“High” is definable in the partial order of the Turing degrees of the recursively enumerable sets

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Synopsis

We will discuss the model theoretic aspects of the partial order of the Turing degrees of the recursively enumerable sets, denoted $\mathcal{R}$. We will give an overview of what is known, and indicate the proofs of the following theorems.

Harrington and Shelah. The first order theory of $\mathcal{R}$ is not decidable.

Harrington and Slaman; Slaman and Woodin. The first order theory of $\mathcal{R}$ is recursively isomorphic to the first order theory of arithmetic.

Nies, Shore, and Slaman. Suppose that A is an arithmetic relation on $\mathcal{R}$ and A is invariant under double-jump equivalence. Then A is definable in $\mathcal{R}$. Further, *high* is definable in $\mathcal{R}$.

Interpretations within $\mathcal{R}$

In arithmetic, Gödel numbers represent first finite sequences and then Turing machines. We will use elements of $\mathcal{R}$ to represent first order structures, in particular countable partial orders.

Definition 1 A *coding* in $\mathcal{R}$ of a partial order P is given by a finite sequence of recursively enumerable degrees $\boldsymbol{p}$ and two formulas $\varphi_1(x, \boldsymbol{y})$ and $\varphi_2(x_1, x_2, \boldsymbol{y})$ with the following properties.

1. Define
   (a) $\boldsymbol{G} = \{g : \mathcal{R} \models \varphi_1(g, \boldsymbol{p})\}$;
   (b) for g_1 and g_2 in $\boldsymbol{G}$, $g_1 \succeq g_2$ if and only if $\mathcal{R} \models \varphi_2(g_1, g_2, \boldsymbol{p})$;
   (c) and let $(\boldsymbol{G}, \succ)$ be the partial order induced by $\succeq$ on the $\succeq$-equivalence classes.
2. $(\boldsymbol{G}, \succ)$ is isomorphic to P.

Undecidability

Theorem 2 *Let T be the set of sentences true in every partially ordered set.*

1. *T is undecidable.*
2. *T is equal to the set of sentences true in every Δ^0_2 partially ordered set.*

Slaman and Woodin Coding

Definition 3 A *Slaman and Woodin coding* of a partial order P is a sequence of recursively enumerable degrees p, q, r and l with the following properties.

1. Define

   (a)
   $$\boldsymbol{G} = \left\{ g : \mathcal{R} \models \begin{array}{c} g \text{ is a minimal element of} \\ \{w : r \geq w \text{ and } w + p \geq q\} \end{array} \right\}$$

   (b) for g_1 and g_2 in $\boldsymbol{G}$, $g_1 \succeq g_2$ if and only if $\mathcal{R} \models g_1 + l \geq g_2$;

   (c) and let $(\boldsymbol{G}, \succ)$ be the partial order induced by $\succeq$ on the $\succeq$-equivalence classes.

2. $(\boldsymbol{G}, \succ)$ is isomorphic to P.

Existence of Partial Order Codings

Theorem 4 (Slaman and Woodin) *Let $(\omega, >)$ be a Δ^0_2 partial ordering of ω. There are recursively enumerable sets P, Q, R and L and a uniformly recursively enumerable sequence of sets $(G_i : i \in \omega)$ such that the following properties hold.*

1. *The degrees of P, Q, R and L are a Slaman and Woodin Coding of $(\omega, >)$.*

2. *Further the map taking the equivalence class of the degree of G_i to that of i in P is an isomorphism between the coded partial order and the partial ordering induced by P.*

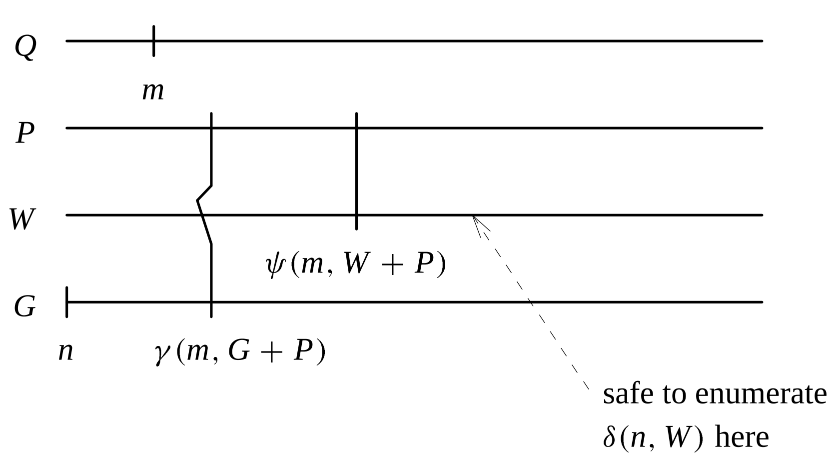
The Proof

Of course, there is a priority construction behind the existence statement for P, Q, R, L and the sequence $(G_i : i \in \omega)$. Without going into too much detail, we can describe the basic logical form to make G minimal below R joining P above Q.

- First, we enumerate Γ and ensure that $\Gamma(G + P) = Q$.
- Then, we consider the following requirement.

  If $W = \Phi(R)$ and $\Psi(W + P) = Q$, then there is a Δ such that $\Delta(W) = G$.

  If $W = \Phi(R)$ and for each n it is not possible to make $\Psi(n, W + P)$ different from $Q(n)$, then we must enumerate Δ and ensure that for each n, $\Delta(n, W) = G(n)$.



By preserving these computations, we can preserve the ability to enumerate n into G within the constraint that $\Delta(W) = A$.

Why a priority construction?

- The requirements to be satisfied are Boolean combinations of Π^0_2-statements.
- The objects to be constructed are recursive enumerations.

Consequently, meeting the dense sets which ensure that the requirement is satisfied must go ahead without knowing which disjunct in the statement of the requirement will be true in the limit.

In the following, we will use various input sets to modulate the way in which we meet these dense sets.

Undecidability

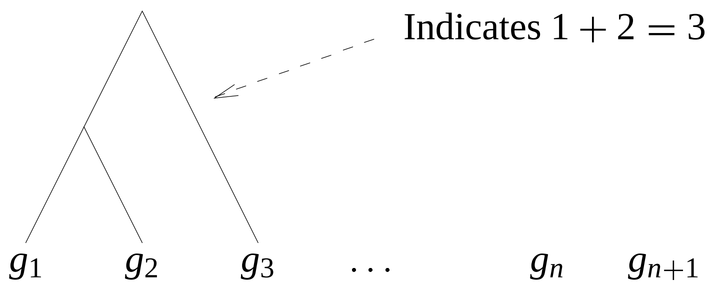
The representation of Δ^0_2 partially ordered sets provides an interpretation of the theory of all partially ordered sets within that of $\mathcal{R}$.

> φ is true in all partially ordered sets if and only if for every partially ordered set P coded in $\mathcal{R}$, φ holds in P.

The latter condition is expressed in $\mathcal{R}$ by quantifying over codes and rewriting φ to be interpreted within the partial orders they define.

Theorem 5 (Harrington and Shelah) *The first order theory of $\mathcal{R}$ is not recursive.*

Partial orders as models of arithmetic



integers are represented by minimal elements of the order

An Observation on Building Codes for Models

In the priority construction to produce a the coded model of arithmetic, the strategies which directly effect the integers of the coded model are simple elaborations on the finite injury method.

Any nonrecursive recursively enumerable set can be used to drive such a construction. That is to say that for any recursively enumerable y, there is a coded model of arithmetic $\mathfrak{M}$ such that all of the integers of $\mathfrak{M}$ are recursive in y.

Recognizing Standard Models

We can use partially ordered sets to represent models of arithmetic, but how can we define a collection of codes for standard models of arithmetic?

Fix T to be finitely axiomitized theory so that any model of T is an end extension of a standard model of arithmetic.

Comparing Models of T

Definition 6 Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are models of T. A *comparison* between initial segments of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is a bijection between the two initial segments which preserves the operations of the structures when viewed as relations.

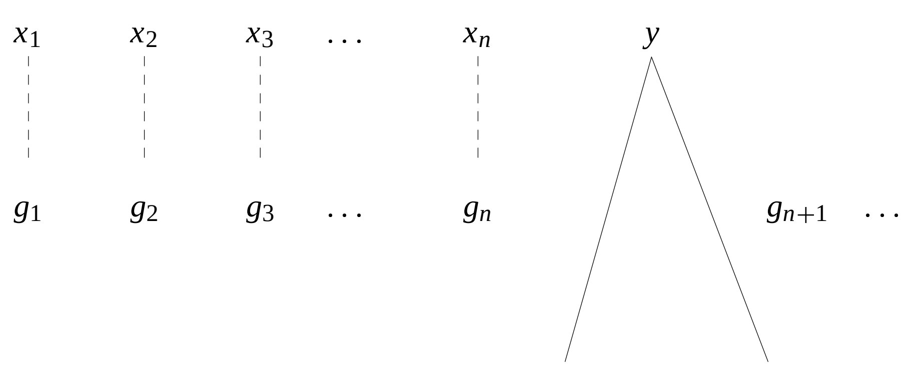
- If $\mathfrak{M}_1$ is standard and $\mathfrak{M}_2 \models T$, then any two comparisons from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ are compatible.
- If $\mathfrak{M}_1$ is standard and $\mathfrak{M}_2 \models T$, then the universe of $\mathfrak{M}_1$ is the domain of a comparison from $\mathfrak{M}_1$ to $\mathfrak{M}_2$.

Comparisons Between Models – In the Codes

Recall, x is *low* if and only if $x' = 0'$. There is a direct way to compare coded models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ all of whose integers are low.

Lemma 7 *Suppose that $x_1, \ldots, x_n$ and y are low recursively enumerable degrees. Then there is a standard model $\mathfrak{M}$ coded by low parameters and with integers represented by $g_1, g_2, \ldots$ such that the following conditions hold.*

1. *For each i less than or equal to n, $x_i \geq_T g_i$.*
2. *For each i and j less than or equal to n, if $x_j \not\geq_T x_i$, then $x_j \not\geq_T g_i$.*
3. *For each k greater than n, $y \not\geq_T g_k$*



Making the Comparison

Suppose that r is low and that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are models whose integers (antichains) are all below r.

- For each $n \in \omega$, we can find a model $\mathfrak{M}$ such that
  - for i between 1 and n, the ith integer of $\mathfrak{M}$ picks out (by being below) the ith integer of $\mathfrak{M}_1$ and
  - for i between $n+1$ and $2n$, the ith integer of $\mathfrak{M}$ picks out the ith integer of $\mathfrak{M}_2$.
- In the codes, we have a system of compatible maps comparing the finite initial segments of $\mathfrak{M}_1$ and $\mathfrak{M}_2$.
- Roughly speaking: In the codes, we have a comparison between initial segments of $\mathfrak{M}_1$ and of $\mathfrak{M}_2$ which includes their standard parts.

A Definable Class of Standard Models

Definition 8 Say that a model $\mathfrak{M}_1$, coded by p, q, r and l as above, is *certifiable* if and only if for every other model $\mathfrak{M}_2$ with integers recursive in r, the above comparison between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ includes all of $\mathfrak{M}_1$ in its domain and is an isomorphism between $\mathfrak{M}_1$ and the restriction of $\mathfrak{M}_2$ to its range.

- Every standard $\mathfrak{M}_1$ with low parameters is certifiable.
- Since every nontrivial parameter bounds the integers of a standard model $\mathfrak{M}$, if $\mathfrak{M}_1$ is certifiable, then $\mathfrak{M}_1$ is isomorphic to a standard model $\mathfrak{M}$.
- Thus, the set of certifiable models is definable in the codes, is not empty, and consists of isomorphic copies of the standard model of arithmetic.

True Arithmetic

Theorem 9 (Harrington and Slaman; Slaman and Woodin) *The first order theory of $\mathcal{R}$ is recursively isomorphic to the first order theory of arithmetic.*

We can reduce the theory of arithmetic to that of $\mathcal{R}$ as follows.

> φ is true in arithmetic if and only if for every certifiable model $\mathfrak{M}$ coded in $\mathcal{R}$, φ holds in $\mathfrak{M}$.

Defining Jump Classes

Suppose that x is a recursively enumerable degree.

- The isomorphism type of $[0, x]$ is a definable invariant of x. In the codes, it contains representations of structures which are definable from x.
- We should obtain a syntactic invariant $\Delta(x)$ from x by proving two technical lemmas.
  - Every structure $\mathfrak{M}$ with a presentation in $\Delta(x)$ is coded by parameters recursive in x.
  - Every structure $\mathfrak{M}$ coded by parameters recursive in x has a presentation in $\Delta(x)$.

Then, we should use the coding of arithmetic to conclude that for a given Δ, the set of x for which $\Delta = \Delta(x)$ is definable in $\mathcal{R}$.

Prompt Simplicity

Problem: Due to subtleties of timing, our construction of codes of structures cannot be implemented below an arbitrary recursively enumerable degree. An arbitrary recursively enumerable set may never change at the right time.

Solution: There is a class of recursively enumerable degrees, the prompt degrees, for which this problem does not occur. Prompt sets are generic with respect to the stages during which they change. Further, by a theorem of Ambos-Spies, Jockusch, Shore, and Soare, the set of prompt degrees is definable in $\mathcal{R}$.

Double Jump Invariance

We will refer to a technical notion, *Effectively Coded Set of Integers*, but not give the full definition for it. An ECSI consists of a code for an enhanced model of arithmetic and a subset A of its integers.

Definition 10 Let $ECSI(x)$ be the set of A such that A is effectively coded by parameters which are recursive in x.

Theorem 11 (Nies, Shore, and Slaman) *The following conditions are equivalent.*

1. *A is $\Sigma^0_3(X)$.*
2. *For every promptly simple degree p, $A \in ECSI(x+p)$.*

The promptly simple degrees are used as generic parameters to construct generic codes; their effect can be integrated away.

For degrees $x'' = y''$ if and only if for their representatives $\Sigma^0_3(X) = \Sigma^0_3(Y)$. Consequently, the double jump classes are invariant within $\mathcal{R}$.

Definability of the Double Jump

Within $\mathcal{R}$, we can define the relation which associated the code $\boldsymbol{p}$ for $\mathbb{N}$ and the integer e to the set of degrees x such that W_e'' has degree x''.

- From $\boldsymbol{p}$, we can define the collection of codes for indices of sets which are $\Sigma^0_3(W_e)$.
- From x, we can define the collection of codes for indices of sets which are in $ECSI(x+p)$ for every promptly simple degree p.
- Using the algebra of comparison maps between such models, we can assert that these sets of indices are the same.

Of course, these two sets of indices are the same if and only if W_e'' has degree x''. It follows that the class of x's such that W_e'' has degree x'' is definable in $\mathcal{R}$.

Defining *High*

Recall x is *high* if and only if $x' = 0''$.

Theorem 12 (Nies, Shore, and Slaman) *$x \in \mathcal{R}$ is high if and only if $\mathcal{R} \models (\forall y)(\exists z \leq x)[z'' = y'']$.*

The left-to-right direction uses the Robinson Jump Interpolation Theorem; the right-to-left direction uses a theorem of Stob and Soare.

Food for Thought

Is the following relation definable?

> $\boldsymbol{p}$ codes arithmetic with the number e and x is the degree of W_e

Claiming to refute a conjecture of Slaman and Woodin, Cooper has asserted that it is not definable.