A simple proof of Clairaut's theorem.

Let \( g \) be continuous on some disk \( D \subseteq \mathbb{R}^2 \), and suppose \( g_{xy} \) and \( g_{yx} \) are continuous on \( D \). We shall show \( g_{xy}(a,b) = g_{yx}(a,b) \) for all \((a,b) \in D\).

The following lemma, which comes directly from Fubini's theorem, is the key. (Fubini's theorem is proved in a note posted earlier.)

**Lemma.** Let \( R \) be a rectangle, with \( R \subseteq D \). Then

\[
\iiint g_{xy} \, dA = \iiint g_{yx} \, dA
\]

**Proof.** Let \( R = [a,b] \times [c,d] \).
Using Fubini + the fundamental thm of Calc:

\[ \iiint g_{xy} \, dA = \iiint g_{xy} \, dy \, dx \]
\[ = \int_{a}^{b} \int_{c}^{d} (g_{x}(x,d) - g_{x}(x,c)) \, dx \]
\[ = \left[ g(x,d) - g(x,c) \right]_{a}^{b} \]
\[ = g(b,d) - g(b,c) - g(a,d) + g(a,c). \]

Similarly

\[ \iiint g_{yx} \, dA = \iiint g_{yx} \, dx \, dy \]
\[ = \int_{c}^{d} \int_{a}^{b} (g_{y}(b,y) - g_{y}(a,y)) \, dy \]
\[ = \left[ g(b,y) - g(a,y) \right]_{c}^{d} \]
\[ = g(b,d) - g(a,d) - g(b,c) + g(a,c). \]

So \( \iiint g_{xy} \, dA = \iiiint g_{yx} \, dA \), as desired.
Lemma 2  Let \( f \) and \( h \) be continuous on \( D \), and suppose \( \int_R^{SSf} dA = \int_R^{SSh} dA \) for all rectangles \( R \subseteq D \). Then \( f = h \) on \( D \).

Proof. Suppose \( f(a, b) \neq h(a, b) \), say \( f(a, b) > h(a, b) \). Since \( f \) and \( h \) are continuous on \( D \), we can find a small rectangle \( R \subseteq D \) such that

\[
f(x, y) - h(x, y) > 0 \quad \text{for all} \quad (x, y) \in R.
\]

But then

\[
\int_R^{SSf-h} dA > 0,
\]

But

\[
\int_R^{SSf-h} dA = \int_R^{SSf} dA - \int_R^{SSH} dA = 0.
\]

This is a contradiction. \( \square \)

Clearly, lemmas 1 and 2 together show \( g_{xy} = g_{yx} \) on \( D \), as desired \( \square \)