ and $\text{crt}(E) \notin \{\rho_{k+1}(N), \eta_k^N\}$. In the case that $\text{crt}(E) < \rho_{k+1}(N)$, one must weaken the conclusion to $\rho_{k+1}(P) = \sup i^{\ulcorner} \rho_{k+1}(N)$, for $i = i_E^N$, but otherwise the conclusions remain the same.²⁹ –

Note also

Lemma 3.10. *Let M be solid, and let $\pi: \mathfrak{C}(M)^- \rightarrow M$ be the anticore map; then π is nearly +1-elementary.*

Proof. Let $k = \deg(M)$; then π is cofinal and Σ_0 elementary, and hence Σ_1 elementary, as a map from $\mathfrak{C}(M)^k$ to M^k . Because we are using pfs fine structure $\rho_k(M) \in \text{ran}(\pi)$, so π is exact. The rest is easy to verify. \square

²⁷See [4, 2.11, 2.12].

²⁸See [13, 4.3.4].

²⁹That $\rho_{k+1}(P) = \sup i^{\ulcorner} \rho_{k+1}(N)$ under these hypotheses is a result of Schlutzenberg; see [13, 9.6.1].

From Lemma 3.8 we get

Lemma 3.11. *Let \mathcal{T} be a 1-maximal plus tree on the solid premouse M , $\alpha <_T \beta$, and suppose that if $\gamma \in D^{\mathcal{T}} \cap [0, \beta]_T$, then γ is a small drop; then $i_{\alpha, \beta}^{\mathcal{T}}$ is nearly (+1)-elementary.*

Nearly +1-elementary embeddings suffice for copying 1-maximal trees.

Definition 3.12. Let M and N be solid premice with degree k , and E and F be extenders over M and N respectively. We say

$$\langle \pi, \varphi \rangle: (M, E) \xrightarrow{*,1} (N, F)$$

iff π is nearly +1-elementary, and $\langle \pi, \varphi \rangle: (M^k, E) \xrightarrow{*} (N^k, F)$ in the sense of [13, 2.5.17]. \dashv

Lemma 3.13. *[Shift Lemma] Suppose that $\langle \pi, \varphi \rangle: (M, E) \xrightarrow{*,1} (N, F)$, where M and N are solid premice of degree k . Let $m = k + 1$ if $\text{crt}(E) < \rho_{k+1}(M)$, and $m = k$ otherwise, and let $n = k + 1$ if $\text{crt}(F) < \rho_{k+1}(N)$ and $n = k$ otherwise. Let*

$$\sigma: \text{Ult}_m(M, E) \rightarrow \text{Ult}_n(N, F)$$

be the completion of the map $\sigma_0([a, f]_E^{M^k}) = [\varphi(a), \pi(f)]_F^{N^k}$. Then

(i) σ is nearly +1-elementary,

(ii) $\sigma \upharpoonright \text{lh}(E) = \varphi \upharpoonright \text{lh}(E)$, and

(iii) $\sigma \circ i_E^M = i_F^N \circ \pi$.

Proof. (Sketch.) Since $\sup \pi \rho_{k+1}(M) \leq \rho_{k+1}(N)$, $m = k + 1$ implies $n = k + 1$.³⁰ Thus the definition of σ_0 makes sense, in that π can be applied to f (even if f is only $\Sigma_1^{M^k}$), and $\pi(f)$ is a function that is used in the N -ultrapower. Note that both i_E^M and i_F^N are exact, so σ is exact by commutativity. The remaining calculations are the usual ones. \square

Definition 3.14. Let $\pi: M \rightarrow N$ be nearly +1-elementary, and let \mathcal{T} be a 1-bounded plus tree on M ; then $\pi\mathcal{T}$ is the 1-bounded copied tree on N . \dashv

Of course we stop the construction of $\pi\mathcal{T}$ if we reach an illfounded model. Letting $\pi_\alpha: M_\alpha \rightarrow N_\alpha$ be the copy map, we prove by induction that if $\beta = T\text{-pred}(\alpha + 1)$, then

$$\langle \pi_\beta \upharpoonright M_{\alpha+1}^*, \pi_\alpha \rangle: (M_{\alpha+1}^*, E_\alpha^{\mathcal{T}}) \xrightarrow{*} (N_{\alpha+1}^*, E_\alpha^{\pi\mathcal{T}}),$$

and if $D^{\mathcal{T}} \cap [0, \beta]_T = \emptyset$, then π_β is nearly +1-elementary. This can be done.³¹

Definition 3.15. A 1-bounded stack on M is a stack s of trees such that

³⁰It is possible that $m = k$ and $n = k + 1$. We could have copied with $n = k$ in this case, but we are not going to do that, because we want to stay in the realm of 1-maximal trees.

³¹See [13, Section 4.5].

- (1) each tree $\mathcal{T}_i(s)$ in s is 1-bounded, and
- (2) for $i > 0$, letting N be the base model of $\mathcal{T}_i(s)$, $\mathcal{T}_i(s)$ is 1-maximal iff M -to- N has no drops of any kind. s is 1-maximal iff $\mathcal{T}_0(s)$ is 1-maximal; otherwise, s is 0-bounded.

+

Let $G^1(M, \lambda, \theta)$ be the variant of $G(M, \lambda, \theta)$ in which the output is a 1-bounded stack on M .

Definition 3.16. A $(\lambda, \theta)^+$ -iteration strategy for M is a winning strategy for player II in $G^1(M, \lambda, \theta)$. Let Σ be such a strategy; then

- (a) (Tail strategy.) For s a 1-bounded stack by Σ with last model N , $\Sigma_{s,N}(t) = \Sigma(s \frown t)$. If M -to- N does not drop in s , then $\Sigma_{s,N}$ is a $(\lambda - \text{lh}(s), \theta)^+$ -iteration strategy for N . Otherwise, it is a $(\lambda - \text{lh}(s), \theta)$ -strategy.
- (b) (Pullback strategy.) If $\pi: N \rightarrow M$ is nearly +1-elementary, then Σ^π is the $(\lambda, \theta)^+$ -strategy for N given by: $\Sigma^\pi(s) = \Sigma(\pi s)$.

+

If Σ is a $(\lambda, \theta)^+$ -strategy for M , then we obtain an ordinary (λ, θ) -strategy for M by restriction Σ to act on 0-bounded stacks s . We call this restricted strategy Σ^- .

Notation: If \mathcal{T} is a tree on (P, Σ) and $\alpha < \text{lh}(\mathcal{T})$, then we may write $\Sigma_\alpha^\mathcal{T}$ for the tail strategy $\Sigma_{\mathcal{T} \upharpoonright \alpha+1, \mathcal{M}_\alpha^\mathcal{T}}$.

3.2. Background-induced +1-strategies

We get $(\lambda, \theta)^+$ -iteration strategies from the same background constructions that gave us (λ, θ) -strategies. The constructions themselves do not change at all, because the strategy predicate of an lpm still has only information about the action of the strategy on 0-bounded, λ -separated trees.

Recall that a *conversion stage* is a tuple $\langle M, \psi, Q, \mathbb{C}, R \rangle$ such that R is an appropriate background universe, \mathbb{C} is a maximal hod pair construction in the sense of R , Q is a level of \mathbb{C} , and $\psi: M \rightarrow Q$ is nearly elementary.³²

Definition 3.17. A *+1-conversion stage* is a conversion stage $\langle M, \psi, Q, \mathbb{D}, R \rangle$ such that M and Q are solid, and ψ is nearly +1-elementary. +

Given a conversion stage $c = \langle M, \psi, Q, \mathbb{C}, R \rangle$ and a 0-bounded plus tree \mathcal{T} on M , [13, Section 4.8] defines the *conversion system*

$$\text{lift}(\mathcal{T}, c) = \langle \mathcal{T}^*, \langle c_\alpha \mid \alpha < \text{lh}(\mathcal{T}) \rangle \rangle.$$

³²See Section 4.8 and Section 9.4 of [13].

Here $c_0 = c$, and $c_\alpha = \langle M_\alpha, \psi_\alpha, Q_\alpha, \mathbb{C}_\alpha, R_\alpha \rangle$ is a conversion stage such that $M_\alpha = \mathcal{M}_\alpha^\mathcal{T}$ and $R_\alpha = \mathcal{M}_\alpha^{\mathcal{T}^*}$. The same construction applies when \mathcal{T} is 1-bounded and c is a +1-conversion stage; we just need to note that when α is a +1-node of \mathcal{T} , then the conversion stage c_α produced by our construction is a +1-conversion stage.

This comes down to adding a few lines to the Shift Lemma for Conversions.³³ Suppose $\alpha + 1$ is a +1-node of \mathcal{T} , $T\text{-pred}(\alpha + 1) = \beta$, and $E = E_\alpha^\mathcal{T}$. Let

$$\begin{aligned}\varphi &= \sigma_{Q_\alpha}^{\mathbb{C}_\alpha} [Q_\alpha | \text{lh}(\psi_\alpha(E^-))] \circ \psi_\alpha, \\ E^* &= B^{\mathbb{C}_\alpha}(\varphi(E^-)),\end{aligned}$$

and

$$E_\alpha^{\mathcal{T}^*} = E^*$$

So E^* is the background extender for the resurrection of $\psi_\alpha(E^-)$. There is enough agreement between φ and ψ_β that E^* should be applied to $R_\beta = \mathcal{M}_\beta^{\mathcal{T}^*}$ in a normal continuation of \mathcal{T}^* . Letting $k = \text{deg}(M)$ and $i^* = i_{E^*}^{R_\beta}$, our next conversion stage is then

$$c_{\alpha+1} = \langle \text{Ult}_{k+1}(M_\beta, E), \pi, i^*(Q_\beta), i^*(\mathbb{C}_\beta), \text{Ult}(R_\beta, E^*) \rangle.$$

Since $\alpha + 1$ is a +1 node, $M_{\alpha+1}^* = M_\beta$. π is (the completion of) the map $\pi: \text{Ult}_1(M_\beta^k, E) \rightarrow i^*(Q_\beta^k)$ given by

$$\pi([a, f]_E^{M_\beta^k}) = [\varphi(a), \psi_\beta(f)]_{E^*}^{R_\beta},$$

where ψ_β moves f by moving its $\Sigma_1^{M_\beta^k}$ definition. Since ψ_β is Th_1 -preserving, π is well defined. It is easy to check that π is Σ_1 -elementary and Th_1 -preserving as a map from $\text{Ult}_1(M_\beta^k, E)$ to $i^*(Q_\beta^k)$, using that

$$\pi \circ i_E^{M_\beta} = i^* \circ \psi_\beta.$$

Commutativity and the fact that $i_E^{M_\beta}$, $i^* \upharpoonright Q_\beta$, and ψ_β are all exact also implies that π is exact. Finally,

$$\begin{aligned}\sup \pi \text{ ``} \rho_{k+1}(M_{\alpha+1}) &= \sup \pi \circ i_E^{M_\beta} \text{ ``} \rho_{k+1}(M_\beta) \\ &= \sup i^* \circ \psi_\beta \text{ ``} \rho_{k+1}(M_\beta) \\ &\leq i^*(\rho_{k+1}(Q_\beta)) = \rho_{k+1}(i^*(Q_\beta)).\end{aligned}$$

Collectively, these calculations show that π is nearly +1-elementary. It agrees with φ on $\text{lh}(E)$, which we need to keep the conversion going.

We lift 1-bounded stacks in the natural way. For example, $\text{lift}(\langle \mathcal{T}, \mathcal{U} \rangle, c) = \langle \text{lift}(\mathcal{T}, c), \text{lift}(\mathcal{U}, d) \rangle$, where d is the last conversion stage in $\text{lift}(\mathcal{T}, c)$. Letting $\mathcal{T}^* = \text{lift}(\mathcal{T}, c)_0$ and $\mathcal{U}^* = \text{lift}(\mathcal{U}, d)_0$,

³³[13, 3.3.2, 4.8.2].

$\langle \mathcal{T}^*, \mathcal{U}^* \rangle$ is the corresponding stack of nice trees on the background universe in c . Abusing notation, we write

$$\text{lift}(\vec{\mathcal{T}}, c)_0 = \langle \mathcal{T}_i^* \mid i < \text{lh}(\vec{\mathcal{T}}) \rangle$$

for the stack of lifted trees on the background universe in c .

Definition 3.18. Let $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ be a coarse strategy pair,³⁴ and let \mathbb{C} be the maximal least branch (w, \mathcal{F}, Ψ) -construction of $(N^*, \in, w, \mathcal{F}, \Psi)$ ³⁵, with models $M_{\nu, k} = M_{\nu, k}^{\mathbb{C}}$ and induced strategies $\Omega_{\nu, k} = \Omega_{\nu, k}^{\mathbb{C}}$.

- (1) Let c be a +1-conversion stage of the form $\langle M, \psi, Q, \mathbb{C}, N^* \rangle$; then $\Omega^+(c, \Psi^*)$ is the $(\omega_1, \omega_1)^+$ -iteration strategy for M given by

$$\vec{\mathcal{T}} \text{ is by } \Omega^+(c, \Psi^*) \text{ iff } \text{lift}(\vec{\mathcal{T}}, c)_0 \text{ is by } \Psi^*.$$

- (2) $(\Omega_{\nu, k}^+)^{\mathbb{C}} = \Omega^+(\langle M_{\nu, k}, \text{id}, M_{\nu, k}, \mathbb{C}, N^* \rangle, \Psi^*)$.

–

Clearly, $\Omega_{\nu, k} = (\Omega_{\nu, k}^+)^-$. As one would expect, $\Omega_{\nu, k+1}$ is the pullback of $\Omega_{\nu, k}^+$ under the anticore map.

Lemma 3.19. *Let \mathbb{C} be the maximal least branch (w, \mathcal{F}, Ψ) -construction of some coarse strategy pair, with associated models $M_{\nu, k}$ and strategies $\Omega_{\nu, k}$ and $\Omega_{\nu, k}^+$. Let $\pi: M_{\nu, k+1}^- \rightarrow M_{\nu, k}$ be the anticore map; then π is nearly +1-elementary, and*

$$\Omega_{\nu, k+1} = (\Omega_{\nu, k}^+)^{\pi}.$$

Proof. π is nearly +1-elementary by 3.10.

Note first that the identity makes sense. $\Omega_{\nu, k+1}$ is a strategy acting on 0-bounded stacks on $M_{\nu, k+1}$, or equivalently, on 1-maximal stacks on $M_{\nu, k+1}^-$. If s is a 1-maximal stack on $M_{\nu, k+1}^-$, then πs is a 1-maximal stack on $M_{\nu, k}$, and so $\Omega_{\nu, k}^+$ acts on it. Thus the identity makes sense.

The identity is true because, letting

$$c = \langle M_{\nu, k+1}, \text{id}, M_{\nu, k+1}, \mathbb{C}, N^* \rangle$$

³⁴See [13, 9.4.14]. Roughly, the requirements are that

- (a) $(N^*, \in, w, \mathcal{F}, \Psi)$ is a *coarse strategy premouse*, i.e. N^* is countable and transitive, $N^* \models \text{ZFC} + \text{“}(w, \mathcal{F}) \text{ is a coherent pair”} + \text{“}\Psi \text{ is a } (\delta^*, \delta^*, \mathcal{F})\text{-iteration strategy for } V \text{ that quasi-normalizes well, has strong hull condensation, and is pushforward consistent”}$, and
- (b) Ψ^* is a complete (ω_1, ω_1) -iteration strategy for $(N^*, \in, w, \mathcal{F}, \Psi)$ that normalizes well and has strong hull condensation, and
- (c) if $i: N^* \rightarrow S$ is the iteration map associated to a stack t according to Ψ^* , then $i(\Psi) \subseteq \Psi_{t, S}^*$.

(w, \mathcal{F}) is a coherent pair iff w is a wellorder of V_δ , where $\delta = \delta(w)$, and $\mathcal{F} \subseteq V_\delta$ is a set of nice extenders such that for $E \in \mathcal{F}$ and $\nu = \text{lh}(E)$, $i_E(w) \cap V_{\nu+1} = w \cap V_{\nu+1}$ and $i_E(\mathcal{F}) \cap V_{\nu+1} = \mathcal{F} \cap V_{\nu+1}$. See [13, 2.9.6].

³⁵Cf. [13, 9.4.10].

and

$$d = \langle M_{\nu,k}, \text{id}, M_{\nu,k}, \mathbb{C}, N^* \rangle,$$

and letting s be a 0-bounded stack on $M_{\nu,k+1}$,

$$\text{lift}(s, c)_0 = \text{lift}(\pi s, d)_0.$$

Thus letting Ψ^* be the strategy for N^* ,

$$\begin{aligned} s \text{ is by } \Omega_{\nu,k} &\text{ iff } \text{lift}(s, c)_0 \text{ is by } \Psi^* \\ &\text{ iff } \text{lift}(\pi s, d)_0 \text{ is by } \Psi^* \\ &\text{ iff } \pi s \text{ is by } \Omega_{\nu,k}^+. \end{aligned}$$

The proof that $\text{lift}(s, c)_0 = \text{lift}(\pi s, d)_0$ is a routine induction, essentially identical to the proof of [13, Lemma 5.4.2]. We omit further detail. \square

3.3. Regularity properties and comparison

The definitions and results of [13] go over to +1-strategies with almost no change.

For example, if \mathcal{T} and \mathcal{U} are 1-maximal trees on M , then Φ is a *tree embedding* from \mathcal{T} to \mathcal{U} iff letting $\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \text{lh}(\mathcal{T}) \rangle, \langle t_\alpha \mid \alpha + 1 < \text{lh}(\mathcal{T}) \rangle \rangle$, Φ has all the properties enumerated in [13, Definition 6.4.1], and in addition

$$\alpha \text{ is a +1-node of } \mathcal{T} \Rightarrow s_\alpha \text{ is +1-elementary.}$$

A +1-strategy Σ has *strong hull condensation* iff whenever $\Phi: \mathcal{T} \rightarrow \mathcal{U}$ is a tree embedding and \mathcal{U} is by Σ , then \mathcal{T} is by Σ ; moreover, whenever $\alpha < \text{lh}(\mathcal{T})$, $v(\alpha) \leq_U \beta$, $\pi = \hat{i}_{v(\alpha), \beta}^{\mathcal{U}} \circ s_\alpha$, and $Q \trianglelefteq \text{dom}(\pi)$, then we have $\Sigma_{\mathcal{T}, Q} = (\Sigma_{\mathcal{U} \upharpoonright_{\beta+1}, \pi(Q)})^\pi$.

If s is a 1-maximal stack, then the quasi-normalization $V(s)$ and embedding normalization $W(s)$ are defined just as before, but so that the trees they produce are 1-maximal. If the trees in s are λ -separated, then $V(s) = W(s)$. This is the only case we care about. A 1-strategy Σ *quasi-normalizes well* iff whenever s is a 1-maximal stack by Σ , then $V(s)$ is by Σ ; moreover if Q and R are the last models of s and $V(s)$, and $\sigma: Q \rightarrow R$ is the quasi-normalization map, then $\Sigma_{s, Q} = (\Sigma_{V(s), R})^\sigma$.³⁶

The definition of internal lift consistency does not change at all.

If P is an lpm and Σ is a $(\lambda, \theta)^+$ -strategy for P , then we say (P, Σ) is *pushforward consistent* iff whenever s is a stack by Σ with last model Q , $\dot{\Sigma}^Q \subseteq \Sigma_{s, Q}$.

³⁶If M -to- Q does not drop in s , then M -to- R does not drop in $V(s)$ and σ is nearly +1-elementary, so $(\Sigma_{V(s), R})^\sigma$ is indeed a strategy for 1-maximal stacks on Q .

Definition 3.20. (AD^+) A $+1$ -hod pair with scope H_{ω_1} is a pair (P, Σ) such that P is a countable lpm, Σ is an $(\omega, \omega_1)^+$ -iteration strategy for P , and (P, Σ) has strong hull condensation, normalizes well, and is internally lift consistent and pushforward consistent. \dashv

Remarks 3.21. (a) If we are not assuming AD^+ we may want to consider uncountable P , and Σ that are defined on uncountable trees.

(b) If (P, Σ) is a $+1$ -hod pair, then (P, Σ^-) is an lbr hod pair in the sense of [13].

(c) If (P, Σ) is a $+1$ -hod pair and P is sound, then (P^+, Σ) is an lbr hod pair in the sense of [13]. \dashv

Since nearly $+1$ -elementary maps suffice to copy 1-maximal trees, we get a Dodd-Jensen lemma:

Lemma 3.22. [Dodd-Jensen] Let (M, Σ) be a $+1$ -mouse pair, s a 1-maximal stack on (M, Σ) with last pair (N, Λ) , and $\pi: (M, \Sigma) \rightarrow (Q, \Omega) \trianglelefteq (N, \Lambda)$ be nearly $+1$ -elementary; then $(Q, \Omega) = (N, \Lambda)$, the branch M -to- N of s does not have a large drop, and $i_s(\eta) \leq \pi(\eta)$ for all $\eta \in o^M$.

Lemma 3.23. Let $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ be a coarse strategy pair, and let \mathbb{C} be the maximal least branch (w, \mathcal{F}, Ψ) -construction of $(N^*, \in, w, \mathcal{F}, \Psi)$ with models $M_{\nu, l}$ and induced $+1$ -strategies $\Omega_{\nu, l}^+$; then for all ν, l , $\Omega_{\nu, l}^+$ has strong hull condensation, quasi-normalizes well, and is internally lift consistent and pushforward consistent.

Proof. (Sketch.) The proof in [13] that $\Omega_{\nu, l}$ has these properties works also for $\Omega_{\nu, l}^+$. \square

We turn now the comparison theorem for $+1$ -hod pairs.

Definition 3.24. Let (P, Σ) be a $+1$ -hod pair; then

(1) (P, Σ) iterates strictly past (Q, Λ) iff there is a 1-maximal, λ -separated tree \mathcal{T} on (P, Σ) with last pair (R, Ψ) such that either $(Q, \Lambda) \triangleleft (R, \Psi)$, or P -to- R has a large drop and $(Q, \Lambda) = (R, \Psi)$.

(2) (P, Σ) iterates to (Q, Λ) iff there is a 1-maximal, λ -separated tree \mathcal{T} on (P, Σ) with last pair (Q, Λ) and such that P -to- Q does not have a large drop. \dashv

In case (1), (Q, Λ) must be an ordinary lbr hod pair, and in case (2) it must be a 1-hod pair. In both cases, the 1-maximal tree \mathcal{T} is uniquely determined by (P, Σ) and (Q, Λ) .

Definition 3.25. (P, Σ) be a $+1$ -hod pair. We say that a coarse strategy pair $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ captures (P, Σ) iff there is an inductive-like pointclass Γ with the scale property such that $\text{Code}(\Sigma) \in \Delta_\Gamma$, and for $\delta^* = \delta(w)$,

(i) $N^* \models \text{“}\delta^* \text{ is Woodin”}$, and

- (ii) $P \in \text{HC}^{N^*}$, and there is a $\text{Coll}(\omega, \delta^*)$ -term τ and a universal Γ -set U such that if $i : N^* \rightarrow S$ is via Σ^* and $g \subseteq \text{Col}(\omega, i(\delta^*))$ is S -generic, then $i(\tau)_g = U \cap S[g]$.

–

Theorem 3.26. (AD^+) *Let (P_0, Σ_0) be a +1-hod pair, let $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ be a coarse strategy pair that captures (P_0, Σ_0) , and let \mathbb{C} be the maximal least branch (w, \mathcal{F}, Ψ) -construction of $(N^*, \in, w, \mathcal{F}, \Psi)$; then there is a $\langle \theta, n \rangle <_{\text{lex}} \langle \delta^*, 0 \rangle$ such that*

- (1) *for all $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$, (P_0, Σ_0) iterates strictly past $(M_{\nu, k}, \Omega_{\nu, k})$, and*
- (2) *(P_0, Σ_0) iterates to $(M_{\theta, n}, \Omega_{\theta, n}^+)$.*

Proof. (Sketch.) The proof is very close to that of [13, Theorem 9.5.2], so we shall just give an outline.

Let $\langle \theta, n \rangle$ be lex least such that (P_0, Σ_0) does not iterate strictly past $(M_{\theta, n}, \Omega_{\theta, n})$. For $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$, let

$$\mathcal{W}_{\nu, k}^* = \lambda\text{-separated tree whereby } (P_0, \Sigma_0) \text{ iterates strictly past } (M_{\nu, k}, \Omega_{\nu, k}).$$

Let

$$(M, \Omega, \Omega^+) = (M_{\theta, n}, \Omega_{\theta, n}, \Omega_{\theta, n}^+),$$

and let \mathcal{T} be the tree on (P_0, Σ_0) formed by iterating away least extender disagreements with M , as follows. We assume by induction

Induction hypothesis $(\dagger)_\alpha$.

If the current last pair of \mathcal{T} is $(Q, \Lambda) = (\mathcal{M}_\alpha^T, \Sigma_\alpha^T)$, then

- (1) (M is passive in extender disagreements) if E is on the M -sequence and $M \parallel \text{lh}(E) = Q \parallel \text{lh}(E)$, then $M \parallel \text{lh}(E) = Q \parallel \text{lh}(E)$.
- (2) (No strategy disagreements) if $S \trianglelefteq Q$ and $S \trianglelefteq M$, then
 - (a) if $S \triangleleft M$, then $S \triangleleft Q$ and $\Lambda_S = \Omega_S$, and
 - (b) if $S = M$, then $S = Q$, α is a +1-node of \mathcal{T} , and $\Lambda = \Omega^+$.

Note that (2)(a) implies that Q is not a proper initial segment of M .

If $(\dagger)_\alpha$ is false, then we stop the construction of \mathcal{T} , and say that \mathcal{T} *fails at* α . Let us assume that \mathcal{T} never fails, and finish the proof.

Claim 3.27. *If $(\dagger)_\alpha$ holds, then either*

- (i) α is a +1-node of \mathcal{T} , $Q = M$, and $\Lambda = \Omega^+$, or

(ii) there is an extender E on the Q -sequence such that $(Q, \Lambda) \parallel \text{lh}(E) = (M, \Omega) \parallel \text{lh}(E)$ but $M \parallel \text{lh}(E)$ is passive.

Proof. Suppose that (ii) fails. Then by $(\dagger)_\alpha(1)$, $Q \triangleleft M$ or $M \trianglelefteq Q$. By (2)(a) then (with $S = Q$), $M \trianglelefteq Q$. But then (2)(b) with $S = M$ implies that (i) holds. \square

If 3.27(i) holds, then (P_0, Σ_0) has iterated to (M, Ω^+) , as desired. In this case we again stop the construction, and say that \mathcal{T} succeeds at α . Assume now that \mathcal{T} neither fails nor succeeds at α , and let E be the unique extender on the Q -sequence such that $Q \parallel \text{lh}(E) = M \parallel \text{lh}(E)$ but $Q \parallel \text{lh}(E) \neq M \parallel \text{lh}(E)$. We set

$$E_\alpha^\mathcal{T} = E^+$$

and continue constructing \mathcal{T} . At limit steps we use Σ_0 to choose a branch, as a tree on (P_0, Σ_0) must do.

The lengths of the $E_\alpha^\mathcal{T}$ are strictly increasing, so if \mathcal{T} never fails, then eventually it must succeed, that is, we must reach α such that $(\dagger)_\alpha$ holds and part (i) of Claim 3.27 holds. This means that (P_0, Σ_0) has iterated to (M, Ω^+) , as desired.

For $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$, let

$\mathcal{W}_{\nu,k}^* = 1$ -maximal λ -separated tree whereby (P_0, Σ_0) iterates strictly past $(M_{\nu,k}, \Omega_{\nu,k})$.

Let us check that the lemma of [13, Section 8.3] on realizing resurrection embeddings as branch embeddings holds for the new $\mathcal{W}_{\nu,k}^*$. Fix $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$ for a while, and suppose that $M_{\nu,k}$ is not sound. Let $\xi_1 = \text{lh}(\mathcal{W}_{\nu,k}^*)$.

By definition, $\mathcal{M}_{\xi_1}^{\mathcal{W}_{\nu,k}^*} \supseteq M_{\nu,k}$, so since $M_{\nu,k}$ is not sound, $\mathcal{M}_{\xi_1}^{\mathcal{W}_{\nu,k}^*} = M_{\nu,k}$. (P_0, Σ_0) iterates strictly past $(M_{\nu,k}, \Omega_{\nu,k})$, so $[0, \xi_1]_{\mathcal{W}_{\nu,k}^*}$ has a large drop. Let

$$\eta_0 = \text{largest } \gamma \text{ in } [0, \xi_1]_{\mathcal{W}_{\nu,k}^*} \cap D^{\mathcal{W}_{\nu,k}^*},$$

$$\eta = W_{\nu,k}^* \text{-pred}(\eta_0),$$

$$i^* = i_{\eta_0, \xi_1}^{\mathcal{W}_{\nu,k}^*} \circ i_{\eta_0}^*,$$

and

$$R = \text{dom}(i^*) = \mathcal{M}_{\eta_0}^{*, \mathcal{W}_{\nu,k}^*}.$$

R is the proper initial segment of $\mathcal{M}_\eta^{\mathcal{W}_{\nu,k}^*}$ that lies on the branch from η to ξ_1 . R has degree k , and it is sound because the drop at η_0 is a large one. Thus

$$R^+ = \mathfrak{C}_{k+1}(R) = \mathfrak{C}_{k+1}(M_{\nu,k}) = M_{\nu,k+1},$$

and i^* is the anticore map. Let $\rho = \rho(M_{\nu,k})$.

Claim 3.28. $\rho < \text{lh}(E_\eta^{\mathcal{W}_{\nu,k}^*})$, and for all $\tau < \eta$, $\text{lh}(E_\tau^{\mathcal{W}_{\nu,k}^*}) \leq \rho$.

Proof. We have $\rho \leq \text{crt}(i^*) < \text{lh}(E_\eta^{\mathcal{W}_{\nu,k}^*})$. Suppose toward contradiction that $\tau < \eta$ and $\rho < \text{lh}(E_\tau^{\mathcal{W}_{\nu,k}^*})$. We can then find such τ with $\tau + 1 <_{W_{\nu,k}^*} \eta$. But this implies that whenever $S \triangleleft \mathcal{M}_\eta^{\mathcal{W}_{\nu,k}^*}$ with $o(S) \geq \text{lh}(E_\tau^{\mathcal{W}_{\nu,k}^*})$, then $\text{lh}(E_\tau^{\mathcal{W}_{\nu,k}^*}) \leq \rho(S)$. Since $\rho(R) = \rho$, we have a contradiction. \square

Sublemma 8.3.1.1 of [13] goes over verbatim when $\langle \nu, k+1 \rangle <_{\text{lex}} \langle \theta, n \rangle$:

Lemma 3.29. Let $\langle \nu, k+1 \rangle <_{\text{lex}} \langle \theta, n \rangle$, and suppose that $M_{\nu,k}$ is not sound. Let $\pi: M_{\nu,k+1}^- \rightarrow M_{\nu,k}$ be the anticore map. Let $\xi = \text{lh}(\mathcal{W}_{\nu,k}^*)$, let η, η_0, R be as above, and let $i^*: R \rightarrow \mathcal{M}_\xi^{\mathcal{W}_{\nu,k}^*} = M_{\nu,k}$ be the branch embedding of $\mathcal{W}_{\nu,k}^*$ as above; then

- (a) $R^+ = M_{\nu,k+1}$ and $i^* = \pi$.
- (b) η is the least γ such that $\text{lh}(E_\gamma^{\mathcal{W}_{\nu,k}^*}) > \rho(M_{\nu,k})$.
- (c) $\mathcal{W}_{\nu,k+1}^* = \mathcal{W}_{\nu,k}^* \upharpoonright \eta + 1$,

Proof. We have already shown (a) and (b), and (c) follows at once. \square

As a result, Lemma 8.3.1 goes over verbatim for resurrections from some $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$:

Corollary 3.30. Let $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, n \rangle$, $P \trianglelefteq M_{\nu,k}$,

$$\eta = \text{least } \xi \text{ such that } P \trianglelefteq \mathcal{M}_\xi^{\mathcal{W}_{\nu,k}^*},$$

and

$$M_{\mu,j} = \text{Res}_{\nu,k}[P];$$

then

$$\mathcal{W}_{\nu,k}^* \upharpoonright (\eta + 1) = \mathcal{W}_{\mu,j}^* \upharpoonright (\eta + 1),$$

$\mathcal{W}_{\mu,j}^*$ has last model $M_{\mu,j}$, and for $\xi = \text{lh}(\mathcal{W}_{\mu,j}^*)$, we have $\eta <_{W_{\mu,j}^*} \xi$, and

$$\sigma_{\mu,j}[P] = \hat{i}_{\eta,\xi}^{\mathcal{W}_{\mu,j}^*}.$$

Proof. The proof of [13, Lemma 8.3.1] also goes over verbatim.³⁷ \square

We can now finish the proof of the theorem in the case $n > 0$.

³⁷The fact that $\mathcal{W}_{\nu,k}^*$ is λ -separated plays a role in the proof.

Claim 3.31. *If $n > 0$, then \mathcal{T} succeeds at some α .*

Proof. Let $n = k + 1$, and suppose toward contradiction that \mathcal{T} fails at α . Let

$$(Q, \Lambda) = (\mathcal{M}_\alpha^\mathcal{T}, \Sigma_\alpha^\mathcal{T}),$$

and let

$$(N, \Psi) = (\mathcal{M}_\xi^{W_{\theta,k}^*}, \Sigma_\xi^{\mathcal{W}_{\theta,k}^*})$$

be the last pair in $W_{\theta,k}^*$.

Suppose first that $M_{\theta,k}$ is sound. Since

$$M_{\theta,k+1} = M_{\theta,k}^+$$

(i.e. they are the same as bare premice), $\mathcal{T} \upharpoonright \alpha + 1 = \mathcal{W}_{\nu,k}^* \upharpoonright \alpha + 1$. But $W_{\nu,k}^*$ never failed, so clearly \mathcal{T} does not fail at α for the “bad extender disagreement” reason; i.e. $(\dagger)_\alpha(1)$ holds. We verify $(\dagger)_\alpha(2)$, i.e. that there is no strategy disagreement between Λ and $\Omega_{\theta,k+1}$, and in fact $(Q, \Lambda) = (M_{\theta,k+1}, \Omega_{\theta,k+1}^+)$.

Let $S \trianglelefteq Q$ and $S \trianglelefteq M_{\theta,k+1}$. If $S \triangleleft M_{\theta,k+1}$ then $S \trianglelefteq M_{\theta,k}$ and

$$(\Omega_{\theta,k})_S = (\Omega_{\theta,k+1})_S.$$

Since P_0 iterated strictly past $M_{\theta,k}$, either $S \triangleleft N$ or $S = N$ and $[0, \xi]_{W_{\theta,k}^*}$ has a large drop. In the latter case $M_{\theta,k}$ is not sound, contradiction. So $S \triangleleft N$, and thus

$$\begin{aligned} (\Omega_{\theta,k+1})_S &= (\Omega_{\theta,k})_S \\ &= \Psi_S \\ &= (\Sigma_\alpha^{W_{\theta,k}^*})_S \\ &= (\Sigma_\alpha^\mathcal{T})_S, \end{aligned}$$

which verifies $(\dagger)_\alpha(2)$ when $S \triangleleft M_{\theta,k+1}$. Line 3 holds by strategy coherence.

Suppose next that $S = M_{\theta,k+1}$.

Subclaim 3.32. *If $S = M_{\theta,k+1} \triangleleft Q$ or $[0, \alpha]_T$ has a large drop, then $\Omega_{\theta,k+1} = \Lambda_S$.*

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works here pretty much word-for-word. □

Subclaim 3.33. *If $S = M_{\theta,k+1}$, then $S = Q$ and $[0, \alpha]_T$ has no large drops.*

Proof. Otherwise (P_0, Σ_0) iterates strictly past $(M_{\theta,n}, \Omega_{\theta,n})$ by Subclaim 3.32. □

Subclaim 3.34. *If $S = M_{\theta,k+1} = Q$ and $[0, \alpha]_T$ has no large drops, then $\Omega_{\theta,k+1}^+ = \Lambda$.*

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works pretty much word-for-word. \square

Putting the subclaims together, we see that (P_0, Σ_0) iterates to $(M_{\theta,k+1}, \Omega_{\theta,k+1}^+)$ in the case that $M_{\theta,k}$ is sound.

So we assume that $M_{\theta,k}$ is not sound. We adopt our previous notation, with $\theta = \nu$. That is,

$$\begin{aligned}\eta_0 &= \text{largest } \gamma \text{ in } [0, \xi_1]_{W_{\theta,k}^*} \cap D^{W_{\theta,k}^*}, \\ \eta &= W_{\theta,k}^* \text{-pred}(\eta_0), \\ i^* &= i_{\eta_0, \xi_1}^{\mathcal{W}_{\theta,k}^*} \circ i_{\eta_0}^*,\end{aligned}$$

and

$$R = \text{dom}(i^*) = \mathcal{M}_{\eta_0}^{*, W_{\theta,k}^*}.$$

R is the proper initial segment of $\mathcal{M}_{\eta}^{\mathcal{W}_{\theta,k}^*}$ that lies on the branch from η to ξ_1 . R has degree k , and it is sound because the drop at η_0 is a large one. Thus

$$R^+ = \mathfrak{C}_{k+1}(R) = \mathfrak{C}_{k+1}(M_{\theta,k}) = M_{\theta,k+1},$$

and i^* is the anticore map.

Let $\rho = \rho_{k+1}(M)$. By 3.28, η is least such that $\text{lh}(E_{\eta}^{\mathcal{W}_{\theta,k}^*}) > \rho$, so

$$\mathcal{T} \upharpoonright \eta + 1 = \mathcal{W}_{\theta,k}^* \upharpoonright \eta + 1.$$

Note that

$$(\Omega_{\theta,k+1})_R = (\Omega_{\theta,k})^{i^*}$$

(by Lemma 3.19)

$$= ((\Sigma_{\xi_1}^{\mathcal{W}_{\theta,k}^*})_{M_{\theta,k}})^{i^*}$$

(because $\mathcal{W}_{\theta,k}^*$ iterates (P_0, Σ_0) strictly past $(M_{\theta,k}, \Omega_{\theta,k})$)

$$= (\Sigma_{\eta}^{\mathcal{W}_{\theta,k}^*})_R$$

(by pullback consistency of Σ_0)

$$= (\Sigma_{\eta}^{\mathcal{T}})_R.$$

Now we proceed as we did when $M_{\theta,k}$ was sound.

Subclaim 3.35. *If $R^+ \triangleleft \mathcal{M}_\eta^\mathcal{T}$ or $[0, \eta]_T$ has a large drop, then $\Omega_{\theta,k+1} = (\Sigma_\eta^\mathcal{T})_{R^+}$.*

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works. \square

Subclaim 3.36. *$R^+ = \mathcal{M}_\eta^\mathcal{T}$ and $[0, \eta]_T$ has no large drops.*

Proof. Otherwise (P_0, Σ_0) iterates strictly past $(M_{\theta,k+1}, \Omega_{\theta,k+1})$ by Subclaim 3.35. \square

Subclaim 3.37. *If $R^+ = \mathcal{M}_\eta^\mathcal{T}$ and $[0, \eta]_T$ has no large drops, then $\Omega_{\theta,k+1}^+ = \Sigma_\eta^\mathcal{T}$.*

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works. \square

Putting these subclaims together, we see that (P_0, Σ_0) iterates to $(M_{\theta,k+1}, \Omega_{\theta,k+1}^+)$ in the case that $M_{\theta,k}$ is unsound.

This completes the proof of Claim 3.31. \square

Now let us prove (†) for \mathcal{T} in the case $n = 0$.

Claim 3.38. *If $n = 0$, then \mathcal{T} succeeds at some α .*

Proof. We cannot have a strategy disagreement involving an $S \triangleleft M_{\theta,-1}$. For if $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, 0 \rangle$ is sufficiently large, then $S \triangleleft M_{\nu,k}$ and $(\Omega_{\theta,-1})_S = (\Omega_{\nu,k})_S$, so this would mean that (P_0, Σ_0) did not iterate strictly past $(M_{\nu,k}, \Omega_{\nu,k})$.³⁸ Similarly, we cannot have a bad extender disagreement involving some E on the $M_{\theta,0}$ -sequence other than its last extender.

Thus we can fix η least such that $M_{\theta,-1} \trianglelefteq \mathcal{M}_\eta^\mathcal{T}$, and we have $(\Omega_{\theta,-1})_S = (\Sigma_\eta^\mathcal{T})_S$ for all $S \triangleleft M_{\theta,-1}$.

Subclaim 3.39. $\Omega_{\theta,-1} = (\Sigma_\eta^\mathcal{T})_{M_{\theta,-1}}$.

Proof. By the proof of [13, 8.4.3]. \square

Subclaim 3.40. *If the last extender E of $M_{\theta,0}$ is nonempty, then E is on the sequence of $\mathcal{M}_\eta^\mathcal{T}$.*

Proof. Let E^* be the background extender $B^{\mathbb{C}}(E)$. The usual proof³⁹ shows that E is an initial segment of the branch extender of $[\kappa_E, i_{E^*}(\kappa_E))$ in $i_{E^*}(\mathcal{T})$, so E^+ is used in $i_{E^*}(\mathcal{T})$, so E is on the $\mathcal{M}_\eta^\mathcal{T}$ -sequence, a contradiction.⁴⁰ \square

By 3.39 and 3.40, $M_{\theta,0} \trianglelefteq \mathcal{M}_\eta^\mathcal{T}$.

Subclaim 3.41. *If $M_{\theta,0} \triangleleft \mathcal{M}_\eta^\mathcal{T}$ or $[0, \eta]_T$ has a large drop, then $\Omega_{\theta,0} = (\Sigma_\eta^\mathcal{T})_{M_{\theta,0}}$.*

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works. \square

³⁸ $M_{\theta,-1}$ is $M_{\theta,0}$ with its last extender, if nonempty, being removed. It is the “lim inf” of the $M_{\nu,k}$ for $\langle \nu, k \rangle <_{\text{lex}} \langle \theta, 0 \rangle$. Its strategy provided by \mathbb{C} is $\Omega_{\theta,-1}$.

³⁹Cf. [13, 8.1.12].

⁴⁰This argument gives a different proof that if $M_{\theta,0}$ is active, then $\Omega_{\theta,-1} = (\Sigma_\eta^\mathcal{T})_{M_{\theta,-1}}$. We simply go to $i_{E^*}(V)$, use the strategy agreement we have there, and then pull it back to V by strategy coherence.

Subclaim 3.42. $M_{\theta,0} = \mathcal{M}_\eta^T$ and $[0, \eta]_T$ has no large drops.

Proof. Otherwise (P_0, Σ_0) iterates strictly past $(M_{\theta,0}, \Omega_{\theta,0})$ by Subclaim 3.41. □

Subclaim 3.43. If $M_{\theta,0} = \mathcal{M}_\eta^T$ and $[0, \eta]_T$ has no large drops, then $\Omega_{\theta,0}^+ = \Sigma_\eta^T$.

Proof. The strategy comparison proof of [13, Theorem 8.4.3] works. □

These subclaims complete the proof of Claim 3.38. □

This in turn completes the proof of Theorem 3.26. □

3.4. The sound case

In the case that $\deg(P) > 0$, Theorem 3.26 implies the statement of $(*)(P, \Sigma)$ that was proved in [13] in the case that P has stable type 1.

For suppose (P, Σ) has stable type 1 and $\deg(P) = k + 1$. Then (P^-, Σ) is a +1-hod pair of degree k . Letting N^*, \mathbb{C} , etc. be as in the statement of $(*)(P, \Sigma)$ in [13], Theorem 3.26 gives us a pair $(M_{\nu,k}, \Omega_{\nu,k})$ such that

- (i) (P^-, Σ) iterates to $(M_{\nu,k}, \Omega_{\nu,k}^+)$, and
- (ii) (P^-, Σ) iterates strictly past $(M_{\eta,j}, \Omega_{\eta,j})$ whenever $\langle \eta, j \rangle <_{\text{lex}} \langle \nu, k \rangle$.

The iterations here are by 1-maximal trees on (P^-, Σ) , which are the same as ordinary (“0-maximal”) trees on (P, Σ) . The iterations in (ii) witness that (P, Σ) iterates strictly past $(M_{\eta,j}, \Omega_{\eta,j})$ whenever $\langle \eta, j \rangle <_{\text{lex}} \langle \nu, k \rangle$.

Let \mathcal{T} be the 1-maximal tree that witnesses that (P^-, Σ) iterates to $(M_{\nu,k}, \Omega_{\nu,k}^+)$, $\xi + 1$ be its length. If $M_{\nu,k}$ is sound, then $(M_{\nu,k+1}, \Omega_{\nu,k+1}) = (M_{\nu,k}^+, \Omega_{\nu,k}^+)$, so \mathcal{T} (regarded as 0-maximal on (P, Σ)) witnesses that (P, Σ) iterates to $(M_{\nu,k+1}, \Omega_{\nu,k+1})$.

So suppose that $M_{\nu,k}$ is not sound. Let $\beta <_T \xi$ be largest such that \mathcal{M}_β^T is $k + 1$ -sound. Since P has stable type 1, β is the largest +1-node on $[0, \xi]_T$. (This is where we use that fact.) Letting $\beta_0 \leq_T \xi$ be such that $T\text{-pred}(\beta_0) = \beta$, \mathcal{T} must have a small drop at β_0 , and no further drops in $(\beta_0, \xi]_T$. So setting

$$i^* = i_{\beta_0, \xi}^T \circ i_{0, \beta_0}^T$$

and

$$R = \text{dom}(i^*)$$

we get that $R^+ = M_{\nu,k+1}$ and i^* is the anticore map. Moreover

$$R^+ = \mathcal{M}_\beta^T$$

because the drop at β_0 was small. We then have

$$\begin{aligned}\Omega_{\nu,k+1} &= (\Omega_{\nu,k}^+)^{i^*} \\ &= (\Sigma_\xi^\mathcal{T})^{i^*} \\ &= (\Sigma_\beta^\mathcal{T}),\end{aligned}$$

by Lemma 3.19 and pullback consistency. This means that (P, Σ) iterates to $(M_{\nu,k+1}, \Omega_{\nu,k+1})$ in the sense required by $(*)(P, \Sigma)$.

3.5. The almost sound case

Now suppose (P, Σ) is not of stable type 1, i.e., it is either of type 2, or of type 1B and not projectum stable. This implies $\deg(P) > 0$; so say $\deg(P) = k + 1$. Again (P^-, Σ) is a +1-hod pair. Let N^*, \mathbb{C} , etc. be as in $(*)(P, \Sigma)$. Let (P^-, Σ) iterate to $(M_{\nu,k}, \Omega_{\nu,k})$ via the 1-maximal tree \mathcal{T} with length $\xi + 1$, as above.

If P has type 2, then P^- is not sound, so $M_{\nu,k}$ is not sound. If P has type 1B and $\eta_{\kappa+1}^P$ is measurable in P , then some total measure on the image of $\eta_{\kappa+1}^P$ is used in $[0, \xi]_T$, producing a type 2 ultrapower. So again $M_{\nu,k}$ is unsound. In both cases, $M_{\nu,k}$ has a measurable cardinal κ such that $\rho_{k+1}(M_{\nu,k}) < \kappa < \rho_k(M_{\nu,k})$. Preimages of κ were hit in $[0, \xi]_T$, so $[0, \xi]_T$ has a small drop.

Thus, letting β be the largest +1-node of \mathcal{T} in $[0, \xi]_T$, we have $\beta < \xi$. Let

$$\begin{aligned}i &= i_{0,\beta}^\mathcal{T}, \\ R &= \mathcal{M}_\beta^\mathcal{T},\end{aligned}$$

and

$$i^* = \hat{i}_{\beta,\xi}^\mathcal{T}.$$

Here $\deg(R) = k$. Since $R = \text{cHull}_1^{R^k}(\sup i^{\rho_1(P^k)} \cup \{p_1(R), \hat{p}_1(R)\})$ and $\sup i^{\rho_1(P^k)} = \rho_1(R^k)$, R^+ is of type 2. $\Sigma_\beta^\mathcal{T}$ is a +1-iteration strategy for R , or equivalently an ordinary strategy for R^+ , moreover

$$\Sigma_\beta^\mathcal{T} = (\Omega_{\nu,k}^+)^{i^*}.$$

$(M_{\nu,k+1}, \Omega_{\nu,k+1})$ is the *type 1 core* of the type 2 pair $(R^+, \Sigma_\beta^\mathcal{T})$, in the following sense.

Definition 3.44. If Q is a pfs premouse, then we set

$$C(Q) = \begin{cases} Q & \text{if } Q \text{ has type 1} \\ \mathfrak{c}_{\deg(Q)}(Q) & \text{if } Q \text{ has type 2.} \end{cases}$$

We call $C(Q)$ the *type 1 core* of Q . Q and $C(Q)$ have the same degree. If Q has type 2, then $C(Q)$

has type 1A, and we let $D(Q)$ be the unique order zero measure on $\hat{\rho}(Q^-)$ such that

$$Q^- = \text{Ult}(C(Q)^-, D(Q)).$$

The *type 1 core* of (Q, Λ) is $(C(Q), \Lambda^i)$, where $i = \text{id}$ if $C(Q) = Q$, and $i = i_{D(Q)}$ otherwise. \dashv

Remark 3.45. Notice that if (Q, Λ) is a type 2 lbr hod pair of degree k , then for $i = i_{D(Q)}$, $(C(Q), \Lambda^i)$ is a type 1 pair of degree k . Λ^i is defined on “level k ” stacks because i is nearly +1-elementary as a map from $C(Q)^-$ to Q^- . \dashv

Claim 3.46. $(M_{\nu, k+1}, \Omega_{\nu, k+1})$ is the *type 1 core* of (R^+, Σ_{β}^T) .

Proof. We have that $k = \text{deg}(R)$, and i^* is nearly +1-elementary as a map from R to $M_{\nu, k}$. It follows that

$$\mathfrak{C}_{k+1}(R) = \mathfrak{C}_{k+1}(M_{\nu, k}) = M_{\nu, k+1}.$$

Moreover, letting $D = D(R)$ and $\pi: M_{\nu, k+1} \rightarrow M_{\nu, k}$ be the anticore map, we have the diagram

$$\begin{array}{ccc} R^+ & \xrightarrow{i^*} & M_{\nu, k} \\ i_D \uparrow & \nearrow \pi & \\ M_{\nu, k+1} & & \end{array}$$

But then

$$\begin{aligned} \Omega_{\nu, k+1} &= (\Omega_{\nu, k}^+)^{\pi} \\ &= (\Omega_{\nu, k}^+)^{i^* \circ i_D} \\ &= (\Sigma_{\beta}^T)^{i_D}, \end{aligned}$$

as required. \square

We extend the mouse order to type 2 pairs by regarding (Q, Λ) as being equivalent to its type 1 core. That is,

Definition 3.47. Let (Q, Λ) and (R, Ω) be mouse pairs with type 1 cores (Q_0, Λ_0) and (R_0, Ω_0) respectively; then $(Q, \Lambda) \leq^* (R, \Omega)$ iff $(Q_0, \Lambda_0) \leq^* (R_0, \Omega_0)$. \dashv

3.6. Generating type 2 pairs

We show that any type 2 pair (Q, Λ) can be recovered from its type 1 core $(C(Q), \Omega)$. Of course, Q can be recovered from $C(Q)$ by taking an ultrapower, the question is how to recover Λ from Ω . The tail strategy $\Omega_{(D(Q))}$ is a strategy for Q^- , not Q . But we can apply the strategy extension method⁴¹ to Ω to obtain a strategy for Q , and this strategy works out to be Λ .

We aren't going to use the result in this subsection later, so we shall omit some details.

⁴¹See [9] and [13, 7.3.11].

Definition 3.48. Suppose that (M, Ω) is an lbr hod pair of type 1A and degree $k + 1$, and D is the order zero measure of M on κ , where $\rho_{k+1}(M) < \kappa < \rho_k(M)$. Suppose also that $\text{cof}_{k+1}^M(\kappa) < \rho_{k+1}(M)$. Let

$$Q^+ = (\text{Ult}(M^-, D), k + 1),$$

and for \mathcal{U} a λ -separated tree on Q^+ of limit length, let

$$\Omega_{\langle D \rangle}^+(\mathcal{U}) = b \text{ iff } V(\langle D \rangle, \mathcal{U} \frown b) \text{ is by } \Omega;$$

then we call $(Q^+, \Omega_{\langle D \rangle}^+)$ the *type 2 pair generated by (M, Ω) and D* . +

Lemma 3.49. *Let (Q, Λ) be a type 2 pair, and let (M, Ω) be its type 1 core; then (Q, Λ) is the type 2 pair generated by (M, Ω) and $D(Q)$.*

Proof. Let $k + 1 = \text{deg}(Q)$, and let $Q^k = \text{Ult}_0(M^k, D)$ where $D = D(Q)$ is the order zero measure of M^k on $\hat{\rho}_k(Q)$. Let

$$\pi = i_D^{M^-}.$$

In order to see that (Q, Λ) is the type 2 pair generated by (M, Ω) and D , we must show that $\Lambda = \Omega_{\langle D \rangle}^+$. Letting \mathcal{U} be a plus tree on Q , the relevant diagram is

$$\begin{array}{ccccc} Q & \xrightarrow{\tau} & R & \xrightarrow{\tau\mathcal{U}} & S \\ \pi \uparrow & & \uparrow \tau & & \uparrow \\ M & \xrightarrow{\pi} & Q & \xrightarrow{\mathcal{U}} & N \end{array}$$

Here τ is the copy map, or what is the same, the canonical embedding from Q to $\text{Ult}(Q^-, \pi(D))$. Since τ is nearly +1-elementary as a map on Q it can be used to lift \mathcal{U} . We then calculate

$$\begin{aligned} \mathcal{U} \text{ is by } \Omega_{\langle D \rangle}^+(\mathcal{U}) & \text{ iff } V(\langle D \rangle, \mathcal{U}) \text{ is by } \Omega \\ & \text{ iff } \pi V(\langle D \rangle, \mathcal{U}) \text{ is by } \Lambda \\ & \text{ iff } V(\langle \pi(D) \rangle, \tau\mathcal{U}) \text{ is by } \Lambda \\ & \text{ iff } \langle \pi(D) \rangle, \tau\mathcal{U} \text{ is by } \Lambda \\ & \text{ iff } \mathcal{U} \text{ is by } \Lambda. \end{aligned}$$

Line 3 holds because quasi-normalizing commutes with lifting, line 4 holds because Λ quasi-normalizes well, and line 5 holds because Λ is pullback consistent. □

4. CONDENSATION LEMMA

The main theorem of this section is Theorem 4.6. This theorem will be used in the \square -construction, but it is more general than is necessary for that application. Its full generality is used in [9].

Our theorem extends Theorem 9.3.2 of [16], which deals with condensation under $\pi: H \rightarrow M$ for pure extender mice H and M . That theorem breaks naturally into two cases: either (1) $H \notin M$, in which case H is the $\text{crt}(\pi)$ -core of M , or (2) $H \in M$, in which case H is a proper initial segment of either M or an ultrapower of M . The proof in case (1) works for least branch hod mice without much change, so we begin with that case.

Definition 4.1. Let M be an lpm or a pure extender premouse, and $n \leq \text{deg}(M)$; then

- (a) \tilde{h}_M^{n+1} is the completion of $h_{\hat{M}^n}^1$, the Σ_1 -Skolem function of \hat{M}^n . We write \tilde{h}_M for $\tilde{h}_M^{\text{deg}(M)+1}$. So $\text{dom}(\tilde{h}_M) \subseteq M$.⁴²
- (b) Let $\rho(M) \leq \alpha$ and $r = p(M) - \alpha$, and suppose that r is solid. Let $\pi: H \rightarrow M$ with H transitive be such that $\text{ran}(\pi) = \tilde{h}_M(\alpha \cup r)$, and suppose that $\pi^{-1}(r)$ is solid over H . Then we call H the α -core of M , and write $H = \text{core}_\alpha(M)$. In addition, if (M, Σ) is a mouse pair, then the α -core of (M, Σ) is (H, Λ) , where $H = \text{core}_\alpha(M)$ and $\Lambda = \Sigma^\pi$ and π is the anticore map.
- (c) M is α -sound iff $M = \text{core}_\alpha(M)$.

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We note that $\text{core}(M) = \mathfrak{C}_{\text{deg}(M)+1}(M)$ is the $\rho(M) + 1$ -core of M . According to this definition, if M is α -sound, then $\rho(M) \leq \alpha$. So M could be sound, but not α -sound because $\alpha < \rho(M)$, which might be confusing at first.

Remark 4.2. Let H be the α -core of M , as witnessed by π . We have $p(M) \subseteq \text{ran}(\pi)$, so the new $\Sigma_{\text{deg}(M)+1}$ subset of $\rho(M)$ is $\Sigma_{\text{deg}(M)+1}$ over H . Thus $\rho(H) = \rho(M)$ and $\pi(p(H)) = p(M)$, and $H \notin M$.

One also gets that if both H, M are of type 1, then H is of type 1A iff M is of type 1A.

One might guess that $P(\alpha)^M \subseteq H$, but this need not be the case, as the following example shows. Let N be sound, and let $M = \text{Ult}(N, E)$, where $\rho(N) \leq \kappa = \text{crt}(E)$, and E has one additional generator α . Let $H = \text{Ult}(N, E \upharpoonright \alpha)$, and let $\pi: H \rightarrow M$ be the factor map. Clearly, π witnesses that H is the α -core of M . But $\alpha = (\kappa^{++})^H < (\kappa^{++})^M$, so H doesn't even have all the bounded subsets of α that are in M .

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Theorem 4.3 (AD^+). Suppose (M, Σ) is a lbr hod pair with scope HC. Suppose H and M are both type 1 premice, $\pi: H \rightarrow M$ is nontrivial⁴³, and letting $n = \text{deg}(M) = \text{deg}(H)$ and $\alpha = \text{crt}(\pi)$,⁴⁴ $\alpha < \rho_n(M)$. Suppose also

- (1) H is α -sound,
- (2) π is nearly elementary, and

⁴²See [13, Definition 2.3.9].

⁴³ π is trivial iff $H = M$ and π is the identity.

⁴⁴Here we allow α to be $o(H)$ and π to be the identity.

(3) H is an lpm of the same kind as M ⁴⁵, and

(4) $H \notin M$.

Then H is the α -core of M .

Proof. Let $r = p(H) - \alpha$.

$$T = \text{Th}_1^{H^n}(\alpha \cup r),$$

so that T codes H . T is sometimes denoted by $\text{Th}_{n+1}^H(\alpha \cup r)$.

Suppose first that $\pi \upharpoonright H^n$ is not cofinal. We have that T is Σ_1 over H^n , and hence T is Σ_1 over some proper initial segment of M^n , so that $T \in M^n$. If $n > 0$, then $M \upharpoonright \rho_n(M) \models \text{KP}$ and $T \in M \upharpoonright \rho_n(M)$, so $H \in M$. If $n = 0$ and H is fully passive, then we have $\pi: H \rightarrow M \upharpoonright \eta$ for some $\eta < o(M)$, and $\text{ran}(\pi)$ in M . Any premouse is closed under transitive collapse, so we again get $H \in M$. If $n = 0$ and H is extender-active, then letting $H^- = H \upharpoonright o(H)$, we get $H^- \in M$ by the argument just given. However, \dot{F}^H can be computed from the fragment $\dot{F}^M \upharpoonright \sup \pi \text{``} o(H)$ and π inside M , so $H \in M$. The case that $n = 0$ and H is branch-active can be handled similarly, noting that the proper initial segments of b^M are in M .

So we may assume $\pi \upharpoonright H^n$ is cofinal Σ_0 , and hence Σ_1 elementary. We then get that $\pi(\eta_n^H) = \eta_n^M$. We claim that $\rho(M) \leq \alpha$. For if not, T is a bounded subset of $\rho(M)$ that is Σ_1 over M^n . Thus $T \in M \upharpoonright \rho(M)$, so $H \in M$.

Suppose $r = \emptyset$. If $\gamma \in (p(M) - \alpha)$, then T can be computed easily from the solidity witness $W_{r,\gamma}^M$ ⁴⁶ so T in M , and with a bit more work, $H \in M$. So we have $p(M) - \alpha = \emptyset$, which implies that H is the α -core of M , as witnessed by π .

Suppose next that $r = \langle \beta_0, \dots, \beta_l \rangle$, and $p(M) - \alpha = \langle \gamma_0, \dots, \gamma_m \rangle$, where $\beta_i > \beta_{i+1}$ and $\gamma_i > \gamma_{i+1}$ for all i . We show by induction on $i \leq l$ that $i \leq m$ and $\pi(\beta_i) = \gamma_i$. Suppose we know it for $i \leq k < l$. Let $W = W_{r,\beta_{k+1}}^H$ be the solidity witness for β_{k+1} in H . Since $\pi \upharpoonright H^n$ is Σ_1 elementary, $\pi(W)$ can be used to compute $\text{Th}_{n+1}^M(\pi(\beta_{k+1}) \cup \{\gamma_0, \dots, \gamma_k\})$ inside M . But $\rho(M) < \pi(\beta_{k+1})$, so we must have $k < m$. Similarly, $\gamma_{k+1} < \pi(\beta_{k+1})$ is impossible, as otherwise $\pi(W)$ could be used in M to compute the Σ_{n+1} theory of $p(M) \cup \rho(M)$. On the other hand, if $\pi(\beta_{k+1}) < \gamma_{k+1}$, then using the solidity witness $W_{p(M),\gamma_{k+1}}^M$ for γ_{k+1} in M , we get $H \in M$.

It follows that $\pi(r) = p(M) - \alpha$, and thus H is the α_0 -core of M . □

Remark 4.4. In the case H is the core of M , we can also get agreement of Σ and Σ^π up to $(\rho^+)^H = (\rho^+)^M$. See [13, Corollary 9.6.6]. It may be possible to prove strategy condensation in the other cases, but we have not tried to do that.

An analog of the above theorem can be stated and proved for type 2 premice. ←

⁴⁵This means: H is passive if and only if M is passive; H is B -active if and only if M is B -active; and H is E -active if and only if M is E -active; in the third case, \dot{F}^H is of type A (B, C) if and only if \dot{F}^M is of type A (B, C) respectively). All but the last clause are implicit in (2).

⁴⁶Here recall that $W_{r,\gamma}$ is the transitive collapse of $\text{Hull}_1^{M^n}(\gamma \cup \{r - \{\gamma\}\})$.

Next we deal with condensation under $\pi: H \rightarrow M$ in the case $H \in M$.⁴⁷ We shall actually prove a stronger result, one that includes condensation for iteration strategies as well as condensation for the mice themselves.

The following is an easy case of condensation for pairs.

Lemma 4.5 (AD^+). *Let (M, Σ) be a mouse pair with scope HC, and let $\pi: (H, \Psi) \rightarrow (M, \Sigma)$ be elementary, with $\pi = \text{identity}$; then either $(H, \Psi) \trianglelefteq (M, \Sigma)$, or $(H, \Psi) \triangleleft \text{Ult}((M, \Sigma), E_\alpha^M)$, where $\alpha = o(H)$.*

Proof. Suppose first H is extender-active. Let $F = \dot{F}^H$ and $G = \dot{F}^M$, and let $\kappa = \text{crt}(F)$. So $\kappa^{+,H} = \kappa^{+,M} < o(H)$, and $i_F^H \text{``}\kappa^{+,H} = i_G^M \text{``}\kappa^{+,M}$. Thus $\text{ran}(\pi)$ is cofinal in $o(M)$, which implies $(H, \Psi) = (M, \Sigma)$.

Next, suppose that H is branch active.⁴⁸ Since π is the identity, $\eta = \eta^H = \eta^M$ and $s = s^H = s^M$. Let $\mathcal{T} = \mathcal{T}(s)$, and let $\nu = \nu^H$, so that $o(H) = \eta + \nu$. Because π preserves \dot{B}^H , $b^H = b^M \cap \nu$. But $b^M \cap \nu = b^{M|o(H)}$ because M is an lpm, so $H = M$. We get $\Sigma^\pi = \Sigma_{l(H)}$ from the internal-lift consistency of (M, Σ) , so $(H, \Psi) \trianglelefteq (M, \Sigma)$.

Finally, suppose that H is fully passive. Clearly, $M|\hat{o}(H)$ is branch-passive, and thus $M|\hat{o}(H) = H$. Using internal-lift consistency for (M, Σ) , we get $(H, \Psi) \trianglelefteq (M, \Sigma)$, unless $M|\hat{o}(H)$ is extender-active. In that case we get $(H, \Psi) \triangleleft \text{Ult}(M, E_\alpha^M)$, where $\alpha = \hat{o}(H)$, using internal-lift consistency and strategy coherence. \square

Our main condensation theorem for mouse pairs is:

Theorem 4.6 (AD^+). *Suppose (M, Σ) is a mouse pair with scope HC. Suppose $\pi: (H, \Psi) \rightarrow (M, \Sigma)$ is nearly elementary, and not the identity. Let $\alpha = \text{crt}(\pi)$, and suppose*

- (1) $\rho(H) \leq \alpha < \rho^-(H)$, and H is α -sound, and
- (2) H is a premouse of the same kind as M , $\text{deg}(H) = \text{deg}(M)$, both H and M are solid pfs premice, and
- (3) $H \in M$.

Let (H_0, Ψ_0) be the type 1 core of (H, Ψ) ; then exactly one of the following holds.

- (a) $(H_0, \Psi_0) \triangleleft (M, \Sigma)$,
- (b) $(H_0, \Psi_0) \triangleleft \text{Ult}_0((M, \Sigma), \dot{E}_\alpha^M)$.

See Footnote 45 for the definition of “same kind”. It does not mean “same type”; we are leaving open the possibility that one of H and M has type 1, while the other has type 2. Note that if $\rho(H) < \alpha$, then conclusion (b) is impossible.

⁴⁷If $\pi: H \rightarrow M$ is elementary, $\alpha = \text{crt}(\pi)$, H is α -sound, and $\alpha < \rho(M)$, then $H \in M$. This is the case with the coarser condensation results of [13, 5.55] and [1, 8.2], where $\alpha = \rho(H)$ and $\pi(\alpha) = \rho(M)$.

⁴⁸Of course, this only applies when M is an lpm. In general, our proofs for pure extender pairs are special cases of the proofs for lbr hod pairs, so it doesn't hurt to assume our mouse pair is an lbr hod pair.

When one applies Theorem 4.6 in the proof of \square_κ in pfs mice, then M has type 1, and one can arrange that H has type 1 as well, so that $(H_0, \Psi_0) = (H, \Psi)$. One also has that $H \in M$. In that proof, $\rho(H) = \rho(M) = \kappa$, and $\alpha = (\kappa^+)^H$, and both H and M are sound. Since $\rho(H) < \alpha$, (b) does not hold. So one gets conclusion (a), that $(H, \Psi) \triangleleft (M, \Sigma)$. In the square proof, what matters then is just that $H \trianglelefteq M$; the full external strategy agreement given by $\Sigma^\pi = \Sigma_H$ is not used.

In the applications of 4.6 to full normalization and positionality in [9], one must consider the case that H and M have different types, and both alternatives in the conclusion are needed.

Remark 4.7. It follows from the theorem that the hypothesis $\alpha < \rho_{\deg(H)}(H)$ can be dropped, if one omits condensation of the external strategy from its conclusion. See 4.14 below. \dashv

By condition (3) and Theorem 4.3 (and the remark after), H is not an α -core of M . If H is α -sound and is not the α -core of M , then by 4.3, $H \in M$.

Notice that one of the alternatives in the conclusion of [16, Theorem 9.3.2] does not occur here. The alternative that $(H, \Psi) \triangleleft \text{Ult}_0((M|\xi, \Sigma), E)$ where $\rho(M|\xi) = \mu$ is the predecessor of α in M and E is an extender on the sequence of M with critical point μ cannot occur here because M is projectum solid.

A relatively coarse special case of Theorem 4.6 is sketched in [13, Theorem 5.55]. In that case, π is assumed to be fully elementary and $\text{crt}(\pi) = \rho(H)$.

Proof of Theorem 4.6. Let $\pi: (H, \Psi) \rightarrow (M, \Sigma)$ be nearly elementary, and let $\alpha_0 = \alpha = \text{crt}(\pi)$ and $\deg(H) = \deg(M) = k_0$. For definiteness, let us assume that H and M are least branch premice. The proof in the case that they are pure extender mice is similar.⁴⁹⁵⁰

Definition 4.8. A tuple $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ is *problematic* iff

- (1) (N, Φ) and (G, Λ) are of the same kind, with scope HC, and $G \in N$,
- (2) $\sigma: (G, \Lambda) \rightarrow (N, \Phi)$ is nearly elementary, with $\text{crt}(\sigma) = \nu$,
- (3) $\rho(G) \leq \nu < \rho_{\deg(G)}(G)$ and G is ν -sound, $\deg(N) = \deg(G)$, G, N are both solid pfs premice, and
- (4) letting (G_0, Λ_0) be the type 1 core of (G, Λ) , both conclusions (a) and (b) of 4.6 fail for the pair $(G_0, \Lambda_0), (N, \Phi)$; that is, it is not the case that $(G_0, \Lambda_0) \triangleleft (N, \Phi)$, and it is not the case that $(G_0, \Lambda_0) \triangleleft \text{Ult}_0((N, \Phi), \dot{E}_\nu^N)$.

\dashv

⁴⁹Even in the pure extender case, one cannot simply quote 9.3.2 of [16], because we are demanding strategy condensation.

⁵⁰Under AD^+ , every countable ω_1 -iterable pure extender mouse M has an complete iteration strategy Σ such (M, Σ) is a pure extender pair. Thus our theorems 4.3, 4.5, and 4.6 together imply the version of 9.3.2 of [16] for pfs mice, modulo some details about where the strategies live, and how elementary π is. See also Remark 4.7.

Claim 1. Let $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be a problematic tuple, and $k = \deg(G)$; then $\sigma(\nu)$ is a cardinal of N , $\sigma(\nu) < \rho_k(N)$, and $G \in N \parallel \sigma(\nu)$.

Proof. By (3), $\rho_{k+1}(G) \leq \nu < \rho_k(G)$, so $\sigma(\nu) < \rho_k(N)$. Since $\nu = \text{crt}(\sigma)$, it is a cardinal of G , so $\sigma(\nu)$ is a cardinal of N . But G is ν -sound, so it is coded by a subset of ν in N , so $G \in N \parallel \sigma(\nu)$. \square

We must show that $\langle (M, \Sigma), (H, \Psi), \pi, \alpha_0 \rangle$ is not problematic. Assume toward contradiction that it is, and that (M, Σ) is minimal in the mouse order such that (M, Σ) is the first term in some problematic tuple, and let $k_0 = \deg(M)$.

We obtain a contradiction by comparing the phalanx (M, H, α_0) with M , as usual. However, since we are comparing strategies, this must be done indirectly, by iterating both into some sufficiently strong background construction \mathbb{C} . It can happen that at some point, the two sides agree with each other (but not with \mathbb{C}). This leads to a problem in the argument that the end model on the phalanx side can't be above M . The solution employed in [13] is to modify how the phalanx is iterated, moving the whole phalanx (including its exchange ordinal) up at certain stages. Our main new problem here is that because of the restricted elementarity of our maps, if we move up naively, the new phalanx and associated embedding may not be problematic. This forces us to drop to a new problematic phalanx on occasion.

Claim 2. Let $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be a problematic tuple, and $k = \deg(G)$; then we can decompose $\sigma \upharpoonright \hat{G}^k$ as

$$\sigma \upharpoonright \hat{G}^k = \bigcup_{\eta < \rho_k(G)} \sigma^\eta,$$

where each σ^η belongs to \hat{N}^k .

Proof. Assume first $k = 0$ (so G, N are of type 1), and that $\hat{\delta}(G)$ is a limit ordinal. For $\eta < \hat{\delta}(G)$, let G^η be $G \parallel \eta$, expanded by I^η , where I^η is the appropriate fragment of \dot{F}^G if G is extender active, the appropriate initial segment of \dot{B}^G if G is branch active, and $I^\eta = \emptyset$ otherwise. Let N^η be $N \parallel \sigma(\eta)$, expanded by $\sigma(I^\eta)$. Let $s = p_1(G) \setminus \nu$ and σ^η be the fragment of σ given by

$$\text{dom}(\sigma^\eta) = h_{G^\eta}^1 \text{``}(\nu \cup s),$$

and

$$\sigma^\eta(h_{G^\eta}^1(\delta, s)) = h_{N^{\sigma(\eta)}}^1(\delta, \sigma(s)),$$

for $\delta < \nu$. We have that $\sigma^\eta \in N$, and $\sigma = \bigcup_{\eta < \hat{\delta}(G)} \sigma^\eta$. If $\hat{\delta}(G)$ is a successor ordinal, we can ramify using the S -hierarchy.

The case $k > 0$ is similar. We have $\hat{G}^k = (G \parallel \rho_k(G), A)$ where $A = \hat{A}_G^k$. For $\eta \leq \rho_k(G)$, let

$$G^\eta = (G \parallel \eta, A \cap G \parallel \eta) = \hat{G}^k \parallel \eta.$$

Let $s = p_1(\hat{G}^k) \setminus \nu$, and let $h_{\hat{G}^k}^1$ be the Σ_1 Skolem function, so that $\hat{G}^k = h_{\hat{G}^k}^1 \text{``}(\nu \cup s)$. For $\eta < \rho_k(G)$,

$$\text{dom}(\sigma^\eta) = h_{G^\eta}^1 \text{``}(\nu \cup s),$$

and for $\gamma < \nu$ in $\text{dom}(\sigma^\eta)$,

$$\sigma^\eta(h_{G^\eta}^1(\gamma, s)) = h_{\hat{N}^k \parallel \sigma(\eta)}^1(\gamma, \sigma(s)).$$

It is easy to see that this works. □

We call $\langle (\sigma^\eta, G^\eta) \mid \nu \leq \eta < \rho_k(G) \rangle$ as above the *natural decomposition* of $\sigma \upharpoonright \hat{G}^k$.

Using claim 2, we can move a problematic tuple $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ up via an iteration map that is continuous at $\rho_k(G)$. When the iteration map is discontinuous at $\rho_k(G)$, we may have to drop.

Definition 4.9. Let $\Phi = \langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be a problematic tuple; then Φ is *extender-active* iff $E_\nu^N \neq \emptyset$. +

When we move up extender-active tuples, the new exchange ordinal is always the image of the old one, so the new tuple is still extender-active.

Claim 3. Let $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be problematic, and suppose that $(N, \Phi) \leq^* (M, \Sigma)$; then there is no proper initial segment (Q, Ω) of (G, Λ) such that $\nu = \rho(Q)$ and either

- (i) $E_\nu^N = \emptyset$, and (Q, Ω) is not an initial segment of (N, Φ) , or
- (ii) $E_\nu^N \neq \emptyset$, and (Q, Ω) is not a proper initial segment of $\text{Ult}((N, \Phi), E_\nu^N)$.

Proof. This follows from the minimality of (M, Σ) in the mouse order. For if (Q, Ω) is a counterexample, then letting $(R, \Gamma) \triangleleft (N, \Phi)$ be such that $R = \sigma(Q)$, we have that $(R, \Gamma) <^* (M, \Sigma)$, and $\langle (R, \Gamma), (Q, \Lambda_Q), \sigma \upharpoonright Q, \nu \rangle$ is problematic. We note that since $Q \triangleleft G$, Q is of type 1 and hence $C(Q) = Q$. □

So under the hypotheses of claim 3, (N, Φ) agrees with (G, Λ) strictly below $\nu^{+.G}$.

We are ready now to enter the phalanx comparison argument of [13]. Fix a coarse strategy pair $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ that captures Σ , and let \mathbb{C} be the maximal (w, \mathcal{F}) construction, with models $M_{\nu, l}$ and induced strategies $\Omega_{\nu, l}$. Let $\delta^* = \delta(w)$. By Theorem 3.26, $(*) (M, \Sigma)$ holds, so we can fix $\langle \eta_0, k_0 \rangle$ lex least such that either

- (i) (M, Σ) iterates to $(M_{\eta_0, k_0}, \Omega_{\eta_0, k_0})$, or
- (ii) (M, Σ) iterates to some type 2 pair generated by $(M_{\eta_0, k_0}, \Omega_{\eta_0, k_0})$.

As we showed in §3, alternative (i) occurs if M has stable type 1, and alternative (ii) occurs otherwise. Let $\mathcal{U}_{\eta_0, k_0}$ be the λ -separated tree on (P_0, Σ_0) that witnesses this. For $\langle \nu, l \rangle <_{\text{lex}} \langle \nu_0, k_0 \rangle$

let $\mathcal{U}_{\nu,l}$ be the unique λ -separated tree on M witnessing that (M, Σ) iterates strictly past $(M_{\mu,l}, \Omega_{\nu,l})$.
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We define λ -separated trees $\mathcal{S}_{\nu,l}$ on (M, H, α_0) for certain $(\nu, l) \leq (\eta_0, k_0)$. Fix $(\nu, l) \leq (\eta_0, k_0)$ for now, and assume $\mathcal{S}_{\nu',l'}$ is defined whenever $(\nu', l') < (\nu, l)$. Let $\mathcal{U} = \mathcal{U}_{\nu,l}$, and for $\tau < \text{lh}(\mathcal{U})$, let

$$\Sigma_{\tau}^{\mathcal{U}} = \Sigma_{\mathcal{U}|(\tau+1)}$$

be the tail strategy for $\mathcal{M}_{\tau}^{\mathcal{U}}$ induced by Σ . We proceed to define $\mathcal{S} = \mathcal{S}_{\nu,l}$, by comparing the phalanx (M, H, α_0) with $M_{\nu,l}$. As we define \mathcal{S} , we lift \mathcal{S} to a padded tree \mathcal{T} on \mathcal{M} , by copying. Let us write

$$\Sigma_{\theta}^{\mathcal{T}} = \Sigma_{\mathcal{T}|(\theta+1)}$$

for the tail strategy for $\mathcal{M}_{\theta}^{\mathcal{T}}$ induced by Σ . For $\theta < \text{lh}(\mathcal{S})$, we will have copy map

$$\pi_{\theta} : \mathcal{M}_{\theta}^{\mathcal{S}} \rightarrow \mathcal{M}_{\theta}^{\mathcal{T}}.$$

The map π_{θ} is nearly elementary. We attach the complete strategy

$$\Lambda_{\theta} = (\Sigma_{\theta}^{\mathcal{T}})^{\pi_{\theta}}$$

to $\mathcal{M}_{\theta}^{\mathcal{S}}$. We also define a non-decreasing sequence of ordinals $\lambda_{\theta} = \lambda_{\theta}^{\mathcal{S}}$ that measure agreement between models of \mathcal{S} , and tell us which model we should apply the next extender to.

The following claim will be useful in pushing up problematic tuples.

Claim 4. Suppose $\xi <_{\mathcal{S}} \theta$ and $(\xi, \theta]_{\mathcal{S}}$ does not drop; then $\Lambda_{\xi} = \Lambda_{\theta}^{i_{\xi, \theta}^{\mathcal{S}}}$.

Proof. Because Σ is pullback consistent, we have $\Sigma_{\xi}^{\mathcal{T}} = (\Sigma_{\theta}^{\mathcal{T}})^{i_{\xi, \theta}^{\mathcal{T}}}$. But then

$$\begin{aligned} \Lambda_{\xi} &= (\Sigma_{\xi}^{\mathcal{T}})^{\pi_{\xi}} \\ &= (\Sigma_{\theta}^{\mathcal{T}})^{i_{\xi, \theta}^{\mathcal{T}} \circ \pi_{\xi}} \\ &= (\Sigma_{\theta}^{\mathcal{T}})^{\pi_{\theta} \circ i_{\xi, \theta}^{\mathcal{S}}} \\ &= \Lambda_{\theta}^{i_{\xi, \theta}^{\mathcal{S}}}, \end{aligned}$$

as desired. □

We start with

$$\mathcal{M}_0^{\mathcal{S}} = M, \mathcal{M}_1^{\mathcal{S}} = H, \lambda_0 = \alpha_0,$$

and

$$\mathcal{M}_0^{\mathcal{T}} = \mathcal{M}_1^{\mathcal{T}} = M, \pi_0 = id, \pi_1 = \pi,$$

⁵¹We can work in N^* from now on, and interpret these statements there. But in fact, the strategies $\Omega_{\nu,l}$ are induced by Σ^* in a way that guarantees they extend to Σ^* -induced strategies $\Omega_{\nu,l}^*$ defined on all of HC. $\mathcal{U}_{\nu,l}$ iterates (M, Σ) past $(M_{\nu,l}, \Omega_{\nu,l}^*)$ in V .

and

$$\Lambda_0 = \Sigma, \quad \Lambda_1 = \Sigma^{\pi_1}.$$

We say that $0, 1$ are distinct roots of \mathcal{S} . We say that 0 is unstable, and 1 is stable⁵². As we proceed, we shall declare additional nodes θ of \mathcal{S} to be unstable. We do so because $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta) = (\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$ for some γ ⁵³, and when we do so, we shall immediately define $\mathcal{M}_{\theta+1}^{\mathcal{S}}$, as well as σ_θ and α_θ such that

$$\Phi_\theta =_{\text{df}} \langle (\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta), (\mathcal{M}_{\theta+1}^{\mathcal{S}}, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$$

is a problematic tuple. Here $\Lambda_{\theta+1} = \Lambda_\theta^{\sigma_\theta}$. In this case, $[0, \theta]_{\mathcal{S}}$ does not drop, and all $\xi \leq_S \theta$ are also unstable. We regard $\theta + 1$ as a new root of \mathcal{S} . This is the only way new roots are constructed.

Let us also write

$$\Phi_\theta^- =_{\text{df}} \langle \mathcal{M}_\theta^{\mathcal{S}}, \mathcal{M}_{\theta+1}^{\mathcal{S}}, \sigma_\theta, \alpha_\theta \rangle$$

for the part of Φ_θ that is definable over $\mathcal{M}_\theta^{\mathcal{S}}$. We say Φ_θ^- is *problematic* iff Φ_θ is problematic for reason that do not involve the external strategies; that is, iff both statements “ $C(\mathcal{M}_{\theta+1}^{\mathcal{S}}) \triangleleft \mathcal{M}_\theta^{\mathcal{S}}$ ” and “ $C(\mathcal{M}_{\theta+1}^{\mathcal{S}}) \triangleleft \text{Ult}_0(\mathcal{M}_\theta^{\mathcal{S}}, E_{\alpha_\theta}^{\mathcal{M}_\theta^{\mathcal{S}}})$ ” are false.

If θ is unstable, then we define

$$\beta_\theta = (\alpha_\theta^+)^{\mathcal{M}_{\theta+1}^{\mathcal{S}}}.$$

If $\xi <_S \theta$, then we shall have $\beta_\theta \leq i_{\xi, \theta}^{\mathcal{S}}(\beta_\xi)$, and

$$\beta_\theta = i_{\xi, \theta}^{\mathcal{S}}(\beta_\xi) \Rightarrow \Phi_\theta = i_{\xi, \theta}^{\mathcal{S}}(\Phi_\xi),$$

in the appropriate sense. In this connection: it will turn out that $i_{\xi, \theta}^{\mathcal{S}}(\beta_\xi) = \beta_\theta$ implies $i_{\xi, \theta}^{\mathcal{S}}$ is continuous at $\rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}})$, where $k = k_0 = \text{deg}(\mathcal{M}_{\xi+1}^{\mathcal{S}})$. So we can set

$$i_{\xi, \theta}^{\mathcal{S}}(\sigma_\xi) = \text{upward extension of } \bigcup_{\eta < \rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}})} i_{\xi, \theta}^{\mathcal{S}}(\sigma_\xi^\eta),$$

where $\langle \sigma_\xi^\eta \mid \eta < \rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}}) \rangle$ is the natural decomposition of σ_ξ . This enables us to make sense of $i_{\xi, \theta}^{\mathcal{S}}(\Phi_\xi^-)$.

The construction of \mathcal{S} takes place in rounds in which we either add one stable θ , or one unstable θ and its stable successor $\theta + 1$. Thus the current last model is always stable, and all extenders used in \mathcal{S} are plus extenders taken from stable models. If γ is stable, then

$$\lambda_\gamma = \hat{\lambda}(E_\gamma^{\mathcal{S}}).$$

⁵²This is different from the notion of “stable pfs premisses”. We shall let context dictate the meaning of the term.

⁵³In the first version of [13] the external strategy agreement was not required for θ to be declared unstable, but it is important to do so here.

In the case γ is unstable and $\gamma + 1 < \text{lh}(\mathcal{S})$, we first define $\lambda_{\gamma+1}$ and set

$$\lambda_\gamma = \inf(\lambda_{\gamma+1}, \alpha_\gamma).$$

See below for more detail.

In sum, we are maintaining by induction that the last node γ of our current \S is stable, and

Induction hypotheses $(\dagger)_\gamma$. If $\theta < \gamma$ and θ is unstable, then

- (1) $0 \leq_S \theta$ and $[0, \theta]_S$ does not drop (in model or degree), and every $\xi \leq_S \theta$ is unstable,
- (2) there is a γ such that $(\mathcal{M}_\theta^S, \Lambda_\theta) = (\mathcal{M}_\gamma^U, \Sigma_\gamma^U)$,
- (3) $\Phi_\theta = \langle (\mathcal{M}_\theta^S, \Lambda_\theta), (M_{\theta+1}^S, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$ is a problematic tuple,
- (4) Φ_θ is extender-active iff Φ_0 is extender-active, and if Φ_θ is extender-active, then $i_{0,\theta}^S(\alpha_0) = \alpha_\theta$,
- (5) if $\xi <_S \theta$, then $\alpha_\theta \leq i_{\xi,\theta}^S(\alpha_\xi)$ and $\beta_\theta \leq i_{\xi,\theta}^S(\beta_\xi)$,
- (6) if $\langle \alpha_\theta, \beta_\theta \rangle = i_{\xi,\theta}^S(\langle \alpha_\xi, \beta_\xi \rangle)$, then
 - (a) $\Phi_\theta^- = i_{\xi,\theta}^S(\Phi_\xi^-)$, and
 - (b) $i_{\xi,\theta}^S$ is continuous at $\rho_{k_0}(\mathcal{M}_{\xi+1}^S)$, where $k_0 = \deg(\mathcal{M}_{\xi+1}^S)$,
- (7) $\mathcal{M}_{\theta+1}^T = \mathcal{M}_\theta^T$, and $\pi_{\theta+1} = \pi_\theta \circ \sigma_\theta$.

Setting $\sigma_0 = \bar{\pi}$, we have $(\dagger)_1$.

For a node γ of \mathcal{S} , we write $\mathcal{S}\text{-pred}(\gamma)$ for the immediate \leq_S -predecessor of γ . For γ a node in \mathcal{S} , we set

$$\text{st}(\gamma) = \text{the least stable } \theta \text{ such that } \theta \leq_S \gamma,$$

and

$$\text{rt}(\gamma) = \begin{cases} \mathcal{S}\text{-pred}(\text{st}(\gamma)) & \text{if } \mathcal{S}\text{-pred}(\text{st}(\gamma)) \text{ exists} \\ \text{st}(\gamma) & \text{otherwise.} \end{cases}$$

The construction of \mathcal{S} ends when we reach a stable θ such that

- (I) $(M_{\nu,l}, \Omega_{\nu,l}) \triangleleft (\mathcal{M}_\theta^S, \Lambda_\theta)$, or $(M_{\nu,l}, \Omega_{\nu,l}) = (\mathcal{M}_\theta^S, \Lambda_\theta)$ and $[\text{rt}(\theta), \theta]_S$ drops, or
- (II) $[\text{rt}(\theta), \theta]_S$ does not drop in model or degree and either $(\mathcal{M}_\theta^S, \Lambda_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l})$, or $(M_{\nu,l}, \Omega_{\nu,l})$ is the type 1 core of $(\mathcal{M}_\theta^S, \Lambda_\theta)$.

We need the second alternative in (II) because models on the phalanx side might have type 2. The comparison arguments of §3 and [13] show that if $\langle \nu, l \rangle$ is least such that (II) holds, and $(\mathcal{M}_\theta^S, \Lambda_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l})$, then $(\mathcal{M}_\theta^S, \Lambda_\theta) = (M_{\nu,l}, \Omega_{\nu,l})$,

If case (I) occurs, then we go on to define $\mathcal{S}_{\nu, l+1}$. If case (II) occurs, we stop the construction. We will sometimes write “case (II)(a)” for the first case of (II) and “case (II)(b)” for the second case.

We now describe how to extend \mathcal{S} one more step. First we assume \mathcal{S} has successor length $\gamma + 1$ and let $\mathcal{M}_\gamma^{\mathcal{S}}$ be the current last model, so that γ is stable. Suppose $(\dagger)_\gamma$ holds. Suppose (I) and (II) above do not hold for γ , so that we have a least disagreement between $\mathcal{M}_\gamma^{\mathcal{S}}$ and $M_{\nu, l}$. By the results of §3, i.e. essentially the proof of [13, Lemma 9.6.5], the least disagreement involves only an extender E on the sequence of $\mathcal{M}_\gamma^{\mathcal{S}}$ as long as $\text{crt}(E) < \rho_l(M_{\nu, l})$.⁵⁴ In this case, letting $\tau = \text{lh}(E)$, we have

- $M_{\nu, l}|(\tau, 0) = \mathcal{M}_\gamma^{\mathcal{S}}|(\tau, -1)$,⁵⁵ and
- $(\Omega_{\nu, l})_{(\tau, 0)} = (\Lambda_\gamma)_{(\tau, -1)}$.

Set

$$\lambda_\gamma^{\mathcal{S}} = \lambda_E,$$

and let ξ be least such that $\text{crt}(E) < \lambda_\xi^{\mathcal{S}}$. We let $\mathcal{S}\text{-pred}(\gamma + 1) = \xi$. Let (β, k) be lex least such that either $\rho(\mathcal{M}_\xi^{\mathcal{S}}|(\beta, k)) \leq \text{crt}(E)$ or $(\beta, k) = (\hat{\sigma}(\mathcal{M}_\xi^{\mathcal{S}}), \text{deg}(\mathcal{M}_\xi^{\mathcal{S}}))$. Set $R = \mathcal{M}_\xi^{\mathcal{S}}|(\beta, k)$ and set

$$\mathcal{M}_{\gamma+1}^{\mathcal{S}} = \text{Ult}(R, E^+),$$

and let $\hat{i}_{\xi, \gamma+1}^{\mathcal{S}}$ be the canonical embedding. Let

$$\mathcal{M}_{\gamma+1}^{\mathcal{T}} = \text{Ult}(\mathcal{M}_\xi^{\mathcal{T}}|(\pi_\xi(\beta), k), \pi_\gamma(E^+)),$$

and let $\pi_{\gamma+1}$ be given by the Shift Lemma. This determines $\Lambda_{\gamma+1}$. We write $E_\gamma^{\mathcal{S}}$ for E^+ and similarly we write $E_\gamma^{\mathcal{T}}$ for $\pi_\gamma(E^+)$. Similar notations apply to extenders $E_\gamma^{\mathcal{U}}$ on the tree \mathcal{U} .

If ξ is stable or $(\beta, k) < (\hat{\sigma}(\mathcal{M}_\xi^{\mathcal{S}}), \text{deg}(\mathcal{M}_\xi^{\mathcal{S}}))$, then we declare $\gamma + 1$ to be stable. $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.⁵⁶

If ξ is unstable, $(\beta, k) = (\hat{\sigma}(\mathcal{M}_\xi^{\mathcal{S}}), \text{deg}(\mathcal{M}_\xi^{\mathcal{S}}))$, and $(\mathcal{M}_{\gamma+1}^{\mathcal{S}}, \Lambda_{\gamma+1})$ is not a model of \mathcal{U} , then again we declare $\gamma + 1$ stable. Again, $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

Finally, suppose ξ is unstable, $(\beta, k) = (\hat{\sigma}(\mathcal{M}_\xi^{\mathcal{S}}), \text{deg}(\mathcal{M}_\xi^{\mathcal{S}}))$, and for some τ ,

$$(\mathcal{M}_{\gamma+1}^{\mathcal{S}}, \Lambda_{\gamma+1}) = (\mathcal{M}_\tau^{\mathcal{U}}, \Sigma_\tau^{\mathcal{U}}).$$

⁵⁴The only extender that has critical point $\geq \rho_l(M_{\nu, l})$, if exists, is the order zero measure $F = D(\mathcal{M}_\gamma^{\mathcal{S}})$ witnessing case (II)(b). This happens precisely when some model on the main branch $[\text{rt}(\theta), \theta]_S$ has type 2. In this case, F is the last extender used in the comparison and is the only extender on the $M_{\nu, l}$ -sequence used.

⁵⁵Recall $\mathcal{M}_\gamma^{\mathcal{S}}|(\tau, -1)$ is the structure obtained from $\mathcal{M}_\gamma^{\mathcal{S}}|_\tau$ by removing E .

⁵⁶It is possible that ξ is unstable, $\lambda_\xi = \alpha_\xi$, $\mathcal{S}\text{-pred}(\gamma + 1) = \xi$, and $\text{crt}(E_\gamma^{\mathcal{S}}) = \lambda_F$ where F is the last extender of $\mathcal{M}_\xi^{\mathcal{S}}|_{\alpha_\xi}$. In this case, $(\beta, k) = (\text{lh}(F), 0)$. The problem then is that $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$ is not an lpm, because its last extender $i_{\xi, \gamma+1}^{\mathcal{S}}(F)$ has a missing whole initial segment, namely F . Schindler and Zeman found a way to deal with this anomaly in [7]. Their method works in the hod mouse context as well. Here we shall not go into the details of this case. The anomaly cannot occur when ξ is stable, because then $\lambda_\xi = \hat{\lambda}(E_\xi^{\mathcal{S}})$ is inaccessible in $\mathcal{M}_\gamma^{\mathcal{S}}$.

We then we declare $\gamma + 1$ to be unstable and $\gamma + 2$ stable. We must define the problematic tuple needed for $(\dagger)_{\gamma+2}$. Let $i = i_{\xi, \gamma+1}^S$, and

$$\langle (N, \Psi), (G, \Phi), \sigma, \alpha \rangle = \langle (\mathcal{M}_\xi^S, \Lambda_\xi), (\mathcal{M}_{\xi+1}^S, \Lambda_{\xi+1}), \sigma_\xi, \alpha_\xi \rangle.$$

We have that $\langle (N, \Psi), (G, \Phi), \sigma, \alpha \rangle$ is problematic. We must define $\Phi_{\gamma+1}$ and verify $(\dagger)_{\gamma+2}$. We break into 2 cases. Let $k = \deg(G)$. (So $k = k_0 = \deg(N) = \deg(M)$.)

Case 1. i is continuous at $\rho_k(G)$.

In this case, we simply let

$$\langle \mathcal{M}_{\gamma+2}^S, \alpha_{\gamma+1} \rangle = \langle i(G), i(\alpha) \rangle.$$

We must define $\sigma_{\gamma+1}$. Note that by our case hypothesis,

$$i(\hat{G}^k) = i(\hat{G})^k.$$

Let $\langle \sigma^\eta \mid \eta < \rho_k(G) \rangle$ be the natural decomposition of $\sigma \upharpoonright \hat{G}^k$, and set

$$i(\sigma \upharpoonright \hat{G}^k) = \bigcup_{\eta < \rho_k(G)} i(\sigma^\eta).$$

Using the continuity of i at $\rho_k(G)$, it is easy to see that $i(\sigma \upharpoonright \hat{G}^k)$ is Σ_0 -elementary and cardinal preserving from $i(\hat{G})^k$ to $i(\hat{N})^k$. We set

$$\sigma_{\gamma+1} = \text{completion of } i(\sigma \upharpoonright \hat{G}^k) \text{ via upward extension of embeddings,}$$

and

$$\Lambda_{\gamma+2} = \Lambda_{\gamma+1}^{\sigma_{\gamma+1}}.$$

This defines $\Phi_{\gamma+1}$, that is

$$\Phi_{\gamma+1} = \langle (i(N), \Lambda_{\gamma+1}), (i(G), \Lambda_{\gamma+2}, i(\sigma), i(\alpha)) \rangle.$$

Abusing notation a bit, let us write

$$\Phi_{\gamma+1}^- = \langle i(N), i(G), i(\sigma), i(\alpha) \rangle.$$

We must see that $\Phi_{\gamma+1}$ is problematic. First, it satisfies the hypotheses of the Condensation Theorem 4.6. For G is α -sound, so $i(G)$ is $i(\alpha)$ -sound. (Note $G \in N$, so $i \upharpoonright G$ is Σ_ω -elementary.) By downward extension of embeddings (cf. [6, Lemma 3.3]), $i(\sigma \upharpoonright \hat{G}^k)$ extends to a unique embedding from some K into $i(N)$, and it is easy to see that $K = i(G)$, because $i(G)$ is k -sound, and that the embedding in question is what we have called $i(\sigma)$. $i(\sigma)$ is nearly elementary: it maps parameters correctly, and $i(\sigma) \upharpoonright i(\hat{G})^k$ is Σ_0 -elementary and cardinal preserving by construction.

Finally, $\text{crt}(i(\sigma)) = i(\alpha)$, because for all sufficiently large $\eta < \rho_k(G)$, $\alpha + 1 \subseteq \text{dom}(\sigma^\eta)$ and $\text{crt}(\sigma^\eta) = \alpha$, so $\text{crt}(i(\sigma^\eta)) = i(\alpha)$.

So we must see that one of the conclusions of 4.6 fails. We show that the conclusion that failed for Φ_ξ fails for $\Phi_{\gamma+1}$.

Suppose first that Φ_ξ^- is problematic. We break into cases. If $E_\alpha^N = \emptyset$ and G is of type 1, then $\neg G \triangleleft N$. But then $E_{i(\alpha)}^{i(N)} = \emptyset$, $i(G)$ is of type 1, and $\neg i(G) \triangleleft i(N)$, so $\Phi_{\gamma+1}^-$ is problematic. If G is of type 1 and $E_\alpha^N \neq \emptyset$, then $\neg G \triangleleft \text{Ult}(N, E_\alpha^N)$. But then $i(G)$ is of type 1, $E_{i(\alpha)}^{i(N)} \neq \emptyset$, and $\neg i(G) \triangleleft \text{Ult}(i(N), E_{i(\alpha)}^{i(N)})$, so $\Phi_{\gamma+1}^-$ is problematic.⁵⁷ If G is of type 2, since Φ_ξ^- is problematic, no $L \triangleleft N$ (or if $E_\alpha^N \neq \emptyset$, no $L \triangleleft \text{Ult}_0(N, E_\alpha^N)$) is such that $L = C(G)$. By elementarity of i the same fact holds for $i(G)$ and therefore, $\Phi_{\gamma+1}^-$ is problematic.

So we may assume Φ_ξ^- is not problematic, and hence also $\Phi_{\gamma+1}^-$ is not problematic. Suppose first $G \triangleleft N$, so $i(G) \triangleleft i(N)$. We must show $\Lambda_{\gamma+1}^{i(\sigma)} \neq (\Lambda_{\gamma+1})_{i(G)}$, so suppose otherwise. Using Claim 4 we get $\Lambda_\xi = \Lambda_{\gamma+1}^i$, so

$$\begin{aligned} \Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i \circ \sigma} \\ &= (\Lambda_{\gamma+1})^{i(\sigma) \circ i} \\ &= (\Lambda_{\gamma+1}^{i(\sigma)})^i \\ &= ((\Lambda_{\gamma+1})_{i(G)})^i \\ &= (\Lambda_\xi)_G, \end{aligned}$$

a contradiction. Equation 2 holds because $i \circ \sigma = i(\sigma) \circ i$, and equation 5 comes from equation 4 using claim 4 again. Thus Φ_ξ is not problematic, contradiction.

Suppose next $G \triangleleft \text{Ult}(N, E_\alpha)$. We want to show $(\Lambda_\xi)_{\langle E_\alpha, G \rangle} = \Lambda_\xi^\sigma$. This comes from the following calculations (see Figure 1).

$$\begin{aligned} \Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i \circ \sigma} \\ &= (\Lambda_{\gamma+1})^{i(\sigma) \circ i} \\ &= ((\Lambda_{\gamma+1})_{\langle i(E_\alpha), i(G) \rangle})^i \\ &= (\Lambda_\xi)_{\langle E_\alpha, G \rangle}. \end{aligned}$$

This first equality follows from Claim 4. The third equality follows from the assumption that $\Phi_{\gamma+1}$ is not problematic. The last equality follows from the fact that $\Lambda_\xi = (\Lambda_{\gamma+1})^i$, so the tail strategy $(\Lambda_{\gamma+1})_{\langle i(E_\alpha), i(N) \rangle}$ is pulled back by the Shift lemma map from $\text{Ult}(N, E_\alpha)$ to $\text{Ult}(i(N), i(E_\alpha))$, which is just i .

⁵⁷In both cases, i has enough elementarity to preserve these facts. For instance, if $\neg(G \triangleleft \text{Ult}(N, E_\alpha^N))$, then this is equivalent to $\neg(G \triangleleft \text{Ult}(N | (\alpha^+)^N, E_\alpha^N))$, and i preserves this fact.

$$\begin{array}{ccccc}
\text{Ult}(i(N), i(E_\alpha)) & \triangleright & i(G) & \xrightarrow{i(\sigma)} & i(N) \\
\uparrow i & & \uparrow i & & \uparrow i \\
\text{Ult}(N, E_\alpha) & \triangleright & G & \xrightarrow{\sigma} & N
\end{array}$$

Figure 1: Diagram illustrating that $(\Lambda_\xi)_{\langle E_\alpha, G \rangle} = \Lambda_\xi^\sigma$.

$$\begin{array}{ccccc}
i(L) & \xrightarrow{l} & i(G) & \xrightarrow{i(\sigma)} & i(N) \\
\uparrow i & & \uparrow i & & \uparrow i \\
L & \xrightarrow{j} & G & \xrightarrow{\sigma} & N
\end{array}$$

Figure 2: Diagram illustrating that $\Lambda_\xi^{\sigma \circ j} = (\Lambda_\xi)_L$.

Suppose next that G is of type 2. Suppose $L \triangleleft N$ is such that $C(G) = L$. (Then case that L is an ultrapower away from N is similar.) Let $j: L \rightarrow G$ be the ultrapower map, and let $l = i(j)$, so that $l: i(L) \rightarrow i(G) = \text{Ult}_0(i(L), i(D))$. $i(G)$ has type 2, so since $\Phi_{\gamma+1}$ is not problematic, $i(L) \triangleleft i(N)$ and $(\Lambda_{\gamma+1})_{i(L)} = \Lambda_{\gamma+1}^{i(\sigma) \circ l}$. We wish to show that $(\Lambda_\xi)_L = \Lambda_\xi^{\sigma \circ j}$.⁵⁸ But we can calculate

$$\begin{aligned}
\Lambda_\xi^{\sigma \circ j} &= \Lambda_{\gamma+1}^{i \circ \sigma \circ j} \\
&= \Lambda_{\gamma+1}^{i(\sigma) \circ l \circ i} = (\Lambda_{\gamma+1}^{i(\sigma) \circ l})^i \\
&= ((\Lambda_{\gamma+1})_{i(L)})^i \\
&= (\Lambda_\xi)_L.
\end{aligned}$$

Line 1 uses Claim 4, as does the step from line 3 to line 4. Line 2 uses the commutativity of the diagram in Figure 2. As we noted, line 3 holds because $\Phi_{\gamma+1}$ is not problematic.

Thus $\langle \mathcal{M}_{\gamma+1}^S, \mathcal{M}_{\gamma+2}^S, \sigma_{\gamma+1}, \alpha_{\gamma+1} \rangle$ is problematic. Setting

$$\mathcal{M}_{\gamma+2}^T = \mathcal{M}_{\gamma+1}^T \text{ and } \pi_{\gamma+2} = \pi_{\gamma+1} \circ \sigma_{\gamma+1},$$

the rest of $(\dagger)_{\gamma+2}$ is clear.

Case 2. i is discontinuous at $\rho_k(G)$.

Set $\kappa = \text{crt}(E_\gamma^S)$. In case 2, $\rho_k(G)$ has cofinality κ in N . Since $\rho(G) \leq \alpha$ and G is α -sound, we have a Σ_1 over \hat{G}^k map of α onto $(\alpha^+)^G$. Ramifying this map, we see that $(\alpha^+)^G$ also has cofinality κ in N . Here recall that $\rho(G) = \rho_1(\hat{G}^k)$ and $p(G) = p_1(\hat{G}^k)$.

⁵⁸As we remarked in 3.45, $\Lambda_{\gamma+1}^{i(\sigma) \circ l}$ and $\Lambda_\xi^{\sigma \circ j}$ are indeed level k strategies.

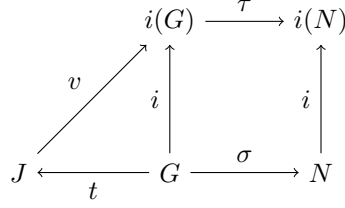


Figure 3: Lift up of (N, G, σ, α) in the case i is discontinuous at $\rho_k(G)$

Let $\langle (\sigma^\eta, G^\eta) \mid \alpha \leq \eta < \rho_k(G) \rangle$ be the natural decomposition of $\sigma \upharpoonright \hat{G}^k$.⁵⁹ Let $s = p(G) - \alpha$, so that

$$\text{dom}(\sigma^\eta) = h_{G^\eta}^1 \text{``}(\alpha \cup s).$$

Let

$$\bar{\tau} = \bigcup_{\eta < \rho_k(G)} i(\sigma^\eta).$$

The domain of $\bar{\tau}$ is no longer all of $i(\hat{G}^k)$, instead

$$\text{dom}(\bar{\tau}) = \bigcup_{\eta < \rho_k(G)} h_{i(G^\eta)}^1 \text{``}(i(\alpha) \cup i(s)).$$

But set

$$J = \text{Ult}(G, E_i \upharpoonright \text{sup } i \text{``}\alpha),$$

and let $t: G \rightarrow J$ be the canonical embedding, and $v: J \rightarrow i(G)$ be the factor map. This is a Σ_k ultrapower, by what may be a long extender. That is, J is the decoding of $\hat{J}^k = \text{Ult}_0(\hat{G}^k, E_i \upharpoonright \text{sup } i \text{``}\alpha)$. Note that t is continuous at α , because α is regular in G (because $\alpha = \text{crt}(\sigma)$), and $\alpha < \rho_k(G)$.

We claim that $\text{ran}(v \upharpoonright \hat{J}^k) \subseteq \text{dom}(\bar{\tau})$. For let $f \in \hat{G}^k$ and $b \subseteq \text{sup } i \text{``}\alpha$ be finite, so that $t(f)(b)$ is a typical element of \hat{J}^k , and $v(t(f)(b)) = i(f)(b)$. We can find $\eta < \rho_k(G)$ such that $f \in \text{dom}(\sigma^\eta)$ and $\eta > \alpha$, so that $i(f) \in \text{dom}(i(\sigma^\eta))$ and $b \subseteq i(\eta)$. Since $f \text{``}(\alpha) \subseteq \text{dom}(\sigma^\eta)$, $i(f) \text{``}i(\alpha) \subseteq \text{dom}(i(\sigma^\eta))$, so $i(f)(b) \in \text{dom}(\bar{\tau})$, as desired.

Let τ be the extension of $\bar{\tau}$ given by: for $a \subseteq \text{sup } i \text{``}\rho_k(G)$ finite,

$$\tau(\tilde{h}_{i(G)}^{k+1}(a, p(i(G)))) = \tilde{h}_{i(N)}^{k+1}(\bar{\tau}(a), p(i(N))).$$

It is easy to check that $\text{ran}(v) \subseteq \text{dom}(\tau)$.

This gives us the diagram in Figure 3.

The map τ here is only partial on $i(G)$, but $\tau \circ v: J \rightarrow i(N)$ is total. Also, $i \text{``}G \subseteq \text{dom}(\tau)$, so $\tau \circ i$ is total on G . For each $\eta < \rho_k(G)$, and $x \in \text{dom}(\sigma^\eta)$,

$$i \circ \sigma^\eta(x) = i(\sigma^\eta)(i(x)),$$

⁵⁹We encourage the reader to focus on the case $k = 0$, which has the main ideas.

so $\tau \circ i$ agrees on \hat{G}^k with $i \circ \sigma$. Since both map $p_k(G)$ to $p_k(i(N))$,

$$\tau \circ i = i \circ \sigma.$$

Clearly $i \upharpoonright G = v \circ t$, so the diagram commutes.

Since $E_i \upharpoonright \sup i^{\alpha}$ is a long extender, we need some care to show that J is a premouse. The worry is that it could be a protomouse, in the the case that G is extender active and $k = 0$. So suppose $k = 0$ and $\mu = \text{crt}(\dot{F}^G)$; it is enough to see that t is continuous at $\mu^{+,G}$. If not, we have $f \in G$ and $b \subseteq \sup i^{\alpha}$ finite such that

$$\sup i^{\alpha} \mu^{+,G} < t(f)(b) < i(\mu^{+,G}).$$

We may assume $\text{dom}(f) = \gamma^{|b|}$, where $\gamma < \alpha$, and by Los, $\text{ran}(f)$ is unbounded in $\mu^{+,G}$. It follows that $\mu^{+,G} < \alpha$, so $\mu^{+,G} = \sigma(\mu^{+,G}) = \mu^{+,N}$. But $\text{cof}(\mu^{+,G}) = \text{cof}(\hat{\sigma}(G)) = \kappa$ in N , so $\mu^{+,G}$ is not a cardinal in N , contradiction.

Thus J is a premouse. We claim that $\tau \circ v$ is nearly elementary. First, τ is a partial Σ_0 map from $i(\hat{G}^k)$ to $i(\hat{N}^k)$, so $\tau \circ v \upharpoonright \hat{J}^k$ is Σ_0 from \hat{J}^k to $i(\hat{N}^k)$. It is also easy to see that $\tau \circ v \upharpoonright \hat{J}^k$ is cardinal preserving.

For $\eta < \rho_k(G)$, we have that σ^η is the identity on $\alpha \cap \eta$, so

$$\sup i^{\alpha} = \sup t^{\alpha} \leq \text{crt}(\tau \circ v).$$

But $\alpha < \sigma(\alpha)$, so $i(\alpha) < i \circ \sigma(\alpha) = \tau \circ v \circ t(\alpha)$. Also, $t(\alpha) \leq i(\alpha)$, so $t(\alpha) < \tau \circ v(t(\alpha))$. Thus $\text{crt}(\tau \circ v) \leq t(\alpha)$, and since $t(\alpha) = \sup t^{\alpha}$, we get

$$\text{crt}(\tau \circ v) = \sup i^{\alpha} = t(\alpha).$$

Remark 4.10. It is possible that i is continuous at α , even though it is discontinuous at $\rho_k(G)$ and hence $J \neq i(G)$. In this case $\text{crt}(v) > i(\alpha) = \sup i^{\alpha}$ and $\text{crt}(\tau) = i(\alpha) = t(\alpha)$. \dashv

We set

$$\begin{aligned} \mathcal{M}_{\gamma+1}^S &= i(N), \\ \mathcal{M}_{\gamma+2}^S &= J, \\ \sigma_{\gamma+1} &= \tau \circ v, \text{ and} \\ \alpha_{\gamma+1} &= \text{crt}(\tau \circ v). \end{aligned}$$

We define then

$$\Phi_{\gamma+1} = \langle (i(N), \Lambda_{\gamma+1}), (J, \Lambda_{\gamma+1}^{\sigma_{\gamma+1}}), \sigma_{\gamma+1}, \alpha_{\gamma+1} \rangle.$$

Claim 5. $\Phi_{\gamma+1}$ is problematic.

Proof. We begin with proving a subclaim.

Subclaim 5a.

- (a) If Φ_ξ^- is problematic, then $\Phi_{\gamma+1}^-$ is problematic.
- (b) Φ_ξ is extender-active iff $\Phi_{\gamma+1}$ is extender-active; moreover, if Φ_ξ is extender-active, then $i_{\xi, \gamma+1}^S(\alpha_\xi) = \alpha_{\gamma+1}$.

Proof. Recall that we write

$$\Phi_{\gamma+1}^- = \langle i(N), J, \tau \circ v, \text{crt}(\tau \circ v) \rangle.$$

It is easy to see that the tuple obeys the hypotheses of 4.6: clearly J is Σ_{k+1} generated by $\sup i^{\text{``}\alpha \cup t(s)}$, and as we showed above, $\text{crt}(\tau \circ v) = \sup i^{\text{``}\alpha}$. t preserves the solidity of s , so $t(s) = p(J) - \sup i^{\text{``}\alpha}$. We have shown that $\tau \circ v$ is nearly elementary. Since $i(G) \in i(N)$, we have $J \in i(N)$, so J is not the $\text{crt}(\tau \circ v)$ -core of $i(N)$.

So what we need to see is that neither of the conclusions (a)-(b) of 4.6 hold for $\langle i(N), J, \tau \circ v, \text{crt}(\tau \circ v) \rangle$, and that $\Phi_{\gamma+1}^-$ is extender-active iff Φ_ξ^- is extender-active. We break into two cases.

Case A. Φ_ξ is not extender-active.

Let us show that $\Phi_{\gamma+1}^-$ is not extender active. For this, we must see that $\text{crt}(\tau \circ v)$ is not an index in $i(N)$. There are two cases. If i is continuous at α , then $\text{crt}(v) > i(\alpha)$ and $\text{crt}(\tau) = i(\alpha)$, so $\text{crt}(\tau \circ v) = i(\alpha)$, which is not an index in $i(N)$. Otherwise, $\text{crt}(\tau \circ v) = \text{crt}(v) = \sup i^{\text{``}\alpha}$. But then $\sup i^{\text{``}\alpha}$ has cofinality κ in $i(N)$, and since κ is a limit cardinal in $i(N)$, it is not the cofinality of the index of a total extender in $i(N)$.

Now we show that $\Phi_{\gamma+1}^-$ is problematic. Suppose toward contradiction that it is not; that is, that

$$C(J) \triangleleft i(N).$$

We have $E_\alpha^N = \emptyset$. Since $(N, \Psi) \leq^* (\mathcal{M}_\xi^T, \Sigma_\xi^T) \leq^* (M, \Sigma)$, Claim 3 gives

$$G|(\alpha^+)^G = N||(\alpha^+)^G.$$

Since $G \in N$, there is a first level P of N such that $P||(\alpha^+)^G = G|(\alpha^+)^G$ and $\rho(P) \leq \alpha$. Because Φ_ξ^- is problematic, $P \neq G$ and if G is of type 2, $G \neq \text{Ult}(P, D)$ where D is the Mitchell order 0 measure on $\hat{\rho}^-(G)$. Letting $n = \text{deg}(P)$, we get by the argument above that in N , $\rho_n(P)$ has the same cofinality as $(\alpha^+)^P = (\alpha^+)^G$, namely κ .

We set

$$Q = \text{Ult}(P, E_i \upharpoonright \sup i^{\text{``}\alpha}),$$

and let $t_0: P \rightarrow Q$ be the canonical embedding, and $v_0: Q \rightarrow i(P)$ be the factor map.⁶⁰ We wish to show that $C(Q) \triangleleft i(N)$. Let us assume that $Q \neq i(P)$, as otherwise this is trivial. This implies that $v_0 \neq \text{id}$ and $\text{crt}(v_0) < \rho_n(Q)$.

⁶⁰More precisely, Q is decoded from $\text{Ult}_0(\hat{P}^n, E_i \upharpoonright \sup i^{\text{``}\alpha})$.

Again, we must see that Q is not a protomouse, in the case P is extender active with $\text{crt}(\dot{F}^P) = \mu$, and $n = 0$. The proof is the same as it was for J . Namely, if $\alpha \leq \mu^{+,P}$ then t_0 is continuous at $\mu^{+,P}$ because any $f \in P$ such that $f: [\gamma]^{|\text{bl}|} \rightarrow \mu^{+,P}$ for $\gamma < \alpha$ has bounded range. If $\mu^{+,P} < \alpha$, then because $P|_{\alpha^{+,P}} = G|_{\alpha^{+,G}}$, $\mu^{+,P}$ is a cardinal of G , and hence of N , so i is continuous at $\mu^{+,P}$, so t_0 is continuous at $\mu^{+,P}$.

It is easy to check that the hypotheses of 4.6 hold for $\langle (i(P), \Omega), (Q, \Omega^{v_0}), v_0, \text{crt}(v_0) \rangle$, where $\Omega = (\Lambda_{\gamma+1})_{i(P)}$. Note here that

$$\sup i^{\text{``}\alpha} = t_0(\alpha) \leq \text{crt}(v_0).$$

t_0 is continuous at α and $\alpha^{+,P}$, and i is discontinuous at $\alpha^{+,P}$ because $\text{cf}^N(\alpha^{+,P}) = \kappa$. Thus

$$\text{crt}(v_0) = \begin{cases} t_0(\alpha) & \text{if } i \text{ is discontinuous at } \alpha, \\ t_0(\alpha)^{+,Q} & \text{otherwise.} \end{cases}$$

Let $s_0 = p(P) \setminus \alpha$; then $P = h_P^{n+1} \text{``}\alpha \cup s_0$ because P is sound and $\rho(P) \leq \alpha$. Thus

$$Q = h_Q^{n+1} \text{``}(\sup i^{\text{``}\alpha} \cup t_0(s_0)).$$

Moreover, t_0 maps the solidity witnesses for s_0 to solidity witnesses for $t_0(s_0)$, so

$$Q \text{ is } \text{crt}(v_0)\text{-sound,}$$

with parameter $t_0(s) \setminus \text{crt}(v_0)$. Also,

$$\rho(Q) \leq t_0(\alpha) \leq \text{crt}(v_0) \leq t_0(\alpha^{+,P}) \leq \rho_n(Q),$$

where the last inequality comes from $\rho_n(Q) = \sup t_0 \text{``}\rho_n(P)$ and the fact that t_0 is continuous at $\alpha^{+,P}$. It is easy to verify that v_0 is nearly elementary. Finally, $i(P)$ is sound, so Q cannot be its $\text{crt}(v_0)$ -core.

Thus the hypotheses of 4.6 hold for $\langle (i(P), \Omega), (Q, \Omega^{v_0}), v_0, \text{crt}(v_0) \rangle$. But note

$$(i(P), \Omega) \triangleleft (i(N), \Lambda_{\gamma+1}) \leq^* (\mathcal{M}_{\gamma+1}^T, \Sigma_{\gamma+1}^T) \leq^* (M, \Sigma).$$

So because (M, Σ) is minimal, one of the conclusions of 4.6 holds for $Q, i(P)$. However, $t_0(\alpha)$ is not an index in $i(N)$ because α is not an index in N , and $t_0(\alpha)^{+,Q}$ is not an index in $i(N)$ because it has cofinality κ in $i(N)$, and κ is a limit cardinal in $i(N)$. So

$$C(Q) \triangleleft i(P) \triangleleft i(N).$$

So $C(J)$ and $C(Q)$ are each the first level of $i(N)$ collapsing $t(\alpha)^{+,J} = t_0(\alpha)^{+,Q}$ to $t(\alpha) = t_0(\alpha)$, so

$$C(J) = C(Q).$$

It follows that $n = k$. We get a contradiction by pulling back to G and P .

Case (i). J, Q are both of type 1.

Then $J = Q$, $t \upharpoonright \alpha = t_0 \upharpoonright \alpha$, and $t(s) = t_0(s_0) = p(J) - t(\alpha)$. It follows that $G = P$, so $G \triangleleft N$, contradicting our assumption that $\Phi_{\bar{\xi}}$ is problematic.

There are some bothersome details in the remaining cases. Some arise from the possibility that t and t_0 do not preserve ρ_k . In that connection, the following lemma is useful. It is implicit (very nearly explicit) in the proof of solidity and universality for pfs mice in [13, Section 4.10].

Lemma 4.11. *(AD⁺) Let (R, Σ) be a sound lbr hod pair of degree k , and suppose that either R has type 1A, or R has type 1B and is not projectum stable; then R_0^k is sound, in the sense that*

$$R_0^k = \text{Hull}_1^{R_0^k}(\rho_1(R_0^k) + 1 \cup p_1(R_0^k)).$$

Proof. (Sketch.) This is implicit in the proof of Theorem 4.10.9 in [13], and in fact it is pointed out in the last 3 lines of that proof. The problem is that R_0^k does not have a name for $\rho_k(R)$ in its language, whereas R^k does.

Let $\rho_k = \rho_k(R)$, $\rho = \rho_{k+1}(R)$, and if R has type 1B, let $\bar{\mathfrak{C}}_k(R)$ be its strong core and $i: \bar{\mathfrak{C}}_k(R) \rightarrow R = \text{Ult}(\bar{\mathfrak{C}}_k(R), D)$ be the canonical embedding. Let

$$\varepsilon = \begin{cases} \text{least } \gamma \text{ s.t. } \rho_k = h_{R^{k-1}}^1(\gamma, p_k(R)) & \text{if } R \text{ has type 1A,} \\ \text{least } \gamma \text{ s.t. } i(D) = h_{R^{k-1}}^1(\gamma, p_k(R)) & \text{if } R \text{ has type 1B.} \end{cases}$$

Our hypotheses on R imply that ε exists⁶¹, and $\varepsilon \in \text{Hull}_1(\rho \cup r)$, where $r = p_1(R_0^k)$.⁶² Letting $p = p_1(R^k)$, this enables us to show that $p \subseteq r$, and to translate $\Sigma_1^{R^k}$ definitions using parameters in $\rho + 1 \cup p$ into $\Sigma_1^{R_0^k}$ definitions using parameters in $\rho + 1 \cup r$.⁶³ It follows that

$$\text{Hull}_1^{R_0^k}(\rho + 1 \cup r) = \text{Hull}_1^{R_0^k}(\rho + 1 \cup p).$$

Since we have assumed that R^k is 1-sound, this is what we need. □

Let us return to Subclaim 5a.

Case (ii). Q has type 1 and J has type 2.

In this case, G must have type 1B or type 2. Suppose first it has type 1B. The relevant diagram is

⁶¹See [13, 4.10.8].

⁶²See [13, 4.10.9, Claim 1].

⁶³See Claims 2 and 3 in the proof of [13, 4.10.9].

$$\begin{array}{ccc}
J & \xleftarrow{t_1(i_D)} & C(J) & = & Q \\
\uparrow t & & \uparrow t_1 & & \uparrow t_0 \\
G & \xleftarrow{i_D} & \bar{\mathfrak{C}}_k(G) & & P
\end{array}$$

Here $i_D: \bar{\mathfrak{C}}_k(G) \rightarrow G$ is the ultrapower by the order zero measure D , and t_1 is the canonical embedding associated to $\text{Ult}(\bar{\mathfrak{C}}_k(G), E_i \upharpoonright \sup i^{\alpha})$. t_1 agrees with t and t_0 on $\alpha^{+,P}$.

Since $C(J)$ is type 1A, Q is type 1A, so P is type 1A. Thus P_0^k is sound by Lemma 4.11. Since G is type 1B and J is type 2, G is not projectum stable, so G_0^k is α -sound by the lemma. Letting $r = p_1(G_0^k) - \alpha$ and $w = p_1(P_0^k) - \alpha$, we get that $t_1(r)$ is the top part of $p_1(C(J)_0^k)$, hence the top part of $p_1(Q_0^k) = t_0(w)$, so $t_1(r) = t_0(w)$. Since t_0 and t_1 agree on $\alpha^{+,P}$, we get that $\bar{\mathfrak{C}}_k(G) = P$. This is a contradiction, because $\rho_k(\bar{\mathfrak{C}}_k(G))$ is measurable in $\bar{\mathfrak{C}}_k(G)$, so $\bar{\mathfrak{C}}_k(G)$ is not a pfs premeasure.

Suppose next that G has type 2. Now the relevant diagram is

$$\begin{array}{ccc}
J & \xleftarrow{t_1(i_D)} & C(J) & = & Q \\
\uparrow t & & \uparrow t_1 & & \uparrow t_0 \\
G & \xleftarrow{i_D} & C(G) & & P
\end{array}$$

The proof in the type 1B case now tells us that $C(G) = P$. That means that Φ_{ξ}^- is not problematic, again a contradiction.

Case (iii). Q has type 2.

In this case, P has type 1B and is not projectum stable. By the Lemma, P_0^k is sound. We take subcases on the type of G .

If G has type 1A, the relevant diagram is

$$\begin{array}{ccccc}
J & = & C(Q) & \xrightarrow{t_0(i_D)} & Q \\
\uparrow t & & \uparrow t_1 & & \uparrow t_0 \\
G & & \bar{\mathfrak{C}}_k(P) & \xrightarrow{i_D} & P
\end{array}$$

As above, the agreement between t and t_0 and the fact that $J = C(Q)$ implies that $G = \bar{\mathfrak{C}}_k(P)$. This is impossible because $\bar{\mathfrak{C}}_k(P)$ is not a pfs premeasure.

If G has type 1B, then J must have type 2, since otherwise $J = C(J) = C(Q)$, so J has type 1A, so G has type 1A. So our diagram is

$$\begin{array}{ccc}
J & \xleftarrow{t_1(i_U)} & C(J) & = & C(Q) & \xrightarrow{t_0(i_D)} & Q \\
\uparrow t & & \uparrow t_1 & & \uparrow t_2 & & \uparrow t_0 \\
G & \xleftarrow{i_U} & \bar{\mathfrak{C}}_k(G) & & \bar{\mathfrak{C}}_k(P) & \xrightarrow{i_D} & P
\end{array}$$

G is not projectum stable, so G_0^k is α -sound. The agreement of t_1 and t_2 on $\alpha^{+,P}$ then implies that $G_0^k = P_0^k$, hence $\bar{\mathfrak{C}}_k(G) = \bar{\mathfrak{C}}_k(P)$, and hence $G = P$. So Φ_ξ^- is not problematic, contradiction.

If G has type 2, the argument above shows that $C(G) = \bar{\mathfrak{C}}_k(P)$. But this is impossible, because $C(G)$ is a pfs premouse and $\bar{\mathfrak{C}}_k(P)$ is not.

Cases (i)-(iii) exhaust the possibilities, so we have proved Subclaim 5a in the case that Φ_ξ is not extender active.

Case B. Φ_ξ is extender-active.

In this case $\sup i^{\alpha} = i(\alpha)$, because α has cofinality $\text{crt}(E_\alpha^N)^{+,N}$ in N . So $i(\alpha)$ is an index in $i(N)$, say of F . Moreover, $i(\alpha) = \text{crt}(\tau \circ v)$, so we have (b) of the subclaim.

Let $R = \text{Ult}(N, E_\alpha^N)$. $G|\alpha^{+,G}$ is an initial segment of R by $(*)(N)$. If $\alpha^{+,R} = \alpha^{+,G}$, then $i(\alpha)^{+,J} = i(\alpha)^{+, \text{Ult}(i(N), F)}$, where $F = i(E_\alpha^N)$, so $\langle i(N), J, \tau \circ v, i(\alpha) \rangle$ is problematic. (Since $\text{crt}(v) > i(\alpha)$, $i(\alpha) = \text{crt}(\tau)$.) Otherwise, we have a first initial segment P of R past $\alpha^{+,G}$ that projects to or below α . We can now use P just as we did in Case A, thereby proving (a) of the subclaim.

This finishes the proof of Subclaim 5a. \square

We are trying to show $\Phi_{\gamma+1}$ is problematic, so by Subclaim 5a, we may assume that Φ_ξ^- and $\Phi_{\gamma+1}^-$ are not problematic. Suppose for example that $G \triangleleft N$ and that $J \triangleleft i(N)$ (so both G, J are of type 1). The relevant diagram is

$$\begin{array}{ccccc}
J & \xrightarrow{v} & i(G) & \xrightarrow{\tau} & i(N) \\
& \searrow t & \uparrow i & & \uparrow i \\
& & G & \xrightarrow{\sigma} & N
\end{array}$$

Since Φ_ξ is problematic, $\Lambda_\xi^\sigma \neq (\Lambda_\xi)_G$. If $\Phi_{\gamma+1}$ is not problematic, then

$$(\Lambda_{\gamma+1})_J = \Lambda_{\gamma+1}^{\tau \circ v}. \quad (4.1)$$

Because $(i(G), (\Lambda_{\gamma+1})_{i(G)}) <^* (M, \Sigma)$, we can apply our induction hypothesis that Theorem 4.6 is true to $v: J \rightarrow i(G)$. Let

$$\Psi = (\Lambda_{\gamma+1})_{i(G)}.$$

Since v is nearly elementary and J is a premouse, Lemma 9.2.3 of [13] implies that (J, Ψ^v) is an lbr hod pair and v is a nearly elementary map of it into $(i(G), \Psi)$ with $\text{crt}(v) \geq \rho(J)$. The arguments

above show that the other hypotheses of 4.6 are satisfied, so we have

$$\Psi_J = \Psi^v,$$

in other words

$$((\Lambda_{\gamma+1})_{i(G)})^v = ((\Lambda_{\gamma+1})_{i(G)})_J = (\Lambda_{\gamma+1})_J. \quad (4.2)$$

The second line here follows from the fact that $\Lambda_{\gamma+1}$ is mildly positional.⁶⁴

We now have:

$$\begin{aligned} \Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i \circ \sigma} \\ &= (\Lambda_{\gamma+1})^{\tau \circ v \circ t} \\ &= (\Lambda_{\gamma+1}^{\tau \circ v})^t \\ &= ((\Lambda_{\gamma+1})_J)^t \\ &= ((\Lambda_{\gamma+1})_{i(G)}^v)^t \\ &= (\Lambda_{\gamma+1})_{i(G)}^i \\ &= (\Lambda_\xi)_G. \end{aligned}$$

The first equality follows from Claim 4. The second equality follows from the fact that $i \circ \sigma = \tau \circ v \circ t$. The fourth equality follows from Equation 4.1. The fifth equality follows from Equation 4.2. We have shown $\Lambda_\xi^\sigma = (\Lambda_\xi)_G$. This contradicts our assumption that Φ_ξ is problematic, and finishes the proof of Claim 5 in the case that $G \triangleleft N$ and $J \triangleleft i(N)$.

The following subclaim will help deal with the type 2 case.

Subclaim 5b. Suppose $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ is a tuple satisfying (1)–(3) of Definition 4.8. Suppose G is of type 2. Let $n = \deg(G)$, D be the measure of Mitchell order 0 on $\hat{\rho}_n(G)$ such that $G = \text{Ult}(C(G), D)$, and let i_D be the ultrapower map. Suppose $C(G) \triangleleft N$ and $\Phi_{C(G)} = \Phi^{\sigma \circ i_D}$. Then

$$(\Phi_{C(G)})_{\langle D \rangle, G} = \Lambda.$$

Proof. Consider the diagram in Figure 4, where we let H be $\text{Ult}(G, i_D(D))$ and j be the corresponding ultrapower map, and Y be the image of H under the copy construction. We let $k : N \rightarrow Y$ be the ultrapower map by $\sigma(i_D(D))$.

⁶⁴See [13, Definition 3.6.1] for the definition of mild positionality. [13, Lemma 4.6.10] shows that background-induced strategies are mildly positional.

$$\begin{array}{ccc}
N & \xrightarrow{k} & Y \\
\sigma \uparrow & & \sigma \uparrow \\
G & \xrightarrow{j} & H \\
i_D \uparrow & & j \uparrow \\
C(G) & \xrightarrow{i_D} & G
\end{array}$$

Figure 4: Diagram illustrating that $(\Phi_{C(G)})_{\langle D \rangle, G} = \Lambda$.

Let $\Sigma = \Phi_{\langle \sigma(i_D(D)) \rangle, Y}$ be the tail of Σ on Y . We have

$$\begin{aligned}
(\Phi_{C(G)})_{\langle D \rangle, G} &= (\Sigma)^{\sigma \circ j} \\
&= \Sigma^{k \circ \sigma} \\
&= \Lambda
\end{aligned}$$

The first equality follows from the definition of pullback strategies on stacks. The second equality follows from commutativity, i.e. $\sigma \circ j = k \circ \sigma$. The third equality follows from our assumption that $\Lambda = \Phi^\sigma$ and the definition of Σ . □

Returning to the proof of Claim 5, i.e. that $\Phi_{\gamma+1}$ is problematic, let us consider the case that $G \triangleleft N$ and J has type 2, so that $J = \text{Ult}(\bar{J}, D)$, where $\bar{J} = C(J) \triangleleft i(N)$ and D is the measure of Mitchell order 0 on $\hat{\rho}_J^k$. The relevant diagram is

$$\begin{array}{ccccccc}
\bar{J} & \xrightarrow{i_D} & J & \xrightarrow{v} & i(G) & \xrightarrow{\tau} & i(N) \\
& & & \swarrow t & \uparrow i & & \uparrow i \\
& & & & G & \xrightarrow{\sigma} & N
\end{array}$$

Again we assume toward contradiction that $\Phi_{\gamma+1}$ is not problematic, so that $(\Lambda_{\gamma+1})_{\bar{J}} = \Lambda_{\gamma+1}^{\tau \circ v \circ i_D}$. Combined with Subclaim 5b, applied with $\Phi = \Lambda_{\gamma+1}$ and $\sigma = \tau \circ v$, this yields

$$((\Lambda_{\gamma+1})_{\bar{J}})_{\langle D, J \rangle} = \Lambda_{\gamma+1}^{\tau \circ v}. \quad (4.3)$$

Since $(i(G), (\Lambda_{\gamma+1})_{i(G)}) <^* (M, \Sigma)$ and $v: (J, ((\Lambda_{\gamma+1})_{i(G)})^v) \rightarrow (i(G), (\Lambda_{\gamma+1})_{i(G)})$ satisfies the hypotheses of Theorem 4.6, we get from 5b and the mild positionality of $\Lambda_{\gamma+1}$:

$$((\Lambda_{\gamma+1})_{\bar{J}})_{\langle D, J \rangle} = (\Lambda_{\gamma+1})_{i(G)}^v. \quad (4.4)$$

$$\begin{array}{ccccc}
i(\bar{G}) & \xrightarrow{l} & i(G) & \xrightarrow{\tau} & i(N) \\
v^* \uparrow & & v \uparrow & & \uparrow \\
H & \xrightarrow{m} & J & & i \\
t^* \uparrow & & t \uparrow & & \uparrow \\
\bar{G} & \xrightarrow{j} & G & \xrightarrow{\sigma} & N
\end{array}$$

Figure 5: Diagram for the case G is of type 2.

Now we calculate

$$\begin{aligned}
\Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i \circ \sigma} \\
&= (\Lambda_{\gamma+1})^{\tau \circ v \circ t} \\
&= ((\Lambda_{\gamma+1})^{\tau \circ v})^t \\
&= (((\Lambda_{\gamma+1})_{\bar{J}})_{\langle D, J \rangle})^t \\
&= ((\Lambda_{\gamma+1})_{i(G)})^v)^t \\
&= (\Lambda_{\gamma+1})_{i(G)}^i \\
&= (\Lambda_\xi)_G.
\end{aligned}$$

Line 1 uses Claim 4. Line 2 uses commutativity. Line 4 uses equation 4.3 and line 5 uses equation 4.4. Line 7 uses Claim 4 again. Since $\Lambda_\xi^\sigma = (\Lambda_\xi)_G$, Φ_ξ is not problematic, contradiction.

Let us consider the case that G is of type 2. Let $(\bar{G}, \Lambda_\xi^{j \circ \sigma})$ be the type 1 core of (G, Λ_ξ^σ) , where $j = i_{\bar{D}}^{\bar{G}}$ for D the order zero measure on $\hat{\rho}_k(G)$. Since Φ_ξ^- is not problematic, $\bar{G} \triangleleft N$ and $i(\bar{G}) \triangleleft i(N)$. J is also of type 2, and setting

$$H = \text{Ult}(\bar{G}, E_i \upharpoonright \sup i^{\alpha})$$

and letting $t^* : \bar{G} \rightarrow H$ be the canonical embedding, we have

$$J = \text{Ult}(H, t^*(D)).$$

So J has type 2, and $H = C(J)$, and $t^*(D) = D(J)$. Let $m : H \rightarrow J$ be the $t^*(D)$ -ultrapower map, and $l : i(\bar{G}) \rightarrow i(G)$ be the $i(D)$ -ultrapower map. Let v^* be the factor map from H to $i(\bar{G})$, so that $v^* \circ t^* = i \upharpoonright \bar{G}$. See Figure 5 for a diagram of the situation.

Assuming toward contradiction that $\Phi_{\gamma+1}$ is not problematic, we have

$$(\Lambda_{\gamma+1})_H = (\Lambda_{\gamma+1})^{\tau \circ v \circ m}. \quad (4.5)$$

As above, the fact that $(i(\bar{G}), (\Lambda_{\gamma+1})_{i(\bar{G})})$ is strictly below (M, Σ) in the mouse order gives

$$(\Lambda_{\gamma+1})_H = ((\Lambda_{\gamma+1})_{i(\bar{G})})_H = ((\Lambda_{\gamma+1})_{i(\bar{G})})^{v^*}. \quad (4.6)$$

We then calculate

$$\begin{aligned}\Lambda_\xi^{\sigma \circ j} &= \Lambda_{\gamma+1}^{\tau \circ v \circ t^*} \\ &= \Lambda_{\gamma+1}^{\tau \circ v \circ m \circ t^*} \\ &= (\Lambda_{\gamma+1})_H^{t^*}\end{aligned}$$

(by equation 4.5)

$$= (\Lambda_{\gamma+1})_{i(\bar{G})}^{v^* \circ t^*}$$

(by equation 4.6)

$$\begin{aligned}&= (\Lambda_{\gamma+1})_{i(\bar{G})}^i \\ &= (\Lambda_\xi)_{\bar{G}}.\end{aligned}$$

This shows that Φ_ξ is not problematic, contradiction.

The cases where $G \triangleleft \text{Ult}(N, E_\alpha^N)$ or $C(G) \triangleleft \text{Ult}(N, E_\alpha^N)$ are handled similarly. For example, let $E = E_\alpha^N$, and suppose $G \triangleleft \text{Ult}(N, E)$ and J is of type 1. In this case, note that i is continuous at α , so $\text{crt}(\tau \circ v) = i(\alpha)$. The relevant diagram is

$$\begin{array}{ccccccc} J & \xleftarrow{v} & i(G) & \xrightarrow{\tau} & i(N) & \xrightarrow{i(E)} & Q & \triangleright & i(G) \\ & & \uparrow i & & \uparrow i & & \uparrow i & & \uparrow i \\ & \searrow t & G & \xrightarrow{\sigma} & N & \xrightarrow{E} & P & \triangleright & G \end{array}$$

Here $P = \text{Ult}(N, E)$ and $Q = \text{Ult}(i(N), i(E))$.

We assume toward contradiction that $\Phi_{\gamma+1}$ is not problematic. This implies

$$((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_J = (\Lambda_{\gamma+1})^{\tau \circ v}. \quad (4.7)$$

As above, the fact that $(i(G), ((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_{i(G)})$ is strictly below (M, Σ) in the mouse order gives

$$((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_J = (((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_{i(G)})_J = ((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_{i(G)}^v. \quad (4.8)$$

We then calculate

$$\begin{aligned}
\Lambda_\xi^\sigma &= (\Lambda_{\gamma+1}^{\tau \circ v})^t \\
&= (((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_J)^t \\
&= (((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_{i(G)}^v)^t \\
&= ((\Lambda_{\gamma+1})_{\langle i(E) \rangle})_{i(G)}^i \\
&= ((\Lambda_\xi)_{\langle E \rangle})_G.
\end{aligned}$$

Thus Φ_ξ is not problematic, contradiction.

So we conclude that $\Phi_{\gamma+1}$ is problematic in all cases. This finishes the proof of Claim 5. \square

Claim 5 finishes the definition of $\Phi_{\gamma+1}$, and the proof that it is a problematic tuple. We have also proved (4) of $(\dagger)_{\gamma+2}$. We now verify the rest of $(\dagger)_{\gamma+2}$. For the first part of (5), note that if i is discontinuous at α , then $\sup i''\alpha = \text{crt}(v) = \text{crt}(\tau \circ v) < i(\alpha)$, and if i is continuous at α , then $\text{crt}(\tau) = i(\alpha) = \text{crt}(\tau \circ v)$. Thus in either case, $\alpha_{\gamma+1} \leq i_{\xi, \gamma+1}^{\mathcal{S}}(\alpha_\xi)$.

For the rest of (5) and (6), it is enough to see that $\beta_{\gamma+1} < i(\beta)$, where $\beta = \beta_\xi = (\alpha^+)^G$. (Recall that we are in the case that i is discontinuous at $\rho_k(G)$.) If i is discontinuous at α , then α is a limit cardinal of G , and $\beta_{\gamma+1} = (\sup i''\alpha)^{+,J} < i(\alpha) < i(\beta)$, as desired. If i is continuous at α , then since $(\alpha^+)^G$ has cofinality κ in N , we get

$$(\alpha_{\gamma+1})^{+,J} = i(\alpha)^{+,J} = \sup i''\beta < i(\beta),$$

as desired.

(7) of $(\dagger)_{\gamma+2}$ is obvious from our definitions.

Remark 4.12. If Case 2 occurs in the passage from $\Phi_\xi^- = \langle N, G, \sigma, \alpha \rangle$ to $\Phi_{\gamma+1}^- = \langle i(N), J, \tau \circ v, \text{crt}(\tau \circ v) \rangle$, then $\rho_k(J) = \sup t''\rho_k(G)$ has cofinality κ in $i(N)$, where $\kappa = \text{crt}(E_\gamma^{\mathcal{S}})$. Along branches of \mathcal{S} containing $\gamma + 1$, κ can no longer be a critical point. It follows that along any given branch, Case 2 can occur at most once. \dashv

If (I) or (II) hold at $\gamma + 2$, then the construction of \mathcal{S} is over. Otherwise, we let $E_{\gamma+2}^{\mathcal{S}}$ be the least disagreement between $\mathcal{M}_{\gamma+2}^{\mathcal{S}}$ and $M_{\nu,l}$, and we set

$$\lambda_{\gamma+1} = \inf(\alpha_{\gamma+1}, \hat{\lambda}(E_{\gamma+2}^{\mathcal{S}})).$$

This completes the successor step in the construction of \mathcal{S} .

Now suppose we are given $\mathcal{S} \upharpoonright \theta$, where θ is a limit ordinal. Let $b = \Sigma(\mathcal{T} \upharpoonright \theta)$.

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix η . There are two subcases.

(A) for all $\gamma \in b - (\eta + 1)$, $\text{rt}(\gamma) = \eta + 1$. In this case, $b - (\eta + 1)$ is a branch of \mathcal{S} . Let \mathcal{S} choose this branch,

$$[\eta + 1, \theta]_{\mathcal{S}} = b - (\eta + 1),$$

and let $\mathcal{M}_{\theta}^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_{\gamma}^{\mathcal{S}}$ for sufficiently large $\gamma \in b - (\eta + 1)$. We define the branch embedding $i_{\gamma, \theta}^{\mathcal{S}}$ as usual and $\pi_{\theta} : \mathcal{M}_{\theta}^{\mathcal{S}} \rightarrow \mathcal{M}_{\theta}^{\mathcal{T}}$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

(B) for all $\gamma \in b - (\eta + 1)$, $\text{rt}(\gamma) = \eta$. Let \mathcal{S} choose

$$[0, \theta]_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and let $\mathcal{M}_{\theta}^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_{\gamma}^{\mathcal{S}}$ for sufficiently large $\gamma \in b$. Branch embeddings $i_{\gamma, \theta}^{\mathcal{S}}$ for $\gamma \geq \eta$ are defined as usual. $\pi_{\theta} : \mathcal{M}_{\theta}^{\mathcal{S}} \rightarrow \mathcal{M}_{\theta}^{\mathcal{T}}$ is given by the fact that copy maps commute with branch embeddings. We declare θ to be stable.

Since θ is stable, $(\dagger)_{\theta}$ follows at once from $\forall \gamma < \theta(\dagger)_{\gamma}$.

Case 2. There are boundedly many unstable ordinals in b but no largest one.

We let η be the sup of the unstable ordinals in b . Let \mathcal{S} choose

$$[0, \theta]_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and define the corresponding objects as in case 1(B). We declare θ stable, and again $(\dagger)_{\theta}$ is immediate.

Case 3. There are arbitrarily large unstable ordinals in b . In this case, b is a disjoint union of pairs $\{\gamma, \gamma + 1\}$ such that γ is unstable and $\gamma + 1$ is stable. We set

$$[0, \theta]_{\mathcal{S}} = \{\xi \in b \mid \xi \text{ is unstable}\},$$

and let $\mathcal{M}_{\theta}^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_{\xi}^{\mathcal{S}}$'s for $\xi \in b$ unstable. There is no dropping in model or degree along $[0, \theta]_{\mathcal{S}}$. We define maps $i_{\xi, \theta}^{\mathcal{S}}, \pi_{\theta}$ as usual. If $(\mathcal{M}_{\theta}^{\mathcal{S}}, \Lambda_{\theta})$ is not a pair of the form $(\mathcal{M}_{\tau}^{\mathcal{U}}, \Sigma_{\tau}^{\mathcal{U}})$, then we declare θ stable and $(\dagger)_{\theta}$ is immediate.

Suppose that $(\mathcal{M}_{\theta}^{\mathcal{S}}, \Lambda_{\theta})$ is a pair of \mathcal{U} . We declare θ unstable. Note that by clauses (4) and (5) of (\dagger) , there is a $\xi <_{\mathcal{S}} \theta$ such that for all γ with $\xi <_{\mathcal{S}} \gamma <_{\mathcal{S}} \theta$, $i_{\xi, \gamma}^{\mathcal{S}}(\langle \alpha_{\xi}, \beta_{\xi} \rangle) = \langle \alpha_{\gamma}, \beta_{\gamma} \rangle$. So we can set

$$\alpha_{\theta} = \text{common value of } i_{\gamma, \theta}^{\mathcal{S}}(\alpha_{\gamma}), \text{ for } \gamma <_{\mathcal{S}} \theta \text{ sufficiently large.}$$

By clause (5), we can set

$$\mathcal{M}_{\theta+1}^{\mathcal{S}} = \text{common value of } i_{\gamma, \theta}^{\mathcal{S}}(\mathcal{M}_{\gamma+1}^{\mathcal{S}}), \text{ for } \gamma <_{\mathcal{S}} \theta \text{ sufficiently large.}$$

We also let

$$\sigma_{\theta} = \text{common value of } i_{\gamma, \theta}^{\mathcal{S}}(\sigma_{\gamma}), \text{ for } \gamma <_{\mathcal{S}} \theta \text{ sufficiently large.}$$

Here

$$i_{\gamma,\theta}^{\mathcal{S}}(\sigma_\gamma) = \text{upward extension of } \bigcup_{\eta < \rho} i_{\gamma,\theta}^{\mathcal{S}}(\sigma_\gamma^\eta),$$

where $\rho = \rho_k(\mathcal{M}_{\gamma+1}^{\mathcal{S}})$ for $k = \text{deg}(\mathcal{M}_{\gamma+1}^{\mathcal{S}})$, and the σ_γ^η are the terms in the natural decomposition of σ_γ . By (6) of (\dagger) and Remark 4.12, $i_{\gamma,\theta}^{\mathcal{S}}$ is continuous at $\rho_k(\mathcal{M}_{\gamma+1})$ for $\gamma <_S \theta$ sufficiently large, so σ_θ is defined on all of $\mathcal{M}_{\theta+1}^{\mathcal{S}}$. It is easy then to see that $\Phi_\theta = \langle (\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta), (\mathcal{M}_{\theta+1}^{\mathcal{S}}, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$ is a problematic tuple.

If (I) holds, then we stop the construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and move on to $\mathcal{S}_{\nu,l+1}$. If (II) holds, we stop the construction of \mathcal{S} and do not move on. If neither holds, we let $E_{\theta+1}^{\mathcal{S}}$ be the extender on the $\mathcal{M}_{\theta+1}^{\mathcal{S}}$ sequence that represents its first disagreement with $M_{\nu,l}$, and set

$$\lambda_{\theta+1} = \hat{\lambda}(E_{\theta+1}^{\mathcal{S}}),$$

and

$$\lambda_\theta = \inf(\lambda_{\theta+1}, \alpha_\theta).$$

It then is routine to verify $(\dagger)_{\theta+1}$.

This finishes our construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and \mathcal{T} . Note that every extender used in \mathcal{S} is taken from a stable node, and every stable node except the last model of \mathcal{S} contributes exactly one extender to \mathcal{S} . The last model of \mathcal{S} is stable.

Claim 6. The construction of $\mathcal{S}_{\nu,l}$ stops for one of the reasons (I) and (II).

Proof sketch. This comes from the proof of Theorem 3.26 and the analysis of the type 2 case in subsection 3.5, together with the method for adapting such results to the comparison of phalanxes with background constructions used in [13]. \square

Claim 7. For some $(\nu, l) \leq (\eta_0, k_0)$, the construction of $\mathcal{S}_{\nu,l}$ stops for reason (II).

Proof. If not, then the construction of $\mathcal{S} = \mathcal{S}_{\eta_0,k_0}$ must reach some stable θ such that $(M_{\eta_0,k_0}, \Omega_{\eta_0,k_0}) \trianglelefteq (\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$, and either $(M_{\eta_0,k_0}, \Omega_{\eta_0,k_0}) \triangleleft (\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$ or the branch of \mathcal{S} leading to θ has a drop. Let

$$Q = \pi_\theta(M_{\eta_0,k_0});$$

then either $(Q, (\Sigma_\theta^{\mathcal{T}})_Q) \triangleleft (M_\theta^{\mathcal{T}}, \Sigma_\theta^{\mathcal{T}})$ or the branch $[0, \theta]_{\mathcal{T}}$ has a drop.

Suppose first that M_{η_0,k_0} is a nondropping iterate of M and let $j : M \rightarrow M_{\eta_0,k_0}$ be the iteration map given by \mathcal{U}_{η_0,k_0} . Letting $\mathcal{T} = \mathcal{T}_{\eta_0,k_0}$, we have $\pi_\theta : \mathcal{M}_\theta^{\mathcal{S}} \rightarrow \mathcal{M}_\theta^{\mathcal{T}}$ from the copying construction. Note that

$$\pi_\theta \circ j : (M, \Sigma) \rightarrow (Q, (\Sigma_\theta^{\mathcal{T}})_Q)$$

is an elementary map, because

$$\Sigma = \Omega_{\eta_0,k_0}^j = ((\Lambda_\theta)_{M_{\eta_0,k_0}})^j = ((\Sigma_\theta^{\mathcal{T}})_Q)^{\pi_\theta \circ j}.$$

Thus $\pi_\theta \circ j$ maps (M, Σ) into an initial segment of $(\mathcal{M}_\theta^T, \Sigma_\theta^T)$ which is either proper or reached along a branch that drops. This is contrary to the Dodd-Jensen Theorem.

Suppose next that $j : M \rightarrow N = \text{Ult}(M_{\eta_0, k_0}, F)$ is the iteration map from $\mathcal{U}_{\eta_0, k_0}$. Let $R = \text{Ult}(Q, \pi_\theta(F))$. We have the diagram

$$\begin{array}{ccccc}
M & \xrightarrow{i_{0,\theta}^T} & Q & \xrightarrow{\pi_\theta(F)} & R \\
& & \uparrow \pi_\theta & & \uparrow \sigma \\
& & M_{\eta_0, k_0} & \xrightarrow{F} & N \\
& & & & \uparrow j \\
& & & & M
\end{array}$$

Here we have drawn the case $Q = \mathcal{M}_\theta^T$ for simplicity. σ is the copy map.

We need to replace M by M^- in calibrating elementarity, because $i_{\pi_\theta(F)} \circ i_{0,\theta}^T$ is not elementary as a map on M , because of the drop at the end. But $i_{\pi_\theta(F)} \circ i_{0,\theta}^T$ is an iteration map associated to a +1-iteration tree on M^- , and $\sigma \circ j$ is +1-elementary as a map on M^- . Thus we can apply Dodd-Jensen in the category of +1-iteration maps, and we have a contradiction as above.⁶⁵

□

By Claim 7, we may fix $(\nu, l) \leq (\eta_0, k_0)$ such that the construction of $\mathcal{S}_{\nu, l}$ terminates for reason (II). Let $\mathcal{S} = \mathcal{S}_{\nu, l}$ and $\text{lh}(\mathcal{S}) = \theta + 1$. Thus θ is stable, $[\text{rt}(\theta), \theta]_{\mathcal{S}}$ does not drop in model or degree (so $l = k_0$), and either

- (a) $(\mathcal{M}_\theta^S, \Lambda_\theta) = (M_{\nu, l}, \Omega_{\nu, l})$, or⁶⁶
- (b) $(\mathcal{M}_\theta^S, \Lambda_\theta)$ has type 2, and $(M_{\nu, l}, \Omega_{\nu, l})$ is its type 1 core.

Let $\mathcal{U} = \mathcal{U}_{\nu, l}$, and $\gamma + 1 = \text{lh}(\mathcal{U})$. The result of comparison via \mathcal{U} is that either

- (a') $(M_{\nu, l}, \Omega_{\nu, l}) \trianglelefteq (\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$, or
- (b') $(\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$ has type 2, and $(M_{\nu, l}, \Omega_{\nu, l})$ is its type 1 core.

In the usual (M, H, α) vs. M comparisons, one shows that the phalanx side does not terminate on a branch above M . The next claim adapts that argument to our current situation, in which the phalanx has been lifted along various branches of \mathcal{S} .

Claim 8. For some unstable ξ , $\text{rt}(\theta) = \xi + 1$.

Proof. If not, then $0 \leq_S \theta$ and the branch $[0, \theta]_{\mathcal{S}}$ does not drop in model or degree. We take cases on how \mathcal{S} and \mathcal{U} end. Let $i = i_{0,\theta}^S$ and $i^* = i_{0,\theta}^T$. Let $\text{lh}(\mathcal{U}) = \gamma + 1$ and $j = i_{0,\gamma}^{\mathcal{U}}$.

⁶⁵Dodd-Jensen applies because we can copy +1 trees on using nearly +1-elementary copy maps. See §3. The more general statement of this fact belongs to Jensen's Σ^* theory.

⁶⁶The comparison arguments of §3 and [13] show that if (ν, l) is least such that (II) holds, and $(\mathcal{M}_\theta^S, \Lambda_\theta) \trianglelefteq (M_{\nu, l}, \Omega_{\nu, l})$, then $(\mathcal{M}_\theta^S, \Lambda_\theta) = (M_{\nu, l}, \Omega_{\nu, l})$.

Case 1. $\mathcal{M}_\theta^S = M_{\nu,l}$ and $M_{\nu,l} \trianglelefteq \mathcal{M}_\gamma^U$.

Proof. We claim that $(\mathcal{M}_\theta^S, \Lambda_\theta) = (\mathcal{M}_\gamma^U, \Sigma_\gamma^U)$. For otherwise i maps (M, Σ) to a proper initial segment of a Σ -iterate of (M, Σ) , contrary to the Dodd-Jensen Theorem. (Note here that Σ is the pullback of Λ_θ under i , by Claim 4.) For the same reason, $[0, \gamma]_U$ does not drop, and thus $(\nu, l) = (\eta_0, k_0)$. The relevant diagram is now

$$\begin{array}{ccccc} \mathcal{M}_\theta^S & \xleftarrow{\pi_\theta} & M_{\nu,l} & \xleftarrow{\text{id}} & M_{\nu,l} \\ \uparrow i^* & & \uparrow i & & \uparrow j \\ M & & M & & M \end{array}$$

Using Dodd-Jensen we get that

$$i = j.$$

To see this, note that both are elementary maps from (M, Σ) to $(\mathcal{M}_\theta^S, \Lambda_\theta) = (\mathcal{M}_\gamma^U, \Sigma_\gamma^U) = (M_{\nu,l}, \Omega_{\nu,l})$. Since j is an iteration map, for all ξ

$$j(\xi) \leq i(\xi).$$

Since i^* is also an iteration map, for all ξ

$$i^*(\xi) = \pi_\theta \circ i(\xi) \leq \pi_\theta \circ j(\xi).$$

Multiplying by π_θ^{-1} , we get that $i(\xi) \leq j(\xi)$ for all ξ . So $i = j$.

Since we can recover branch extenders from branch embeddings, we get then that

$$e_\theta^S = e_\gamma^U. \text{ }^{67}$$

Let $\eta \leq_S \theta$ be least such that η is stable. Then $e_\eta^S = e_\theta^S \upharpoonright \delta = e_\gamma^U \upharpoonright \delta$ for some δ . But there is τ such that $e_\tau^U = e_\gamma^U \upharpoonright \delta$. Thus $\mathcal{M}_\eta^S = \mathcal{M}_\tau^U$. We have also

$$\Lambda_\eta = \Lambda_\theta^{i_{\eta,\theta}^S} = (\Sigma_\gamma^U)^{i_{\tau,\gamma}^U} = \Sigma_\tau^U,$$

by pullback consistency, since $i_{\eta,\theta}^S = i_{\tau,\gamma}^U$.

If η is a limit ordinal, then by the rules at limit stages of \mathcal{S} above, we declare η unstable. This contradicts our assumption. If $\mathcal{S}\text{-pred}(\eta) = \mu$, then μ is unstable by our minimality assumption on η ; but then we declare η unstable by our rules at successor stages. Again, we reach a contradiction. This finishes Case 1. \square

Case 2. \mathcal{M}_θ^S has type 2, and $M_{\nu,l} \trianglelefteq \mathcal{M}_\gamma^U$.

Proof. Let $(\mathcal{M}_\theta^S, \Lambda_\theta) = \text{Ult}((M_{\nu,l}, \Omega_{\nu,l}), F)$, where F is the order zero measure on $\hat{\rho}_l(\mathcal{M}_\theta^S)$.

⁶⁷ e_θ^S is the sequence of extenders used along the branch $[0, \theta]_S$ and similarly for e_γ^U .

We claim that $\mathcal{M}_\gamma^{\mathcal{U}} = M_{\nu,l}$ and $[0, \gamma]_U$ does not drop. For if $M_{\nu,l} \triangleleft \mathcal{M}_\gamma^{\mathcal{U}}$ or $[0, \gamma]_U$ drops, then $\mathcal{U} \frown \langle F \rangle$ is a +1-iteration tree on (M^-, Σ) whose last model is $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$ and whose main branch has a large drop, while i is a +1-elementary map from (M^-, Σ) to $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$. This contradicts Dodd-Jensen, in the form of Lemma 3.22. The relevant diagram is now

$$\begin{array}{ccccc} \mathcal{M}_\theta^{\mathcal{T}} & \xleftarrow{\pi_\theta} & \mathcal{M}_\theta^{\mathcal{S}} & \xleftarrow{i_F} & M_{\nu,l} \\ \uparrow i^* & & \uparrow i & & \uparrow j \\ M & & M & & M \end{array}$$

From the claim, we have that $i_F \circ j$ is defined (and total on M). Dodd-Jensen yields $i = i_F \circ j$. For $i_F \circ j$ is a +1-iteration map on (M^-, Σ) , so $i_F \circ j(\eta) \leq i(\eta)$ for all η . But i^* is a +1-iteration map on (M^-, Σ) , so $i^*(\eta) = \pi_\theta \circ i(\eta) \leq \pi_\theta \circ i_F \circ j(\eta)$, so $i(\eta) \leq i_F \circ j(\eta)$. Thus $i = i_F \circ j$.

On the other hand, the generators of the extender of i are contained in $\sup i \restriction \rho_l(M) = \rho_l(M_{\nu,l})$, while $i_F \circ j$ has the generator $\text{crt}(F) > \rho_l(M_{\nu,l})$. So $i \neq i_F \circ j$, contradiction. \square

Case 3. $\mathcal{M}_\theta^{\mathcal{S}} = M_{\nu,l}$ and $\mathcal{M}_\gamma^{\mathcal{U}}$ has type 2.

Proof. Let $(\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}}) = \text{Ult}((M_{\nu,l}, \Omega_{\nu,l}), F)$, where F is the order zero measure on $\hat{\rho}_l(\mathcal{M}_\gamma^{\mathcal{U}})$. The relevant diagram is now

$$\begin{array}{ccccc} & & Q & & \\ & \nearrow \pi_\theta(F) & & \nwarrow \sigma & \\ \mathcal{M}_\theta^{\mathcal{T}} & \xleftarrow{\pi_\theta} & M_{\nu,l} & \xrightarrow{i_F} & \mathcal{M}_\gamma^{\mathcal{U}} \\ \uparrow i^* & & \uparrow i & & \uparrow j \\ M & & M & & M \end{array}$$

Here $Q = \text{Ult}(\mathcal{M}_\theta^{\mathcal{T}}, \pi_\theta(F))$ and $\sigma: \text{Ult}(M_{\nu,l}, F) \rightarrow \text{Ult}(\mathcal{M}_\theta^{\mathcal{T}}, \pi_\theta(F))$ is the copy map.

We shall use Dodd-Jensen to show that $i \restriction \rho_l(M) = j \restriction \rho_l(M)$.

First, j is a +1-iteration map on (M^-, Σ) , and $i_F \circ i: (M^-, \Sigma) \rightarrow (\text{Ult}(M_{\nu,l}, F), \Sigma_\gamma^{\mathcal{U}})$ is +1-elementary, so $j(\eta) \leq i_F \circ i(\eta)$ for all η . But $\text{crt}(F) > \rho_l(M_{\nu,l})$, so

$$\forall \eta < \rho_l(M) (j(\eta) \leq i(\eta)).$$

On the other hand, $i_{\pi_\theta(F)} \circ i^*$ is a +1-iteration map from (M^-, Σ) to (Q, Ω) , where $\Omega = \Sigma_{\mathcal{T} \frown \langle \pi_\theta(F) \rangle}$. Here Ω is a level l strategy, so (Q, Ω) is a type 2 pair generated by $(\mathcal{M}_\theta^{\mathcal{T}}, \Sigma_\theta^{\mathcal{T}})$. Ω is a +1-strategy for Q^- . Since

$$\begin{aligned} \Omega_{\nu,l} &= \Omega^{i_{\pi_\theta(F)} \circ i^*} \\ &= \Omega^{\sigma \circ i_F} \end{aligned}$$

we have

$$\Sigma_{\gamma}^{\mathcal{U}} = \Omega^{\sigma}.$$

So $\sigma \circ j$ is elementary in the category of mouse pairs. By Dodd-Jensen,

$$i_{\pi_{\theta}(F)} \circ i^*(\eta) = i_{\pi_{\theta}(F)} \circ \pi_{\theta} \circ i(\eta) \leq \sigma \circ i_F \circ i(\eta) = \sigma \circ j(\eta)$$

for all η . But $i_{\pi_{\theta}(F)}$ and i_F are the identity on $\rho_l(M_{\nu,l})$, and σ agrees with π_{θ} on $\rho_l(M_{\nu,l})$. Thus

$$\forall \eta < \rho_l(M)(i(\eta) \leq j(\eta)).$$

Thus $i \upharpoonright \rho_l(M) = j \upharpoonright \rho_l(M)$. But the generators of i and j are all below $\rho_l(M_{\nu,l}) = \sup i \upharpoonright \rho_l(M) = \sup j \upharpoonright \rho_l(M)$. It follows that $\mathcal{M}_{\theta}^S = \mathcal{M}_{\gamma}^{\mathcal{U}}$, contrary to our case hypothesis. \square .

Case 4. \mathcal{M}_{θ}^S and $\mathcal{M}_{\gamma}^{\mathcal{U}}$ are both of type 2.

Proof. We shall show that $i = j$. This then leads to the same contradiction we arrived at in case 1.

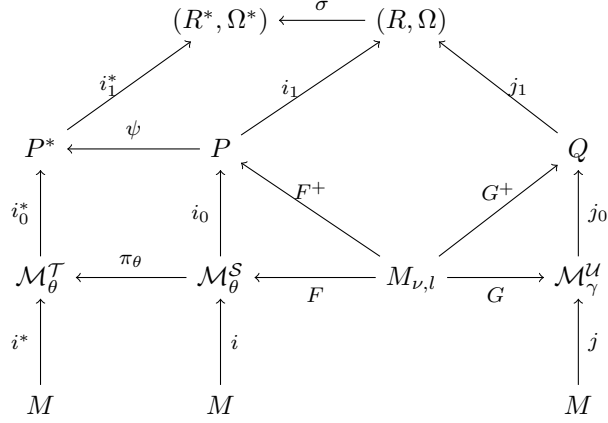
Let F and G be the order zero measures on the $M_{\nu,l}$ sequence such that $\mathcal{M}_{\theta}^S = \text{Ult}(M_{\nu,l}, F)$ and $\mathcal{M}_{\gamma}^{\mathcal{U}} = \text{Ult}(M_{\nu,l}, G)$ respectively. Let $F_0 = i_F(F)$ and $G_0 = i_G(G)$, and

$$\begin{aligned} P &= \text{Ult}(\mathcal{M}_{\theta}^S, F_0) = \text{Ult}(M_{\nu,l}, F^+), \\ Q &= \text{Ult}(\mathcal{M}_{\gamma}^{\mathcal{U}}, G_0) = \text{Ult}(M_{\nu,l}, G^+), \\ R &= \text{Ult}(P, i_{F^+}(G^+)) = \text{Ult}(Q, i_{G^+}(F^+)), \end{aligned}$$

and

$$\Omega = (\Omega_{\nu,l})_{\langle F^+ \rangle, \langle i_{F^+}(G^+) \rangle} = (\Omega_{\nu,l})_{\langle G^+ \rangle, \langle i_{G^+}(F^+) \rangle}.$$

$\Omega_{\nu,l}$ quasi-normalizes well when considered as a +1-strategy for $M_{\nu,l}^-$, and $\langle F^+, i_{F^+}(G^+) \rangle$ is a stack of λ -separated trees, so the identity on the last displayed line is justified. Ω is a level l strategy for R , that is, a +1-strategy for R^- . The following diagram describes our situation:



Here $i_0, i_1, j_0,$ and j_1 are the ultrapower maps. i_0^* and i_1^* come from copying the ultrapowers giving rise to i_0 and i_1 . The copy maps are ψ and σ . We claim that $i_1 \circ i_0 \circ i = j_1 \circ j_0 \circ j$. For $j_1 \circ j_0 \circ j(\eta) \leq i_1 \circ i_0 \circ i(\eta)$ for all η , because $j_1 \circ j_0 \circ j$ is an iteration map and $\Sigma = \Omega^{i_1 \circ i_0 \circ i}$, so that $i_1 \circ i_0 \circ i$ is +1-elementary as a map from (M^-, Σ) to (R, Ω) . On the other hand,

$$\begin{aligned}
i_1^* \circ i_0^* \circ i^*(\eta) &= \sigma \circ i_1 \circ i_0 \circ i(\eta) \\
&\leq \sigma \circ j_1 \circ j_0 \circ j(\eta)
\end{aligned}$$

because $i_1^* \circ i_0^* \circ i^*$ is a +1-iteration map on (M^-, Σ) and $\sigma \circ j_1 \circ j_0 \circ j$ is nearly +1-elementary on (M^-, Σ) .

Thus $i_1 \circ i_0 \circ i = j_1 \circ j_0 \circ j$. Let

$$\rho = \rho_l(M_{\nu,l});$$

then the generators of i and j are contained in ρ , while the generators of $i_0, i_1, j_0,$ and j_1 are all strictly above ρ . So

$$\begin{aligned}
E_i \upharpoonright \rho &= E_{i_1 \circ i_0 \circ i} \upharpoonright \rho \\
&= E_{j_1 \circ j_0 \circ j} \upharpoonright \rho \\
&= E_j \upharpoonright \rho.
\end{aligned}$$

Thus $i = j$ and $\mathcal{M}_\theta^S = \mathcal{M}_\gamma^U$. The stability of θ then leads to a contradiction, as in case 1. \square

This completes the proof of Claim 8. \square

Let ξ be as in Claim 8, and let τ be such that $(\mathcal{M}_\xi^S, \Lambda_\xi) = (\mathcal{M}_\tau^U, \Sigma_\tau^U)$. We have $e_\xi^S = e_\tau^U$ by the proof in claim 8.

Claim 9. $\tau < \gamma$, and $\alpha_\xi \leq \text{lh}(E_\tau^U) < \alpha_\xi^{+, \mathcal{M}_\tau^U}$.

Proof. Suppose $\tau = \gamma$. Let B code $\mathcal{M}_{\xi+1}^S$ as a subset of α_ξ , namely for $Q = \mathcal{M}_{\xi+1}^S$,

$$B = \text{Th}_1^{\hat{Q}^l}(\alpha_\xi \cup p_1(\hat{Q}^l)).$$

Then $B \in \mathcal{M}_\xi^S$ because Φ_ξ is problematic, so $B \in \mathcal{M}_\tau^U$. But $B \notin \mathcal{M}_{\xi+1}^S$, so $B \notin \mathcal{M}_\theta^S$, so $B \notin M_{\nu,l}$. Suppose $\theta > \xi + 1$, then $\mathcal{M}_\theta^S = M_{\nu,l}$ or $M_{\nu,l}$ is a type 1 core of \mathcal{M}_θ^S ; in either case, $M_{\nu,l}$ is not $l+1$ -sound. This means either $M_{\nu,l} = \mathcal{M}_\tau^U$ or $M_{\nu,l}$ is a type 1 core of \mathcal{M}_τ^U . In both cases, $\mathcal{M}_\theta^S, M_{\nu,l}$, and \mathcal{M}_γ^U agree to their common value for $\rho_l > \alpha_\xi$, so $B \in M_\theta^S$. Contradiction.

So $\theta = \xi + 1$ and therefore, $(\mathcal{M}_{\xi+1}^S, \Lambda_{\xi+1}) = (M_{\nu,l}, \Omega_{\nu,l}) \triangleleft (\mathcal{M}_\tau^U, \Sigma_\tau)$ (the possibility that $(\mathcal{M}_{\xi+1}^S, \Lambda_{\xi+1})$ is a type 1 core of $(\mathcal{M}_\tau^U, \Sigma_\tau)$ cannot occur because $M_{\xi+1}^S \in \mathcal{M}_\tau^U$). This contradicts the problematicity of Φ_ξ . So we must have $\tau < \gamma$.

Note that $(\mathcal{M}_\xi^S, \Lambda_\xi)$, $(\mathcal{M}_{\xi+1}^S, \Lambda_{\xi+1})$, and $(\mathcal{M}_\tau^U, \Sigma_\tau^U)$ all agree with $M_{\nu,l}$ below α_ξ . (Possibly not at α_ξ .) This is because otherwise $\lambda_\xi < \alpha_\xi$, and $\xi + 1$ is a dead node in \mathcal{S} . It follows that $\alpha_\xi \leq \text{lh}(E_\tau^U)$.

If $\alpha_\xi < \text{lh}(E_\tau^U)$, then $\mathcal{M}_{\tau+1}^U$ agrees with \mathcal{M}_γ^U up to their common value η for α_ξ^+ . Suppose for contradiction that $\text{lh}(E_\tau^U) > \alpha_\xi^{+, \mathcal{M}_\tau^U}$. So $\eta = \alpha_\xi^{+, \mathcal{M}_\tau^U}$. Suppose $M_{\nu,l}$ is \mathcal{M}_γ^U or its type 1 core, then note that $B \in \mathcal{M}_\tau^U - \mathcal{M}_\gamma^U$, so $\eta < \alpha_\xi^{+, \mathcal{M}_\tau^U}$. Contradiction. Now suppose $M_{\nu,l} \triangleleft \mathcal{M}_\gamma^U$. Then by the way our comparison works, $o(M_{\nu,l}) \geq \text{lh}(E_\tau^U) \geq \alpha_\xi^{+, \mathcal{M}_\tau^U}$. But $\rho(M_{\nu,l}) = \rho(\mathcal{M}_{\xi+1}^S) = \rho(\mathcal{M}_\theta^S) \leq \alpha_\xi$ and $M_{\nu,l}$ is sound. Contradiction.

Thus $\text{lh}(E_\tau^U) < \alpha_\xi^{+, \mathcal{M}_\tau^U}$ as desired. \square

Now let

$$\rho = \rho(\mathcal{M}_{\xi+1}^S) = \rho_{l+1}(\mathcal{M}_{\xi+1}^S).$$

Thus $\rho = \rho(\mathcal{M}_\theta^S) = \rho(M_{\nu,l})$ as well.

Claim 10. Either $\rho = \alpha_\xi$, or $\rho^{+, \mathcal{M}_{\xi+1}^S} = \alpha_\xi$. Moreover, $\text{lh}(E_\beta^T) \leq \rho$ for all $\beta < \tau$.

Proof. $\rho \leq \alpha_\xi$ because $\mathcal{M}_{\xi+1}^S$ is α_ξ -sound, so we are done unless $\rho < \alpha_\xi$.

Suppose $\rho < \alpha_\xi$. Since ρ is a cardinal in $\mathcal{M}_{\xi+1}^S$ and $\sigma_\xi \upharpoonright \alpha_\xi = \text{id}$, ρ is a cardinal in \mathcal{M}_ξ^S . But $|\alpha_\xi| \leq \rho$ in \mathcal{M}_ξ^S , so $|\alpha_\xi| = \rho$ in \mathcal{M}_ξ^S . Since α_ξ is a cardinal in $\mathcal{M}_{\xi+1}^S$, $\alpha_\xi = \rho^{+, \mathcal{M}_{\xi+1}^S}$. \square

The following elementary fact will help:

Proposition 4.13. *Let \mathcal{W} be a λ -separated tree, let $\mu + 1 \leq_W \eta$, and let $\mathcal{M}_\eta^{\mathcal{W}} \upharpoonright \text{lh}(E_\mu^{\mathcal{W}}) \trianglelefteq N \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}}$; then*

(a) $\text{lh}(E_\mu^{\mathcal{W}}) \leq \rho^-(N)$, and

(b) $\rho(N)$ is not in the open interval $(\text{crt}(E_\mu^{\mathcal{W}}), \text{lh}(E_\mu^{\mathcal{W}}))$.

Proof. Let $E_\alpha = E_\alpha^{\mathcal{W}}$ and $M_\alpha = \mathcal{M}_\alpha^{\mathcal{W}}$. If $\rho(M_{\mu+1}^*) \leq \text{crt}(E_\mu)$ then $\rho(M_{\mu+1}) = \rho(M_{\mu+1}^*)$. If $\text{crt}(E_\mu) < \rho(M_{\mu+1}^*)$, then

$$\text{lh}(E_\mu) < \sup i_{\mu+1}^* \text{“} \rho(M_{\mu+1}^*) = \rho(M_{\mu+1}) \text{”}.$$

We use here that E_μ has plus type; otherwise $\text{lh}(E_\mu) = \rho(M_{\mu+1})$ is possible.⁶⁸ Note also that $\text{lh}(E_\mu) < \rho^-(M_{\mu+1})$ in both cases. So the proposition holds when $\eta = \mu + 1$ and $N = \mathcal{M}_{\mu+1}$. If $\mathcal{M}_{\mu+1} | \text{lh}(E_\mu^{\mathcal{W}}) \trianglelefteq N \triangleleft \mathcal{M}_\eta^{\mathcal{W}}$, then $\text{lh}(E_\mu) \leq \rho(N)$ because $\text{lh}(E_\mu)$ is a cardinal in $M_{\mu+1}$ and $\text{lh}(E_\mu) \leq \rho^-(N)$. Thus the proposition holds when $\eta = \mu + 1$.

We now proceed by induction on η . Let $\mu + 1 <_W \beta + 1$ and $W\text{-pred}(\beta + 1) = \alpha$. By induction, (a) and (b) hold at $\eta = \alpha$. We claim that they hold at $\eta = \beta + 1$. For since $M_\alpha | \text{lh}(E_\mu) \trianglelefteq M_{\beta+1}^* \trianglelefteq M_\alpha$, we have $\text{lh}(E_\mu) < \rho^-(M_{\beta+1}^*)$ and $\rho(M_{\beta+1}^*)$ is not in the open interval $(\text{crt}(E_\mu), \text{lh}(E_\mu))$.⁶⁹ But then, taking the E_β ultrapower, we see that $\text{lh}(E_\mu) \leq \text{lh}(E_\beta) \leq \rho^-(M_{\beta+1})$ and $\rho(M_{\beta+1})$ is not in the open interval $(\text{crt}(E_\mu), \text{lh}(E_\mu))$. The same is true for $N \trianglelefteq M_{\beta+1}$ such that $\text{lh}(E_\beta) \leq o(N)$ because $\text{lh}(E_\beta)$ is a cardinal in $M_{\beta+1}$, and hence for $N \trianglelefteq M_{\beta+1}$ such that $\text{lh}(E_\mu) \leq o(N)$ by coherence.

The induction hypotheses (a) and (b) clearly pass through limits. \square

Claim 11. $E_{\alpha_\xi}^{\mathcal{M}_\xi^{\mathcal{S}}} \neq \emptyset$.

Proof. Suppose $E_{\alpha_\xi}^{\mathcal{M}_\xi^{\mathcal{S}}} = \emptyset$. Thus

$$\rho \leq \alpha_\xi \leq \hat{\lambda}(E_\tau^{\mathcal{U}}) < \text{lh}(E_\tau^{\mathcal{U}}).$$

Let $F = E_\mu^{\mathcal{U}}$ be the first extender used in $[0, \gamma]_U$ such that $\text{lh}(F) \geq \alpha_\xi$. (Thus $\text{lh}(F) > \alpha_\xi$.) We must have $\text{crt}(F) < \hat{\lambda}(E_\tau^{\mathcal{U}})$, since otherwise $U\text{-pred}(\mu + 1) \geq \tau + 1$, so some extender used in $[0, U\text{-pred}(\mu + 1)]_U$ has length $\geq \alpha_\xi$.

We claim that $\rho \leq \text{crt}(F)$. First notice that there is an $N \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}}$ such that $\text{lh}(F) \leq o(N)$ and $\rho(N) = \rho$; namely, $N = \mathcal{M}_\gamma^{\mathcal{U}}$ if $\mathcal{M}_\gamma^{\mathcal{U}}$ has type 2 with $M_{\nu,l}$ as its type 1 core, and $N = M_{\nu,l}$ otherwise. So by Proposition 4.13, ρ is not in the open interval $(\text{crt}(F), \text{lh}(F))$. But $\rho < \text{lh}(F)$ because $\alpha_\xi < \text{lh}(F)$, so

$$\rho \leq \text{crt}(F).$$

By Claim 10, $\text{lh}(E_\beta^{\mathcal{T}}) \leq \rho$ for all $\beta < \tau$, so we get

$$U\text{-pred}(\mu + 1) = \tau.$$

By Claim 9, $\alpha_\xi^{+, \mathcal{M}_\gamma^{\mathcal{U}}} = \alpha_\xi^{+, \mathcal{M}_{\tau+1}^{\mathcal{U}}} < \alpha_\xi^{+, \mathcal{M}_\tau^{\mathcal{U}}}$. It follows that $\mathcal{M}_{\mu+1}^{*, \mathcal{U}} \triangleleft \mathcal{M}_\tau^{\mathcal{U}} | \alpha_\xi^{+, \mathcal{M}_\tau^{\mathcal{U}}}$, so \mathcal{U} drops at $\mu + 1$. This implies that $\mathcal{M}_\gamma^{\mathcal{U}}$ has type 1 and is unsound, so $M_{\nu,l} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}}$. If $M_{\nu,l} \triangleleft \mathcal{M}_\gamma^{\mathcal{U}}$, then $\text{lh}(F) \leq \rho(M_{\nu,l}) = \rho$ by 4.13. Thus

$$\mathcal{M}_\gamma^{\mathcal{U}} = M_{\nu,l}.$$

⁶⁸But even in the λ -separated case, it is possible that $\rho(M_\eta) = \text{lh}(E_\mu)$ for $\mu + 1 <_W \eta$.

⁶⁹But $\rho(M_{\beta+1}^*) = \text{lh}(E_\mu)$ is possible, which is why the proposition allows $\rho(M_\eta^{\mathcal{W}}) = \text{lh}(E_\mu)$.

Moreover $\rho = \rho(M_{\nu,l})$, so there can be no further dropping along $[\mu + 1, \gamma]\mathcal{U}$, and thus

$$\begin{aligned}\rho &= \rho_{l+1}(\mathcal{M}_{\mu+1}^{*\mathcal{U}}), \\ \mathcal{M}_{\mu+1}^{*\mathcal{U}} &= \mathfrak{C}_{l+1}(M_{\nu,l}),\end{aligned}$$

and

$$i_0 =_{\text{df}} i_{\mu+1,\gamma} \circ i_{\mu+1}^{*\mathcal{U}}$$

is the anticore map.⁷⁰

Taking the type 1 core of a type 2 premouse of degree l commutes with taking its standard $l+1$ core, so

$$\begin{aligned}C(\mathcal{M}_{\xi+1}^S) &= C(\mathfrak{C}_{l+1}(\mathcal{M}_{\theta}^S)) \\ &= \mathfrak{C}_{l+1}(C(\mathcal{M}_{\theta}^S)) \\ &= \mathfrak{C}_{l+1}(M_{\nu,l}) \\ &= \mathcal{M}_{\mu+1}^{*\mathcal{U}}.\end{aligned}$$

So $C(\mathcal{M}_{\xi+1}^S) \triangleleft \mathcal{M}_{\tau}^{\mathcal{U}} = \mathcal{M}_{\xi}^S$, which means that Φ_{ξ}^- is not problematic. To see that Φ_{ξ} is not problematic, consider the diagram

$$\begin{array}{ccccc} M_{\theta}^S & \xleftarrow{i_D} & M_{\nu,l} & = & \mathcal{M}_{\gamma}^{\mathcal{U}} \\ \uparrow i_{\xi+1,\theta}^S & & \uparrow i_0 & & \uparrow i_0 \\ \mathcal{M}_{\xi+1}^S & \xleftarrow{i_{\bar{D}}} & C(\mathcal{M}_{\xi+1}^S) & = & \mathcal{M}_{\eta+1}^{*\mathcal{U}} \end{array}$$

Here $\bar{D} = D(\mathcal{M}_{\xi+1}^S)$ if $\mathcal{M}_{\xi+1}^S$ has type 2, and \bar{D} is principal otherwise. $D = i_{\xi+1,\theta}^S(\bar{D})$. Note \mathcal{M}_{θ}^S has type 2 iff $\mathcal{M}_{\xi+1}^S$ has type 2, because $\mathcal{M}_{\xi+1}^S$ is stable and $\rho(\mathcal{M}_{\xi+1}^S) \leq \text{crt}(i_{\xi+1,\theta}^S)$. We now calculate

$$\begin{aligned}(\Lambda_{\xi})_{C(\mathcal{M}_{\xi+1}^S)} &= (\Sigma_{\tau}^{\mathcal{U}})_{\mathcal{M}_{\mu+1}^{*\mathcal{U}}} \\ &= \Omega_{\nu,l}^{i_0} \\ &= \Lambda_{\theta}^{i_D \circ i_0} \\ &= \Lambda_{\theta}^{i_{\xi+1,\theta}^S \circ i_{\bar{D}}} \\ &= \Lambda_{\xi+1}^{i_{\bar{D}}}.\end{aligned}$$

⁷⁰At this point we know that $\mathcal{M}_{\gamma}^{\mathcal{U}} = M_{\nu,l}$ is not α_{ξ} -sound, so $\xi + 1 <_S \theta$.

Line 1 holds because $(\mathcal{M}_\xi^S, \Lambda_\xi) = (\mathcal{M}_\tau^U, \Sigma_\tau^U)$. Line 3 holds by pullback consistency for Σ and the fact that $\Sigma_\tau^U = \Omega_{\nu,l}$. Line 3 holds because $\Lambda_\theta^{iD} = \Omega_{\nu,l}$. The last line holds by Claim 4. Thus Φ_ξ is not problematic, contradiction. \square

The final claim completes our proof by contradiction.

Claim 12. $E_{\alpha_\xi}^{\mathcal{M}_\xi^S} = \emptyset$.

Proof. Otherwise $E_{\alpha_\xi}^{\mathcal{M}_\xi^S} = (E_\tau^U)^-$, so $\text{lh}((E_\tau^U)^-) = \alpha_\xi$. By Claim 10,

$$\hat{\lambda}(E_\tau^U) \leq \rho.$$

Let $F = E_\mu^U$ be the first extender used in $[0, \gamma]_U$ such that $\text{lh}(F) \geq \alpha_\xi$. We claim that $\mu = \tau$. For if not,

$$\alpha_\xi < \text{lh}(F),$$

and F is incompatible with E_τ^U , so that

$$\text{crt}(F) < \hat{\lambda}(E_\tau^U) \leq \rho \leq \alpha_\xi.$$

But this means that there is an $N \trianglelefteq \mathcal{M}_\gamma^U$ such that $\text{lh}(F) \leq o(N)$ and $\rho(N)$ is in the open interval $(\text{crt}(F), \text{lh}(F))$, contrary to Proposition 4.13.⁷¹

Thus $F = E_\tau^U$ and $\tau + 1 \leq_U \gamma$. But

$$\rho(\mathcal{M}_{\tau+1}) \notin (\text{crt}(F), \text{lh}(F)],$$

and

$$\rho^-(\mathcal{M}_{\tau+1}^U) > \text{lh}(F).$$

(Here we use that F has plus type to rule out $\rho(\mathcal{M}_{\tau+1}^U) = \text{lh}(F)$. This implies that whenever $N \trianglelefteq \mathcal{M}_{\tau+1}^U$, $\text{lh}(F) \leq o(N)$, and $\rho(N) = \text{lh}(F)$, then $N \in \mathcal{M}_{\tau+1}^U$. That in turn implies that $\tau + 1 \neq \gamma$.⁷²

Now let $G = E_\eta^U$, where $\eta + 1 \leq_U \gamma$ and $U\text{-pred}(\eta + 1) = \tau + 1$. $\text{crt}(G) > \hat{\lambda}(F)$, so $\text{crt}(G) > \alpha_\xi \geq \rho$. Since $\rho = \rho(N)$ where either $N = \mathcal{M}_\gamma^U$ or $N = M_{\nu,l} \trianglelefteq \mathcal{M}_\gamma^U$, we get $\rho = \rho(\mathcal{M}_{\tau+1}^U)$, from 4.13. Moreover, U must drop at $\eta + 1$, with $\rho(\mathcal{M}_{\eta+1}^{*\mathcal{U}}) = \rho$, and have no further drops. It follows that \mathcal{M}_γ^U has type 1 and is not $l + 1$ -sound, $\mathcal{M}_\gamma^U = M_{\nu,l}$, and

$$\begin{aligned} \mathcal{M}_{\eta+1}^{*\mathcal{U}} &= \mathfrak{C}_{l+1}(M_{\nu,l}) \\ &= \mathfrak{C}_{l+1}(C(\mathcal{M}_\theta^S)) \\ &= C(\mathcal{M}_{\xi+1}^S). \end{aligned}$$

⁷¹Namely, $N = M_{\nu,l}$ if $M_{\nu,l} \triangleleft \mathcal{M}_\gamma^U$, and $N = \mathcal{M}_\gamma^U$ otherwise.

⁷²Otherwise $N = M_{\nu,l}$ or $N = \mathcal{M}_\gamma^U$ is a counterexample.

Moreover the anticore map from $\mathcal{M}_{\eta+1}^{*,\mathcal{U}}$ to $M_{\nu,l}$ is

$$i_0 = i_{\eta+1,\gamma}^{\mathcal{U}} \circ i_{\eta+1}^{*,\mathcal{U}}.$$

There is enough agreement between $\mathcal{M}_{\tau+1}^{\mathcal{U}}$ and $\text{Ult}_0(\mathcal{M}_{\tau}^{\mathcal{U}}, F)$ that $\mathcal{M}_{\eta+1}^{*,\mathcal{U}} \triangleleft \text{Ult}(\mathcal{M}_{\tau}^{\mathcal{U}}, F)$. This implies that Φ_{ξ}^- is not problematic, because the ‘‘ultrapower away’’ conclusion holds. To see that Φ_{ξ} is not problematic, consider first the diagram

$$\begin{array}{ccccc} M_{\theta}^{\mathcal{S}} & \longleftarrow & M_{\nu,l} & = & \mathcal{M}_{\gamma}^{\mathcal{U}} \\ \uparrow i_{\xi+1,\theta}^{\mathcal{S}} & & \uparrow i_0 & & \uparrow i_0 \\ \mathcal{M}_{\xi+1}^{\mathcal{S}} & \longleftarrow & C(\mathcal{M}_{\xi+1}^{\mathcal{S}}) & = & \mathcal{M}_{\eta+1}^{*,\mathcal{U}} \\ & & \downarrow i_{\bar{D}} & & \end{array}$$

(Again, $\bar{D} = D(\mathcal{M}_{\xi+1}^{\mathcal{S}})$ if $\mathcal{M}_{\xi+1}^{\mathcal{S}}$ has type 2 and \bar{D} is principal otherwise, and $D = i_{\xi+1,\theta}^{\mathcal{S}}(\bar{D})$.) Calculating as above, we get

$$\begin{aligned} (\Sigma_{\tau+1}^{\mathcal{U}})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}} &= \Omega_{\nu,l}^{i_0} \\ &= \Lambda_{\theta}^{i_D \circ i_0} \\ &= \Lambda_{\theta}^{i_{\xi+1,\theta}^{\mathcal{S}} \circ i_{\bar{D}}} \\ &= \Lambda_{\xi+1}^{i_{\bar{D}}}. \end{aligned}$$

Since $\Lambda_{\xi} = \Sigma_{\tau}^{\mathcal{U}}$, what we must see is that

$$((\Sigma_{\tau}^{\mathcal{U}})_{\langle F \rangle})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}} = (\Sigma_{\tau+1}^{\mathcal{U}})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}}. \quad (4.9)$$

That is, the tail of Σ after the length 2 stack $\langle \mathcal{U} \upharpoonright \tau + 1, \langle F \rangle \rangle$ agrees with the tail of Σ after the length 1 stack $\mathcal{U} \upharpoonright \tau + 2$, as far as trees based on $\mathcal{M}_{\eta+1}^{*,\mathcal{U}}$ goes. This follows from the fact that Σ normalizes well. For let $\mathcal{W} = W(\mathcal{U} \upharpoonright \tau + 1, F)$ and consider the embedding normalization diagram

$$\begin{array}{ccccc} M & \xrightarrow{i_{0,\tau}^{\mathcal{U}}} & \mathcal{M}_{\tau}^{\mathcal{U}} & \xrightarrow{i_F} & \text{Ult}(\mathcal{M}_{\tau}^{\mathcal{U}}, F) \\ & \searrow i_{0,\beta}^{\mathcal{W}} & & & \downarrow \sigma \\ & & & & \mathcal{M}_{\beta}^{\mathcal{W}} \end{array}$$

$E_{\tau}^{\mathcal{W}} = F = E_{\tau}^{\mathcal{U}}$, so $\mathcal{W} \upharpoonright \tau + 2 = \mathcal{U} \upharpoonright \tau + 2$. Let

$$\alpha = \alpha_{\xi}^{+, \text{Ult}(\mathcal{M}_{\tau}^{\mathcal{U}}, F)}.$$

It is not hard to see that $\alpha < \text{lh}(E_{\tau+1}^{\mathcal{W}})$, so since Σ is strategy coherent,

$$(\Sigma_{\tau+1}^{\mathcal{U}})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}} = (\Sigma_{\beta}^{\mathcal{W}})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}}.$$

Because Σ normalizes well,

$$(\Sigma_{\tau}^{\mathcal{U}})_{\langle F \rangle} = (\Sigma_{\beta}^{\mathcal{W}})^{\sigma}.$$

But $\sigma \upharpoonright \alpha = \text{id}$ by the elementary properties of embedding normalization, so

$$((\Sigma_{\tau}^{\mathcal{U}})_{\langle F \rangle})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}} = (\Sigma_{\beta}^{\mathcal{W}})_{\mathcal{M}_{\eta+1}^{*,\mathcal{U}}}.$$

Putting these equalities together, we get equation 4.9.

Thus Φ_{ξ} is not problematic, a contradiction. \square

Claims 11 and 12 are the contradiction that finishes our proof of the Condensation Theorem, 4.6. \square

We can drop the hypothesis that $\text{crt}(\pi) < \rho_{\text{deg}(H)}(H)$ from Theorem 4.6, at the cost of omitting its conclusions concerning condensation of the external strategies. This will be useful in the proof of square and full normalization.

Theorem 4.14. *Assume AD^+ , and let (M, Λ) be a mouse pair with scope HC. Let H be a sound premouse, $\pi: H \rightarrow M$ be nearly elementary, and suppose that*

- (1) $\rho(H) \leq \text{crt}(\pi)$, and
- (2) $H \in M$.

Then either

- (a) $C(H) \triangleleft M$, or
- (b) $C(H) \triangleleft \text{Ult}(M, E_{\alpha}^M)$, where $\alpha = \text{crt}(\pi)$.

Proof. Let $\alpha = \text{crt}(\pi)$, n be largest such that $\alpha < \rho_n(H)$, and $n \leq \text{deg}(H)$. Let G and N be the same as H and M , except that $\text{deg}(G) = n = \text{deg}(N)$. Let $\Psi = \Sigma_N^{\pi}$. The hypotheses of 4.6 hold of (G, Ψ) , (N, Σ_N) , and π . (We have $H \in M$ by 4.3, hence $G \in N$, hence G is not the α -core of N .) Hence one of the conclusions of 4.6 holds of them.

If it is conclusion (a), then $C(G) \triangleleft N$, which easily implies $C(H) \triangleleft M$. If it is (b), then $C(G) \triangleleft \text{Ult}_0(N, \dot{E}_{\alpha}^M)$ yields $C(H) \triangleleft \text{Ult}_0(M, \dot{E}_{\alpha}^M)$. \square

References

- [1] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

- [2] G. Sargsyan. *Hod Mice and the Mouse Set Conjecture*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2015.
- [3] Grigor Sargsyan and Nam Trang. *The largest Suslin axiom*. Submitted. Available at math.rutgers.edu/~gs481/lisa.pdf.
- [4] Ernest Schimmerling and John R. Steel. Fine structure for tame inner models. *Journal of Symbolic Logic*, 61(01):621–639, 1996.
- [5] Ernest Schimmerling and Martin Zeman. Characterization of \square_κ in core models. *Journal of Mathematical Logic*, 4(01):1–72, 2004.
- [6] Ralf Schindler and Martin Zeman. Fine structure. In *Handbook of set theory*, pages 605–656. Springer, 2010.
- [7] Ralf-Dieter Schindler, John Steel, and Martin Zeman. Deconstructing inner model theory. *The Journal of Symbolic Logic*, 67(2):721–736, 2002.
- [8] F. Schlutzenberg and N. Trang. Scales in hybrid mice over \mathbb{R} . 2016. Submitted. Available at math.uci.edu/~ntrang.
- [9] Benjamin Siskind and John R. Steel. Full normalization for mouse pairs. *arXiv preprint arXiv:2207.11065*, 2022.
- [10] John R. Steel. Local hod computation. Handwritten notes, available at math.berkeley.edu/~steel/papers/Publications.html.
- [11] John R. Steel. $\text{HOD}^{L(\mathbb{R})}$ is a core model below θ . *Bulletin of Symbolic Logic*, 1(1):75–84, 1995.
- [12] John R. Steel. Ordinal definability in models of determinacy. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*. Cambridge University Press, 2016. Cambridge Books Online.
- [13] John R. Steel. A comparison process for mouse pairs. 2022. Available at math.berkeley.edu/~steel/papers/Publications.html.
- [14] John R. Steel and W. Hugh Woodin. HOD as a core model. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*, pages 257–346. Cambridge University Press, 2016. Cambridge Books Online.
- [15] Nam Trang. Hod in natural models of AD^+ . *Annals of Pure and Applied Logic*, 165(10):1533–1556, 2014.
- [16] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.