Long games

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Contents:

Fix $\alpha < \omega_1$. $\mathbb{R}^\alpha$ denotes $\mathbb{N}^{\omega\cdot\alpha}$. Let $C \subset \mathbb{R}^\alpha$ be given. $G_\alpha(C)$ is played as follows:

I	n_0		n_2		$\cdots$	n_ω		$\cdots$	
II		n_1		$\cdots$			$n_{\omega+1}$		$\cdots$

I and II alternate playing natural numbers n_ξ for $\xi < \omega \cdot \alpha$. I plays first at limits.

I wins if $\langle n_\xi \mid \xi < \omega \cdot \alpha \rangle \in C$. Otherwise II wins.

$G_\alpha(C)$ is **determined** if one of the players has a winning strategy.

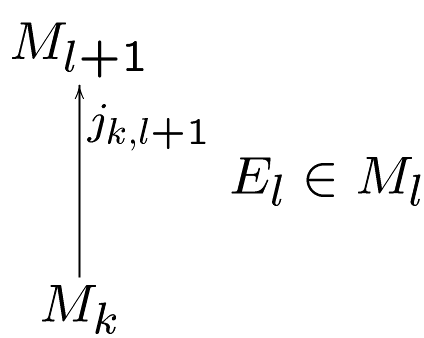
Aim to show (using roughly α Woodin cardinals) that for $\mathbf{\Pi}^1_1$ sets C, $G_\alpha(C)$ is determined.

This continues a line of results of Martin, Steel, and Woodin. There are more results along these lines, on games of variable countable length.

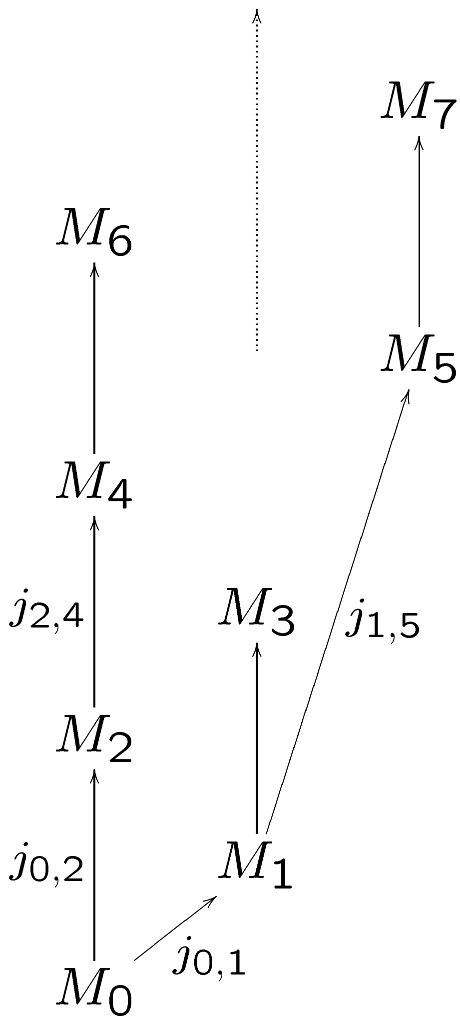
An **iteration tree** $\mathcal{T}$ of length ω consists of

- a tree order T on ω,
- a sequence of models $\langle M_k \mid k < \omega \rangle$, and
- embeddings $j_{k,l}: M_k \to M_l$ for $k \, T \, l$.

Each model M_{l+1} for $l + 1 > 0$ is an ultrapower of a preceeding model. More precisely: $M_{l+1} = \mathrm{Ult}(M_k, E_l)$, where E_l an extender picked from M_l, and k is the T predecessor of $l + 1$. $j_{k,l+1}$ is the ultrapower embedding.



An iteration tree on M is a tree with $M_0 = M$.

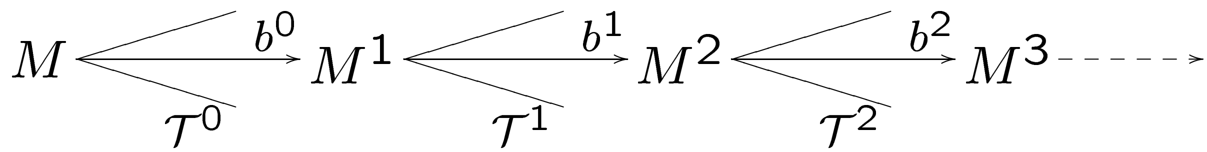


Our trees will always have an **even branch**, $M_0, M_2, M_4, \ldots$, giving rise to the direct limit M_{even}.

The tree structure on the odd models will usually be some permutation of $\omega^{<\omega}$. With each **odd branch** b we associate the direct limit M_b.

(In this example, $0\ T\ 1$, $0\ T\ 2$, $1\ T\ 3$, etc.)

In the **iteration game*** on M, players "good" and "bad" collaborate to produce:



"Bad" plays iteration trees $\mathcal{T}^i$ on M^i. "Good" plays branches b^i through $\mathcal{T}^i$. We let M^{i+1} be the direct limit model determined by b^i.

Once the game ends, we let M^ω be the direct limit of the models M^i.

"Good" wins if all models created (including M^ω) are wellfounded. Otherwise "bad" wins.

M is **iterable** if the good player has a winning strategy. An **iteration strategy** for M is a winning strategy for the good player.

Elementary substructures of V are iterable in this sense (Martin–Steel).

*The definition given here is specialized to our context.

Suppose $M \models$ "δ is a Woodin cardinal," and in V there are M–generics for col(ω, δ). Let $\dot{A}$ name a set of reals in $M^{\text{col}(\omega,\delta)}$. Define a game $\mathcal{G}(M, \delta, \dot{A})$ as follows:

I and II collaborate playing a real x as usual. In addition I plays an iteration tree $\mathcal{T}$ on M, and ordinals witnessing M_{even} is illfounded.

I wins if for every odd branch b, there exists g so that

1. g is col$(\omega, j_b(\delta))$–generic/M_b, and
2. $x \in j_b(\dot{A})[g]$.

Otherwise II wins.

Lemma I. Either II wins $G_\omega(\dot{A})$ in $M^{\text{col}(\omega,\delta)}$; or else I wins $\mathcal{G}(M, \delta, \dot{A})$ in V.

The Lemma is proved by methods similar to those introduced by Martin and Steel in “A proof of projective determinacy.” (More on this later.) It can be used to prove the determinacy of $G_{\omega\cdot\omega}(C)$, for $\mathbf{\Pi}^1_1$ sets C.

Let $\mathcal{A}(x_n)$ denote the “auxiliary” part of the game in Martin’s proof of $\mathbf{\Pi}^1_1$ determinacy. (I plays ordinals witnessing the linear order associated to (x_n) is wellfounded; II plays descending ordinals.)

The determinacy proof for $G_{\omega\cdot\omega}(C)$ breaks into parts as follows:

- Round 1 in $\mathcal{A}$, followed by
- An application of Lemma I, followed by
- Round 2 in $\mathcal{A}$, followed by
- An application of Lemmas I, followed by
- Round 3 in $\mathcal{A}$, etc.

Fix $C \subset \mathbb{R}^\omega$, $\mathbf{\Pi}^1_1$. Fix a continuous embedding $(x_n) \mapsto L(x_n)$, so that

- $L(x_n)$ is a linear order on ω and
- $(x_n) \in C$ iff $L(x_n)$ is wellfounded.

Continuous means that $L(x_n) \restriction k+1$ depends only on $x_0 \restriction k, \ldots, x_{k-1} \restriction k$. $L(x_0 \ldots, x_{k-1})$ stands for $L(x_n) \restriction k+1$.

Fix M and $\langle \delta_k \mid k < \omega \rangle$ so that

- M is a class model.
- Each δ_k is a Woodin cardinal in M, and countable in V.
- M is iterable.
- The continuous embedding L belongs to M.

Fix some $\mu \in M$, an ordinal much greater than $\sup\{\delta_k\}$. For expository simplicity, fix

- g_0 which is col(ω, δ_0)–generic/M;
- g_1 which is col(ω, δ_1)–generic/$M[g_0]$;
- g_2 which is col(ω, δ_2)–generic/$M[g_0 * g_1]$;

etc.

Work to specify definable predicates P_0, Q_0 over M; P_1, Q_1 over $M[g_0]$; P_2, Q_2 over $M[g_0 * g_1]$; etc.

P_k and Q_k will be predicates on tuples of the form $(x_0, \ldots, x_{k-1}, \alpha_0, \ldots, \alpha_{k-1}, \gamma)$ where

- Each x_i is a real in $M[g_0 * \cdots * g_i]$,
- The α_i-s are ordinals below μ, matching the order given by $L(x_0, \ldots, x_{k-1}) \restriction k$, and
- γ is an ordinal greater than μ.

(Let us call tuples of this form **k-sequences**.)

To define P_k and Q_k, consider the following game F_k:*

$$\begin{array}{c|cc} \text{I} & \alpha_k & \\ \hline \text{II} & & \gamma^* \end{array} \quad x_k$$

- I plays $\alpha_k < \mu$ so that $\alpha_0, \dots \alpha_{k-1}, \alpha_k$ match the order given by $L(x_0, \dots, x_{k-1})$.
- II plays $\gamma^* > \mu$ so that $\gamma^* < \gamma$.
- The two players then collaborate as usual to play $x_k \in M[g_0 * \dots * g_k]$.

The game is played in $M[g_0 * \dots g_k]$.

Observe that a run of F_k produces a $k+1$-sequence

$$(x_0, \dots, x_{k-1}, x_k, \alpha_0, \dots, \alpha_{k-1}, \alpha_k, \gamma^*)$$

*Defined with respect to a k-sequence $x_0, \dots, x_{k-1}, \alpha_0, \dots, \alpha_{k-1}, \gamma$.

Put a k-sequence $(x_0, \ldots, x_{k-1}, \alpha_0, \ldots, \alpha_{k-1}, \gamma)$ in P_k iff (in $M[g_0, * \cdots * g_k]$) II has a strategy in F_k to produce a $k+1$-sequence which belongs to P_{k+1}.

Note that this can be decided in

$$M[g_0, * \cdots * g_{k-1}],$$

since g_k is added by a homogenous forcing. So P_k is indeed a predicate over $M[g_0 * \cdots * g_{k-1}]$.

Put a k-sequence $(x_0, \ldots, x_{k-1}, \alpha_0, \ldots, \alpha_{k-1}, \gamma)$ in Q_k iff I has a strategy in F_k to produce a $k+1$-sequence which belongs to Q_{k+1}.

(Note the change from "II has a strategy" to "I has a strategy.")

Again this can be decided in $M[g_0 * \cdots * g_{k-1}]$.

The predicates P_k, Q_k are defined by induction, not on k but on γ: To determine whether $(\cdots, \cdots, \gamma)$ belongs to P_k we require knowledge of membership of P_{k+1}, but only for $k+1$-sequences $(\cdots, \cdots, \gamma^*)$ with $\gamma^* < \gamma$.

Claim 1. For any γ, either $(\emptyset, \emptyset, \gamma) \notin P_0$, or $(\emptyset, \emptyset, \gamma) \notin Q_0$.

Proof. Assume for contradiction $(\emptyset, \emptyset, \gamma)$ belongs to both P_0 and Q_0.

Then in $M[g_0]$ we have two strategies in F_0: A strategy for II to enter P_1, and a strategy for I to enter Q_1.

Letting the strategies play each other we obtain α_0, γ_0, x_0 so that

1. $(\alpha_0, x_0, \gamma_0)$ belongs to both P_1 and Q_1; and
2. $\gamma_0 < \gamma$.

Using 2 we may now repeat the process. We obtain α_1, γ_1, x_1 so that $(\alpha_0, \alpha_1, x_0, x_1, \gamma_1)$ belongs to both P_1 and Q_1; and $\gamma_1 < \gamma_0$.

Continuing this way get an infinite descending sequence of γ_k-s, a contradiction. $\square$

Wlog, M has nothing but L above the Woodin cardinals. Precisely, $M = \mathsf{L}(\mathsf{V}_\tau^M)$ where $\tau = \sup\{\delta_k \mid k < \omega\}$. We can talk about Silver indiscernibles for M.

Claim 2. Suppose that for all γ, $(\emptyset, \emptyset, \gamma) \notin P_0$. Then (in V) I wins $G_{\omega\cdot\omega}(C)$.

Proof. Let γ be an indiscernible for M. We have $(\emptyset, \emptyset, \gamma) \notin P_0$.

Let $\lambda > \gamma$ be another indiscernible. Observe that

$$\begin{aligned}&(\alpha_0, \ldots, \alpha_{k-1}, x_0, \ldots, x_{k-1}, \gamma) \in P_k \iff \\ &(\alpha_0, \ldots, \alpha_{k-1}, x_0, \ldots, x_{k-1}, \lambda) \in P_k\end{aligned}$$

for k-sequences $(\alpha_0, \ldots, \alpha_{k-1}, x_0, \ldots, x_{k-1}, \gamma)$.

This will allow us to play the move γ^* in F_k: Instead of playing the game F_k associated to $(\ldots\ldots, \gamma)$, we will switch to the game associated to $(\ldots\ldots, \lambda)$ and play $\gamma^* = \gamma < \lambda$.

Let us play against some imaginary opponent in $G_{\omega\cdot\omega}(C)$. We describe how to play for I and win.

By assumption $(\emptyset, \emptyset, \gamma) \notin P_0$. By indiscernibility, $(\emptyset, \emptyset, \lambda) \notin P_0$.

So, in $M^{\text{col}(\omega,\delta_0)}$: $\exists \alpha_0$ (a valid move in F_0) $\forall$ (valid) γ^*, II does not have a strategy to produce x_0 so that $(x_0, \alpha_0, \gamma^*) \in P_1$.

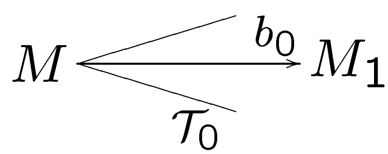
Fix α_0 witnessing this, and apply the statement with $\gamma^* = \gamma$.

In $M^{\text{col}(\omega,\delta_0)}$, II does not have a strategy to produce $x_0 \in \mathbb{R}$ so that $(x_0, \alpha_0, \gamma) \in P_1$.

Using Lemma I we have a strategy Σ_0 for I in the corresponding game $\mathcal{G}$.

Let the imaginary opponent play against Σ_0. We obtain:

- $x_0 \in \mathbb{R}$,
- An iteration tree as follows:



- g_0 so that

  1. g_0 is col$(\omega, \delta_0{}^{\mathsf{s}})$–generic/$M_1$, and
  2. $(x_0, \alpha_0{}^{\mathsf{s}}, \gamma^{\mathsf{s}}) \notin P_1{}^{\mathsf{s}}$.

($\alpha_0{}^{\mathsf{s}}$ denotes α_0 “shifted” to M_1, i.e., $j_{0,1}(\alpha_0)$. Similarly with $P_1{}^{\mathsf{s}}$ etc.)

(b_0 is given by the iteration strategy for M.)

So far we covered the first ω moves in $G_{\omega\cdot\omega}(C)$.

$(x_0, {\alpha_0}^{\mathsf{s}}, \lambda^{\mathsf{s}}) \notin {P_1}^{\mathsf{s}}$ by 2 and indiscernibility.

So, in $M_1[g_0]^{\mathrm{col}(\omega, {\delta_1}^{\mathsf{s}})}$: $\exists \alpha_1$ (valid in F_1) $\forall$ (valid) γ^*, II does not have a strategy to produce x_1 so that $(x_0, x_1, {\alpha_0}^{\mathsf{s}}, \alpha_1, \gamma^*) \in {P_2}^{\mathsf{s}}$.

Fix α_1 witnessing this, and apply the statement with $\gamma^* = \gamma^{\mathsf{s}}$.

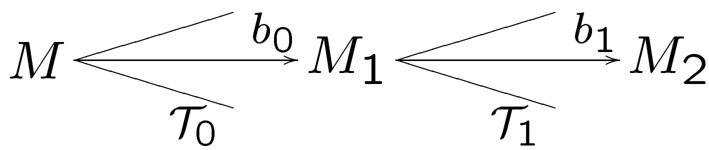
In $M[g_0]^{\mathrm{col}(\omega, {\delta_1}^{\mathsf{s}})}$, II does not have a strategy to produce $x_1 \in \mathbb{R}$ so that

$$(x_0, x_1, {\alpha_0}^{\mathsf{s}}, \alpha_1, \gamma^{\mathsf{s}}) \in {P_2}^{\mathsf{s}}.$$

Using Lemma I we have a strategy Σ_1 for I in the corresponding game $\mathcal{G}$ over $M_1[g_0]$.

Let the imaginary opponent play against Σ_1. We obtain:

- $x_1 \in \mathbb{R}$,
- A second iteration tree as follows:



- g_1 so that

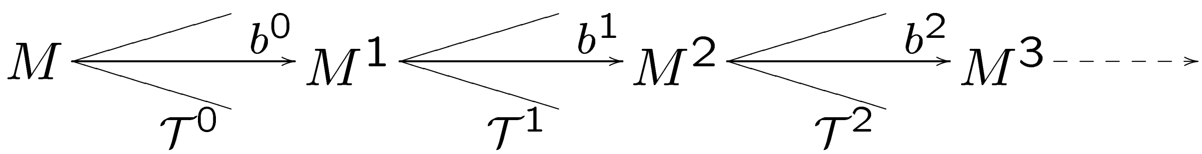
  1. g_1 is col$(\omega, \delta_1{}^{\mathrm{s}\,\mathrm{s}})$–generic/$M_2$, and
  2. $(x_0, x_1, \alpha_0{}^{\mathrm{s}\,\mathrm{s}}, \alpha_1{}^{-\,\mathrm{s}}, \gamma^{\mathrm{s}\,\mathrm{s}}) \notin P_2{}^{\mathrm{s}\,\mathrm{s}}$.

(A second $^{\mathrm{s}}$ denotes shifting from M_1 to M_2. b_1 is again obtained using the iteration strategy for M.)

So far we covered the first $\omega{\cdot}2$ moves in $G_{\omega\cdot\omega}(C)$.

Continuing this way we obtain:

1. $x_0 {}^\frown x_1 {}^\frown x_2 \cdots$, covering all moves in $G_{\omega\cdot\omega}(C)$.
2. Iteration trees, branches, and models as follows:



3. Ordinals $\alpha_i \in M_i$. At each stage k we know that

$$\alpha_0^{\mathsf{s\,s\,s}\cdots\mathsf{s}}, \alpha_1^{-\mathsf{s\,s}\cdots\mathsf{s}}, \alpha_2^{--\mathsf{s}\cdots\mathsf{s}}, \cdots, \alpha_k^{---\cdots-}$$

   (a tuple of ordinals in M_k) matches the order given by $L(x_0, \ldots, x_{k-1})$.

Let M^∞ be the direct limit of the models M_k. Note that M^∞ is wellfounded, because of our use of an iteration strategy in picking b_i.

Let $\alpha_n^\infty = j_{n,\infty}(\alpha_n)$. By 3, $(\alpha_n^\infty)_{n<\omega}$ matches the order given by $L(x_n)$. So this order is wellfounded, and $(x_n) \in C$.

$\square$(Claim 2)

Recall that μ was a fixed parameter in the definitions of P_k. Let us in retrospect fix a μ which is an indiscernible for M.

Claim 3. Suppose that there exists a γ so that $(\emptyset, \emptyset, \gamma) \in P_0$. Then (in V) II wins $G_{\omega \cdot \omega}(C)$.

Proof. By Claim 1, $(\emptyset, \emptyset, \gamma) \notin Q_0$. We can now mirror the argument of the previous Claim, showing that II wins $G_{\omega \cdot \omega}(C)$. □

Corollary. $G_{\omega \cdot \omega}(C)$ is determined. □

Recall the structure of the proof:

- Round 1 in $\mathcal{A}$, followed by
- An application of Lemma I, followed by
- Round 2 in $\mathcal{A}$, followed by
- An application of Lemma I, followed by
- Round 3 in $\mathcal{A}$, etc.

where $\mathcal{A}$ includes the auxiliary moves in the proof of $\mathbf{\Pi}^1_1$ determinacy.

If the proof of determinacy for $G_{\omega\cdot\omega}(C)$ gave us auxiliary moves ($\mathcal{A}^*$ say) with properties similar enough to those in $\mathcal{A}$, we could replace $\mathcal{A}$ by $\mathcal{A}^*$ in the scheme above, and get determinacy for games of length $\omega \cdot \omega + \omega \cdot \omega$.

In fact, by "breaking into parts and composing" we could prove determinacy for games of any fixed countable length. (This process of "breaking and composing" is similar in flavor to the one used by Martin to propagate the scale property under game quantifiers of fixed countable length.)

So, does the determinacy proof for $G_{\omega\cdot\omega}(C)$ give auxiliary moves, with properties similar to those in $\mathcal{A}$?

Does Lemma I give such moves?

Note: $\mathcal{A}$ includes the auxiliary moves in the proof of $\mathbf{\Pi}^1_1$ determinacy. The auxiliary moves in the proof of determinacy for **homogeneously Suslin** sets are just as good.

So if Lemma I gave us a homogeneously Suslin representation, that would be enough.

Lemma I draws on the following result of Martin and Steel:

Work in M, assuming δ is a Woodin cardinal.

Suppose $B \subset \mathbb{R} \times \mathbb{R}$ is δ^+ homogeneously Suslin. Let T be a tree witnessing this. We have $p[T] = B$.

Let
$$\begin{aligned} A &= \{x \in \mathbb{R} \mid \exists y \ (x, y) \in B\} \\ &= \{x \in \mathbb{R} \mid \exists y \exists f \ (x, y, f) \in [T]\}. \end{aligned}$$
Let $B^* = \neg A$.

Let T^* be the Martin–Solovay tree obtained from T. We have $p[T^*] = B^*$.

Theorem. (Martin–Steel) B^* is homogeneously Suslin. This is witnessed by T^*.

How to get from the Theorem to Lemma I?

Fix a name $\dot{A} \in M^{\text{col}(\omega,\delta)}$.

In M, let T be the tree of attempts to construct:

- $x = \langle x_0, x_1 \ldots \rangle \in R$,
- $\vec{p} = \langle p_0, p_1, \ldots \rangle$ a sequence of conditions in col(ω, δ), increasing in strength,
- $\dot{x}$ (or a sequence $\langle \dot{x}, 0, 0, 0, \ldots \rangle$), a name in $M^{\text{col}(\omega,\delta)}$,

so that

1. p_0 forces "$\dot{x} \in \dot{A}$," and
2. for each $n < \omega$, p_n forces "$\dot{x}(\check{n}) = \check{x_n}$."

Apply methods similar to the ones used by Marin and Steel to this tree. There are two main differences.

First, T is not homogeneously Suslin.

Secondly, what we need is not

$$A = \{x \in \mathbb{R} \mid \exists \vec{p}, \exists \dot{x},\ (x, \vec{p}, \dot{x}) \in [T]\},$$

but

$$A = \{x \in \mathbb{R} \mid \exists \textbf{generic } \vec{p}, \exists \dot{x},\ (x, \vec{p}, \dot{x}) \in [T]\}.$$

The second difference is the most serious.

Because of these differences we don't get a **tree** T^* which projects to the complement of A; we get a **game**.

Think of a tree as a game with just one player. Here we have to add a second player, to keep the first one honest about $\vec{p}$.

At any rate, what we get is close enough to a homogeneously Suslin representation, and can be used in the determinacy proof in place of the $\mathbf{\Pi}^1_1$ auxiliary moves.