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How to Write Proofs Part II [1]

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"PROOFS" Problem Set #2

Use Mathematical Induction to prove the following problems.

1. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$.
2. $n(n^2+5)$ is divisible by 6 for all natural numbers n.
3. For every positive integer n, the square of the sum of the first n positive integers is equal to the sum of the first n cubes.
4. For every positive integer n, $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer.
5. For every positive integer n, $6^{n+2} + 7^{2n+1}$ has 43 as a factor.
6. For all positive integers n, $n+1$ is either a prime or can be factored into primes.
7. For each natural number $n > 1$, let U_n be a real number with the property that for at least one pair of natural numbers p and q with $p+q = n, U_n = U_p + U_q$. (When $n = 1$, we define $U_1 = a$, where a is some given real number.) Prove that $U_n = na$ for all n.
8. For all natural numbers n, $2^{n-1} \leq n!$.
9. Prove that $(cos\theta)(cos2\theta)(cos4\theta)\cdots(cos2^{n-1}\theta) = \frac{sin2^n\theta}{2^n sin\theta}$ for every natural number n.
10. (Fermat's Little Theorem) $a^p \equiv a \pmod{p}$ where p is a prime.
11. A map can be colored with 2 colors iff all of its vertices have even degree.
12. (1998 IMO) If a, b and $q = \frac{a^2+b^2}{ab+1}$ are positive integers, then $q = \gcd(a,b)^2$.
13. $\frac{[(1+\sqrt{5})^n - (1-\sqrt{5})^n]}{2^n\sqrt{5}}$ is an integer for all natural numbers n.
14. Let p be any polynomial of degree m. Let $q(n)$ denote the sum $q(n) = p(1)+p(2)+p(3)+\cdots+p(n)$. Prove that there is a polynomial q of degree $m+1$ satisfying this equation.
15. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by n straight lines of which no three meet in a single common point and no two are parallel. Prove your result. Can you obtain a more general result when parallellism is permitted? If concurrence is permitted? If both are permitted?
16. (DeMoivre's Theorem) $[cos\theta + isin\theta]^n = cos(n\theta) + isin(n\theta)$ for any non-negative integers n.
17. The Fermat numbers $F_n = 2^{2^n} + 1$ are pairwise relatively prime for any nonnegative integers n.