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Arithmetics [1]

Part I. Integers and Polynomials

1. A grasshopper can jump p or q inches right or left on the line. Find all points on the line the grasshopper can reach starting from the origin.

2. Prove that the minimal positive integer d of the form $d = mp + nq$ coincides with the Greatest Common Divisor of p and q, and that $GCD(p, q)$ can be found by the following *Euclidean algorithm:*

Divide p by $q > 0$ with the remainder r: $p = lq + r$ where $q > r \geq 0$. If $r > 0$, proceed with q, r instead of p, q. If $r = 0$, stop. The last non-zero remainder d equals $GCD(p, q)$.

3. Find $GCD(2^{120} - 1, 2^{100} - 1)$, $GCD(n^{30} - 1, n^4 - 1)$.

4. *Polynomials.* A function of the form $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ with $a_0 \neq 0$ is called a *polynomial of degree n.*

Given two polynomials $p(x)$ and $q(x) \neq 0$, prove that the minimal degree polynomial of the form $m(x)p(x)+n(x)q(x)$ is the Greatest Common Divisor of $p(x)$ and $q(x)$ and can be found by the following algorithm:

Divide p by q with the remainder r: $p(x) = l(x)q(x) + r(x)$ where $\deg r(x) < \deg q(x)$. If $r \neq 0$, proceed with q and r instead of p and q. If $r = 0$, stop. The last non-zero remainder $d(x)$ is $GCD(p(x), q(x))$.

Notation: $\mathbb{Z}$ — the set of all integer numbers.
$\mathbb{Q}$ — the set of all rational numbers.
$\mathbb{R}$ — the set of all real numbers.
$\mathbb{R}[x]$ — the set of all polynomials with real coefficients.
$\mathbb{Q}[x]$ — the set of all polynomials with rational coefficients.
$\mathbb{Z}[x]$ — the set of all polynomials with integer coefficients.

5. In $\mathbb{Z}[x]$, find the minimal degree polynomial of the form $m(x)(x + 2) + n(x)x$. Apply the Euclidean algorithm to $p = x + 2, q = x$.

6. (a) An invertible element is called a *unit.* Find all units in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}[x]$.

(b) An element $a \neq 0$ is called *composite* if it can be factored as $a = bc$ where none of b, c is a unit. Is $x^3 - 2$ composite in $\mathbb{Q}[x]$? Which polynomials of the form $ax^3 + bx^2 + cx + d$ are composite in $\mathbb{R}[x]$?

(c) Prove that a polynomial equation of degree n has at most n solutions.

The Fundamental Theorem of Algebra says that any polynomial in $\mathbb{R}[x]$ *of degree* > 2 *is composite.*

Homework.

(a) Find the Greatest Common Divisor of 11...11 (100 ones), and 11...11 (60 ones).

(b) A Heffelump is a chess piece which moves like Knight but with p steps in one direction and q steps in the perpendicular direction. Determine for which p and q the Heffelump, starting from one cell on the infinite chess board, can reach any other cell.

(c) Prove that there exists an integer a such that $a+1, a+2, ..., a+1998$ are all composite.

(d) Prove that if ab is divisible by c but neither a nor b is divisible by c then c is composite. The same — for a, b, c in $\mathbb{R}[x]$ or $\mathbb{Q}[x]$. Can you prove the same statement for $\mathbb{Z}[x]$?

(e) Prove that any polynomial $x^n + a_1 x^{n-1} + ... + a_n$ of degree $n > 0$ can be factored into a product of linear and quadratic real polynomials (of the form $x - b$ or $x^2 + px + q$), and the factorization is unique up to a permutation of the factors.

(f) Formulate and prove a "unique prime factorization theorem" for $\mathbb{Q}[x]$.

(g) Prove a "unique prime factorization theorem" for $\mathbb{Z}[x]$.

Part II. Arithmetics modulo m

1. Find the last digit of 2^{1998}.

2. Given a positive integer m, two integers a and b are called *congruent modulo* m (write: $a \equiv b (\mod m)$ if $a - b$ is divisible by m (in other words, if a and b have the same remainder upon division by m).

(a) Suppose $ac \equiv bc (\mod 6)$ and $c \not\equiv 0 (\mod 6)$. Does it mean that $a \equiv b (\mod 6)$? The same — modulo 7?

(b) Prove that c is invertible modulo m if and only if $GCD(c, m) = 1$.

(c) Find the inverse to each remainder modulo 7.

(d) Compute 5^{103} modulo 7.

(e) Find all solutions of equation $x^2 = 1$ modulo 7.

Wilson's Theorem: For any prime integer p $1...(p-1) \equiv -1 (\mod p)$.

Fermat's Little Theorem: If p *is a prime integer then* $a^p \equiv a (\mod p)$ *for any* a.

3. Prove that $7^{120} - 1$ is divisible by 143.

4. Let p be prime.

(a) For any $a \not\equiv 0(\mod p)$ the sequence $a^k(\mod p), k = 0, 1, 2, ...,$ is periodic. If $r(a)$ is the minimal period then the remainders of $a, a^2, ..., a^{r(a)}$ are distinct.

(b) If the minimal periods $r(a)$ and $r(b)$ of the sequences $a^k(\mod p)$ and $b^k(\mod p)$ are relatively prime, then the minimal period of $(ab)^k(\mod p)$ equals $r(a)r(b)$.

(c) Let r be the Least Common Multiple of the minimal periods $r(a)$ and $r(b)$. Then there exists a remainder c with the minimal period $r(c) = r$.

(d) Let s be the Least Common Multiple of the minimal periods $r(a)$ for all $a = 1, 2, ..., p-1$. Then there exist a with $r(a) = s$. Deduce that $s < p$.

5.Let s be the same as in $4(d)$. Prove that $x^s - 1 \equiv (x-1)(x-2)...(x-(p-1))(\mod p)$. Deduce that $s = p-1$ and that all remainders $1, 2, ..., p-1$ are powers $a, ..., a^{p-1}(\mod p)$ of the same a.

6. For which prime numbers p the equation $x^2 \equiv -1(\mod p)$ has solutions? Find such a solution when it exists.

Homework

(a) Find 3^{100} modulo 7 and 7^{7^7} modulo 11.

(b) Find $1^2 + ... + 36^2$ modulo 37.

(c) Given a polynomial $p(x)$ with integer coefficients such that $p(1) = 2$. Show that $p(7)$ is never a perfect square.

(d) Could a perfect cube end with 0...01 (100 zeroes)?

(e) Let A be the sum of digits of 4444^{4444}, B be the sum of digits of A. Find the sum of digits of B.

(f) Prove that there are infinitely many prime numbers congruent to 3 modulo 4.

(g) Can you generalize Fermat's little theorem for a composite p.

(h) Prove that the equation $x^2 + y^2 = 3$ has no rational solutions, and $x^2 + y^2 = 1$ has infinitely many rational solutions.

(i) Prove that a spot of area > 1 on the lattice paper can be translated in such a way that it hits at least two points of the lattice.

(k) Prove that any convex spot of area > 4 centrally symmetric with respect to the origin of the lattice paper contains a non-zero lattice point.