PROBLEMS OF MARRIAGE

Eugene Mukhin

1. The best strategy to find the best spouse.

A person A is looking for a spouse, so A starts dating. After A dates the person B, A decides whether s/he wants to marry B or to reject B and start dating somebody else. Of course, after having dated two different persons B and C, A can tell which one was a better choice (but not before).

Assume that during life time A will meet n people who potentially can become A's spouse. A plays it honest – A can get married only once and A cannot get married to a person who was already rejected by A. In the end A does not think s/he is a looser only if A got married to the best person out of all n people s/he has met.

Question 1. What is the best strategy for A to get the best partner?

Question 2. What are A's chances of getting the best partner?

Example. The person A has a simple strategy – s/he marries the first date. Then the chances of A to win are $1/n$.

Let us call St_k the following strategy. A meets first k people and rejects them all. Then A marries the first person who is better than all the previous ones. Of course, here k can be $0, 1, 2, \dots, n-1$.

Question 1'. Which of the strategies St_k is the best?

Question 2'. What are the chances of A of getting the best partner if s/he follows strategy St_k?

Theorem 1. *If A uses strategy St_k then his/her chances are*

$$C(k) = \frac{1}{n}\left(1 + \frac{k}{k+1} + \dots + \frac{k}{n-1}\right).$$

Proof: Let us fix k, n and compute the chances of A to win.

a) There is a $1/n$ chance that the best partner will be number $k+1$ – in this case A wins.

b) There is a $1/n$ chance that the best partner is number $k+2$. In this case, A wins if s/he rejects the date number $k+1$. It means that the best partner out of the first $k+1$ partners was not number $k+1$. Chances for that are $k/(k+1)$. Total chances in this case are $(1/n)*(k/(k+1))$.

c) There is a $1/n$ chance that the best partner is number $k+3$. In this case, A wins if s/he rejects the dates number $k+1$ and $k+2$. It means that the best partner out of the first $k+2$ partners was neither number $k+1$ nor $k+2$. Chances for that are $k/(k+2)$. Total chances in this case are $(1/n)*(k/(k+2))$.

Continuing this argument we prove the Theorem. $\square$

Example.

a) $n=2$: $C_0 = C_1 = 1/2$.
b) $n=3$: $C_0 = 1/3$, $C_1 = 1/2$, $C_2 = 1/3$.
c) $n=4$: $C_0 = 1/4$, $C_1 = 11/24$, $C_2 = 10/24$, $C_3 = 1/4$.
d) $n=5$: $C_0 = 1/5$, $C_1 = 25/60$, $C_2 = 26/60$, $C_3 = 21/60$, $C_4 = 1/5$.

Now, we want to estimate, given n, for which k the number C_k is the greatest. To do that consider the difference

$$D_k := C_k - C_{k-1} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1} - 1\right).$$

Clearly, D_k are positive for $k = 1, \ldots, k_0$ and negative for $k = k_0+1, k_0+2, \ldots, n-1$. So, C_k is the greatest when $k = k_0$.

For k_0 we have (approximately)

$$\frac{1}{k_0} + \frac{1}{k_0+1} + \cdots + \frac{1}{n-1} \approx 1. \tag{1}$$

Lemma 1.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = \ln n + \gamma + a(n),$$

where $0 < \gamma < 1$ is some constant and $a(n)$ is a small number for large n.

Proof: a) Consider the graph of function $y = 1/x$. Then $1/2 + 1/3 + \cdots + 1/(n-1)$ is less than the area under the graph on the segment $1 \leqslant x \leqslant n$. This area equals $\ln n$, so $1 + 1/2 + 1/3 + \cdots + 1/n < \ln(n) + 1$.

b) On the same graph we see that $1/2 + 1/3 + \cdots + 1/(n-1)$ is greater than the area under the graph on segment $2 \leqslant x \leqslant n$, which equals $\ln(n) - \ln(2)$. So, $1 + 1/2 + 1/3 + \cdots + 1/(n-1) > \ln(n) - \ln(2) + 1 > \ln(n)$. Moreover, the difference between $1/2 + 1/3 + \cdots + 1/(n-1)$ and $\ln(n) - \ln(2)$ is growing with n.

The Lemma follows from combining a) and b). $\square$

Note that Lemma 1 implies that $1 + 1/2 + 1/3 + \cdots = \infty$. The constant γ is called *the Euler constant*, cf. Exercise 1.

Thus, for large n, formula (1) can be rewritten as $\ln(n) - \ln(k_0) \approx 1$, that is $\ln(n/k_0) \approx 1$. So, k_0 is approximately n/e.

In fact, one can show that for any n, k_0 is either integer part of n/e, $[n/e]$, or $[n/e]+1$.

In the same way C_{k_0} is approximately $1/e$. So, the chances of A to win decline with n from $1/2$ ($n = 2$ case) to $1/e$ ("$n = \infty$" case).

Fact. The strategy St_{k_0} is indeed the best strategy of all possible strategies.

2. Stable marriage

2.1. **Simple stable marriage.** Consider n men and n women. We would like to marry all of them to each other, meaning to pair them up, so that in each pair there is a man and a woman. We will call such pairing *a matching*. There are $n!$ matchings.

Let M be a matching. Suppose we have married couples (m_1, w_1) and (m_2, w_2). Suppose m_1 prefers w_2 to w_1 and w_2 prefers m_1 to m_2. In this case, the matching is not stable since m_1 and w_2 would leave their partners and marry each other. Such pair (m_1, w_2) is called *a blocking pair*. A matching is called *stable* if there are no blocking pairs.

Question. Given lists of preferences, does there exist a stable matching?

Examples. a) Suppose all men and all women have the same list of preferences. Then there exists a unique stable matching – the best man marries the best woman, the second best man marries the second best woman, etc.

b) Suppose all men have the same preferences. Then there is a unique stable matching. Namely, the best woman marries her first preference, then the second best woman marries her first preference out of men left, then the third marries from men left, and so on.

c) Suppose any two men have different first preferences. Then there is at least one stable matching: all men marry their first choices. If, in addition, all women have different first preferences then there is another stable matching.

d) Let $n = 3$, let we have preferences:

men	*women*
$1, 2, 3$	$2, 3, 1$
$3, 2, 1$	$2, 1, 3$
$2, 1, 3$	$3, 1, 2$

That is the first man prefers the first woman to the second and the second to the third, the second man prefers the third women to the second and the second to the first, etc. There are 3 stable matchings: $\{(1,1),(2,3),(3,2)\}$, $\{(1,2),(2,3),(3,1)\}$, $\{(1,3),(2,2),(3,1)\}$.

The algorithm.

Start with all men and women free.
While there is a single man m, do:
  Let w be the best woman in m list who m did not propose to yet.
  Make m propose to w.(but not before)
  If w is free then m and w become engaged.
  If w is engaged and likes her party better than m then m is rejected.
  If w is engaged and likes m better than her party then w breaks up with her partner and gets engaged to m.
If all men are engaged, marry all the engaged couples.

Theorem 2. *The algorithm always stops at some point.*

Proof: Each man cannot propose more than n times. If a man proposes the nth time (to the worst women on his list), it means that all other women have rejected him (or broke up with him for somebody else). Hence, all other women are engaged and the woman he proposes to is bound to be free. □

Theorem 3. *The resulting matching is stable.*

Proof: Suppose (m, w) is a blocking pair. It means that m has proposed to w and either was rejected or their engagement was broken. In both cases w got a better partner then m. Notice that after a woman got engaged her next partners get only better. So, (m, w) cannot be a blocking pair and we have a contradiction. □

Note that we proved that the algorithm stops and produces a stable matching in no more than n^2 cycles. This is very fast: even to check that some given matching is stable one has to check n^2 possible blocking pairs!

Corollary 4. *For any list of preferences there is at least one stable matching.*

Theorem 5. *Any use of the algorithm (with men as proposers) yields the same result.*

Proof: Suppose we have a result of our algorithm, M_1, and another stable matching M_2. Suppose a man m is married to w_1 in M_1 and to w_2 in M_2. We claim that m prefers w_1 to w_2.

Suppose m prefers w_2 to w_1. It means that w_2 has rejected m at some point when the algorithm produced M_1. Thus some men were rejected by stable partners. Without loss of generality, assume that it happened the first time when w_2 rejected m.

Then the partner of w_2 in M_1 at the time when she rejected m, we call him m_1, can have no stable partners better than w_2. Then (m_1, w_2) is a blocking pair in M_2. Contradiction.

Thus, in any matching we can get from our algorithm, each man obtains the best partner he can get in ANY stable matching. Obviously, such a matching is unique. □

Theorem 6. *In the resulting matching each man gets the best partner he can have in a stable matching, and each woman gets the worst partner she can get in a stable matching.*

Proof: We have already proved the first part. Now, let M_0 be the matching obtained from the algorithm. Let M be some other stable matching. Suppose a woman w likes her partner in M_0, m_1, more than her partner in M_1, m_2. Then the pair (m_2, w) is blocking for M_1. □

Corollary 7. *Pair up each man with his best stable partner. Then it is a matching, moreover, this matching is stable.*

We will call the stable matching produced by the algorithm with man as proposers the men-oriented matching and denote it M_0.

Note that the stable matching is unique if and only if the men-oriented version and the women-oriented version of the algorithm give the same result.

Theorem 8. *Let M_1, M_2 be stable matchings. Let m and w be matched in M_1 but not in M_2. Then one of them prefers M_1 and another one prefers M_2.*

Proof: Denote X_1 the set of men who like M_1 better, Y_1 the set of women who like M_1 better. Also denote X_2 the set of men who like M_2 better, Y_2 the set of women who like M_2 better.

Then in M_1 there is no pair (m, w), such that $m \in X_1$, $w \in Y_1$. Otherwise such pair would block M_2. In particular $|X_1| \leqslant |Y_2|$. Similarly, in M_2 there is no pair (m, w), such that $m \in X_2$, $w \in Y_2$ because such pair would block M_1. In particular $|Y_2| \leqslant |X_1|$.

It means that $|Y_2| = |X_1|$ and all men from X_1 are matched to women in Y_2, and all men from X_2 are matched to women from Y_1. □

Corollary 9. *Let M_1, M_2 be stable matchings. Then the number of people prefering M_1 to M_2 equals the number of people prefering M_2 to M_1.*

Let M_1, M_2 be two stable matchings. We will say $M_1 \leqslant M_2$ if no man prefers M_1 to M_2. Then $M_2 \geqslant M_1$ if no woman prefers M_2 to M_1. Some remarkable properties of this ordering are described in Exercise 5.

2.2. **Some generalizations.** Suppose we have the same problem with less number of men than women. If we apply the same (men-oriented) version of the algorithm, we obtain a stable pairing. So, a stable pairing always exists. In fact, all Theorems 2-8 still hold true, cf. Exercise 8.

Theorem 10. *In all stable matchings the set of unmatched women is the same.*

Proof: Let w be unmatched in some stable matching M_1 but matched in M_0 with m. Then w is the best stable partner to m (analogously to Theorem 6), and pair (m, w) blocks M_1. □

Let us now allow people declare some partners totally unacceptable. We think that a person would rather stay single than marry an unacceptable partner. Again, in this case one can prove analogs of Theorems 2-8, cf. Exercise 9.

Theorem 11. *In all stable matchings the sets of married men and women are always the same.*

Proof: Let M_1, M_2 be two stable matchings. Let's construct a directed graph. Draw a vertex for each man and for each woman. Draw an edge from a man m to a woman w if m and w are married in M_1. Also draw an edge from a woman w to her partner in M_2. So, in each vertex, at most one edge is coming in and at most one edge is coming out.

Suppose there is a man m matched in M_1 but not matched in M_2. Then starting at m there is a path. This path cannot have loops, so it is unique and must end. Each man on this path prefers M_1 (applying the analog of Theorem 8), so it cannot end on a man. Similarly, each woman on this path prefers M_2, so the path cannot end on a woman. Contradiction. □

Some other generalization are described in Exercises 11 and 12.

2.3. **False preferences.** Suppose some group of people found out the lists of preferences submitted by others. These people want to falsify their lists of preferences in order to obtain a better partner. Curiously, one can prove

Theorem 12. *Let M_0 be the men-optimal matching with honest preference lists. Then no coalition of people can falsify their preferences in such a way that all members of the coalition get a better partner in some stable matching based on false list of preferences than in M_0.*

Suppose we have a women oriented society, meaning the women-oriented version of algorithm is used.

Theorem 13. *Men can falsify their preferences in such a way, that women oriented version of the algorithm produces the same result as men oriented version of algorithm based on honest preferences.*

Proof: Men can declare all women except (worse then) their best stable partner unacceptable. □

3. Exercises

1. Recall that the Euler constant γ is given by

$$\gamma = \lim_{n\to\infty}(1 + 1/2 + 1/3 + \cdots + 1/n - \ln n); \tag{2}$$

Use you calculator/computer to show that the Euler constant is $\gamma = 0.577216...$. What should be n in (2) to get the kth decimal digit of γ correct?

2. Recall that the number e is given by

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots \frac{1}{n!} + \ldots . \tag{3}$$

Use your calculator/computer to show that $e = 2.718281828...$. What should be n in (3) to get the kth decimal digit of e correct? Compare to Exercise 1.

3. A person meets n possible future spouses one by one in some order. S/he can either marry or reject the person s/he's just met; then meet another person. S/he can get married only once and s/he cannot marry a person which s/he has already rejected. We say s/he wins if s/he marries one of the best two choices out of all n options. Find a strategy which gives more than fifty percent chance to succeed.

4. Let F_n be the Fibonacci numbers, that is $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Prove that there exist lists of preferences for n men and n women, such that there are at least F_n stable matchings.

5. Consider a problem of stable marriage.

a) Prove that for any two stable matchings M_1, M_2, there exists a unique stable matching M, such that $M_1 \leqslant M$, $M_2 \leqslant M$ and there is no stable matching N such that $M_1 \leqslant N$, $M_2 \leqslant N$ and $N \leqslant M$. Denote $M = M_1 + M_2$.

b) Prove that for any two stable matchings M_1, M_2, there exists a unique stable matching M such that $M_1 \geqslant M$, $M_2 \geqslant M$ and there is no stable matching N such that, $M_1 \geqslant N$, $M_2 \geqslant N$ and $N \geqslant M$. Denote $M = M_1 * M_2$.

c) Let M_1, M_2, M_3 are stable matchings. Prove $M_1*(M_2+M_3) = (M_1*M_2)+(M_1*M_3)$.

d) Let M_1, M_2, M_3 are stable matchings. Prove $M_1+(M_2*M_3) = (M_1+M_2)*(M_1+M_3)$.

6. Let $n = 4$. In this case we have at most 10 stable matchings. The following lists or preferences allow exactly 10 stable matchings:

men	*women*
$1, 2, 3, 4$	$4, 3, 2, 1$
$2, 1, 4, 3$	$3, 4, 1, 2$
$3, 4, 1, 2$	$2, 1, 4, 3$
$4, 3, 2, 1$	$1, 2, 3, 4$

Find all 10 stable matchings and compute products and sums of all stable matchings.

7. Prove Theorem 12.

8. Prove Theorems 2-8 when numbers of men and women are different.

9. Prove Theorems 2-8 when unacceptable partners are allowed.

10. Consider **the** following **stupid algorithm**. It starts at a random matching. Given a matching, the stupid algorithm finds some blocking pair (m_1, w_2) (say, m_1 is matched to w_1 and w_2 is matched to m_2), and forms a new matching which has all the same pairs except for $(m_1, w_1)$and (m_2, w_2) which are changed to (m_1, w_2), (m_2, w_1). It stops if the matching is stable. Is it true that the stupid algorithm always stops?

11. **The hospital/residents problem.** Suppose there are several hospitals and several residents. Each hospital can allocate some number of residents (may be more than one and different hospitals can have different sizes). Both residents and hospitals have lists of preferences. The definition of stable matching is the same as in the stable marriage problem. Formulate and prove Theorems 2-13 adjusted to this setting.

12. **The roommate problem.** Suppose we have $2n$ people which we want to allocate to n rooms (each room is for two people). Each person has a list of preferences. The stable matching is defined as in the stable marriage problem. Find an example when no stable matching is possible.

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