1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f(x) + f(y) \neq 0$ and

$$\frac{f(x) - f(x - y)}{f(x) + f(x + y)} + \frac{f(x) - f(x + y)}{f(x) + f(x - y)} = 0$$

for all $x, y \in \mathbb{R}$.

**Solution:** Any function of the form $f(x) = c$, where $c$ is a nonzero constant, clearly satisfies the requirements; we will show these are the only solutions. Bringing the original equation to a common denominator yields $2f(x)^2 - f(x - y)^2 - f(x + y)^2 = 0$ for all $x, y$. Define $g(x) = f(x)^2$; clearly $g(x) \geq 0$ for all $x$. Our equation states that $2g(x) = g(x - y) + g(x + y)$. We will show this forces $g$ to be constant.

If $g$ is not constant, there exist some $a, b$ with $g(a) < g(b)$. We claim that $g(na - (n - 1)b) = ng(a) - (n - 1)g(b)$ for all nonnegative integers $n$. This is shown by induction: the base cases $n = 0, 1$ are clear; if $n > 1$ then our equation (with $x = (n - 1)a - (n - 2)b, y = a - b$) yields

$$2g((n - 1)a - (n - 2)b) = g(na - (n - 1)b) + g((n - 2)a - (n - 3)b)$$

$$\Rightarrow 2(n - 1)g(a) - 2(n - 2)g(b) = g(na - (n - 1)b) + (n - 2)g(a) - (n - 3)g(b)$$ (by induction hypothesis)

$$\Rightarrow g(na - (n - 1)b) = ng(a) - (n - 1)g(b)$$

and the induction step is complete. However, if $n > g(b)/(g(b) - g(a))$ then $g(na - (n - 1)b) = ng(a) - (n - 1)g(b) < 0$ and we have a contradiction. This shows that $g$ must be constant.

So $g(x) = d$ for some constant $d \geq 0$; then $f(x) = \pm \sqrt{d}$ for all $x$. If both values occur, then the $f(x) + f(y) \neq 0$ constraint is violated; hence we must have $f(x) = \sqrt{d}$ for all $x$ or $f(x) = -\sqrt{d}$ for all $x$ (and $d \neq 0$).

2. Prove that for $n \geq 1$, the last $n + 2$ digits of $11^{10^n}$ are 6000...0001, with $n$ zeros between the 6 and the final 1.

**Solution:** The statement is that $11^{10^n} = k \cdot 10^{n+2} + 6 \cdot 10^{n+1} + 1$ for some positive integer $k$; we prove it by induction on $n$. For the base case $n = 1$, we expand $(10 + 1)^{10}$ by the binomial theorem and obtain

$$(\text{terms divisible by } 10^3) + \left(\binom{10}{2}\right)10^2 + \left(\binom{10}{1}\right)10^1 + 1 = (\text{terms divisible by } 10^3) + 4500 + 100 + 1$$

whose last three digits are indeed 601. Now, for $n \geq 2$, if the statement holds for $n - 1$, then to prove it for $n$, we continue the process, using the binomial theorem twice:

$$11^{10^n} = (11^{10^{n-1}})^{10} = (k \cdot 10^{n+1} + [6 \cdot 10^n + 1])^{10}$$

$$= (\text{terms divisible by } 10^{2n+2}) + \left(\binom{10}{1}\right)(k \cdot 10^{n+1})(6 \cdot 10^n + 1)^9 + (6 \cdot 10^n + 1)^{10}$$

and all the terms except the last are divisible by $10^{n+2}$. Therefore it suffices to find the last $n + 2$ digits of $(6 \cdot 10^n + 1)^{10}$, which equals

$$(\text{terms divisible by } 10^{2n}) + \left(\binom{10}{1}\right)6 \cdot 10^n + 1 = (\text{terms divisible by } 10^{2n}) + 6 \cdot 10^{n+1} + 1.$$ 

Since $2n \geq n + 2$, we have what we need.
3. Let $A_1, B_1, C_1$ be points in the interior of sides $BC, CA, AB$, respectively, of equilateral triangle $ABC$. Prove that if the radii of the inscribed circles of $\triangle C_1AB_1, \triangle B_1CA_1, \triangle A_1BC_1, \triangle A_1B_1C_1$ are equal, then $A_1, B_1, C_1$ are the midpoints of the sides of $\triangle ABC$ on which they lie.

**Solution:** First, suppose that $BA_1 > CB_1$. We claim this forces $CB_1 > AC_1$. If we rotate triangle $A_1B_1C_1$ by $2\pi/3$ radians about the center of $\triangle ABC$, we get $\triangle A_2B_2C_2(A_2 \in CA, B_2 \in AB)$ whose incirle coincides with that of $\triangle B_1C_1A_1$ (since they have the same radius and are both tangent to $AB$ and $CA$). But by our assumption, $AB_1 > A_2B_2$; now if $AC_1 > CB_1 = AB_2$ then segment $C_1B_1$ lies outside triangle $A_2B_2C_2$ and cannot be tangent to its incircle, a contradiction. Hence $BA_1 > CB_1$ does indeed imply $CB_1 > AC_1$ and, likewise, $AC_1 > BA_1$; combining yields $BA_1 > CB_1 > AC_1 > BA_1$, impossible. We conclude that our supposition was wrong, so $BA_1 \leq CB_1$; likewise $CB_1 \leq AC_1 \leq BA_1$ and we have equality throughout. This implies (by rotational symmetry) that $\triangle A_1B_1C_1$ is equilateral, and that $A_1B_1 + A_1C_1 = BA_1 + CB_1 = AC_1$.

If the incenter of $\triangle B_1A_1C_1$ is $I$, then $\angle B_1IC_1 = (\pi + \angle B_1AC_1)/2 = 2\pi/3$ and $I$ must lie on the arc of a circle passing through $B_1, C_1$; the unique point on this arc which is at maximal distance from $B_1C_1$ - i.e. the position of $I$ which gives the largest radius for the incircle - is the midpoint of the arc, and $I$ is located there iff $\triangle B_1A_1C_1$ is equilateral. However, looking at triangle $B_1A_1C_1$ which actually is equilateral, we see that the maximum possible radius is attained there; since it has the same radius as $\triangle B_1A_1C_1$, this latter is also equilateral, and $B_1A = AC_1$. By our symmetry this yields $B_1A = AC_1 = C_1B = BA_1 = A_1C = CB_1$ as needed.

4. Let $0 < \alpha < 1$. Prove that there exists a real number $x, 0 < x < 1$, such that $\alpha^n < \{nx\}$ for every positive integer $n$. (Here $\{nx\}$ is the fractional part of $nx$.)

**Solution:** It suffices to find some real (noninteger) $x$ with this last property; we then replace $x$ by $\{x\}$ to satisfy the $0 < x < 1$ condition (note that the other condition is still satisfied, since $\{nx\} = \{n\{x\}\}$). We claim that $x = \sqrt{m^2 - 1}$ for a sufficiently large integer $m$ meets our requirements.

Note that, for all $n$ sufficiently large, $1/2n^2 > \alpha^n$. (Proof: Choose integer $s > 0$ large so that $1/s < \alpha^{-1/4} - 1$; then, for $t \geq s, 1/t \leq 1/s < \alpha^{-1/4} - 1 \Rightarrow (t + 1)/t^2 = (1 + 1/t)^2 < \alpha^{-1/2}$ and, writing $n^2/s^2 = (\frac{m}{n})^2(\frac{n}{m})^2 \cdots (\frac{m}{n})^2$ for $n > s$, we see $n^2/s^2 < \alpha^{-(n-s)/2} \Rightarrow 2n^2\alpha^n < 2s^2\alpha^{(n+s)/2}$ which is less than 1 when $n$ is large enough.) We claim that, if $n \geq m$, then $\{nx\} > 1/2n^2$. Indeed, letting $k = [nx] = \lfloor n\sqrt{m^2 - 1} \rfloor$, we have

\[
(n\sqrt{m^2 - 1} - k)(n\sqrt{m^2 - 1} + k) = n^2(m^2 - 1) - k^2 \geq 1
\]

(since it is an integer and, from the first factorization, is positive)

\[
\Rightarrow \{nx\} = n\sqrt{m^2 - 1} - k \geq 1/(n\sqrt{m^2 - 1} + k) > 1/2nm \geq 1/2n^2.
\]

These results show that, if $m$ is large enough, then $\{nx\} > 1/2n^2 > \alpha^n$ for all $n \geq m$.

Now let $s$ be chosen so that $\alpha^s < 1/3$; we claim that, if $m > \max(1, 2s/3(1 - \alpha))$, then $\{nx\} > \alpha^n$ for all $n \leq m$. Indeed, we have $m - \frac{2}{\alpha m} < \sqrt{m^2 - 1}$ (this is verified by squaring both sides), so $nm - 1 < n(m - \frac{2}{\alpha m}) < n\sqrt{m^2 - 1} < nm$ and hence $\{nx\} > n(m - \frac{2}{\alpha m}) - (nm - 1) = 1 - \frac{2}{\alpha m}$. When $s \leq n < m$, we have $1 - \frac{2}{\alpha m} > 1/3 > \alpha^n$ and our condition is met; when $n < s, 1 - \frac{2}{\alpha m} > 1 - \frac{2}{\alpha m} > \alpha > \alpha^n$ and we are again done.

Thus, by choosing $x = \sqrt{m^2 - 1}$ for $m$ sufficiently large, we can ensure $\{nx\} > \alpha^n$ for both all $n \geq m$ and $n < m$.

**Remark:** A cleaner alternative would be the following “probabilistic” solution: If one considers the unit interval $(0, 1)$, then, for each $n$, the set of $x$ for which $\{nx\} \leq \alpha^n$ is a finite union of subintervals of total length $\alpha^n$. For fixed $n_0$, the sum of all these lengths for $n \geq n_0$ is $\alpha^{n_0}/(1 - \alpha)$, which is less than $(1 - \alpha)/n_0$ when $n_0$ is large enough. Then, by pigeonhole, one can choose $x$ with $1 - (1 - \alpha)/n_0 < x < 1$ so that $\{nx\} > \alpha^n$ for all $n > n_0$; since any $x$ in this interval also works with $n \leq n_0$, existence is proven. However, this solution is difficult in full rigor, since one must know that an infinite sequence of
intervals whose total length converges to less than \( l \) cannot cover an interval of length \( l \). Fortunately, there are results in analysis (e.g. Cantor’s theorem) from which this follows.

5. Pentagon \( ABCDE \) is cyclic, i.e., inscribed in a circle. Diagonals \( AC \) and \( BD \) meet at \( P \), and diagonals \( AD \) and \( CE \) meet at \( Q \). Triangles \( ABP, AEQ, CDP \), and \( APQ \) have equal areas. Prove that the pentagon is regular.

**Counterexample:** (Thanks to Maksim Maydanskiy) The statement of the problem is false. Consider the transformation from the left figure to the right figure below. \( ABCD \) and \( ABCE \) are isosceles trapezoids, and \( ABCQ \) is a parallelogram; \( AB = CD = CQ = 1 \) and \( AE = AQ = BC = 2 \). Throughout the transformation, \( \angle ABC \) is decreased so that \( \angle AQE \), initially of measure \( \pi/4 \), approaches \( \pi/2 \). \( ABCDE \) is always cyclic (because isosceles trapezoids are cyclic), and, by properties of trapezoids and parallelograms (see below), the areas of \( \triangle ABP, CDP, APQ \) stay equal to each other. The difference between areas of \( \triangle AEQ, \triangle ABP \) is initially \( > 1 \) and ultimately \( < 0 \); by continuity it must be 0 at some point in between. At this point, all the requirements of the problem are satisfied, but \( AB = 1 \) while \( BC = 2 \), so the pentagon is not regular.

**Remark:** The problem statement is, indeed, incorrect. The following is the intended version.

Problem: Pentagon \( ABCDE \) is cyclic, i.e., inscribed in a circle. Diagonals \( AC \) and \( BD \) meet at \( P \), and diagonals \( AD \) and \( CE \) meet at \( Q \). Triangles \( ABP, AEQ, CDP, CDQ \), and \( APQ \) have equal areas. Prove that the pentagon is regular.

Solution: Adding the area of triangle \( BCP \) to those of \( \triangle ABP, \triangle CDP \), we see that \( \triangle ABC, \triangle DBC \) have equal areas, so \( AD, BC \) are parallel. Then \( ABCD \) is a cyclic trapezoid, so it is isosceles and \( AB = CD \). Similarly, \( ACDE \) is a trapezoid and \( CD = EA \). Now construct parallelogram \( ABCR \); we see that \( R \) lies on ray \( AD \) and \( \triangle ARP \) has the same area as \( \triangle ABP \) (because they have equal base \( AP \) and, by symmetry, equal altitudes). Since these properties uniquely determine \( R \), we conclude that \( R = Q \), so lines \( AB, CQ = CE \) are parallel. Then \( ABCE \) is a trapezoid, so it is isosceles and \( EA = BC \). Similarly, \( DEAP \) is shown to be a parallelogram, so \( ABDE \) is a trapezoid and \( AB = DE \). We have now shown that \( DE = AB = CD = EA = BC \), so all sides of the pentagon are equal. Since it is cyclic, all sides subtend arcs of equal measure \( \theta \); then every angle subtends an arc of measure \( 3\theta \), so each angle is \( 3\theta/2 \) and the pentagon is regular.

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