

# Homogeneously Suslin sets

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I. What is (pure) descriptive set theory about?

Answer #1 Definable sets of reals.

- but "definable" is vague
- DST is about projective,  $L(\mathbb{R})$ -definable, etc., sets, but it goes further.
- So far, DST is not about arbitrary,  $OD(\mathbb{R})$  sets of reals.

Answer #2 Homogeneously Suslin sets.

- So far, yes.
- Maybe not forever.

## Plan

1. Definition of "homogeneously Suslin", and related concepts.
2. • Closure properties of  $\text{Hom}$   
• Generic absoluteness and  $\text{Hom}$
3. • Iteration strategies in  $\text{Hom}$   
• Generic absoluteness from iterable mice
4. Does  $\text{Hom}$  go "all the way"?

Rmks Will assume large cardinal hypotheses without regret.

This is a survey, but not a history.

Def For  $T$  a tree on  $\omega \times \mathbb{Z}$  and  $x \in \omega$

$x \in p[T]$  iff  $\exists f \in \mathbb{Z}^\omega \forall n (x|n, f|n) \in T$

Def A homogeneity system for  $T$  is a map  $s \mapsto \mu_s$  ( $s \in \omega^{<\omega}$ ) s.t.

- $\mu_s$  is a countably complete u.f.

s.t.  $\mu_s(T_s) = 1$

- $s \leq t \Rightarrow \mu_s$  compatible with  $\mu_t$

- $x \in p[T]$  iff  $\langle \mu_{x|n} / n \in \omega \rangle$  is a countably complete tower

(iff  $\text{dir lim } \text{Ult}(V, \mu_{x|n})$  is wellfounded)

Def  $T$  is  $\kappa$ -homogeneous iff  $T$  admits a homogeneity system consisting of  $\kappa$ -complete measures.

Also call  $p[T]$   $\kappa$ -homogeneous

Def  $\text{Hom}_\kappa = \{A \subseteq {}^\omega \omega \mid A \text{ is } \kappa\text{-homogeneous}\}$

$$\text{Hom}_{<\lambda} = \bigcap_{\kappa < \lambda} \text{Hom}_\kappa$$

$$\text{Hom}_\infty = \text{Hom}_{<\text{ORD}}$$

### Rmks

- $\text{Hom}_\kappa$  is closed downward under  $\leq_w$ .
- $\alpha \leq \beta \Rightarrow \text{Hom}_\beta \subseteq \text{Hom}_\alpha$ . Since  $\leq_w$  is wellfounded, there are only finitely many different pointclasses  $\text{Hom}_\alpha$ .
- The concept was explicitly isolated by Martin and Kechris.

Thm (Martin '68) Every  $\text{Hom}_\kappa$  set is determined.

Thm (Martin '68) If  $\kappa$  is measurable, then every  $\Pi^1_1$  set is  $\text{Hom}_\kappa$ .

Prop. (K. Windseus) Equivalents are:

1.  $A \in \text{Hom}_K$

2. There are  $i_{st} : M_s \rightarrow M_t$ , for  $s \leq t$ , such that  $i_{su} = i_{tu} \circ i_{st}$  for  $s \leq t \leq u$ , and each  $M_s$  is transitive and closed under  $2^{\aleph_0}$  sequences, and

$x \in A \iff \text{dir lim}_n M_{x \upharpoonright n}$  is wellfounded

and  $\text{crit } i_{st} \geq K$  for all  $s \leq t$ , and  $M_0 = V$ .

Rmk So  $A$  is homogeneously Suslin iff  $A$  is continuously reducible to wellfoundedness of towers of measures.

Def  $A$  is  $\kappa$ -weakly homogeneous iff  
for some  $\kappa$ -homogeneous  $B \subseteq {}^\omega \omega \times {}^\omega \omega$ ,

$$A(x) \iff \exists y B(x, y)$$

If  $T$  on  $(\omega \times \omega) \times \mathbb{Z}$  witnesses  $B$   
is  $\kappa$ -homogeneous, then regarding  $T$   
as a tree on  $\omega \times (\omega \times \mathbb{Z})$ , we call  
 $T$   $\kappa$ -weakly homogeneous.

Prop  $T$  is  $\kappa$ -weakly homogeneous iff  
there is a countable set  $\sigma$  of  $\kappa$ -comp  
measures s.t.

$x \in p[T]$  iff  $\exists$  tower  $\langle \mu_i \mid i < \omega \rangle$  s.t.

- each  $\mu_i \in \sigma$  and  $\mu_i(T_{x \restriction i}) = 1$   
and
- $\langle \mu_i \mid i < \omega \rangle$  is countably complete

## The Martin-Solovay tree

Let  $\mu_{s,t}$  for  $s, t \in \omega^{<\omega}$  witness that  $T$  is weakly homogeneous. Let  $\langle t_i \mid i < \omega \rangle$  enumerate  $\omega^{<\omega}$  naturally. Fix

$$(s, \langle \alpha_0, \dots, \alpha_{|hs|-1} \rangle) \in U$$

iff  $\alpha_0 < \gamma$  and  $\forall i < j \leq |hs|-1$

$$t_i \subsetneq t_j \Rightarrow \pi(d_i) > \alpha_j,$$

for

$$\pi: U_{\pi}(V, \mu_{s \upharpoonright t_i, t_i}) \rightarrow U_{\pi}(V, \mu_{s \upharpoonright t_j, t_j})$$

We write

$$U = ms(T, \vec{\mu}, \gamma)$$

For  $\gamma$  sufficiently large

$x \notin p[T]$  iff  $\forall y \ U_{\pi}(V, \langle \mu_{x \upharpoonright n, y \upharpoonright n} \mid n < \omega \rangle)$   
is illfounded

$$\text{iff } x \in p[ms(T, \vec{\mu}, \gamma)]$$

Thm (Marrin, Solovay) Let  $\vec{\mu}$  witness that  $T$  is  $\kappa$ -weakly homogeneous, and  $G$  be  $V$ -generic /  $\mathbb{P}$ , where  $|\mathbb{P}| < \kappa$ . Then for  $\delta > |T|$

$$V[G] \models p[ms(T, \vec{\mu}, \delta)] = {}^\omega \omega - p[T]$$

Rmk We say that  $T$  and  $ms(T, \vec{\mu}, \delta)$  are  $\kappa$ -absolute complements, and call  $p[T]$   $\kappa$ -universally Baire. (Feng, Mag Woodin)

Cor (Marrin, Solovay) If  $\kappa$  is measurable  $G$  is  $V$ -generic /  $\mathbb{P}$  for  $|\mathbb{P}| < \kappa$ , and  $\varphi$  is  $\Sigma_3'$ , then for all  $x \in {}^\omega \omega \cap V$

$$V \models \varphi[x] \text{ iff } V[G] \models \varphi[x]$$

Prt homogeneous  $T$  for  $\Pi_1'$  yields weakly homogeneous  $T$  for  $\Sigma_2'$  yields  $ms(T, \vec{\mu}, \delta)$  for  $\Pi_2'$ .



Thm (Martin, Steel '85) Let  $T$   
be  $\delta^+$ -weakly homogeneous via  $\vec{\mu}$ ,  
where  $\delta$  is Woodin. Then for all  
sufficiently large  $\delta$

$ms(T, \vec{\mu}, \delta)$  is  $\alpha$ -homogeneous  
for all  $\alpha < \delta$ .

Cor If  $\lambda$  is a limit of Woodin cardinals  
then  $Hom_{<\lambda}$  is closed under  $\mathcal{F}^R, \mathcal{V}^R$ .

Cor If  $\lambda$  is a limit of Woodins, then  
for any  $\Sigma_n^1 \varphi$ , any  $x \in {}^\omega \omega \cap V$ :  
 $V \models \varphi[x]$  iff  $V[G] \models \varphi[x]$

for  $G \ V\text{-gen}/P$ , with  $|P| < \lambda$ .

Rmk The corollary was first proved in '84  
by Woodin, using generic embedding  
techniques of Foreman, Magidor, and Shelah.

Thm (Woodin, late 80's) Let  $\delta$  be Woodin, and  $T, T^*$  be  $\delta^+$ -absolute complements. Then  $T$  is  $\alpha$ -weakly homogeneous, for all  $\alpha < \delta$ .

Rmk First proved using stationary tower forcing. There is a proof using iteration trees and iterations to make reals generic. Shows  $\text{Hom}_\infty = \infty$ -univ. Baire.

Woodin proved a "tree production lemma", which yields absolutely complementing trees projecting to  $\{x \mid \varphi(x)\}$  and  $\{x \mid \neg \varphi(x)\}$  from a certain generic absoluteness property of  $\varphi$ .

Stationary tower forcing implies that kind of absoluteness for the  $\varphi$  defining, e.g.,  $\mathbb{R}^\#$ .

Thm (Woodin)  $\mathbb{R}^\# \in \text{Hom}_\infty$ ; in fact,  
 $A \in \text{Hom}_\infty \Rightarrow A^\# \in \text{Hom}_\infty$ . (Provided  
 there are arbitrarily large Woodin  
 cardinals.)

Thm (S.) Every  $\text{Hom}_\infty$  set admits  
 a  $\text{Hom}_\infty$  scale. (Provided there  
 are arbitrarily large Woodin cardinals.)

Rmk In sum

generic absoluteness of  $\varphi(\cdot)$   $\Rightarrow$

$\{x \mid \varphi(x)\}$  univ. Baire  $\Rightarrow$

$\{x \mid \varphi(x)\}$  weakly homog.  $\Rightarrow$

$\{x \mid \varphi(x)\}$  homogeneous

## The Derived model theorem

Let  $\lambda$  be a limit of Woodin,  
and  $G$  generic for  $\text{Col}(\omega, < \lambda)$   
and

$$R^* = \bigcup_{\alpha < \lambda} R \cap V[G \cap \text{Col}(\omega, < \alpha)]$$

Put

$$A \in \text{Hom}^* \iff A = p[T] \cap R^*, \text{ for some } T \text{ s.t. } \exists \alpha < \lambda$$

$V[G \cap \text{Col}(\omega, < \alpha)] \models T$  is  
 $< \lambda$  weakly homogeneous

Thm (Woodin)

$$(a) \quad L(R^*, \text{Hom}^*) \models \text{AD}^+$$

$$(b) \quad \text{Hom}^* = \text{Suslin-co-Suslin sets of } L(R^*, \text{Hom}^*)$$

(c) The Suslin-co-Suslin sets of any model of  $\text{AD}^+$  (eg any initial seg. of  $\text{Hom}_\infty$ ) can be realized this way).

## Reduction to $(\Sigma_1^2)^{Hom_\infty}$

Def. A  $(\Sigma_1^2)^{Hom_\infty}$  sentence (formula) is one of the form

$$\exists A \in Hom_\infty (HC, \epsilon, A) \models \varphi$$

A  $(\Sigma_1^2)^{Hom_\infty}$  set is one defined by such a formula.

Rmk. Let  $A \in \text{Hom}_{<\lambda}$ , and  $G$  be  $V$ -generic for  $\mathbb{P}$  s.t.  $|\mathbb{P}| < \lambda$ . We

set

$$A^{V[G]} = p[T]^{V[G]}, \text{ where } T \text{ witnesses that } A \text{ is } |\mathbb{P}|^+-\text{homogeneous}$$

(Independent of  $T$ !)

If  $\lambda$  is a limit of Woodins, we get

$$(HC^V, \epsilon, A) \prec (HC^{V[G]}, \epsilon, A^{V[G]})$$

Theorem (Woodin)  $(\Sigma_1^2)^{\text{Hom}_{<\infty}}$  sentences are absolute for set forcing, granted that there are arbitrarily large Woodin cardinals.

Proof Upward absoluteness was just done.

Now let

$$V[g] \models \exists A \in \text{Hom}_\infty \psi$$

Pick  $\delta$  Woodin,  $\delta > |P|$  where  $g$  is  $\mathbb{P}$ -generic

Let

$$j: V \rightarrow M \subseteq V[g]$$

be the generic embedding associated to  
a stationary tower generic  $G$  s.t.

$g \in V[g]$ . Then

$$V[g] \models \exists A \in \text{Hom}_\infty \psi$$

and

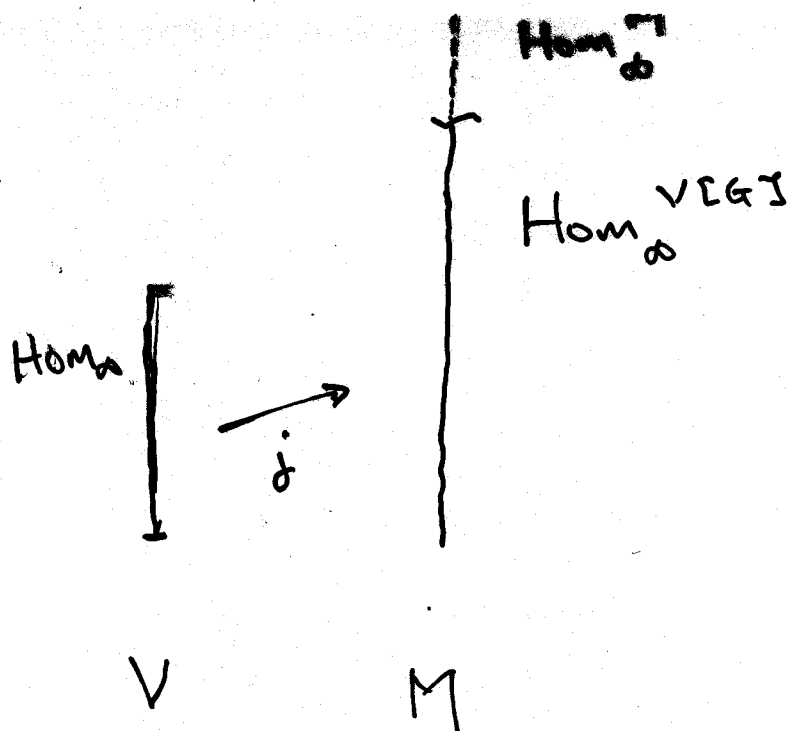
$$\text{Hom}_\infty^{V[g]} \subseteq j(\text{Hom}_\infty)$$

so

$$M \models \exists A \in \text{Hom}_\infty \psi$$

so

$$V \models \exists A \in \text{Hom}_\infty \psi$$



- $\mathbb{R}^M = \mathbb{R}^{VEG}$
- $\text{Hom}_{\infty}^{VEG}$  is a wedge initial segment of  $\text{Hom}_{\infty}^M$

Rmk All generic-absoluteness-of- $\varphi$  proofs so far can be cast: large card. hypothesis  $H$  is s.t.

$$H \vdash \varphi \leftrightarrow \varphi^*$$

where  $\varphi$  is  $(\Sigma_1^2)^{\text{Hom}_{\infty}}$ .

(Almost, anyway.) E.g.  $L(\mathbb{R})$  truth, or truth in the minimal model of  $AD_{\mathbb{R}}$ , or ... is provably (from some large cardinal)  $(\Sigma_1^2)^{\text{Hom}_{\infty}}$ .



Question Does some large cardinal hypo. (e.g. proper class of supercompact) imply statements of the form

$$\forall x \in {}^\omega \omega \exists A \in \text{Hom}_\infty \psi$$

(where  $\psi$  has only real quantifiers) are absolute for set forcing?

Question Does some large cardinal hypothesis (e.g. proper class of supercompact) imply

"There is a real which is not

$(\Sigma_1^2)^{\text{Hom}_\infty}$  in a countable ordinal".

The current limits of generic absoluteness:

Def Let

$$x \in A \iff \exists n_0 \forall n_1 \exists n_2 \dots \bigwedge_{\alpha < \omega_1} R(x, \langle n_\beta \mid \beta < \alpha \rangle)$$

where  $R$  is  $\Pi_1'$ . Then we say

$A$  is  $\exists^{w_1}(\text{closed-}\Pi_1')$

Prop  $\text{CH} \Rightarrow \exists^{w_1}(\text{closed-}\Pi_1') = \Sigma_1^2$

Theorem (Woodin '88?) Assume there are arbitrarily large measurable Woodin cardinals. Then

(a)  $\exists^{w_1}(\text{closed-}\Pi_1') \subseteq \text{Hom}_\infty$

(b)  $\exists^{w_1}(\text{closed-}\Pi_1')$  statements are absolute for set forcings.

Remarks • We don't yet have a reduction of  $\mathfrak{L}^{\omega_1}$  (closed- $\Pi_1'$ ) statements to  $(\Sigma_1^2)^{\text{Hom}_\infty}$  statements.

A proof of the determinacy of the long games would probably give this.

Alternatively, it would follow from the existence of a  $\text{Hom}_\infty$  iteration strategy for a countable

$M \models \exists$  measurable Woodin.

- The analog of the theorem for  $\mathfrak{L}^{\omega_1}$  (open- $\Pi_1'$ ) is an open problem

Corollary (Woodin '88) Assume there are arbitrarily large measurable Woodin cardinals. Let  $V[g_1]$  and  $V[g_2]$  be set generic extensions satisfying CH. Then for any  $\Sigma_1^2$   $\varphi$ :

$$V[g_1] \models \varphi \text{ iff } V[g_2] \models \varphi.$$

### Questions

- Is  $\diamond$  a "complete invariant" at the  $\Sigma_2^2$  level?
- Are there "conditional generic absoluteness" theorems at the  $\Sigma_n^2$  level, for all  $n$ ?

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Theorem (Woodin, late 90's)

The universal  $(\Sigma_1^2)^{\text{Hom}_\infty}$  set  
is  $\Sigma_3^2$ .

Cor.  $\Pi_3^2$  truth cannot be reduced  
to  $(\Sigma_1^2)^{\text{Hom}_\infty}$  truth.

Remark So the existence of conditional  
generic absoluteness theorems at the  
 $\Sigma_3^2$  level contradicts Woodin's  
 $\Omega$  conjecture.

### III Hom<sub>∞</sub> iteration strategies

#### The Iteration game

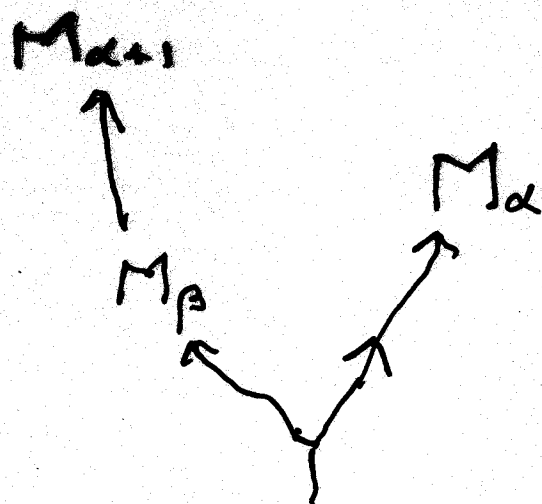
Given a transitive  $M$  and ordinal  $\Theta$  we say  $M$  is  $\Theta$ -iterable iff  $\text{II}$  has a winning strategy in the following game of  $\Theta$  moves:

Set  $M_0 = M$ .

Move  $\alpha+1$ : Player  $\text{I}$  picks an extender  $F_\alpha$  from  $M_\alpha$ , and some  $\beta < \alpha$  s.t.

$$M_{\alpha+1} = \text{Ult}(M_\beta, F_\alpha)$$

makes sense.  $\text{I}$  wins if  $M_{\alpha+1}$  is illfounded. Otherwise, the game continues.



Move 1 II picks a branch  $b$  of the tree built so far s.t.

$$M_\lambda = \text{dir lim}_{\alpha \in b} M_\alpha$$

is wellfounded, and s.t.  $b$  is cofinal in  $\lambda$ . If he doesn't, he loses.

If II hasn't lost after  $\Theta$  moves then he wins.

A winning strategy for II is a  $\Theta$ -iteration strategy for  $M$ .

Question. Is every countable  $M \prec V$   
 $\omega_1 + 1$  - iterable?

Question. Let  $L[\vec{E}]$  be a canonical  
inner model constructed using full  
background extenders. Is every countable  
 $M \prec L_\alpha[\vec{E}]$   $\omega_1 + 1$  iterable?

Remark In each case, can lift plays  
of the iteration game (iteration  
trees) on  $M$  to iteration trees  
on  $V$ . Suppose we try to iterate  $M$   
by making sure that the corresponding  
branch of the lifted tree "is wellfounded".  
(I.e. yields a wellfounded direct limit model".

Thm (Morris, S. 1986) This works so  
long as there is exactly one cofinal  
wellfounded branch in the lifted tree on  $V$ .



Theorem (Woodin) Assume  $\exists j: L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1}^j)$

Then there is a countable iteration tree on  $V$  having distinct cofinal wellfounded branches.

For  $L[\vec{E}]$  models which are not too complicated, we have had some success. E.g.

Theorem (Mitchell, S. 1990) Assume there are arbitrarily large Woodin cardinals. Then there is a canonical minimal  $L[\vec{E}]$  model

$$M_w^\# \models \exists \omega \text{ Woodin cardinals}$$

which has a unique  $\Theta^+$ -iteration strategy for all regular  $\Theta$ .

Remark An  $\omega_1$ -iteration strategy for a countable  $M$  is essentially a set of reals. The known iterability proofs give  $\text{Hom}_\infty$  iteration strategies.  $\text{Hom}_\infty$  strategies extend to uncountable trees.

Cor to prt Granted arbitrarily large Woodin cardinals,  $M_\omega^\#$  has a  $\text{Hom}_\infty$  iteration strategy.

Correctness of iterable mice ;  
generic absoluteness from "iterable  
mice :

Thm (Woodin, S. early 90's) There is a recursive  $\varphi \mapsto \varphi^*$  s.t.

$$L(R) \models \varphi \text{ iff } M_\omega^\# \models \varphi^*$$

(Granted that  $M_\omega^\#$  is  $\omega_1+1$  iterable - and exists)

Remark Thus  $L(R)$  truth is  $(\Sigma_1^2)^{\text{Hom}_{\aleph_0}}$ ,  
and absolute for set forcing,  
granted arbitrarily large Woodin  
cardinals.

Remark A similar result holds for  
 $\aleph_1$ , (closed- $\Pi_1'$ ) truth vis-a-vis  
truth in the minimal  $L[\vec{E}]$  model  
satisfying "there is a measurable  
Woodin" which is  $\aleph_1+1$  iterable.

(Provided there is such a model.)

(Woodin, <sup>early 90's</sup> ~~1990~~)

This would reduce  $\aleph_1$ , (closed- $\Pi_1'$ )  
truth to  $(\Sigma_1^2)^{\text{Hom}_{\aleph_0}}$  truth, provided  
we got  $\text{Hom}_{\aleph_0}$  iteration strategies.

Question How far to  $\text{Hom}_{\omega}$  iteration strategies go?

Some warning signs at the level of supercompacts:

- 1) There are iteration trees on  $V$  of length  $\omega$  having no cofinal branch if there is an  $\omega$ -extendible card. and "long extenders" are allowed in our trees. (Neeman)
- 2) In trees with long extenders, it is impossible to completely avoid "moving generators", so the proof of comparison ~~at the~~ seems to break down.

In sum, we don't really ~~know~~ have even a good iterability conjecture at the

level of supercompacts.

Can some other  $(\Sigma, {}^2)$   $\text{Hom}_\infty$  condition  
identify the canonical mice at the  
supercompact level?

Thm (Woodin, late 90's) Let  $\kappa$  be  
extendible, and  $G$  be  $V$ -generic  
for  $\text{Col}(\omega, \kappa)$ . Then

$$L(\text{Hom}_\infty, \mathbb{R})^{V[G]} \equiv L(\text{Hom}_\infty, \mathbb{R})^{V[G][H]}$$

for all  $H$  generic over  $V[G]$ , moreover

$$L(\text{Hom}_\infty, \mathbb{R})^{V[G]} \models \text{AD}_\mathbb{R}.$$

Cor ~~ANALYSIS~~ No wellorder of  $P(\kappa) \cap V$   
can be  $(\Sigma, {}^2)$   $\text{Hom}_\infty$  in  $V[G]$ .

Seven conjectures Assume there are arb. large supercompact. 25

1. There is a  $\text{Hom}_{\infty}$  iteration strategy for an  $L[\vec{E}]$  model with a superstrong
2. There is a  $\text{Hom}_{\infty}$  set of reals coding an  $L[\vec{E}, \mathbb{R}]$  model with a superstrong
3. No countable  $M \prec V$  has a  $\text{Hom}_{\infty}$  iteration strategy
4. The  $\Omega$  conjecture is false.
5. There are  $\Sigma_n^2$  conditional generic absoluteness theorems, for all  $n$ .
6. There is no ~~most~~ canonical inner model with an extendible containing all the reals and coded by a  $\text{Hom}_{\infty}$  set of reals.
7. There is a canonical minimal model with an extendible, but its truth set is not  $(\Sigma_1^2)^{\text{Hom}_{\infty}}$ .

In this picture,  $\text{Hom}$  does not go "all the way".

There are generic absoluteness theorems, and notions of canonicity for mice, which are not ~~even~~ enforced by  $\text{Hom}_\infty$  sets.